Research Article


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1. Introduction

The information that are obtained for any kind of computation have some kinds of vagueness and uncertainty. In this regard, fuzzy set (f-set) [1] and intuitionistic fuzzy set (if-set) [2] are initiated to tackle such type of informational uncertainty and vagueness. In f-set, the condition “well defined” of classical set is characterized by a membership function μf defined by a membership grade μf(u) within [0, 1] for all members u of initial universe U, whereas the if-set characterizes such condition by two functions, that is, membership function μf and nonmembership function μf defined by membership grade μf(u) and nonmembership grade μf(u), respectively, within [0, 1] for all u ∈ U. Subject to conditions that both μf(u) and μf(u) are dependent and their sum μf(u) + μf(u) must lie within [0, 1] with hesitation grade μf(u) = 1 − (μf(u) + μf(u)). Both f-set and if-set are not capable to tackle the situations in which a neutral grade along with refusal grade is required to be emphasized, so picture fuzzy set (pf-set) [3] is conceptualized to manage such scenarios. In pf-set, objective belongingness is further characterized by three membership functions, that is, μf, μf, and μf such that for all u ∈ U, their sum μf(u) + μf(u) + μf(u) and refusal grade μf(u) = 1 − μf(u) − μf(u) − μf(u) lie within [0, 1]. Recently, the uncertain complexities in different daily-life problems have been investigated and addressed by the authors [4–8] through employing various multicriteria decision-making techniques based on pf-set. In order to equip f-set, if-set and pf-set with parameterization tool, Molodtsov [9] introduced soft set (s-set) for dealing with uncertainties and vagueness. The fuzzy soft set (f s-set) [10], intuitionistic fuzzy

In various real-world scenarios, the classification of attributes into subattributive values in the form of non-overlapping sets is necessary. The s-sets and its hybrids are incompatible with such scenarios, so Smarandache [19] introduced hypersoft sets (hs-sets) to address the insufficiency of s-set-like models. In hs-set, a new approximate function, multiargument approximate function (maa-function), is employed that maps Cartesian product of attribute-valued disjoint sets to the power set of initial universe. The rudiments and elementary axioms of hs-sets have been discussed in [20] and elaborated with numerical examples. Ilhan et al. [21] discussed the validity of hs-sets for the entitlement of multidecursive opinions under expert set environment. Rahman et al. [22–24] investigated the hybridized properties of hs-sets under the environments of convexity and concavity, parameterization, and bijection. They employed decision-making algorithmic approaches to solve real-world problems. The hybridized structures of hs-set with f-set and if-set are fuzzy hypersoft set (fhs-set) [25, 26] and intuitionistic fuzzy hypersoft set (ifhs-set) [27], respectively. Saeed et al. [28] characterized the concept of hypersoft graphs and discussed its some properties.

In order to manage s-set information in graphs, Thumbakara and George [29] characterized soft graphs in 2014. As its extension, fuzzy soft graph (fis-graph) [30], intuitionistic fuzzy soft graph (ifs-graph) [31], and picture fuzzy soft graph (pfis-graph) [32] are conceptualized. In order to evaluate uncertain nature of approximate elements as a whole in f-set and ifs-set, Alkhazaleh et al. [33] and Bashir et al. [34] characterized possibility fuzzy soft set (pf -set) and possibility intuitionistic fuzzy soft set (poifs-set), respectively, by assigning possibility degree to each approximate element collectively in these structures. Akram and Shahzadi [35] developed possibility intuitionistic fuzzy soft graph (poifs-graph) and investigated some of its properties and operations.

In poifs-graph, neutral membership grade is ignored, and it lacks the consideration of maa-function. Consequently, various daily-life scenarios such as recruitment process, medical diagnosis, optimal product selection, and so on are not tackled by poifs-graph that demands a novel graphical structure to be characterized in literature. We, therefore, present a new graphical model, that is, possibility picture fuzzy hypersoft graph (popfs-graph) in this study. Its advantageous aspects are as follows: it provides due status to neutral membership degree while dealing uncertainties, and it employs maa-function to cope with the scenarios having further partitioning of attributes into their respective attribute-valued sets. The proposed graphical model is more flexible as it overcomes the shortcomings of existing relevant models for dealing with uncertainties. It assigns a possibility degree to each approximate element of its maa-function to deal with its uncertain behavior. The major contributions of the paper are outlined as follows:

(1) A novel graphical hybrid popfs-graph is characterized that is capable to cope with the following situations collectively:

(a) The situation in which the categorization of opted parameters into their respective disjoint subclasses having their relevant attributable values is necessary

(b) The situation in which the consideration of multiargument parameterization in the domain of approximate mapping is mandatory to have reliable approximation of alternatives

(c) The situation that demands a mode for the assessment of uncertain nature of approximate elements to assess the level of acceptance

(2) The novel notions of pfhs-set, popfs-set, and popfs-graph are introduced, and then some essential fundamental properties such as subgraph, spanning subgraph, strong subgraph; aggregation operations such as union, intersection, and complement; and products such as Cartesian product, cross product, lexicographic product, strong product, and composition of popfs-graph are investigated.

(3) A decision-support system is constructed based on the proposal of an algorithm that is implemented in real-life multiattributed decision-making problem for the selection of suitable candidate.

The layout of the remaining paper is as follows: some essential definitions relevant to main work are recalled from existing literature for proper understanding of proposed study in Section 2. Some fundamentals, that is, properties and set-theoretic operations of popfs-graph, are characterized with graphical representation-based examples in Section 3. Section 4 presents the analytical cum graphical exploration of some products and compositions of popfs-graphs. In Section 5, the concept of popfs-graph is applied in decision-making for HRM-scenario. The comparison analysis is presented in Section 6. Lastly, paper is summarized with more future directions in Section 7.

2. Preliminaries

This portion of the paper presents some elementary terms and definitions by reviewing the existing literature for vivid understanding of the proposed study.

In literature, Cuong [3] characterized the following concept of pf-set as a generalization of if-set [2] by introducing a new grade called neutral grade $\mu_N (\tilde{t})$ to furnish the neutrality of decision-makers.
Definition 1 (see [3]). A pf-set $P_a$ is defined as $P_a = \{ (u, < \mu_T(u), \mu_N(u), \mu_F(u)) | u \in \mathcal{U} \}$ such that $\mu_T(u), \mu_N(u), \mu_F(u)$ represent positive, neutral, and negative membership grades, respectively, of $u \in \mathcal{U}$ subject to the condition that $0 \leq \mu_T(u) + \mu_N(u) + \mu_F(u) \leq 1$ with refusal membership grade $\mu_R(u) = 1 - \mu_T(u) - \mu_N(u) - \mu_F(u)$. The collection of all pf-sets over $\mathcal{U}$ is denoted as $P_a(\mathcal{U})$.

The parameters play a key role in reliable and authentic decision-making process. The existing fuzzy set-like models are inadequate with any kind of parameterization tool; therefore, Molodtsov [9] initiated the following idea of s-set to address the limitations of predefined uncertain models with the provision of parameterization tool.

Definition 2 (see [9]). A s-set $S$ over $\mathcal{U}$ is a pair $(\psi_S, \mathfrak{V})$, where $\psi_S : \mathcal{U} \rightarrow P(\mathcal{Y})$ is an approximate function of $S$ and $\mathfrak{V} \subseteq \mathfrak{E}$ (a set of parameters). For any $u \in \mathfrak{V}$, $\psi_S(a)$ is called an approximate element of $S$. In this definition, the symbol $P(\mathcal{Y})$ is meant for power set of $\mathcal{U}$.

In 2015, Yang et al. [12] developed the following novel model, that is, pf-s-set to tackle the insufficiencies of pf-set for parameterization context. In short, they combined the theory of pf-set with Molodtsov’s theory of s-set to carry out a parameterized family of universal set with picture fuzzy setting.

Definition 3 (see [12]). A pf-s-set $P_a$ over $\mathcal{U}$ is a pair $(\psi_p, Z)$, where $\psi_p : \mathcal{U} \rightarrow P_a(\mathcal{U})$ and $Z \subseteq \mathfrak{E}$. For any $z \in Z, \psi_p(z)$ is a pf-s-subset and known as approximate element of pf-s-set $P_a$ that, $\psi_p(z)$ can be represented as pf-s-set over $\mathcal{U}$ such that $\psi_p(z) = \{ (u, < \mu_T^z(u), \mu_N^z(u), \mu_F^z(u)) | u \in \mathcal{U} \}$, where $\mu_T^z(u), \mu_N^z(u), \mu_F^z(u)$ represent positive, neutral, and negative membership grades, respectively, of $u \in \mathcal{U}$ subject to the condition that $0 \leq \mu_T^z(u) + \mu_N^z(u) + \mu_F^z(u) \leq 1$ with refusal membership grade $\mu_R^z(u) = 1 - \mu_T^z(u) - \mu_N^z(u) - \mu_F^z(u)$.

In 2018, Smarandache [19] extended the concept of s-set and developed a novel model, that is, hs-set that utilizes a novel mapping known as maa-function to deal with the shortcomings of s-set regarding the categorization of parameters into their relevant parameter valued subclasses.

Definition 4 (see [19]). An hs-set $H$ over $\mathcal{U}$ is a pair $(\mathfrak{H}, \mathfrak{D})$, where $\mathfrak{D}$ is the Cartesian product of $\mathfrak{D}^i = \{ i, i = 1, 2, 3, \ldots, n \}, \mathfrak{D}^j \cap \mathfrak{D}^l = \emptyset$ having attribute values of attributes $\tilde{h}, \breve{i} = 1, 2, 3, \ldots, n, \breve{h}, \tilde{i} \neq j$, respectively, and $\mathfrak{H} : \mathfrak{D} \rightarrow P(\mathcal{Y})$ is called approximate function (so-called maa-function) of $H$, and for all $\tilde{a} \in \mathfrak{D}, \mathfrak{H}(\tilde{a})$ is called approximate element of $H$. The hs-set over $\mathcal{U}$ is said to be fs-set and if hs-set if $\mathfrak{H} : \mathfrak{D} \rightarrow P(\mathcal{Y})$ and $\mathfrak{H} : \mathfrak{D} \rightarrow IF(\mathcal{Y})$, respectively, where $IF(\mathcal{Y})$ and $IF(\mathcal{Y})$ denote the family of all fuzzy subsets and intuitionistic fuzzy subsets, respectively.

In 2021, Chellamani et al. [32] explored the graphical notations of pf-s-sets to handle pf-s-information efficiently.

Definition 5 (see [32]). Let $M' = (\mathfrak{E}, E)$ be a simple graph with $\mathfrak{E}$ as set of vertices and $E$ as set of edges and $\mathfrak{C}$ be a nonvoid set of parameters. By a picture fuzzy soft graph (pf-s-graph), we mean a four-tuple $G = (\mathfrak{E}, G, L, M)$ with $(L, G)$ and $(M, G)$ that are pf-s-sets over $\mathfrak{E}$ and $E$, respectively; for all $e \in E, (L, G, M) (G, L, M)$ is a pf-s-graph of $G$ if $\mu_T^G(\tilde{e}, \tilde{b}_1) \leq \min \{ \mu_T^G(\tilde{b}_1), \mu_T^G(\tilde{b}_2) \}, \mu_N^G(\tilde{b}_1, \tilde{b}_2) \leq \min \{ \mu_N^G(\tilde{b}_1, \tilde{b}_2), \mu_N^G(\tilde{b}_1, \tilde{b}_2) \}$ and $\mu_F^G(\tilde{b}_1, \tilde{b}_2) \leq \min \{ \mu_F^G(\tilde{b}_1), \mu_F^G(\tilde{b}_2) \}$ such that $0 \leq \mu_T^G(\tilde{b}_1, \tilde{b}_2) + \mu_N^G(\tilde{b}_1, \tilde{b}_2) + \mu_F^G(\tilde{b}_1, \tilde{b}_2) \leq 1$ for all $\tilde{b}_1, \tilde{b}_2 \in \mathfrak{E}$.

3. Possibility Picture Fuzzy Hypersoft Graphs

This portion presents the characterization of popfhs-graph along with some essential properties and results. First, we present the definitions of picture fuzzy hypersoft set $pfhs$-set and possibility picture fuzzy hypersoft set $popfhs$-set, which are missing in existing literature.

Definition 6. A pfhs-set $H_p$ over $\mathcal{U}$ is a pair $(\pi_{hs}, Z)$, where $\pi_{hs} : Z \rightarrow P_a(\mathcal{U})$ and $Z = \Pi^*_i Z_i$, where $Z_i (i = 1, 2, \ldots, n)$ is nonoverlapping attribute-valued sets with respect to distinct attributes. For any $z \in Z, \pi_{hs}(z)$ is a pfhs-set and known as approximate element of $pfhs$-set $H_p$, that is, $\pi_{hs}(z)$ can be represented as $pfhs$-sets over $\mathcal{U}$ such that $\pi_{hs}(z) = \{ (u, < \mu_T^z(u), \mu_N^z(u), \mu_F^z(u)) | u \in \mathcal{U} \}$.

Definition 7. A possibility picture fuzzy hypersoft set (popfhs-set) $H_p$ over $\mathcal{U}$ is stated as $H_p = \{ (z, \pi_{hs}(z)) | z \in Z \}$, where $\pi_{hs}(z)$ is a pfhs-set as defined in Definition 6 and $\delta : \mathcal{U} \rightarrow [0, 1]$ with $\delta(z)$ is the possibility degree of $z$ to $H_p$. The collection of all $popfhs$-sets over $\mathcal{U}$ is represented by $\Omega_{popfhs}(\mathcal{U})$.

Throughout the remaining paper, $\mathfrak{A} = (\mathfrak{E}, \mathfrak{G})$ is a simple graph with $\mathfrak{E}$ as set of vertices and $\mathfrak{G}$ as set of edges; $E$ denotes set of parameters; and $\mathfrak{Q}$ denotes the disjoint sets containing subparametric values for distinct parameters $\breve{\theta}_i, i = 1, 2, \ldots, n$ of $E$. Also, $\mathfrak{Q} = \mathfrak{Q}^1 \times \mathfrak{Q}^2 \times \mathfrak{Q}^3 \times \cdots \times \mathfrak{Q}^n$.

Definition 8. A popfhs-graph is a four-tuple $A = (\mathfrak{A}', \mathfrak{Q}, J, K)$, where $J : \mathfrak{A}' \rightarrow P_a(\mathfrak{E}), K : \mathfrak{A}' \rightarrow P_a(\mathfrak{E} \times \mathfrak{E})$ given by $J(a) = \mathfrak{J}_a = \{ (\tilde{v}_1, \tilde{v}_2), \tilde{v}_1 \subseteq \mathfrak{E} \} \mathfrak{J}_a(a) \subseteq \mathfrak{E} \} \times \mathfrak{E} \} \mathfrak{J}_a(a) \subseteq \mathfrak{E} \} \times \mathfrak{E} \}$ and $K(a) = \mathfrak{K}_a = \{ (\tilde{v}_1, \tilde{v}_2), \tilde{v}_1 \subseteq \mathfrak{E} \} \mathfrak{K}_a(a) \subseteq \mathfrak{E} \} \times \mathfrak{E} \} \mathfrak{K}_a(a) \subseteq \mathfrak{E} \} \times \mathfrak{E} \}$.

popfhs-sets over $\mathfrak{E}$ and $\mathfrak{E} \times \mathfrak{E}$ with $\mathfrak{J}_a(a) \subseteq \mathfrak{E} \} \mathfrak{K}_a(a) \subseteq \mathfrak{E} \} \times \mathfrak{E} \} \mathfrak{J}_a(a) \subseteq \mathfrak{E} \} \mathfrak{K}_a(a) \subseteq \mathfrak{E} \} \times \mathfrak{E} \}$.

Note: The collection of all popfhs-graphs is represented by $\Omega_{popfhs}(\mathfrak{E})$. 
Example 1. Let \( \mathbf{H} = (\mathbf{B}, \mathbf{C}) \) be a simple graph with \( \mathbf{B} = \{v_1, v_2, v_3\} \) and \( \mathbf{Q} = \{q_1, q_2\} \), \( \mathbf{Q}_2 = \{q_1, q_2\} \), and \( \mathbf{Q}_3 = \{q_1, q_2, q_3\} \) such that \( \mathbf{Q} = \mathbf{Q}_1 \times \mathbf{Q}_2 \times \mathbf{Q}_3 \). Let \( \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \) be the geometrical depiction, respectively.

**Definition 9.** A popfhs-graph \( \mathbf{G} = (\mathbf{H}, \mathbf{Q}, J, K) \) is called a popfhs-subgraph of \( \mathbf{H} = (\mathbf{H}^*, \mathbf{Q}^*, J^1, K^1) \) if

\[
\begin{align*}
(1) & \quad \mathbf{Q}^* \subseteq \mathbf{Q} \\
(2) & \quad J^1 \subseteq J \\
(3) & \quad K^1 \subseteq K
\end{align*}
\]

Example 2. Repeating Example 1 with \( \mathbf{Q}_1 = \{a_1, a_2\} \), \( \mathbf{Q}_2 = \{a_1\} \), and \( \mathbf{Q}_3 = \{a_1\} \) we have a new popfhs-graph \( \mathbf{G} = (\mathbf{H}^*, \mathbf{Q}^*, J^1, K^1) \) which is a subgraph of the popfhs-graph given in Example 1.

**Definition 10.** A popfhs-subgraph \( \mathbf{G} = (\mathbf{H}^*, \mathbf{Q}^*, J^1, K^1) \) is called a popfhs-subgraph of popfhs-graph \( \mathbf{G} = (\mathbf{H}, \mathbf{Q}, J, K) \) when

\[
J^1(\mathbf{G}) = J^1(\mathbf{G}) \forall \mathbf{G} \in \mathbf{H}, \sigma \in \mathbf{Q}.
\]

**Definition 11.** A popfhs-subgraph \( \mathbf{G} = (\mathbf{H}^*, \mathbf{Q}^*, J^1, K^1) \) is called a strong popfhs-subgraph (SPFHS-subgraph) of popfhs-graph \( \mathbf{G} = (\mathbf{H}, \mathbf{Q}, J, K) \) when

\[
J^1(\mathbf{G}) = J^1(\mathbf{G}) \forall \mathbf{G} \in \mathbf{H}, \sigma \in \mathbf{Q}.
\]

### 3.1. Set-Theoretic Operations of popfhs-Graphs

This segment of the paper investigates the basic set theoretic operations of popfhs-graphs along with their geometrical interpretations.

**Definition 12.** The union of two popfhs-graphs \( \mathbf{H}_1 = (\mathbf{H}_1^*, \mathbf{Q}_1^*, J^1, K^1) \) and \( \mathbf{H}_2 = (\mathbf{H}_2^*, \mathbf{Q}_2^*, J^2, K^2) \), denoted by \( \mathbf{H}_1 \cup \mathbf{H}_2 \), is a popfhs-graph \( \mathbf{H} = (\mathbf{H}, \mathbf{Q}, J, K) \) such that \( \mathbf{Q} = \mathbf{Q}_1 \cup \mathbf{Q}_2 \). In this graph,

\[
\text{Table 1: Numerical computation of Example 1 with (a) } \mathbf{P}_1(\sigma_1), \text{ (b) } \mathbf{P}_2(\sigma_2), \text{ (c) } \mathbf{P}_3(\sigma_3), \text{ and (d) } \mathbf{P}_4(\sigma_4)
\]

<table>
<thead>
<tr>
<th>J</th>
<th>( \tilde{v}_1 )</th>
<th>( \tilde{v}_2 )</th>
<th>( \tilde{v}_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 )</td>
<td>(0.2, 0.1, 0.3, 0.2)</td>
<td>(0.1, 0.3, 0.2, 0.3)</td>
<td>(0.0, 1.0)</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>(0.1, 0.3, 0.2, 0.3)</td>
<td>(0.2, 0.4, 0.1, 0.1)</td>
<td>(0.0, 1.0)</td>
</tr>
<tr>
<td>( \sigma_3 )</td>
<td>(0.1, 0.2, 0.2, 0.3)</td>
<td>(0.3, 0.1, 0.2, 0.2)</td>
<td>(0.1, 0.2, 0.3, 0.1)</td>
</tr>
<tr>
<td>( \sigma_4 )</td>
<td>(0.2, 0.2, 0.1, 0.3)</td>
<td>(0.3, 0.2, 0.1, 0.4)</td>
<td>(0.1, 0.3, 0.2, 0.2)</td>
</tr>
</tbody>
</table>

Also, the popfhs-components for \( K \) are given as follows:
Table 2: Tabular notation of Example 2 with (a) $P_F(\sigma_1)$, (b) $P_F(\sigma_2)$, and (c) $P_F(\sigma_3)$.

<table>
<thead>
<tr>
<th>$\mathcal{K}$</th>
<th>$\mathcal{V}_1$</th>
<th>$\mathcal{V}_2$</th>
<th>$\mathcal{V}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>(0.1, 0.3, 0.1, 0.1)</td>
<td>(0.2, 0.2, 0.1, 0.2)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>(0.1, 0.2, 0.1, 0.1)</td>
<td>(0.2, 0.1, 0.3, 0.1)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>$\sigma_3$</td>
<td>(0.1, 0.4, 0.1, 0.2)</td>
<td>(0.2, 0.2, 0.1, 0.1)</td>
<td>(0.2, 0.1, 0.2, 0.1)</td>
</tr>
</tbody>
</table>

Figure 1: Geometrical interpretation of Table 1.

Figure 2: Graphical representation of Table 2.

$$T_{K_1}(\mathcal{V}_1) = \begin{cases} 
if \sigma \in Q^1 - Q^2 \land (\mathcal{V}_1, \mathcal{V}_2) \in (\mathcal{B}_1 \times \mathcal{B}_2) - (\mathcal{B}_2 \times \mathcal{B}_1) \\
K_{\mathcal{V}_1}(\mathcal{V}_1) & \text{or if } \sigma \in Q^1 - Q^3 \land (\mathcal{V}_1, \mathcal{V}_2) \in (\mathcal{B}_1 \times \mathcal{B}_1) \cap (\mathcal{B}_2 \times \mathcal{B}_2) \\
K_{\mathcal{V}_1}(\mathcal{V}_1) & \text{or if } \sigma \in Q^1 \cap Q^2 \land (\mathcal{V}_1, \mathcal{V}_2) \in (\mathcal{B}_1 \times \mathcal{B}_1) - (\mathcal{B}_2 \times \mathcal{B}_2), \\
K_{\mathcal{V}_1}(\mathcal{V}_1) & \text{or if } \sigma \in Q^2 - Q^1 \land (\mathcal{V}_1, \mathcal{V}_2) \in (\mathcal{B}_2 \times \mathcal{B}_2) - (\mathcal{B}_1 \times \mathcal{B}_1) \\
K_{\mathcal{V}_1}(\mathcal{V}_1) & \text{or if } \sigma \in Q^2 \land Q^3 \land (\mathcal{V}_1, \mathcal{V}_2) \in (\mathcal{B}_2 \times \mathcal{B}_2) \cap (\mathcal{B}_1 \times \mathcal{B}_1) \\
\max\{K_{\mathcal{V}_1}(\mathcal{V}_1), K_{\mathcal{V}_1}(\mathcal{V}_1)\} & \text{if } \sigma \in Q^1 \cap Q^2 \land (\mathcal{V}_1, \mathcal{V}_2) \in (\mathcal{B}_1 \times \mathcal{B}_1) \cap (\mathcal{B}_2 \times \mathcal{B}_2), \\
0, \text{ otherwise,} 
\end{cases}$$
Example 3. Let \( \mathcal{A}_1 = (\mathcal{A}^1_1, \Omega_1, \mathcal{J}_1, \mathcal{K}_1) \) be a pofhs-graph where \( \mathcal{A}^1_1 = (B_1, C_1) \) with \( B_1 = \{ v_1, v_2, v_3 \} \) and \( C_1 = \{ a_{11} \} \). 
\( \Omega_1 = \{ \alpha_1 \} \), and \( \mathcal{J}_1 = \{ v_1, v_2, v_3 \} \) such that \( \Omega_1 = \Omega_1 \times \Omega_2 \times \Omega_3 = \{ \alpha_1, \alpha_2, \alpha_3 \} \) and \( \mathcal{J}_1 = \mathcal{J}_1 \times \mathcal{J}_2 \times \mathcal{J}_3 = \{ \mathcal{J}_1 \} \). 
\( \mathcal{K}_1 = \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3 = \{ \mathcal{K}_1 \} \). 
\( \mathbb{V}_k (v_i, v_j) = 0, \mathbb{L}_k (v_i, v_j) = 1, \mathbb{G}_k (v_i, v_j) = 1, \mathbb{F}_k (v_i, v_j) = 0, \mathbb{M}_k (v_i, v_j) = 1, \mathbb{K}_k (v_i, v_j) = 1 \). 
Its tabulation is given in Table 3. Also, let \( \mathcal{A}_2 = (\mathcal{A}^2_1, \Omega_2, \mathcal{J}_2, \mathcal{K}_2) \) be a pofhs-graph, where \( \mathcal{A}^2_1 = (B_2, C_1) \) with \( B_2 = \{ v_1, v_2, v_3 \} \) and \( C_1 = \{ a_{11} \} \). 
\( \Omega_2 = \{ \alpha_1 \} \), and \( \mathcal{J}_2 = \mathcal{J}_2 \times \mathcal{J}_2 \times \mathcal{J}_2 = \{ \mathcal{J}_2 \} \). 
\( \mathcal{K}_2 = \mathcal{K}_2 \times \mathcal{K}_2 \times \mathcal{K}_2 = \{ \mathcal{K}_2 \} \). 
\( \mathbb{V}_k (v_i, v_j) = 0, \mathbb{L}_k (v_i, v_j) = 0, \mathbb{G}_k (v_i, v_j) = 0, \mathbb{F}_k (v_i, v_j) = 0, \mathbb{M}_k (v_i, v_j) = 0, \mathbb{K}_k (v_i, v_j) = 0 \). 
Its tabulation is given in Table 3. Now let \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \) with \( \Omega = \Omega_1 \cup \Omega_2 \) and \( \mathbb{V}_k (v_i, v_j) = 0, \mathbb{L}_k (v_i, v_j) = 0, \mathbb{G}_k (v_i, v_j) = 0, \mathbb{F}_k (v_i, v_j) = 0, \mathbb{M}_k (v_i, v_j) = 0, \mathbb{K}_k (v_i, v_j) = 0 \). 
Its tabulation is given in Table 5. The graphical interpretations of Tables 3–5 are presented in Figures 3–5, respectively.
Table 3: Tabulation of Example 3.

<table>
<thead>
<tr>
<th>(\mathcal{J})</th>
<th>(\tilde{\mathcal{V}}_1)</th>
<th>(\tilde{\mathcal{V}}_2)</th>
<th>(\tilde{\mathcal{V}}_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_1)</td>
<td>(0.2, 0.1, 0.2, 0.2)</td>
<td>(0.3, 0.1, 0.1, 0.3)</td>
<td>(0.2, 0.2, 0.1, 0.3)</td>
</tr>
<tr>
<td>(\sigma_2)</td>
<td>(0.2, 0.3, 0.1, 0.2)</td>
<td>(0.2, 0.2, 0.1, 0.5)</td>
<td>(0.1, 0.2, 0.1, 0.5)</td>
</tr>
<tr>
<td>(\sigma_3)</td>
<td>(0.2, 0.1, 0.1, 0.6)</td>
<td>(0.1, 0.1, 0.4, 0.4)</td>
<td>(0.1, 0.2, 0.1, 0.7)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\mathcal{K})</th>
<th>(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)</th>
<th>(\tilde{\mathcal{V}}_2, \tilde{\mathcal{V}}_3)</th>
<th>(\tilde{\mathcal{V}}_3, \tilde{\mathcal{V}}_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_1)</td>
<td>(0.2, 0.3, 0.2, 0.2)</td>
<td>(0.2, 0.1, 0.3, 0.2)</td>
<td>(0.2, 0.1, 0.2, 0.2)</td>
</tr>
<tr>
<td>(\sigma_2)</td>
<td>(0.2, 0.1, 0.3, 0.2)</td>
<td>(0.2, 0.2, 0.1, 0.2)</td>
<td>(0.2, 0.3, 0.1, 0.2)</td>
</tr>
<tr>
<td>(\sigma_3)</td>
<td>(0.0, 0.1, 0.0)</td>
<td>(0.1, 0.2, 0.1, 0.3)</td>
<td>(0.1, 0.2, 0.1, 0.2)</td>
</tr>
</tbody>
</table>

Table 4: Tabulation of popfs-graph \(\mathfrak{M}_2 = (\mathfrak{M}_2, \mathfrak{X}^2, \mathfrak{Y}^2, \mathfrak{Z}^2)\) according to Example 3.

<table>
<thead>
<tr>
<th>(\mathcal{J})</th>
<th>(\tilde{\mathcal{V}}_3)</th>
<th>(\tilde{\mathcal{V}}_4)</th>
<th>(\tilde{\mathcal{V}}_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_2)</td>
<td>(0.2, 0.1, 0.2, 0.3)</td>
<td>(0.2, 0.1, 0.2, 0.2)</td>
<td>(0.1, 0.2, 0.1, 0.5)</td>
</tr>
<tr>
<td>(\sigma_4)</td>
<td>(0.2, 0.1, 0.1, 0.6)</td>
<td>(0.1, 0.2, 0.1, 0.4)</td>
<td>(0.2, 0.1, 0.2, 0.4)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\mathcal{K})</th>
<th>(\tilde{\mathcal{V}}_3, \tilde{\mathcal{V}}_4)</th>
<th>(\tilde{\mathcal{V}}_4, \tilde{\mathcal{V}}_5)</th>
<th>(\tilde{\mathcal{V}}_3, \tilde{\mathcal{V}}_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_2)</td>
<td>(0.2, 0.3, 0.1, 0.2)</td>
<td>(0.3, 0.1, 0.2, 0.3)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>(\sigma_4)</td>
<td>(0.2, 0.1, 0.3, 0.2)</td>
<td>(0.1, 0.3, 0.2, 0.3)</td>
<td>(0.1, 0.2, 0.2, 0.3)</td>
</tr>
</tbody>
</table>

Table 5: Tabulation of \(\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2\).

<table>
<thead>
<tr>
<th>(\mathcal{J})</th>
<th>(\tilde{\mathcal{V}}_1)</th>
<th>(\tilde{\mathcal{V}}_2)</th>
<th>(\tilde{\mathcal{V}}_3)</th>
<th>(\tilde{\mathcal{V}}_4)</th>
<th>(\tilde{\mathcal{V}}_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_1)</td>
<td>(0.2, 0.1, 0.2, 0.2)</td>
<td>(0.3, 0.1, 0.1, 0.3)</td>
<td>(0.2, 0.2, 0.1, 0.3)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>(\sigma_2)</td>
<td>(0.2, 0.3, 0.1, 0.2)</td>
<td>(0.2, 0.2, 0.1, 0.5)</td>
<td>(0.2, 0.1, 0.2, 0.2)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>(\sigma_3)</td>
<td>(0.2, 0.1, 0.1, 0.6)</td>
<td>(0.1, 0.1, 0.4, 0.4)</td>
<td>(0.1, 0.2, 0.1, 0.7)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>(\sigma_4)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 1.0)</td>
<td>(0.2, 0.1, 0.1, 0.6)</td>
<td>(0.1, 0.2, 0.1, 0.4)</td>
<td>(0.2, 0.1, 0.2, 0.4)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\mathcal{K})</th>
<th>(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)</th>
<th>(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_3)</th>
<th>(\tilde{\mathcal{V}}_2, \tilde{\mathcal{V}}_3)</th>
<th>(\tilde{\mathcal{V}}_3, \tilde{\mathcal{V}}_4)</th>
<th>(\tilde{\mathcal{V}}_3, \tilde{\mathcal{V}}_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_1)</td>
<td>(0.2, 0.3, 0.2, 0.2)</td>
<td>(0.2, 0.1, 0.2, 0.2)</td>
<td>(0.2, 0.1, 0.3, 0.2)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>(\sigma_2)</td>
<td>(0.2, 0.3, 0.2, 0.2)</td>
<td>(0.2, 0.2, 0.1, 0.2)</td>
<td>(0.2, 0.3, 0.1, 0.2)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>(\sigma_3)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.1, 0.2, 0.1, 0.2)</td>
<td>(0.1, 0.2, 0.1, 0.3)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 0.1, 0)</td>
</tr>
<tr>
<td>(\sigma_4)</td>
<td>(0.0, 0.1, 0)</td>
<td>(0.0, 1.0)</td>
<td>(0.2, 0.1, 0.3, 0.2)</td>
<td>(0.1, 0.2, 0.2, 0.3)</td>
<td>(0.1, 0.3, 0.2, 0.3)</td>
</tr>
</tbody>
</table>

Figure 3: Graphical representation of Table 3.

Figure 4: Geometrical depiction of Table 4.
Theorem 2. If $\mathcal{A}_1, \mathcal{A}_2 \in \Omega_{PFISHG}$ then $\mathcal{A}_1 \cap \mathcal{A}_2 \in \Omega_{PFISHG}$.

Proof. It can easily be proved with the help of axioms used in Definition 13. Therefore, its proof is omitted. $\square$

Example 4. Let $\mathcal{A}_1 = (\mathcal{W}_1, \mathcal{Q}_1, \mathcal{J}_1, \mathcal{K}_1)$ be a pophs-graph where $\mathcal{W}_1 = (\mathcal{B}_1, \mathcal{G}_1)$ with $\mathcal{B}_1 = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ and $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are subparametric nonoverlapping sets with respect to distinct attributes $\alpha_1, \alpha_2, \alpha_3$, where $\mathcal{Q}_1 = \{\alpha_{11}\}$, $\mathcal{Q}_2 = \{\alpha_{21}\}$, and $\mathcal{Q}_3 = \{\alpha_{31}, \alpha_{32}\}$. $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 = \{\alpha_1, \alpha_2\}$ and $T_{K_1}(\bar{v}_1, \bar{v}_2) = 0, \mu_{K_1}(\bar{v}_1, \bar{v}_2) = 0, \mu_{K_1}(\bar{v}_1, \bar{v}_2) = 1$, and $(\forall (\bar{v}_1, \bar{v}_2) \in \mathcal{B}_1 \times \mathcal{B}_1)$. Table 6 and Figure 6 elaborate its tabulation and geometrical depiction, respectively. Also, let $\mathcal{A}_2 = (\mathcal{W}_2, \mathcal{Q}_2, \mathcal{J}_2, \mathcal{K}_2)$ be a pophs-graph where $\mathcal{W}_2 = (\mathcal{B}_2, \mathcal{G}_2)$ with $\mathcal{B}_2 = \{\bar{v}_3, \bar{v}_4, \bar{v}_5\}$ and $\mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4$ are subparametric nonoverlapping sets with respect to distinct attributes $\alpha_2, \alpha_3, \alpha_4$, where $\mathcal{Q}_2 = \{\alpha_{21}\}$, $\mathcal{Q}_3 = \{\alpha_{31}, \alpha_{32}\}$, and $\mathcal{Q}_4 = \{\alpha_1, \alpha_2\}$. $\mathcal{Q} = \mathcal{Q}_2 \times \mathcal{Q}_3 \times \mathcal{Q}_4 = \{\alpha_2, \alpha_3\}$ and $\mu_{K_2}(\bar{v}_3, \bar{v}_4) = 0, \mu_{K_2}(\bar{v}_3, \bar{v}_4) = 0, \mu_{K_2}(\bar{v}_3, \bar{v}_4) = 1$, and $(\forall (\bar{v}_3, \bar{v}_4) \in \mathcal{B}_2 \times \mathcal{B}_2)$. Figure 6 and Figure 7 elaborate its tabulation and geometrical depiction, respectively.
Table 6: Tabulation of pophs-graph $\mathfrak{A}_1 = (\mathfrak{G}_1, \mathfrak{Q}_1, J_1, K_1)$ for Example 4.

<table>
<thead>
<tr>
<th>$J$</th>
<th>$\tilde{v}_1$</th>
<th>$\tilde{v}_2$</th>
<th>$\tilde{v}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>$\langle 0, 1, 2, 0, 2 \rangle$</td>
<td>$\langle 1, 0, 2, 1, 0, 3 \rangle$</td>
<td>$\langle 2, 0, 1, 0, 2, 0, 2 \rangle$</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$\langle 0, 1, 0, 3, 0, 3 \rangle$</td>
<td>$\langle 0, 1, 2, 0, 1, 0, 5 \rangle$</td>
<td>$\langle 0, 2, 0, 2, 0, 0, 4, 0, 4 \rangle$</td>
</tr>
</tbody>
</table>

$K = (\tilde{v}_1, \tilde{v}_2)$

|$\sigma_1$ | $\langle 0, 2, 0, 2, 0, 1, 0, 2 \rangle$ | $\langle 0, 2, 0, 1, 0, 2, 0, 2 \rangle$ | $\langle 0, 2, 0, 1, 0, 3, 0, 2 \rangle$ |
| $\sigma_2$ | $\langle 0, 2, 0, 1, 0, 3, 0, 3 \rangle$ | $\langle 0, 2, 0, 1, 0, 3, 0, 4 \rangle$ | $\langle 0, 3, 0, 1, 0, 2, 0, 3 \rangle$ |

Figure 6: Geometrical interpretation of Table 6.

Its tabulation and pictorial representation are provided in Table 7 and Figure 7, respectively. Consider $\mathfrak{A} = \mathfrak{A}_1 \cap \mathfrak{A}_2$ with $\mathfrak{Q} = \mathfrak{Q}_1 \cap \mathfrak{Q}_2$. Its tabulation and geometrical interpretation are stated in Table 8 and Figure 8, respectively.

**Definition 14.** The compliment $\mathfrak{K} = (\mathfrak{K}^*, \mathfrak{Q}, J, K)$ of SFPFHS-subgraph $\mathfrak{M} = (\mathfrak{M}^*, \mathfrak{Q}, J, K)$ with $K_{\tilde{v}_1, \tilde{v}_2} = J_{\tilde{v}_1} \cap J_{\tilde{v}_2}$ is

\[
\begin{align*}
(1) \quad & \mathfrak{K}^* = \mathfrak{Q}^* \\
(2) \quad & \text{T}_{\mathfrak{K}}(\tilde{v}) = 0, 0, \text{otherwise if } \mu_{\tilde{v}_1}(\tilde{v}) = 0, 0, \text{otherwise if } \mu_{\tilde{v}_2}(\tilde{v}) = 0, 0, \text{otherwise if } \mu_{\tilde{v}_1, \tilde{v}_2}(\tilde{v}) = 0, 0, \text{otherwise if } \mu_{\tilde{v}_1}(\tilde{v}) = 0, 0, \text{otherwise.}
\end{align*}
\]

Figure 7: Graphical representation of Table 7.

### 4. Some Products and Composition of pophs-graphs

In this section, we discussed some products and composition of pophs-graphs with the help of graphical representation and numerical examples.

**Definition 15.** For two pophs-graphs, $\mathfrak{A}_1 = (\mathfrak{G}_1^*, \mathfrak{Q}_1^*, J_1^*, K_1^*)$ and $\mathfrak{A}_2 = (\mathfrak{G}_2^*, \mathfrak{Q}_2^*, J_2^*, K_2^*)$ with respect to $\mathfrak{G}_1^* = (\mathfrak{B}_1, \mathfrak{C}_1)$ and $\mathfrak{G}_2^* = (\mathfrak{B}_2, \mathfrak{C}_2)$. Let $\mathfrak{A}_1 \times \mathfrak{A}_2$ where $\mathfrak{A} = (J, K, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ and $(J = J_1 \times J_2, K = K_1 \times K_2, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is pophs-set over $\mathfrak{G} = \mathfrak{B}_1 \times \mathfrak{B}_2$, $K = (K_1 \times K_2, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is pophs-set over $\mathfrak{G} = (\mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{B}_2, \mathfrak{C}_2)$, and $K = (J, K, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is pophs-graphs where

\[
\begin{align*}
(1) \quad & \text{T}_{\mathfrak{A}(\tilde{v})}(\tilde{v}) = \text{T}_{\mathfrak{A}_1}(\tilde{v}) \land \text{T}_{\mathfrak{A}_2}(\tilde{v}) \land \text{T}_{\mathfrak{B}(\tilde{v})}(\tilde{v}) = I_{\mathfrak{A}_1}(\tilde{v}) \land I_{\mathfrak{A}_2}(\tilde{v}) \land I_{\mathfrak{B}(\tilde{v})}(\tilde{v}) = 0, 0, \text{otherwise}
\end{align*}
\]

Figure 8: Graphical representation of Table 8.

Example 5. Let $\mathfrak{G}_1^* = (\mathfrak{B}_1, \mathfrak{C}_1)$ be a simple graph with $\mathfrak{B}_1 = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ and $\mathfrak{C}_1 = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$, and $\mathfrak{Q}_1 = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$, and $\mathfrak{Q}_2 = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$, and $\mathfrak{Q}_3 = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$. Let $\mathfrak{G}_1^* = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3$. The $\mathfrak{A}_1 = (\mathfrak{G}_1^*, \mathfrak{Q}_1^*, J_1^*, K_1^*)$. $\mathfrak{A}_2 = (\mathfrak{G}_2^*, \mathfrak{Q}_2^*, J_2^*, K_2^*)$.
Table 9: pophfs-graph $\mathcal{A}^1 = (\mathfrak{A}'^1, \mathfrak{C}'^1, J^1, K^1)$ (Example 5).

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$\pi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>(0.3,0.2,0.3)</td>
<td>(0.2,0.1,0.3,0.5)</td>
<td>(0.2,0.3,0.3,0.5)</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>(0.2,0.1,0.3,0.4)</td>
<td>(0.2,0.3,0.1,0.6)</td>
<td>(0.2,0.3,0.3,0.5)</td>
</tr>
</tbody>
</table>

$\mathfrak{A}'^1$ is a pophfs-graph, which is stated in Table 9.

Let $\mathfrak{A}'^1 = (\mathfrak{B}'^1, \mathfrak{C}'^1)$ be a simple graph with $\mathfrak{B}'^1 = \{\tau_1, \tau_2, \tau_3\}$ and $\mathfrak{C}'^1 = \{\tau_1, \tau_2, \tau_3, \tau_4\}$, where $\mathfrak{B}'^1 = \{\alpha_1, \alpha_2, \alpha_3\}$, $\mathfrak{C}'^1 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$, and $\mathfrak{K}'^1 = \{\gamma_1, \gamma_2, \gamma_3\}$. Let $\mathfrak{A}'^1 = (\mathfrak{B}'^1, \mathfrak{C}'^1)$ be a cross product $\mathfrak{A}'^1 = (\mathfrak{B}'^1, \mathfrak{C}'^1)$ over $\mathfrak{K}'^1 = \{\gamma_1, \gamma_2, \gamma_3\}$.

**Proof.** It can easily be proved by following the axiomatic concepts provided in Definition 15 and Example 5. Therefore, we have omitted its proof.

**Definition 16.** For two pophfs-graphs, $\mathfrak{A}'^1 = (\mathfrak{X}'^1, \mathfrak{Y}'^1, J^1, K^1)$ and $\mathfrak{A}'^2 = (\mathfrak{X}'^2, \mathfrak{Y}'^2, J^2, K^2)$ with respect to $\mathfrak{K}'^1 = (\mathfrak{X}'^1, \mathfrak{Y}'^1)$ and $\mathfrak{K}'^2 = (\mathfrak{X}'^2, \mathfrak{Y}'^2)$.

**Theorem 4.** The cross product of two pophfs-graphs is pophfs-graph.

**Proof.** It can easily be proved by following the axiomatic concepts provided in Definition 16. Therefore, we have omitted its proof.

**Definition 17.** For two pophfs-graphs, $\mathfrak{A}'^1 = (\mathfrak{X}'^1, \mathfrak{Y}'^1, J^1, K^1)$ and $\mathfrak{A}'^2 = (\mathfrak{X}'^2, \mathfrak{Y}'^2, J^2, K^2)$ with respect to $\mathfrak{K}'^1 = (\mathfrak{X}'^1, \mathfrak{Y}'^1)$ and $\mathfrak{K}'^2 = (\mathfrak{X}'^2, \mathfrak{Y}'^2)$.

**Theorem 3.** The Cartesian product of two pophfs-graphs is pophfs-graph.

$\mathfrak{K}' = (\mathfrak{K}'^1 \ominus \mathfrak{K}'^2, \mathfrak{X}'^1 \times \mathfrak{X}'^2)$ is a pophfs-graphs where

$\mathfrak{A}'^1 \ominus \mathfrak{K}' = (\mathfrak{X}'^1, \mathfrak{Y}'^1, J^1, \mathfrak{K}'^1)$ and $\mathfrak{A}'^2 \ominus \mathfrak{K}' = (\mathfrak{X}'^2, \mathfrak{Y}'^2, J^2, \mathfrak{K}'^2)$ are pophfs-graphs.

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 16. Therefore, we have omitted its proof.

**Theorem 1.** The cross product of two pophfs-graphs is pophfs-graph.

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 16. Therefore, we have omitted its proof.

**Theorem 2.** The cross product of two pophfs-graphs is pophfs-graph.

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 16. Therefore, we have omitted its proof.

$\mathfrak{K}' = (\mathfrak{K}'^1 \ominus \mathfrak{K}'^2, \mathfrak{X}'^1 \times \mathfrak{X}'^2)$ is a pophfs-graphs where

$\mathfrak{A}'^1 \ominus \mathfrak{K}'^1 = (\mathfrak{X}'^1, \mathfrak{Y}'^1, J^1, \mathfrak{K}'^1)$ and $\mathfrak{A}'^2 \ominus \mathfrak{K}'^2 = (\mathfrak{X}'^2, \mathfrak{Y}'^2, J^2, \mathfrak{K}'^2)$ are pophfs-graphs.

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 16. Therefore, we have omitted its proof.
Table 10: pophs-graph $\mathfrak{G}^2 = (\mathfrak{X}, \mathfrak{Q}^2, \mathfrak{J}, \mathfrak{K}^2)$ (Example 5).

<table>
<thead>
<tr>
<th>$\mathfrak{K}$</th>
<th>$\tilde{\tau}_1$</th>
<th>$\tilde{\tau}_2$</th>
<th>$\tilde{\tau}_3$</th>
<th>$\tilde{\tau}_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\omega}_3$</td>
<td>(0.3, 0.2, 0.2, 0.5)</td>
<td>(0.2, 0.1, 0.2, 0.4)</td>
<td>(0.3, 0.1, 0.1, 0.4)</td>
<td>(0.3, 0.1, 0.2, 0.6)</td>
</tr>
<tr>
<td>$\tilde{\omega}_4$</td>
<td>(0.1, 0.1, 0.3, 0.5)</td>
<td>(0.3, 0.2, 0.2, 0.7)</td>
<td>(0.1, 0.2, 0.1, 0.5)</td>
<td>(0.2, 0.3, 0.1, 0.8)</td>
</tr>
</tbody>
</table>

Table 11: pophs-graph $\mathbb{W}(\tilde{\omega}_1, \tilde{\omega}_3) = \mathbb{W}_1(\tilde{\omega}_1) \times_\mathbb{P} \mathbb{W}_2(\tilde{\omega}_3)$ of $\mathfrak{G} = \mathfrak{X} \times_\mathbb{P} \mathfrak{G}^2$ (Example 5).

<table>
<thead>
<tr>
<th>$\mathfrak{K}$</th>
<th>$\tilde{\eta}_{11}$</th>
<th>$\tilde{\eta}_{12}$</th>
<th>$\tilde{\eta}_{13}$</th>
<th>$\tilde{\eta}_{14}$</th>
<th>$\tilde{\eta}_{21}$</th>
<th>$\tilde{\eta}_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\omega}_1$</td>
<td>(0.3, 0.2, 0.2, 0.3)</td>
<td>(0.2, 0.1, 0.2, 0.3)</td>
<td>(0.3, 0.1, 0.2, 0.3)</td>
<td>(0.3, 0.1, 0.1, 0.3)</td>
<td>(0.2, 0.1, 0.3, 0.5)</td>
<td>(0.2, 0.1, 0.3, 0.5)</td>
</tr>
<tr>
<td>$\tilde{\omega}_3$</td>
<td>(0.2, 0.1, 0.3, 0.4)</td>
<td>(0.2, 0.1, 0.3, 0.5)</td>
<td>(0.2, 0.2, 0.3, 0.5)</td>
<td>(0.2, 0.1, 0.3, 0.4)</td>
<td>(0.2, 0.1, 0.3, 0.4)</td>
<td>(0.2, 0.1, 0.3, 0.5)</td>
</tr>
</tbody>
</table>

Figure 9: Graphical representation of Table 9 with (a) $\mathbb{W}(\tilde{\omega}_1)$ and (b) $\mathbb{W}(\tilde{\omega}_3)$.

Figure 10: Geometrical depiction of Table 10 with (a) $\mathbb{W}(\tilde{\omega}_1)$ and (b) $\mathbb{W}(\tilde{\omega}_3)$.

Figure 11: Geometrical interpretation of $\mathbb{W}(\tilde{\omega}_2, \tilde{\omega}_3) = \mathbb{W}_1(\tilde{\omega}_1) \times_\mathbb{P} \mathbb{W}_2(\tilde{\omega}_3)$. 
Here, $\forall (\bar{\pi}, \bar{\tau}) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ and $\mathcal{W}(y, \zeta) = \mathcal{W}_1(y) \otimes_p \mathcal{W}_2(\zeta)$ are pophs-graphs of $\mathcal{A}$.

Theorem 5. The lexicographical product of two pophs-graphs is pophs-graph.

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 17. Therefore, we have omitted its proof. □

$$
\mathcal{C} = \{(\bar{\pi}, \bar{\tau}) | \bar{\pi} \in \mathcal{B}_1, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_1, \bar{\tau} \in \mathcal{B}_2 \}
$$

and $\mathbb{K} = (\mathcal{K}_1 \times \mathcal{K}_2, \mathcal{Q}_1 \times \mathcal{Q}_2)$ are pophs-graphs where

(1) $T_{J(y, C)}(\bar{\pi}, \bar{\tau}) = T_{J_y(y)}(\bar{\pi}) \wedge$
$T_{J_y(y)}(\bar{\pi}) \wedge T_{J_y(y)}(\bar{\tau}) = I_{J_y(y)}(\bar{\pi}) \wedge$
$I_{J_y(y)}(\bar{\pi}) \wedge I_{J_y(y)}(\bar{\tau}) = F_{J_y(y)}(\bar{\tau})$
$(\bar{\pi}) \wedge F_{J_y(y)}(\bar{\tau}) = F_{J_y(y)}(\bar{\pi})$
$(\bar{\pi}) \wedge F_{J_y(y)}(\bar{\tau}) = \mu_{J_y(y)}(\bar{\tau})$
$\mu_{J_y(y)}(\bar{\tau}) \wedge (\bar{\pi}, \bar{\tau}) \in \mathcal{B}_1, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_1, \bar{\tau} \in \mathcal{B}_2$

(2) $T_{K(\bar{\pi})}(\bar{\pi}, \bar{\tau}_2) = T_{K(\bar{\pi})}(\bar{\pi}) \wedge$
$T_{K(\bar{\pi})}(\bar{\pi}) \wedge T_{K(\bar{\pi})}(\bar{\tau}_2) = \mu_{K(\bar{\pi})}(\bar{\tau}_2)$
$(\bar{\pi}) \wedge \mu_{K(\bar{\pi})}(\bar{\tau}_2) \in \mathcal{E}_2, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_2$

(3) $T_{K(\bar{\pi})}(\bar{\pi}, \bar{\tau}_2) = T_{K(\bar{\pi})}(\bar{\pi}) \wedge$
$T_{K(\bar{\pi})}(\bar{\pi}) \wedge T_{K(\bar{\pi})}(\bar{\tau}_2) = \mu_{K(\bar{\pi})}(\bar{\tau}_2)$
$(\bar{\pi}) \wedge \mu_{K(\bar{\pi})}(\bar{\tau}_2) \in \mathcal{E}_2, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_2$

Definition 18. For two pophs-graphs, $\mathcal{A}_1 = (\mathcal{A}_1, \mathcal{Q}_1, J_1, K_1)$ and $\mathcal{A}_2 = (\mathcal{A}_2, \mathcal{Q}_2, J_2, K_2)$ with respect to $\mathcal{A}_1 = (\mathcal{B}_1, \mathcal{E}_1)$ and $\mathcal{A}_2 = (\mathcal{B}_2, \mathcal{E}_2)$. Let $\mathcal{A} = \mathcal{A}_1 \otimes_p \mathcal{A}_2$ be strong product of $\mathcal{A}_1$ and $\mathcal{A}_2$, where $\mathcal{A} = (J_1, \mathcal{Q}_1 \times \mathcal{Q}_2)$ is pophs-set over $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, $\mathbb{K} = (\mathcal{K}_1 \times \mathcal{K}_2, \mathcal{Q}_1 \times \mathcal{Q}_2)$ is pophs-set over

$$
\mathcal{C} = \{(\bar{\pi}, \bar{\tau}) | \bar{\pi} \in \mathcal{B}_1, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_1, \bar{\tau} \in \mathcal{B}_2 \}
$$

and $\mathbb{K} = (\mathcal{K}_1 \times \mathcal{K}_2, \mathcal{Q}_1 \times \mathcal{Q}_2)$ is pophs-graphs where

(1) $T_{J(y, C)}(\bar{\pi}, \bar{\tau}) = T_{J_y(y)}(\bar{\pi}) \wedge$
$T_{J_y(y)}(\bar{\pi}) \wedge T_{J_y(y)}(\bar{\tau}) = I_{J_y(y)}(\bar{\pi}) \wedge$
$I_{J_y(y)}(\bar{\pi}) \wedge I_{J_y(y)}(\bar{\tau}) = F_{J_y(y)}(\bar{\tau})$
$(\bar{\pi}) \wedge F_{J_y(y)}(\bar{\tau}) = F_{J_y(y)}(\bar{\pi})$
$(\bar{\pi}) \wedge F_{J_y(y)}(\bar{\tau}) = \mu_{J_y(y)}(\bar{\tau})$
$\mu_{J_y(y)}(\bar{\tau}) \wedge (\bar{\pi}, \bar{\tau}) \in \mathcal{B}_1, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_1, \bar{\tau} \in \mathcal{B}_2$

(2) $T_{K(\bar{\pi})}(\bar{\pi}, \bar{\tau}_2) = T_{K(\bar{\pi})}(\bar{\pi}) \wedge$
$T_{K(\bar{\pi})}(\bar{\pi}) \wedge T_{K(\bar{\pi})}(\bar{\tau}_2) = \mu_{K(\bar{\pi})}(\bar{\tau}_2)$
$(\bar{\pi}) \wedge \mu_{K(\bar{\pi})}(\bar{\tau}_2) \in \mathcal{E}_2, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_2$

(3) $T_{K(\bar{\pi})}(\bar{\pi}, \bar{\tau}_2) = T_{K(\bar{\pi})}(\bar{\pi}) \wedge$
$T_{K(\bar{\pi})}(\bar{\pi}) \wedge T_{K(\bar{\pi})}(\bar{\tau}_2) = \mu_{K(\bar{\pi})}(\bar{\tau}_2)$
$(\bar{\pi}) \wedge \mu_{K(\bar{\pi})}(\bar{\tau}_2) \in \mathcal{E}_2, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_2$

(4) $T_{K(\bar{\pi})}(\bar{\pi}, \bar{\tau}_2) = T_{K(\bar{\pi})}(\bar{\pi}) \wedge$
$T_{K(\bar{\pi})}(\bar{\pi}) \wedge T_{K(\bar{\pi})}(\bar{\tau}_2) = \mu_{K(\bar{\pi})}(\bar{\tau}_2)$
$(\bar{\pi}) \wedge \mu_{K(\bar{\pi})}(\bar{\tau}_2) \in \mathcal{E}_2, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_2$

Here, $\forall (\bar{\pi}, \bar{\tau}) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ and $\mathcal{W}(y, \zeta) = \mathcal{W}_1(y) \otimes_p \mathcal{W}_2(\zeta)$ are pophs-graphs of $\mathcal{A}$.

Theorem 6. The strong product of two pophs-graphs is pophs-graph.

Definition 19. For two pophs-graphs, $\mathcal{A}_1 = (\mathcal{A}_1^s, \mathcal{Q}_1^s, J_1^s, K_1^s)$ and $\mathcal{A}_2 = (\mathcal{A}_2^s, \mathcal{Q}_2^s, J_2^s, K_2^s)$ with respect to $\mathcal{A}_1^s = (\mathcal{B}_1, \mathcal{E}_1)$ and $\mathcal{A}_2^s = (\mathcal{B}_2, \mathcal{E}_2)$. Let $\mathcal{A} = \mathcal{A}_1 \otimes^s \mathcal{A}_2$ be composition of $\mathcal{A}_1^s$ and $\mathcal{A}_2^s$, where $\mathcal{A} = (J_1^s, \mathcal{Q}_1^s \times \mathcal{Q}_2^s)$ is pophs-set over $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, $\mathbb{K} = (\mathcal{K}_1 \times \mathcal{K}_2, \mathcal{Q}_1 \times \mathcal{Q}_2)$ is pophs-set over

$$
\mathcal{C} = \{(\bar{\pi}, \bar{\tau}) | \bar{\pi} \in \mathcal{B}_1, (\bar{\tau}_1, \bar{\tau}_2) \in \mathcal{E}_1, \bar{\tau} \in \mathcal{B}_2 \}
$$
Theorem 7. The composition of two popf-graphs is popf-graph.

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 19. Therefore, we have omitted its proof. \qed

Definition 20. For two popf-graphs, \( \mathcal{A}^1 = (\mathcal{A}^1, \mathcal{Q}^1, \mathcal{J}^1, \mathcal{K}^1) \) and \( \mathcal{A}^2 = (\mathcal{A}^2, \mathcal{Q}^2, \mathcal{J}^2, \mathcal{K}^2) \) with respect to \( \mathcal{A}^{1*} = (\mathcal{B}_1, \mathcal{C}_1) \) and \( \mathcal{A}^{2*} = (\mathcal{B}_2, \mathcal{C}_2) \). Let \( \mathcal{A} = \mathcal{A}^1 \cap \mathcal{A}^2 \) be the intersection of \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \), where \( \mathcal{A} = (\mathcal{J}, \mathcal{K}, \mathcal{Q}^1 \cup \mathcal{Q}^2) \) is popf-graph set over \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \) and \( \mathcal{K} = (\mathcal{K}^1 \cup \mathcal{K}^2, \mathcal{Q}^1 \cup \mathcal{Q}^2) \) is popf-graph set over \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \), where for \( \tilde{\pi}, \tilde{\tau} \in \mathcal{B} \), popf-components can be given by

\[
\begin{align*}
T_{\tilde{\pi}}(\tilde{\tau}) &= \begin{cases} 
T_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
T_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
T_{\tilde{\pi}'}(\tilde{\tau}) \cup T_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases} \\
I_{\tilde{\pi}}(\tilde{\tau}) &= \begin{cases} 
I_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
I_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
I_{\tilde{\pi}'}(\tilde{\tau}) \cup I_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases} \\
F_{\tilde{\pi}}(\tilde{\tau}) &= \begin{cases} 
F_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
F_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
F_{\tilde{\pi}'}(\tilde{\tau}) \cup F_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases} \\
\mu_{\tilde{\pi}}(\tilde{\tau}) &= \begin{cases} 
\mu_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
\mu_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
\mu_{\tilde{\pi}'}(\tilde{\tau}) \cup \mu_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases}
\end{align*}
\]

Definition 21. For two popf-graphs, \( \mathcal{A}^1 = (\mathcal{A}^{11}, \mathcal{Q}^1, \mathcal{J}^1, \mathcal{K}^1) \) and \( \mathcal{A}^2 = (\mathcal{A}^{21}, \mathcal{Q}^2, \mathcal{J}^2, \mathcal{K}^2) \) with respect to \( \mathcal{A}^{1*} = (\mathcal{B}_1, \mathcal{C}_1) \) and \( \mathcal{A}^{2*} = (\mathcal{B}_2, \mathcal{C}_2) \). Let \( \mathcal{A} = \mathcal{A}^1 \cup \mathcal{A}^2 \) be the union of \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \), where \( \mathcal{A} = (\mathcal{J}, \mathcal{K}, \mathcal{Q}^1 \cup \mathcal{Q}^2) \) is popf-graph set over \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \) and \( \mathcal{K} = (\mathcal{K}^1 \cup \mathcal{K}^2, \mathcal{Q}^1 \cup \mathcal{Q}^2) \) is popf-graph set over \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \), where for \( \tilde{\pi}, \tilde{\tau} \in \mathcal{B} \), popf-components can be given by

\[
\begin{align*}
\epsilon_{T_{\tilde{\pi}}(\tilde{\tau})} &= \begin{cases} 
T_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
T_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
T_{\tilde{\pi}'}(\tilde{\tau}) \cup T_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases} \\
\epsilon_{I_{\tilde{\pi}}(\tilde{\tau})} &= \begin{cases} 
I_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
I_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
I_{\tilde{\pi}'}(\tilde{\tau}) \cup I_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases} \\
\epsilon_{F_{\tilde{\pi}}(\tilde{\tau})} &= \begin{cases} 
F_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
F_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
F_{\tilde{\pi}'}(\tilde{\tau}) \cup F_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases} \\
\epsilon_{\mu_{\tilde{\pi}}(\tilde{\tau})} &= \begin{cases} 
\mu_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cup \mathcal{Q}^2 \\
\mu_{\tilde{\pi}'}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2 \\
\mu_{\tilde{\pi}'}(\tilde{\tau}) \cup \mu_{\tilde{\pi}''}(\tilde{\tau}) & \tilde{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2
\end{cases}
\end{align*}
\]
Start
Input
(1) Input $B$ and $Q$ as universal set and set of parametric tuples, respectively.
(2) Input popfhs-sets $\{I, Q, J\}$ and $\{K, \bar{Q}\}$.

Construction
(3) Construct popfhs-graph $\mathbf{A} = (\mathbf{A}', \mathcal{Q}, J, K)$.
(4) Construct resultant popfhs-graph for $\bar{\omega} = \land \omega$, $\forall$ values of $\kappa$.
(5) Construct popfhs-graph $\mathbf{A}(\bar{\omega})$ along with $I$-Matrix.

Computation
(6) Calculate score $S_x$ of $C_x$ if $\kappa$ and its average by the formula $S_x = T_x + I_x - F_x + \mu_x + 1/4$.
(7) Opt $C_x$ if $C_x = \max C_x$.
(8) Opt any one of $C_x$ if $\kappa$ bears multiple values.

End

Algorithm 1: Optimal selection of candidate by using popfhs-graph.

when $\bar{\omega} \in \mathcal{Q}^1 \cap \mathcal{Q}^2$ and $\bar{\pi}\bar{\tau} \in \mathcal{E}$, and uncertain parts are as follows:

\[
\begin{align*}
\mathbb{T}_{K'K''K'}(\bar{\omega}) &= \min \{\mathbb{T}_{J'K''K''}(\bar{\pi}\bar{\tau}), \mathbb{T}_{J'K'K''}(\bar{\pi}\bar{\tau})\}, \\
\mathbb{I}_{K'K''K''}(\bar{\omega}) &= \min \{\mathbb{I}_{J'K''K''}(\bar{\pi}\bar{\tau}), \mathbb{I}_{J'K'K''}(\bar{\pi}\bar{\tau})\}, \\
\mathbb{F}_{K'K''K''}(\bar{\omega}) &= \min \{\mathbb{F}_{J'K''K''}(\bar{\pi}\bar{\tau}), \mathbb{F}_{J'K'K''}(\bar{\pi}\bar{\tau})\}, \\
\mu_{K'K''K''}(\bar{\omega}) &= \min \{\mu_{J'K''K''}(\bar{\pi}\bar{\tau}), \mu_{J'K'K''}(\bar{\pi}\bar{\tau})\}.
\end{align*}
\]

Definition 23. The complement $\mathbf{A}^c = (\mathbf{A}'^c, \mathcal{Q}^c, J^c, K^c)$ of popfhs-graph $\mathbf{A} = (\mathbf{A}', \mathcal{Q}, J, K)$ is a popfhs-graph for which $\bar{\pi}, \bar{\tau} \in \mathcal{B}$ and $\bar{\omega} \in \mathcal{Q}$, and it satisfies the following conditions:

| (1) $\mathcal{Q}^c = \mathcal{Q}$ |
| (2) $J^c(\bar{\omega}) = J(\bar{\omega})$ |
| (3) $\mathbb{T}_{K'K''K'}^c(\bar{\pi}, \bar{\tau}) = \mathbb{T}_{J'K''K''}(\bar{\pi}) \land \mathbb{T}_{J'K'K''}(\bar{\tau}) - \mathbb{T}_{K'K''K''}(\bar{\pi}, \bar{\tau})$ |
| (4) $\mathbb{I}_{K'K''K''}^c(\bar{\pi}, \bar{\tau}) = \mathbb{I}_{J'K''K''}(\bar{\pi}) \land \mathbb{I}_{J'K'K''}(\bar{\tau}) - \mathbb{I}_{K'K''K''}(\bar{\pi}, \bar{\tau})$ |
| (5) $\mathbb{F}_{K'K''K''}^c(\bar{\pi}, \bar{\tau}) = \mathbb{F}_{J'K''K''}(\bar{\pi}) \land \mathbb{F}_{J'K'K''}(\bar{\tau}) - \mathbb{F}_{K'K''K''}(\bar{\pi}, \bar{\tau})$ |
| (6) $\mu_{K'K''K''}^c(\bar{\pi}, \bar{\tau}) = \mu_{J'K''K''}(\bar{\pi}) \lor \mu_{J'K'K''}(\bar{\tau}) - \mu_{K'K''K''}(\bar{\pi}, \bar{\tau})$ |

Definition 24. If $\mathbf{A}^c = \mathbf{A}$ where $\mathbf{A} = (\mathbf{A'}, \mathcal{Q}, J, K)$ is a popfhs-graph, then $\mathbf{A}$ is self-complementary.

Definition 25. If $K(\bar{\omega})$ is popfhs-graph of $\mathbf{A}$, $\forall \bar{\omega} \in \mathcal{Q}$, then $\mathbf{A}$ is complete, and one can write
Table 12: popths-graph Ξ = (W', Q, J, K).

<table>
<thead>
<tr>
<th>J</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
<th>C₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>(0.2, 0.1, 0.3, 0.4)</td>
<td>(0.3, 0.1, 0.1, 0.3)</td>
<td>(0.2, 0.1, 0.3, 0.5)</td>
<td>(0.1, 0.2, 0.2, 0.6)</td>
<td>(0.2, 0.1, 0.1, 0.5)</td>
<td>(0.2, 0.1, 0.2, 0.3)</td>
</tr>
<tr>
<td>a₂</td>
<td>(0.1, 0.1, 0.3, 0.7)</td>
<td>(0.2, 0.3, 0.2, 0.7)</td>
<td>(0.1, 0.3, 0.2, 0.3)</td>
<td>(0.3, 0.1, 0.3, 0.8)</td>
<td>(0.0, 0.0, 0)</td>
<td>(0.2, 0.3, 0.1, 0.7)</td>
</tr>
<tr>
<td>a₃</td>
<td>(0.3, 0.4, 0.1, 0.7)</td>
<td>(0.2, 0.3, 0.1, 0.6)</td>
<td>(0.1, 0.1, 0.3, 0.5)</td>
<td>(0.2, 0.1, 0.2, 0.6)</td>
<td>(0.3, 0.2, 0.1, 0.7)</td>
<td>(0.2, 0.1, 0.3, 0.5)</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccccccc}
\text{Table 13: Score values with choice values.} \\
\hline
& C₁ & C₂ & C₃ & C₄ & C₅ & C₆ & C₇ \\
\hline
\text{C₁} & 0.250 & 0.300 & 0.225 & 0.175 & 0.200 & 0.250 & 1.400 \\
\text{C₂} & 0.300 & 0.250 & 0.200 & 0.325 & 0.225 & 0.225 & 1.525 \\
\text{C₃} & 0.225 & 0.200 & 0.250 & 0.150 & 0.200 & 0.325 & 1.400 \\
\text{C₄} & 0.175 & 0.325 & 0.150 & 0.250 & 0.175 & 0.250 & 1.325 \\
\text{C₅} & 0.200 & 0.225 & 0.200 & 0.175 & 0.250 & 0.150 & 1.200 \\
\text{C₆} & 0.250 & 0.325 & 0.325 & 0.250 & 0.150 & 0.250 & 1.450 \\
\hline
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{Table 14: Advantageous aspects of proposed model over relevant existing models.} \\
\hline
\text{Models} & \text{Grade of true membership} & \text{Grade of false membership} & \text{Grade of neutrality} & \text{Mapping with single argument} & \text{Mapping with multiargument} & \text{Entitlement of possibility degree} & \text{Project selection ranking} \\
\hline
\text{fs-graph [30]} & ✓ & × & × & ✓ & × & × & × \\
\text{i fs-graph [31]} & ✓ & ✓ & × & ✓ & × & ✓ & × \\
\text{pf s-graph [32]} & ✓ & ✓ & ✓ & ✓ & × & ✓ & × \\
\text{po f s-set [33]} & ✓ & × & × & ✓ & × & ✓ & × \\
\text{pof s-set [34]} & ✓ & ✓ & × & ✓ & ✓ & ✓ & ✓ \\
\text{po f s-graph [35]} & ✓ & ✓ & × & ✓ & ✓ & ✓ & ✓ \\
\text{Proposed model} & ✓ & ✓ & ✓ & ✓ & ✓ & ✓ & ✓ \\
\hline
\end{array}
\]
\[ \mu_{K(\bar{\omega})}(\bar{\pi}, \bar{\tau}) = \begin{cases} 
\min\{\mu_{J(\bar{\omega})}(\bar{\pi}, \bar{\tau}), \mu_{J(\bar{\omega})}(\bar{\pi}, \bar{\tau})\} : \mu_{K(\bar{\omega})}(\bar{\pi}, \bar{\tau}) = 0 \\
0 : \mu_{K(\bar{\omega})}(\bar{\pi}, \bar{\tau}) > 0 
\end{cases} \]

Theorem 10. The complement \( \mathcal{A}^c = (\mathcal{A}^c, \mathcal{Q}^c, \mathcal{J}^c, \mathcal{K}^c) \) of strong popfhs-graph \( \mathcal{A} = (\mathcal{A}, \mathcal{Q}, \mathcal{J}, \mathcal{K}) \) is popfhs-graph.

Theorem 11. If \( \mathcal{A} = (\mathcal{A}, \mathcal{Q}, \mathcal{J}, \mathcal{K}) \) and its complement \( \mathcal{A}^c = (\mathcal{A}^c, \mathcal{Q}^c, \mathcal{J}^c, \mathcal{K}^c) \) are strong popfhs-graphs \( \forall \bar{\omega} \in \mathcal{A}, \bar{\pi}, \bar{\tau} \in \mathcal{B} \), then \( \mathcal{A} \cup \mathcal{A}^c \) is complete popfhs-graph.

5. Application in Decision-Making

In order to validate the proposed study, an algorithm-based application is discussed for reliable decision-making process (The brief graphical description of steps involved in Algorithm 1 is presented in Figure 12).

Example 6. Suppose an organization intends to recruit a candidate to fill a vacant post of assistant manager. Six candidates, that is, \( \mathcal{B} = \{c_1, c_2, c_3, c_4, c_5, c_6\} \), have been scrutinized by recruitment committee. The committee further requires evaluation to select one of these candidates. The evaluation indicators are qualification \( (\beta_1) \), relevant experience \( (\beta_2) \), and computer skill \( (\beta_3) \). Their subparametric disjoint sets are \( \mathcal{Q}_1 = \{b_{11}\}, \mathcal{Q}_2 = \{b_{21}, b_{22}, b_{23}\}, \) and \( \mathcal{Q}_3 = \{b_{31}\} \), respectively, such that \( \mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 = \{\bar{c}_1, \bar{c}_2, \bar{c}_3\} \) and \( \mathcal{A} = \{(W, Q) \} \{W(\bar{c}_1), W(\bar{c}_2), W(\bar{c}_3)\} \) are popfhs-graph. This selection is accomplished by proposing an algorithm (i.e. Algorithm 1 which is presented in Figure 1).

The graphical explanation of Table 12 is provided in Figure 13.

The popfhs-graphs \( W(\bar{c}_1), W(\bar{c}_2), \) and \( W(\bar{c}_3) \) with respect to subparametric values are given in Table 12 and stated in Figure 2. The 1-Matrices of popfhs-graphs are

\[
W(\bar{w}_i) = \begin{pmatrix}
(0,0,0,0) & (0,2,0,0,1) & (0,0,0,0) & (0,3,2,0,3) & (0,0,0,0) & (0,0,0,0) \\
(0,2,0,0,1) & (0,0,0,0) & (0,0,0,0) & (0,3,2,0,2) & (0,3,1,0,1) & (0,0,0,0) \\
(0,0,0,0) & (0,3,1,0,3) & (0,0,0,0) & (0,0,0,0) & (0,2,0,1,0) & (0,4,0,0,0) \\
(0,3,0,0,3) & (0,2,0,0,2) & (0,0,0,0) & (0,0,0,0) & (0,1,0,2,0) & (0,0,0,0) \\
(0,0,0,0) & (0,3,0,1,0,3) & (0,2,0,1,0,2) & (0,1,0,2,3,0,4) & (0,0,0,0) & (0,3,0,1,0,3) \\
(0,0,0,0) & (0,0,0,0) & (0,2,0,3,0,2) & (0,0,0,0) & (0,3,0,1,0,3) & (0,0,0,0)
\end{pmatrix}
\]
7. Conclusions

In this research, authors have managed the real-world decision-making situation that demands: (i) the categorization of opted parameters into their respective disjoint subclasses having their relevant attributive values, (ii) the consideration of multiargument parameterization in the domain of approximate mapping to have reliable approximation of alternatives, and (iii) a mode for the assessment of uncertain nature of approximate elements to have level of acceptance, collectively with the development of novel context of popfhs-graph. As a conceptual framework, essential rudiments, operations, and products are characterized with the help of elaborated instances. A real-world decision-making problem from human resource management for the optimal selection of candidate is resolved with the proposal of an intelligent algorithm. Since the sum of values of uncertain components in proposed model is considered within \([0,1]\), but it is not sufficient for the situations where decision-makers provide their opinions as uncertain values whose sum exceeds 1; therefore, future work may include the extension of this work to possibility neutrosophic hypersoft setting to deal with above describe limitation. Moreover, many other notions of classical graph may also be characterized by utilizing the proposed context.

Data Availability

This study has no associated data.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

References