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Research Article

The Analytical Solutions for the Stochastic-Fractional Broer–Kaup Equations

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We take into account here the stochastic-fractional Broer–Kaup equations (SFBKEs) perturbed by the multiplicative Wiener process. To get rational, hyperbolic, and elliptic stochastic solutions for SBKEs, we utilize the Jacobi elliptic function method. The derived solutions are significantly more useful and effective in comprehending various important challenging physical phenomena due to the important of SFBKEs in describing the propagation of shallow water waves. Also, we use the MATLAB Package to create 2D and 3D graphs for certain solutions of SFBKEs in order to discuss the impact of fractional order and the Wiener process on the solutions of SFBKEs.

1. Introduction

Partial differential equations (PDEs) have become increasingly popular due to their wide range of applications in nonlinear science such as engineering [1], civil engineering [2], quantum mechanics [3], thermoelasticity [4], soil mechanics [5], statistical mechanics [6], population ecology [7, 8], economics [9], and biology [10, 11]. As a result, it is vital to find accurate solutions in order to have a better understanding of nonlinear phenomena. Many approaches, such as Darboux transformation [12], Hirota's function [13], sine-cosine [14, 15], (G'/G)-expansion [16–18], perturbation [19, 20], Riccati-Bernoulli sub-ODE [21], $\exp(-\phi(\varsigma))$-expansion [22], tanh-sech [23, 24], and the Jacobi elliptic function [25, 26], have been employed to determine analytical solutions of these equations.

Until the 1950s, deterministic models of differential equations were commonly utilized in various applications to describe system dynamics. However, it is obvious that most natural physical phenomena are not deterministic. On the other hand, fractional derivatives is used to describe a variety of physical phenomena in engineering disciplines, mathematical biology, signal processing, electromagnetic theory and various scientific studies. Because fractional order integrals and derivatives allow for the representation of different substances' memory and heredity properties, these new fractional-order models are better suited than the earlier utilized integer-order models. Fractional-order models have the most significant advantage in compared to integer-order models, where such effects are ignored.

As a result, when modeling these phenomena, we need to take into account certain random fluctuations. To achieve a better level of qualitative agreement, we investigate the stochastic-fractional BroerKaup equations (SFBKEs).

$$\sigma U dW + \left[2U\mathbb{D}_x^\alpha U + \mathbb{D}_x^\alpha V\right]dt = \sigma\ dU\ W \text{ and} \tag{1}$$

$$dv + \left[\mathbb{D}_x^\alpha(\mathrm{UV}) + \mathbb{D}_x^\alpha U + \mathbb{D}_{xxx}^\alpha U\right]dt = \sigma V\ d\mathscr{W}, \tag{2}$$

where the field of horizontal velocity is denoted by $U(x,t)$, $V(x,t)$ is the height that deviate from equilibrium position of liquid, $\mathbb{D}^\alpha$ is the conformable derivative (CD) [27], $\mathscr{W}(t)$ is a standard Wiener process (SWP), and σ is the noise strength.

The Broer–Kaup equations (BKEs) (1-2), with $\sigma = 0$, is used to model the bidirectional propagation of long waves in shallow water [28]. The shallow water equations describe the motion of water bodies wherein the depth is short relative to the scale of the waves propagating on that body and are derived from the depth-averaged Navier–Stokes equations [29]. Due to the significance of BKEs, many researchers have developed analytical solutions for this system utilizing various approaches such as improved (G'/G)-expansion [30], the bifurcation method and qualitative theory of dynamical systems [31], $\exp(-\phi(\varsigma))$-expansion [32], (G'/G)-expansion method [33, 34], sine-cosine [35], He's variational principle [36], and first integral method [37]. While the exact stochastic solutions of SFBKEs (1-2) have never been investigated before.

The Jacobi elliptic function approach is used to get a wide range of solutions, including elliptic, hyperbolic, and rational functions. This is the first investigation to get exact solutions to SFBKEs with combination of a stochastic term and fractional derivative. Furthermore, we use MATLAB to create 2D and 3D diagrams for a few of the solutions of the SFBKEs (1-2) produced in this work to show how the SWP effects on these solutions.

The following is the format of this paper: In Section 2, to establish the wave equation for SFBKEs (1-2), we apply a powerful wave transformation. In Section 3, the analytic stochastic-fractional solutions of SFBKEs (1-2) is created using the Jacobi elliptic function approach. While, the influence of the SWP and the fractional order on the obtained solutions is investigated in Section 4. The paper's conclusion is provided in Section 5.

2. Wave Equation for SFBKEs

The following wave transformation is applied:

$$\begin{aligned} U(x,t) &= u(\xi)e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)},\\ V(x,t) &= v(\xi)e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)},\\ \xi &= \frac{1}{\alpha}x^{\alpha}+\lambda t. \end{aligned} \tag{3}$$

To get the wave equation of SFBKEs (1-2), where λ is a constant and u and v are deterministic functions. Putting (3) into equations (1) and (2) and using the following equations:

$$\begin{aligned} dU &= [\lambda u' dt + \sigma u\, d\mathscr{W}]e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)},\\ dV &= [\lambda v' dt + \sigma v\, d\mathscr{W}]e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)},\\ \mathbb{D}_x^{\alpha}U &= u' e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)}, \mathbb{D}_x^{\alpha}V = v' e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)},\\ \mathbb{D}_{xxx}^{\alpha}U &= u''' e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)}, \mathbb{D}_x^{\alpha}(\mathrm{UV}) = (uv)' e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)}, \end{aligned} \tag{4}$$

we attain the following equations:

$$\lambda u' + 2uu' e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)} + v' = 0 \text{ and} \tag{5}$$

$$\lambda v' + (uv)' e^{\left(\sigma\mathscr{W}(t)-1/2\sigma^2 t\right)} + u' + u''' = 0. \tag{6}$$

Taking expectation $\mathbb{E}(\cdot)$ for equations (5) and (6), we get the following equations:

$$\lambda u' + 2uu' e^{-1/2\sigma^2 t}\mathbb{E}\left(e^{\sigma\mathscr{W}(t)}\right) + v' = 0 \text{ and} \tag{7}$$

$$\lambda v' + (uv)' e^{-1/2\sigma^2 t}\mathbb{E}\left(e^{\sigma\mathscr{W}(t)}\right) + u' + u''' = 0. \tag{8}$$

Since $\mathscr{W}(t)$ is a normal distribution, then $\mathbb{E}(e^{\sigma\mathscr{W}(t)}) = e^{\sigma^2/2t}$. Now equations (7) and (8) take the type of the following equations:

$$\lambda u' + 2uu' + v' = 0 \text{ and} \tag{9}$$

$$\lambda v' + (uv)' + u' + u''' = 0. \tag{10}$$

Integrating equations (9) and (10) and putting the constants of integration equal zero, we get the following equations:

$$v = -\lambda u - u^2 \text{ and} \tag{11}$$

$$\lambda v + (uv) + u + u'' = 0. \tag{12}$$

Plugging equations (11) into (12), we obtain the following equation:

$$u'' - u^3 - 2\lambda u^2 - \left(\lambda^2 - 1\right)u = 0. \tag{13}$$

3. Exact Solutions of SFBKEs

We are using the Jacobi elliptic functions approach [38] to get the solutions to equation (13). As a consequence, we can find the exact solution of SFBKEs (1-2).

3.1. Jacobi Elliptic Functions Method. We assume that the solutions to equation (13) is as follows:

$$u(\xi) = \sum_{i=1}^{M} a_i H^i, \tag{14}$$

where H solves

$$H' = \sqrt{\frac{1}{2}pH^4 + qH^2 + r}, \tag{15}$$

where p, q, and r are real parameters.

We notice that equation (15) has a variety of solutions depending on p, q, and r (See Table 1). where $sn(\xi) = sn(\xi,m), cn(\xi) = cn(\xi,m)$, and $dn(\xi,m) = dn(\xi,m)$ are the Jacobi elliptic functions (JEFs) for $0 < m < 1$. When $m \longrightarrow 1$, the JEFs are converted into the hyperbolic functions shown below.

Table 1

Case	p	q	r	H
1	$2m^2$	$-(1+m^2)$	1	$sn(\xi)$
2	2	$2m^2-1$	$-m^2(1+m^2)$	$ds(\xi)$
3	2	$2-m^2$	$(1+m^2)$	$cs(\xi)$
4	$-2m^2$	$2m^2-1$	$(1+m^2)$	$cn(\xi)$
5	-2	$2-m^2$	(m^2-1)	$dn(\xi)$
6	$m^2/2$	$(m^2-2)/2$	$1/4$	$sn(\xi)/1\pm dn(\xi)$
7	$m^2/2$	$(m^2-2)/2$	$m^2/4$	$sn(\xi)/1\pm \mathrm{dn}(\xi)$
8	$-1/2$	$(m^2+1)/2$	$-(1-m^2)^2/4$	$mcn(\xi)\pm dn(\xi)$
9	$(m^2-1)/2$	$(m^2+1)/2$	$(m^2-1)/4$	$\mathrm{d}n(\xi)/1\pm sn(\xi)$
10	$(1-m^2)/2$	$(1-m^2)/2$	$(1-m^2)/4$	$cn(\xi)/1\pm sn(\xi)$
11	$(1-m^2)^2/2$	$(1-m^2)^2/2$	$1/4$	$sn(\xi)/1\pm sn(\xi)$
12	2	0	0	c/ξ
13	0	1	0	ce^{ξ}

$$cn(\xi)\longrightarrow \operatorname{sech}(\xi), sn(\xi)\longrightarrow \tanh(\xi), cs(\xi)\longrightarrow \operatorname{csch}(\xi),$$
$$ds\longrightarrow \operatorname{csch}(\xi), dn(\xi)\longrightarrow \operatorname{sech}(\xi). \tag{16}$$

3.2. Solutions of SFBKEs. Now, let us determine the parameter M by balancing u'' with u^3 in equation (13) as follows:

$$M+2=3M\Rightarrow M=1. \tag{17}$$

Rewriting equation (14) with $M=1$ as follows:

$$u=a_0+a_1H. \tag{18}$$

Differentiating equation (18) twice, we have, by using (15),

$$u''=a_1qH+a_1pH^3. \tag{19}$$

Plugging equations (18) and (19) into equation (13) we have the following equation:

$$\begin{aligned}&\left(a_1p-8a_1^3\right)H^3-\left(24a_0a_1^2+12\lambda a_1^2\right)H^2\\&+\left(a_1q-24a_0^2a_1-24\lambda a_0a_1-4\lambda^2a_1\right)H\\&-\left(8a_0^3+12\lambda a_0^2+4\lambda^2a_0\right)=0.\end{aligned} \tag{20}$$

Balancing each coefficient of H^k to zero, we get for $k=0,1,2,3$

$$\begin{aligned}a_1p-a_1^3&=0,\\-3a_0a_1^2-2\lambda a_1^2&=0,\\a_1q-3a_0^2a_1-4\lambda a_0a_1-\left(\lambda^2-1\right)a_1&=0,\end{aligned} \tag{21}$$

and

$$a_0^3+2\lambda a_0^2+\left(\lambda^2-1\right)a_0=0. \tag{22}$$

Solving these equations, we get for $q\leq 0$ and $p\geq 0$,

$$\begin{aligned}\text{Set I}: a_0&=-\sqrt{-q},\\a_1&=\sqrt{p},\\\lambda&=+\frac{3}{2}\sqrt{-q},\end{aligned} \tag{23}$$

and

$$\begin{aligned}\text{Set II}: a_0&=\sqrt{-q},\\a_1&=-\sqrt{p},\\\lambda&=-\frac{3}{2}\sqrt{-q}.\end{aligned} \tag{24}$$

For the first Set I: the solution of (13), for $p\geq 0$ and $q\leq 0$, is as follows:

$$u(\xi)=-\sqrt{-q}+\sqrt{p}H(\xi). \tag{25}$$

As a result, by using (3) and (11), the solution of the SFBKEs (2-1) reads as follows:

$$\begin{aligned}U(x,t)&=\left[-\sqrt{-q}+\sqrt{p}H\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2t\right)},\\V(x,t)&=\left[\frac{5}{2}q+\frac{5}{2}\sqrt{-pq}H-pH^2\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2t\right)}.\end{aligned} \tag{26}$$

By using the previous table, there are many cases for $p\geq 0$, $q\leq 0$, and r as follows.

3.2.1. First Case. If $P=2m^2$, $q=-(1+m^2)$, and $r=1$, then $H(\xi)=sn(\xi)$. So, the solution of the SFBKEs (1-2), using (26), is as follows:

$$U(x,t)=\left[-\sqrt{-q}+\sqrt{p}\,sn\left(\frac{1}{\alpha}x^\alpha+\frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2t\right)}\text{ and} \tag{27}$$

$$\begin{aligned}V(x,t)=\Bigg[&\frac{5q}{2}+\frac{5}{2}\sqrt{-pq}\,sn\left(\frac{1}{\alpha}x^\alpha+\frac{3}{2}\sqrt{-q}\,t\right)\\&-psn^2\left(\frac{1}{\alpha}x^\alpha+\frac{3}{2}\sqrt{-q}\,t\right)e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2t\right)}\Bigg].\end{aligned} \tag{28}$$

If $m \longrightarrow 1$, then equations (27) and (28) degenerates as follows:

$$U(x,t)=\left[-\sqrt{-q}+\sqrt{p}\tanh\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right)\right] \text{ and}$$
$$V(x,t)=\left[\frac{5q}{2}+\frac{5}{2}\sqrt{-pq}\tanh\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right) - p\tanh^2\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}. \quad (29)$$

3.2.2. Second Case. If $P=2$, $q=2m^2-1$, and $r=-m^2(1-m^2)$ for $m\le 1/\sqrt{2}$, then $H(\xi)=ds(\xi)$. Thus, the solution of the SFBKEs (1-2), using (26), is as follows:

$$U(x,t)=-\left[\sqrt{-q}+\sqrt{p}\,ds\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and} \quad (30)$$

$$V(x,t)=\left[\frac{5q}{2}+\frac{5}{2}\sqrt{-pq}\,ds\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right) - p\,ds^2\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right)e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}.\right. \quad (31)$$

If $m \longrightarrow 1$, then equations (30) and (31) degenerates as follows:

$$U(x,t)=\left[-\sqrt{-q}+\sqrt{p}\,\operatorname{csch}\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right)\right] e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and}$$
$$V(x,t)=\left[-\frac{5q}{2}+\frac{5}{2}\sqrt{-pq}\,\operatorname{csch}\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right) - p\operatorname{csch}^2\left(\frac{1}{\alpha}x^{\alpha}+\frac{3}{2}\sqrt{-q}\,t\right)e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}.\right. \quad (32)$$

3.2.3. Third Case. If $p=m^2/2$, $q=(m^2-2)/2$, and $r=1/4$ (or $r=(m^2/4)$, then $H(\xi)=sn(\xi)/1\pm dn(\xi)$. Thus, the solution of the SFBKEs (1-2), using (26), is as follows:

$$U(x,t)=\left[-\sqrt{-q}+\sqrt{p}\;\frac{sn\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}{1\pm dn\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}\right] e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and} \quad (33)$$

$$V(x,t)=\left[\frac{5q}{2}+\frac{5}{2}\sqrt{-pq}\;\frac{sn\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}{1\pm dn\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)} + -p\frac{sn^2\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}{\left(1\pm dn\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)\right)^2} e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}.\right. \quad (34)$$

If $m \longrightarrow 1$, then equations (33) and (34) degenerates as follows:

$$U(x,t)=\left[-\sqrt{-q}+\sqrt{p}\;\frac{\tanh\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}{1\pm \operatorname{sech}\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}\right] e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and}$$
$$V(x,t)=\left[\frac{5q}{2}+\frac{5}{2}\sqrt{-pq}\;\frac{\tanh\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}{1\pm \operatorname{sech}\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)} + -p\frac{\tanh^2\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)}{\left(1\pm \operatorname{sech}\left(1/\alpha x^{\alpha}+3/2\sqrt{-q}\,t\right)\right)^2} e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}.\right. \quad (35)$$

3.2.4. Fourth Case. If $p=2$, $q=0$,and $r=0$, then $H(\xi)=C/\xi$ So, the solution of the SFBKEs (1-2), using (26), is as follows:

$$U(x,t)=\left[\sqrt{2}\,\alpha C x^{-\alpha}\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and}$$
$$V(x,t)=\left[-2C^2\alpha^2 x^{-2\alpha}\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}. \quad (36)$$

For the second Set II, the solution of equation (13) is as follows:

$$u(\xi)=-\sqrt{-q}+\sqrt{p}H(\xi), \text{for } p\ge 0, q\le 0. \quad (37)$$

As a result, by using (3) and (11), the solution of the SFBKEs (2-1) reads as follows:

$$U(x,t)=\left[-\sqrt{-q}+\sqrt{p}\,H\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and}$$
$$V(x,t)=\left[\frac{5}{2}q+\frac{5}{2}\sqrt{-pq}\,H-pH^2\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}. \quad (38)$$

By using the previous table, there are many cases for $p\ge 0$, $q\le 0$ and r as follows.

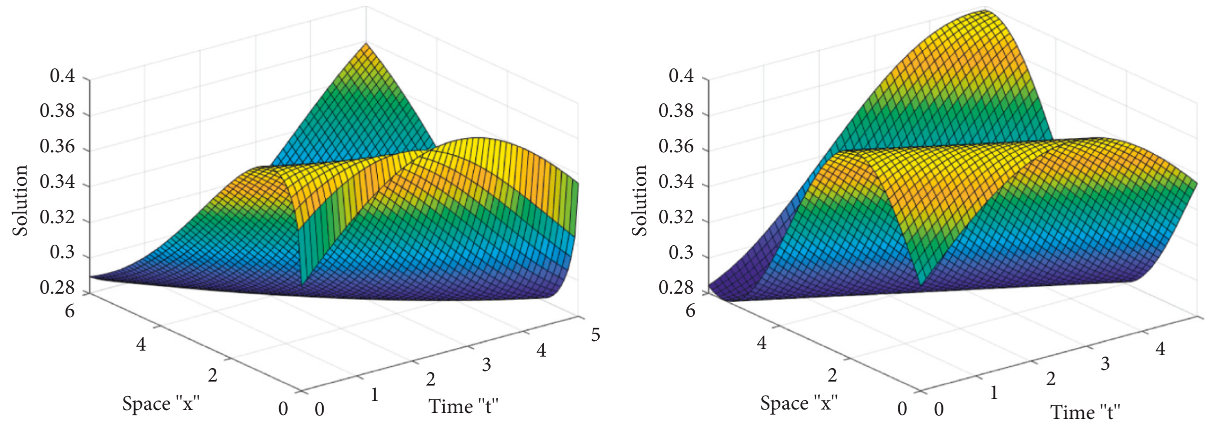
3.2.5. First Case. If $P=2m^2$, $q=-(1+m^2)$, and $r=1$, then $H(\xi)=sn(\xi)$. So, the solution of the SFBKEs (1-2), using (38), is as follows:

$$U(x,t)=\left[\sqrt{-q}-\sqrt{p}\,sn\left(\frac{1}{\alpha}x^{\alpha}-\frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and} \quad (39)$$

$$V(x,t)=\left[\frac{-q}{2}+\frac{1}{2}\sqrt{-pq}\,sn\left(\frac{1}{\alpha}x^{\alpha}-\frac{3}{2}\sqrt{-q}\,t\right) - p\,sn^2\left(\frac{1}{\alpha}x^{\alpha}-\frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}. \quad (40)$$

If $m \longrightarrow 1$, then equations (39) and (40) degenerates as follows:

$$U(x,t)=\left[\sqrt{-q}-\sqrt{p}\tanh\left(\frac{1}{\alpha}x^{\alpha}-\frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)} \text{ and}$$
$$V(x,t)=\left[\frac{-q}{2}+\frac{1}{2}\sqrt{-pq}\tanh\left(\frac{1}{\alpha}x^{\alpha}-\frac{3}{2}\sqrt{-q}\,t\right) - p\tanh^2\left(\frac{1}{\alpha}x^{\alpha}-\frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma\mathcal{W}(t)-1/2\sigma^2 t\right)}. \quad (41)$$



$\sigma = 0$, $\alpha = 0.5$ $\sigma = 0$, $\alpha = 1$

Figure 1: 3D plot of equation (33) with $\sigma = 0$ and different α.

3.2.6. Second Case. If $P = 2$, $q = 2m^2 - 1$, and $r = -m^2(1 - m^2)$ for $m \leq 1/\sqrt{2}$, then $H(\xi) = ds(\xi)$. Thus, the solution of the SFBKEs (1-2), using (38), is as follows:

$$U(x,t) = [\sqrt{-q} - \sqrt{-p}\, ds\left(\frac{1}{\alpha}x^{\alpha} - \frac{3}{2}\sqrt{-q}\,t\right)e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)}. \tag{42}$$

$$V(x,t) = \left[\frac{-q}{2} + \frac{1}{2}\sqrt{-pq}\, ds\left(\frac{1}{\alpha}x^{\alpha} - \frac{3}{2}\sqrt{-q}\,t\right) - p\, ds^2\left(\frac{1}{\alpha}x^{\alpha} - \frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)}. \tag{43}$$

If $m \longrightarrow 1$, then equations (42)and (43) degenerates as follows:

$$U(x,t) = \left[\sqrt{-q} - \sqrt{p}\, \mathrm{csch}\left(\frac{1}{\alpha}x^{\alpha} - \frac{3}{2}\sqrt{-q}\,t\right)\right] e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)} \text{ and}$$

$$V(x,t) = \left[\frac{-q}{2} + \frac{1}{2}\sqrt{-pq}\, \mathrm{csch}\left(\frac{1}{\alpha}x^{\alpha} - \frac{3}{2}\sqrt{-q}\,t\right) - p\mathrm{csch}^2\left(\frac{1}{\alpha}x^{\alpha} - \frac{3}{2}\sqrt{-q}\,t\right)\right]e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)}. \tag{44}$$

3.2.7. Third Case. If $p = m^2/2$, $q = (m^2 - 2)/2$, and $r = 1/4$ (or $r = m^2/4$, then $H(\xi) = sn(\xi)/1 \pm dn(\xi)$. Thus, the solution of the SFBKEs (1-2), using (38), is as follows:

$$U(x,t) = \left[\sqrt{-q} - \sqrt{p}\, \frac{sn\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}{1 \pm dn\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}\right] e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)} \text{ and} \tag{45}$$

$$V(x,t) = \left[\frac{-q}{2} + \frac{1}{2}\sqrt{-pq}\, \frac{sn\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}{1 \pm dn\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)} + -p\frac{sn^2\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}{\left(1 \pm dn\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)\right)^2}\right] e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)}. \tag{46}$$

If $m \longrightarrow 1$, then equations (45) and (46) degenerates as follows:

$$U(x,t) = \left[\sqrt{-q} - \sqrt{p}\, \frac{\tanh\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}{1 \pm \mathrm{sech}\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}\right] e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)} \text{ and}$$

$$V(x,t) = \left[\frac{-q}{2} + \frac{1}{2}\sqrt{-pq}\, \frac{\tanh\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}{1 \pm \mathrm{sech}\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)} + -p\frac{\tanh^2\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)}{\left(1 \pm \mathrm{sech}\left(1/\alpha x^{\alpha} - 3/2\sqrt{-q}\,t\right)\right)^2}\right] e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)}. \tag{47}$$

3.2.8. Fourth Case. If $p = 2$, $q = 0$, and $r = 0$, then $H(\xi) = C/\xi$ So, the solution of the SFBKEs (1-2), using (26), is as follows:

$$U(x,t) = \left[-\alpha\sqrt{2}\, C x^{-\alpha}\right] e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)} \text{ and}$$
$$V(x,t) = \left[-2\alpha^2 C^2 x^{-2\alpha} x^{-2}\right] e^{\left(\sigma \mathscr{W}(t) - 1/2\sigma^2 t\right)}. \tag{48}$$

4. The Impact of Fractional Order and Noise on the Solutions

The effect of the noise on the obtained solution of the SFBKEs (1-2) is discussed. The MATLAB tools are used to provide some graphs for different values of σ (noise strength).

Firstly, from the effect of fractional order in Figures 1 and 2, if $\sigma = 0$ and $m = 0.4$, we can see that the surface expands when α is increasing:

4.1. Secondly the Effect of Noise. In Figures 3 and 4, when noise is added, the surface gets much flattered if its strength is increased $\sigma = 1, 2$.

In Figure 5, we introduce 2D plot of the $U(x,t)$ in (33) with $\sigma = 0, 0.5, 1, 2$ and with $\alpha = 1$, which emphasize the previous results.

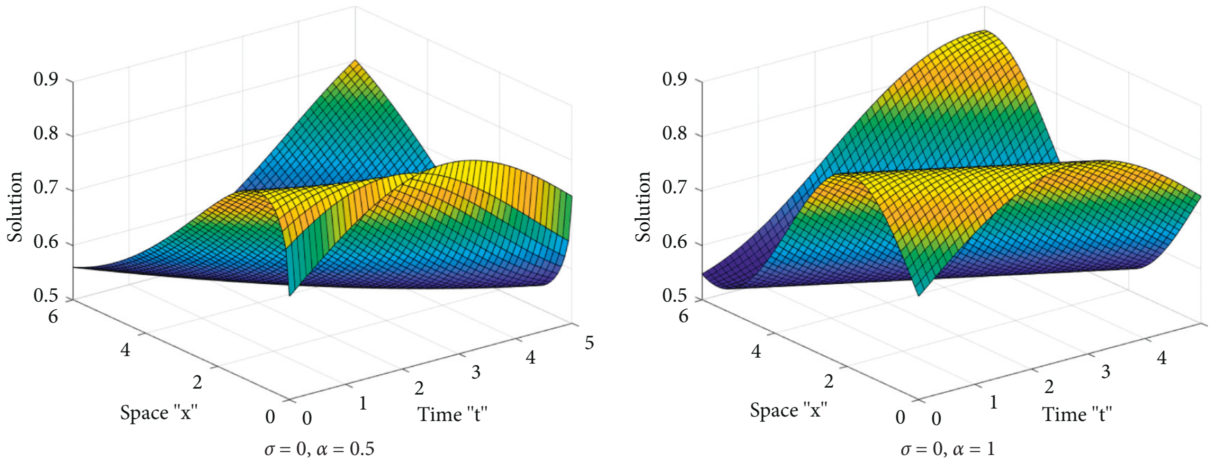


Figure 2: 3D plot of equation (34) with $\sigma = 0$ and different α.

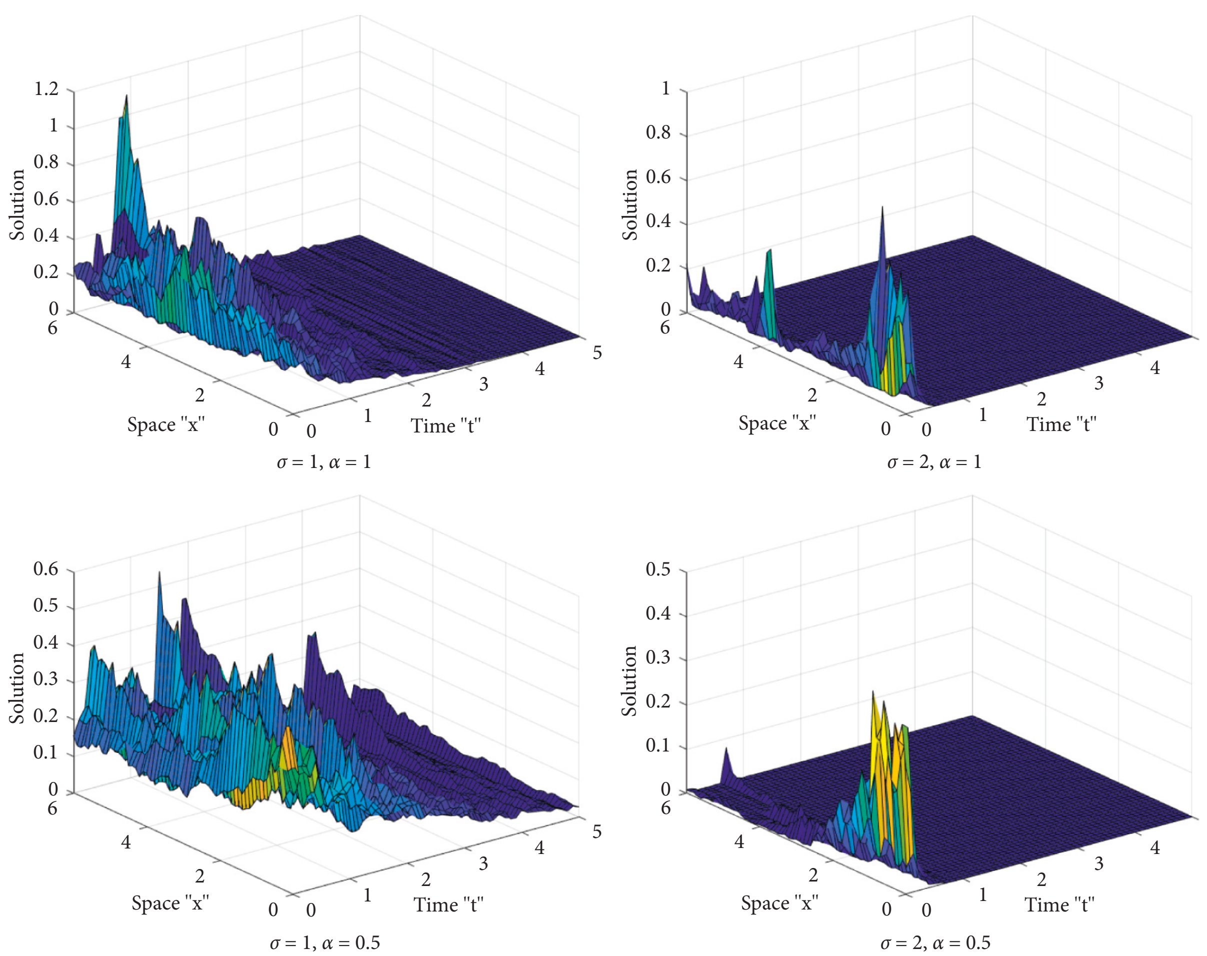


Figure 3: 3D plot of equation (33) with $\sigma = 1, 2$ and $\alpha = 1, 0.5$.

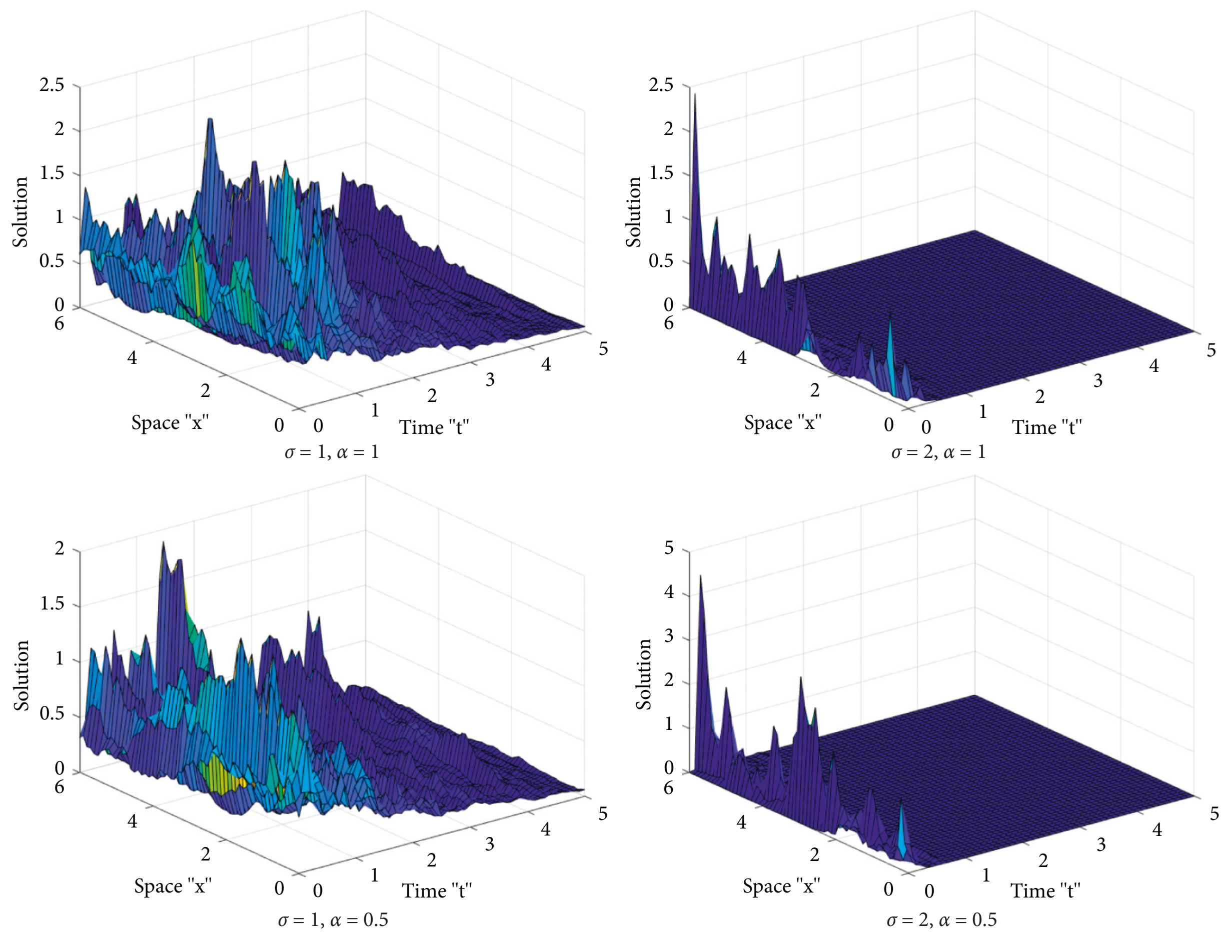


Figure 4: 3D plot of equation (34) with $\sigma = 1, 2$ and $\alpha = 1, 0.5$.

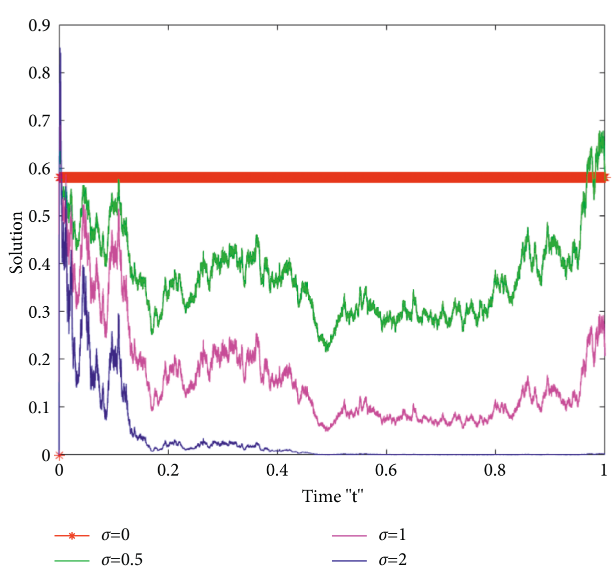


Figure 5: 2D graph of equation (33).

5. Conclusions

By using the Jacobi elliptic function method, the exact solutions of the stochastic-fractional Broer–Kaup equations (1-2) were successfully achieved. These obtained solutions are far more helpful and effective in comprehending several crucial difficult physical phenomena due to the importance of SFBKEs in describing the propagation of shallow water waves. Also, we used MATLAB package to show how the multiplicative noise influenced the solutions of SFBKEs. In future work, we may use additive noise to treat the SFBKEs (1-2) [39].

Data Availability

All data are available in this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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