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Research Article

Application of ARA-Residual Power Series Method in Solving Systems of Fractional Differential Equations

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In this research, systems of linear and nonlinear differential equations of fractional order are solved analytically using the novel interesting method: ARA- Residual Power Series (ARA-RPS) technique. This approach technique is based on the combination of the residual power series scheme with the ARA transform to establish analytical approximate solutions in a fast convergent series representation using the concept of the limit. The proposed method needs less time and effort compared with the residual power series technique. To prove the simplicity, applicability, and reliability of the presented method, three numerical examples are proposed and simulated. The obtained results show that the ARA-RPS technique is applicable, simple, and effective to get solutions to linear and nonlinear engineering and physical problems.

1. Introduction

Numerous phenomena in various fields of science can be fruitfully formulated by the use of fractional derivatives. This is because the sensible modeling for a physical phenomenon depends on instantaneous time as well as on prior time history; hence, we may use fractional calculus to deal with these problems [1–10].

Many physical and engineering problems can be formulated by fractional differential equations (FDEs) and obtaining the solutions of these equations have been the theme of many interesting investigations and have attracted the attention of researchers. Recently, there have been a large number of techniques dedicated to get solutions of FDEs [11–16]. These techniques can be classified into two categories, approximate and analytical, such as transform methods, homotopy analysis methods, and residual power series methods. [17–25].

On the other hand, there are a lot of contributions related to solving systems of FDEs and since the finding of the analytical solution for a system of FDEs sometimes required a complex computation, analytical, and numerical techniques were established and developed to search for solutions of linear and nonlinear FDEs, see [26–36].

In the literature, various integral transforms such as Laplace, Fourier, Sumudu, and ARA transforms have been introduced to deal with differential equations. ARA transform was introduced first in 2020, which is a powerful technique in solving differential equations and systems. One advantage of this transform is that it can solve differential equations with singular points around zero. Also, ARA transform is applicable for some functions that the Laplace transform cannot be applied to; see [37].

In this paper, we employ the ARA-RPS method to investigate analytical and approximate solutions of linear and nonlinear systems of FDEs. The ARA-RPS technique that has been introduced in [39] is a combination of the ARA transform [37–39] and the residual power series method [40–47].

To achieve our goal of finding the solutions of systems of FDEs, we operate the ARA transform on the original system and implement appropriate series expansions to solve the new system. The expansion coefficients are determined based on the concepts of limits at infinity, unlike the fractional power series (FPS) method which may take more time in the solution procedure.

This work is organized as follows: in the second section, we introduce some basic definitions and theorems of ARA transform and fractional derivatives. The methodology of the ARA-RPS method is presented in section three. Numerical examples and simulations are introduced in section four.

2. Basic Definitions and Theorems

In this section, we introduce the definition of the fractional operators Riemann–Liouville and Caputo. Definition of the ARA transform and some properties and theorems of the fractional ARA-Tayler's series representations are also revisited.

Definition 1. The Riemann–Liouville fractional integral of the function $y(t)$ of order $\alpha > 0$ is defined by

$$I_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} y(\tau) \mathrm{d}\tau, \quad (1)$$

where $\Gamma(\alpha)$ is the gamma function.

Definition 2. The Caputo fractional derivative of the function $y(t)$ of order $\alpha > 0$ is defined by

$$D^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{y^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau, & m-1 < \alpha < m, \\ y^{(m)}(t), & \alpha = m, \end{cases} \quad (2)$$

where m is a nonnegative integer.

Definition 3. [37] The ARA transform of order n of the continuous function $y(t)$ on the interval $(0, \infty)$, is defined by

$$\mathscr{G}_n[y(t)](s) = s \int_0^{\infty} t^{n-1} e^{-st} y(t) \mathrm{d}t, \quad s > 0. \quad (3)$$

The inverse of the ARA transform is given by

$$y(t) = \frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \left((-1)^n \left(\frac{1}{s\Gamma(n-1)} \int_0^s (s-x)^{n-1} \mathscr{G}_{n+1}[y(t)](x) \mathrm{d}x + \sum_{k=0}^{n-1} \frac{s^k}{k!} \frac{\partial^k Y(0)}{\partial s^k} \right) \right) \mathrm{d}s, \quad (4)$$

where $c = \mathrm{Re}(s)$ and

$$Y(s) = \int_0^{\infty} e^{-st} y(t) \mathrm{d}t. \quad (5)$$

In the following arguments, we present some basic properties of the ARA transform [39] that are essential in our research.

Let $y(t)$ and $h(t)$ be two continuous functions on the interval $(0, \infty)$ for which the ARA transform exists. Then we have for $s > 0$

(1) $\mathscr{G}_n[\alpha y(t) + \beta h(t)](s) = \alpha \mathscr{G}_n[y(t)](s) + \beta \mathscr{G}_n[h(t)](s)$, where α and β are nonzero constants.
(2) $\lim_{s \longrightarrow \infty} \mathscr{G}_1[y(t)](s) = y(0)$.
(3) $\mathscr{G}_1[y'(t)](s) = s\mathscr{G}_1[y(t)](s) - sy(0)$.
(4) $\mathscr{G}_2[y(t)](s) = -s(\mathrm{d}/(\mathrm{d}s))((\mathscr{G}_1[y(t)](s))/s)$.
(5) $\mathscr{G}_2[y'(t)](s) = -s(\mathrm{d}/(\mathrm{d}s))(\mathscr{G}_1[y(t)](s))$.
(6) $\mathscr{G}_2[t^{\alpha}](s) = ((\Gamma(\alpha+2))/s^{\alpha+1})$, $s > 0$, $\alpha > 0$.
(7) $\mathscr{G}_1[D^{\alpha} y(t)](s) = (1/s^{m-\alpha}) \mathscr{G}_1[y^{(m)}(t)](s)$, $m - 1 < \alpha \leq m$.
(8) $\mathscr{G}_2[D^{\alpha} y(t)](s) = s^{\alpha} \mathscr{G}_2[y(t)](s) - \alpha s^{\alpha-1} \mathscr{G}_1[y(t)](s) + (\alpha - 1) s^{\alpha-1} y(0)$, $0 < \alpha \leq 1$.
(9) $\lim_{s \longrightarrow \infty} s\mathscr{G}_2[y(t)] = y(0)$.

Theorem 1. *[26] Suppose that the FPS representation of the function $y(t)$ at $t = 0$ has the form*

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n\alpha}, \quad m-1 < \alpha \leq m, \quad m = 1, 2, \ldots, 0 \leq t \leq \beta. \quad (6)$$

If $y(t)$ and $D^{n\alpha} y(t)$ are continuous on $[0, \beta]$, for each $n = 0, 1, \ldots$, then the coefficients a_n's have the form $a_n = D^{n\alpha} y(0)/\Gamma(n\alpha + 1)$, for $n = 0, 1, 2, \ldots$, where $D^{n\alpha} = D^{\alpha} \cdot D^{\alpha} \ldots D^{\alpha}$ (n– times).

Theorem 2. *[39] Let $y(t)$ be a continuous function on every finite interval $[0, \beta]$ in which the ARA transform of order two exists and has the FPS representation:*

$$\mathscr{G}_2[y(t)](s) = \sum_{n=0}^{\infty} \frac{h_n}{s^{n\alpha+1}}; \quad 0 < \alpha \leq 1 \; s > 0. \quad (7)$$

Then

$$h_n = (n\alpha + 1) D^{n\alpha} y(0). \quad (8)$$

Remarks

(1) The inverse ARA transform of order two for the FPS representation in Theorem 2 has the following form

$$y(t) = \sum_{n=0}^{\infty} \frac{D^{n\alpha} y(0) t^{n\alpha}}{\Gamma(n\alpha + 1)}. \quad (9)$$

(2) The ARA transform of order one of the function $y(t)$ has the following presentation

$$\mathscr{G}_1[y(t)](s) = \sum_{n=0}^{\infty} \frac{h_n}{(n\alpha + 1) s^{n\alpha}}. \quad (10)$$

For the convergence analysis and conditions of the previous theorem, we propose the following results.

Theorem 3. *[39] Let $y(t)$ be a continuous function on every finite interval $[0,\beta]$ in which the ARA transform exists. Assume that $\mathcal{G}_1[y(t)](s)$ has the following series representation:*

$$\mathcal{G}_1[y(t)](s) = \sum_{n=0}^{\infty} \frac{C_n}{s^{n\alpha}}, \tag{11}$$

where

$$C_n = D^{n\alpha} y(0). \tag{12}$$

If $|\mathcal{G}_1[D^{(n+1)\alpha} y(t)](s)| \le M$ on $0 < s \le d$, then the remainder $R_n(s)$, of the above series, when we consider only the first n terms, satisfies the following inequality:

$$\left|R_n(s)\right| \le \frac{M}{s^{(n+1)\alpha}}, \quad 0 < s \le d. \tag{13}$$

3. Methodology of the ARA-RPSM

Many phenomena in the fields of engineering and physics could be regulated by fractional initial value problems to simulate and describe some features of these phenomena. Analytical methods are used to get approximate solutions for some fractional initial value problems that we could not find out their exact solutions for. One of these techniques is the ARA-RPS method that depends on coupling the residual power series method with the ARA transform.

To perform the ARA-RPS method, consider the following system of FDEs in the sense of the Caputo derivative

$$D^{\alpha} y_i(t) = f_i(t, y_1(t), y_2(t), \ldots, y_m(t)), \quad t \ge 0,\ 0 < \alpha \le 1, \tag{14}$$

with the initial conditions (ICs)

$$y_i(0) = c_i, \quad i = 1, \ldots, m, \tag{15}$$

where f_i are continuous real valued functions defined on $[0,\infty] \times \mathbb{R}^m$ and $y_i(t)$ are unknown analytical functions to be determined.

Firstly, we apply the ARA transform of order two $\mathcal{G}_2$ on both sides of each equation in system (14)

$$\mathcal{G}_2[D^{\alpha} y_i(t)](s) = \mathcal{G}_2[f_i(t, y_1(t), y_2(t), \ldots, y_m(t))](s). \tag{16}$$

Using property 8 and the ICs in (15), we get

$$s^{\alpha}\mathcal{G}_2[y_i(t)](s) - \alpha s^{\alpha-1}\mathcal{G}_1[y_i(t)](s) + (\alpha-1)s^{\alpha-1}c_i = F_i(s), \tag{17}$$

where $F_i(s) = \mathcal{G}_2[f_i(t, y_1(t), y_2(t), \ldots, y_m(t))](s)$.

Hence (17) can be written as

$$\mathcal{G}_2[y_i(t)](s) = \frac{\alpha}{s}\mathcal{G}_1[y_i(t)](s) - \frac{(\alpha-1)}{s}c_i + \frac{F_i(s)}{s^{\alpha}}, \quad \text{for } i = 1, \ldots, m. \tag{18}$$

Assuming that the ARA-RPS solutions of equations in (18) have the series expansions

$$\mathcal{G}_1[y_i(t)](s) = \sum_{n=0}^{\infty} \frac{h_{i,n}}{(n\alpha+1)s^{n\alpha}}, \tag{19}$$

$$\mathcal{G}_2[y_i(t)](s) = \sum_{n=0}^{\infty} \frac{h_{i,n}}{s^{n\alpha+1}}, \quad \text{for } i = 1, 2, \ldots, m. \tag{20}$$

Using the fact that

$$\lim_{s\longrightarrow\infty} s\mathcal{G}_2[y_i(t)](s) = y_i(0) = c_i, \tag{21}$$

we have $h_{i,0} = c_i$ for $i = 1, \ldots m$ and so the k^{th} truncated series expansions of (19) and (20) have the form

$$\mathcal{G}_1[y_i(t)]_k(s) = \sum_{n=0}^{k} \frac{h_{i,n}}{(n\alpha+1)s^{n\alpha}} = c_i + \sum_{n=1}^{k} \frac{h_{i,n}}{(n\alpha+1)s^{n\alpha}}, \tag{22}$$

$$\mathcal{G}_2[y_i(t)]_k(s) = \sum_{n=0}^{k} \frac{h_{i,n}}{s^{n\alpha+1}} = \frac{c_i}{s} + \sum_{n=1}^{k} \frac{h_{i,n}}{s^{n\alpha+1}}. \tag{23}$$

To find the coefficients of the series expansions in (22) and (23), we define the ARA-residual functions of the (18) as follows:

$$\mathcal{G}_2\mathrm{Res}^i(s) = \mathcal{G}_2[y_i(t)](s) - \frac{\alpha}{s}\mathcal{G}_1[y_i(t)](s) + \frac{(\alpha-1)}{s}c_i - \frac{F_i(s)}{s^{\alpha}}, \quad \text{for } i = 1, \ldots, m. \tag{24}$$

and the k^{th} ARA-residual functions are

$$\mathcal{G}_2 Res_k^i(s) = \mathcal{G}_2[y_i(t)]_k(s) - \frac{\alpha}{s}\mathcal{G}_1[y_i(t)]_k(s) + \frac{(\alpha-1)}{s}c_i - \frac{F_i(s)}{s^{\alpha}}, \quad \text{for } i = 1, \ldots, m,\ k = 1, 2, \cdots. \tag{25}$$

The following facts are necessary to obtain the ARA-RPS solutions:

(i) $\mathcal{G}_2 Res^i(s) = 0$.

(ii) $\lim_{k\longrightarrow\infty}\mathcal{G}_2 Res_k^i(s) = \mathcal{G}_2 Res^i(s)$, for $s > 0$.

(iii) $\lim_{s\longrightarrow\infty} s\mathcal{G}_2 Res^i(s) = 0$ and $\lim_{s\longrightarrow\infty} s\mathcal{G}_2 Res_k^i(s) = 0$.

(iv) $\lim_{s\longrightarrow\infty} s^{k\alpha+1}\mathcal{G}_2 Res^i(s) = \lim_{s\longrightarrow\infty} s^{k\alpha+1}\mathcal{G}_2 Res_k^i(s) = 0$, for $0 < \alpha \le 1, i = 1, 2, \ldots, m,\ k = 1, 2, \ldots$.

To find the coefficients $h_{i,n}$'s in the series expansion (24) and (25), we solve the system

$$\lim_{s\longrightarrow\infty} s^{k\alpha+1}\mathcal{G}_2 Res_k^i(s) = 0, \tag{26}$$

for $i = 1, \ldots, m$ and $k = 1, 2, \ldots$.

Then the obtained coefficients $h_{i,n}$ are substituted in the series solution (23) and then apply the inverse ARA transform of order two $\mathcal{G}_2^{-1}$ to get the solution of system (14) in the original space.

4. Numerical Examples

In this section, we introduce three interesting examples on systems of linear and nonlinear FDEs. These examples demonstrate the applicability and efficiency of the presented method.

Example 1. Consider the linear system of FDEs

$$\begin{cases} D^{\alpha} y_1(t) = y_1(t) + y_2(t), \\ D^{\alpha} y_2(t) = -y_1(t) + y_2(t), \end{cases} \quad 0 \le t \le 1,\ 0 < \alpha \le 1, \tag{27}$$

with the ICs

$$y_1(0) = 0, y_2(0) = 1. \tag{28}$$

Solution: Applying the ARA transform of order two $\mathcal{G}_2$, on both sides of the equations in system (27), we have

$$\begin{cases} \mathcal{G}_2[D^{\alpha} y_1(t)](s) = \mathcal{G}_2[y_1(t)](s) + \mathcal{G}_2[y_2(t)](s), \\ \mathcal{G}_2[D^{\alpha} y_2(t)](s) = \mathcal{G}_2[-y_1(t)](s) + \mathcal{G}_2[y_2(t)](s). \end{cases} \tag{29}$$

Running the ARA transform on the equations in system (29), we have

$$\begin{cases} s^{\alpha}\mathcal{G}_2[y_1(t)](s) - \alpha s^{\alpha-1}\mathcal{G}_1[y_1(t)](s) + (\alpha-1)s^{\alpha-1}y_1(0) \\ = \mathcal{G}_2[y_1(t)](s) + \mathcal{G}_2[y_2(t)](s), \\ s^{\alpha}\mathcal{G}_2[y_2(t)](s) - \alpha s^{\alpha-1}\mathcal{G}_1[y_2(t)](s) + (\alpha-1)s^{\alpha-1}y_2(0) \\ = -\mathcal{G}_2[y_1(t)](s) + \mathcal{G}_2[y_2(t)](s). \end{cases} \tag{30}$$

Substituting the ICs (28) and simplifying the equations in system (30), we get

$$\begin{cases} \mathcal{G}_2[y_1(t)](s) - \dfrac{\alpha}{s}\mathcal{G}_1[y_1(t)](s) - \dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_1(t)](s) \\ -\dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_2(t)] = 0, \\ \mathcal{G}_2[y_2(t)](s) - \dfrac{\alpha}{s}\mathcal{G}_1[y_2(t)](s) + \dfrac{\alpha-1}{s} \\ +\dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_1(t)](s) - \dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_2(t)](s) = 0. \end{cases} \tag{31}$$

Assume that the ARA-RPS solutions of (31) have the following series representations:

$$\begin{aligned} \mathcal{G}_1[y_1(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{1,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathcal{G}_2[y_1(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{1,n}}{s^{n\alpha+1}}, \\ \mathcal{G}_1[y_2(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{2,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathcal{G}_2[y_2(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{2,n}}{s^{n\alpha+1}}. \end{aligned} \tag{32}$$

Using the fact in property 9

$$\lim_{s\longrightarrow\infty} s\mathcal{G}_2[y_i(t)](s) = y_i(0), \quad i = 1, 2, \ldots, \tag{33}$$

we get the coefficients $h_{1,0} = 0, h_{2,0} = 1$, and so the k^{th} ARA-RPS solutions of system (31) have the form

$$\begin{aligned} \mathcal{G}_1[y_1(t)]_k(s) &= \sum_{n=1}^{k} \frac{h_{1,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathcal{G}_2[y_1(t)]_k(s) &= \sum_{n=1}^{k} \frac{h_{1,n}}{s^{n\alpha+1}}, \\ \mathcal{G}_1[y_2(t)]_k(s) &= 1 + \sum_{n=1}^{k} \frac{h_{2,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathcal{G}_2[y_2(t)]_k(s) &= \frac{1}{s} + \sum_{n=1}^{k} \frac{h_{2,n}}{s^{n\alpha+1}}. \end{aligned} \tag{34}$$

Define the ARA-residual functions of the equations in system (31) as follows:

$$\begin{cases} \mathcal{G}_2 Res^1(s) = \mathcal{G}_2[y_1(t)](s) - \dfrac{\alpha}{s}\mathcal{G}_1[y_1(t)](s) \\ -\dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_1(t)](s) - \dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_2(t)](s), \\ \mathcal{G}_2 Res^2(s) = \mathcal{G}_2[y_2(t)](s) - \dfrac{\alpha}{s}\mathcal{G}_1[y_2(t)](s) + \dfrac{\alpha-1}{s} \\ +\dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_1(t)](s) - \dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_2(t)](s), \end{cases} \tag{35}$$

and the k^{th} ARA-residual functions of system (35)

$$\begin{cases} \mathcal{G}_2 Res_k^1(s) = \mathcal{G}_2[y_1(t)]_k(s) - \dfrac{\alpha}{s}\mathcal{G}_1[y_1(t)]_k(s) \\ -\dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_1(t)]_k(s) - \dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_2(t)]_k(s), \\ \mathcal{G}_2 Res_k^2(s) = \mathcal{G}_2[y_2(t)]_k(s) - \dfrac{\alpha}{s}\mathcal{G}_1[y_2(t)]_k(s) + \dfrac{\alpha-1}{s} \\ +\dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_1(t)]_k(s) - \dfrac{1}{s^{\alpha}}\mathcal{G}_2[y_2(t)]_k(s). \end{cases} \tag{36}$$

To find the first unknown coefficients $h_{1,1}$ and $h_{2,1}$ of the series expansions (34), we substitute the first truncated expansions $\mathcal{G}_1[y_1(t)]_1(s)$, $\mathcal{G}_1[y_2(t)]_1(s)$, $\mathcal{G}_2[y_1(t)]_1(s)$, and $\mathcal{G}_2[y_2(t)]_1(s)$ into the first ARA-residual functions $\mathcal{G}_2 Res_1^1(s)$ and $\mathcal{G}_2 Res_1^2(s)$ in system (36) to get

$$\begin{cases} \mathcal{G}_2\mathrm{Res}_1^1(s) = \dfrac{h_{1,1}}{s^{\alpha+1}} - \dfrac{\alpha h_{1,1}}{(\alpha+1)s^{\alpha+1}} - \dfrac{h_{1,1}}{s^{2\alpha+1}} - \dfrac{1}{s^{\alpha+1}} - \dfrac{h_{2,1}}{s^{2\alpha+1}}, \\ \mathcal{G}_2\mathrm{Res}_1^2(s) = \dfrac{1}{s} + \dfrac{h_{2,1}}{s^{\alpha+1}} - \dfrac{\alpha}{s} - \dfrac{\alpha h_{2,1}}{(\alpha+1)s^{\alpha+1}} + \dfrac{\alpha-1}{s} \\ + \dfrac{h_{1,1}}{s^{2\alpha+1}} - \dfrac{1}{s^{\alpha+1}} - \dfrac{h_{2,1}}{s^{2\alpha+1}}, \end{cases} \quad (37)$$

which can be written as

$$\begin{cases} \mathcal{G}_2\mathrm{Res}_1^1(s) = \dfrac{1}{s^{\alpha+1}}\left(h_{1,1} - \dfrac{\alpha h_{1,1}}{\alpha+1} - 1\right) - \dfrac{1}{s^{2\alpha+1}}\left(h_{1,1} - h_{2,1}\right), \\ \mathcal{G}_2\mathrm{Res}_1^2(s) = \dfrac{1}{s^{\alpha+1}}\left(h_{2,1} - \dfrac{\alpha h_{2,1}}{\alpha+1} - 1\right) + \dfrac{1}{s^{2\alpha+1}}\left(h_{1,1} - h_{2,1}\right). \end{cases} \quad (38)$$

Simplifying the equations in system (38) and multiplying the resulting equations by $s^{\alpha+1}$, we obtain

$$\begin{cases} s^{\alpha+1}\mathcal{G}_2 Res_1^1(s) = h_{1,1} - \dfrac{\alpha h_{1,1}}{\alpha+1} - 1 - \dfrac{1}{s^{\alpha}}\left(h_{1,1} - h_{2,1}\right), \\ s^{\alpha+1}\mathcal{G}_2 Res_1^2(s) = h_{2,1} - \dfrac{\alpha h_{2,1}}{\alpha+1} - 1 + \dfrac{1}{s^{\alpha}}\left(h_{1,1} - h_{2,1}\right). \end{cases} \quad (39)$$

Taking the limit as $s \longrightarrow \infty$ to both sides of equations in system (39), we have

$$\begin{aligned} h_{1,1} &= \alpha + 1, \\ h_{2,1} &= \alpha + 1. \end{aligned} \quad (40)$$

To find the second unknown coefficients $h_{1,2}$ and $h_{2,2}$ of the series expansions (34), we substitute the second truncated expansion $\mathcal{G}_1[y_1(t)]_2(s)$, $\mathcal{G}_1[y_2(t)]_2(s)$, $\mathcal{G}_2[y_1(t)]_2(s)$, and $\mathcal{G}_2[y_2(t)]_2(s)$ into the second ARA-residual functions $\mathcal{G}_2 Res_2^1(s)$ and $\mathcal{G}_2 Res_2^2(s)$ in system (39) to get

$$\begin{cases} \mathcal{G}_2 Res_2^1(s) = \dfrac{h_{1,1}}{s^{\alpha+1}} + \dfrac{h_{1,2}}{s^{2\alpha+1}} - \dfrac{\alpha h_{1,1}}{(\alpha+1)s^{\alpha+1}} - \dfrac{\alpha h_{1,2}}{(2\alpha+1)s^{2\alpha+1}} - \dfrac{h_{1,1}}{s^{2\alpha+1}} \\ - \dfrac{h_{1,2}}{s^{3\alpha+1}} - \dfrac{1}{s^{\alpha+1}} - \dfrac{h_{2,1}}{s^{2\alpha+1}} - \dfrac{h_{2,2}}{s^{3\alpha+1}}, \\ \mathcal{G}_2 Res_2^2(s) = \dfrac{h_{2,1}}{s^{\alpha+1}} + \dfrac{h_{2,2}}{s^{2\alpha+1}} - \dfrac{\alpha h_{2,1}}{(\alpha+1)s^{\alpha+1}} - \dfrac{\alpha h_{2,2}}{(2\alpha+1)s^{2\alpha+1}} + \dfrac{h_{1,1}}{s^{2\alpha+1}} \\ + \dfrac{h_{1,2}}{s^{3\alpha+1}} - \dfrac{1}{s^{\alpha+1}} - \dfrac{h_{2,1}}{s^{2\alpha+1}} - \dfrac{h_{2,2}}{s^{3\alpha+1}}. \end{cases} \quad (41)$$

Simplifying the equations in system (41) and multiplying both sides by $s^{2\alpha+1}$, we get

$$\begin{cases} s^{2\alpha+1}\mathcal{G}_2 Res_2^1(s) = s^{\alpha}\left(h_{1,1} - \dfrac{\alpha h_{1,1}}{\alpha+1} - 1\right) + h_{1,2} - \dfrac{\alpha h_{1,2}}{2\alpha+1} \\ - h_{1,1} - h_{2,1} - \dfrac{h_{1,2}}{s^{\alpha}} - \dfrac{h_{2,2}}{s^{\alpha}}, \\ s^{2\alpha+1}\mathcal{G}_2 Res_2^2(s) = s^{\alpha}\left(h_{2,1} - \dfrac{\alpha h_{2,1}}{\alpha+1} - 1\right) + h_{2,2} - \dfrac{\alpha h_{2,2}}{2\alpha+1} \\ + h_{1,1} - h_{2,1} + \dfrac{h_{1,2}}{s^{\alpha}} - \dfrac{h_{2,2}}{s^{\alpha}}. \end{cases} \quad (42)$$

Substituting the values of $h_{1,1}$ and $h_{2,1}$ then taking the limit as $s \longrightarrow \infty$ to both sides of equations in system (42), we have

$$\begin{aligned} h_{1,2} &= 2(2\alpha + 1), \\ h_{2,2} &= 0. \end{aligned} \quad (43)$$

By following the same steps as before, and based on the fact $\lim_{s\to\infty} s^{k\alpha+1}\mathcal{G}_2 Res_k^i(s) = 0$ for $k = 1, 2, \ldots, i = 1, 2.$ We can conclude that

$$\begin{aligned} &h_{1,3} = 2(3\alpha+1), h_{2,3} = -2(3\alpha+1), \\ &h_{1,4} = 0, h_{2,4} = -4(4\alpha+1), \\ &h_{1,5} = -4(5\alpha+1), h_{2,5} = -4(5\alpha+1), \\ &h_{1,6} = -8(6\alpha+1), h_{2,6} = 0, \\ &h_{1,7} = -8(7\alpha+1), h_{2,7} = 8(7\alpha+1). \end{aligned} \quad (44)$$

Consequently, the ARA-RSP solutions of system (31) are

$$\begin{cases} \mathcal{G}_2[y_1(t)](s) = \dfrac{\alpha+1}{s^{\alpha+1}} + \dfrac{2(2\alpha+1)}{s^{2\alpha+1}} + \dfrac{2(3\alpha+1)}{s^{3\alpha+1}} - \dfrac{4(5\alpha+1)}{s^{5\alpha+1}} \\ - \dfrac{8(6\alpha+1)}{s^{6\alpha+1}} - \dfrac{8(7\alpha+1)}{s^{7\alpha+1}} + \ldots, \\ \mathcal{G}_2[y_2(t)](s) = \dfrac{1}{s} + \dfrac{\alpha+1}{s^{\alpha+1}} - \dfrac{2(3\alpha+1)}{s^{3\alpha+1}} - \dfrac{4(4\alpha+1)}{s^{4\alpha+1}} \\ - \dfrac{4(5\alpha+1)}{s^{5\alpha+1}} + \dfrac{8(7\alpha+1)}{s^{7\alpha+1}} + \ldots. \end{cases} \quad (45)$$

Furthermore, to get the series solutions in the original space of system (27), we apply the inverse ARA transform of order two $\mathcal{G}_2^{-1}$ to both sides of equations in system (45) to get

$$\begin{cases} y_1(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4t^{5\alpha}}{\Gamma(5\alpha+1)} \\ -\frac{8t^{6\alpha}}{\Gamma(6\alpha+1)} - \frac{8t^{7\alpha}}{\Gamma(7\alpha+1)} + \ldots, \\ y_2(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4t^{4\alpha}}{\Gamma(4\alpha+1)} \\ -\frac{4t^{5\alpha}}{\Gamma(5\alpha+1)} + \frac{8t^{7\alpha}}{\Gamma(7\alpha+1)} + \ldots. \end{cases} \tag{46}$$

For $\alpha = 1$, the series expansions (46) have the form, $y_1(t) = e^t \sin t$ and $y_2(t) = e^t \cos t$ which are coincide with the exact solutions for the ordinary system (27) and (28).

Table 1, shows a comparison of the ARA-RPS solution and the exact solution of Example 1 with $\alpha = 1$ at various values of t and $k = 8$. The absolute error is also presented in Table 1. Numerical results of Example 1 with $k = 8$, and different values of α are listed in Table 2. In addition, the comparison of the results obtained for the exact solution corresponding to, $\alpha = 1$ and the numerical solutions given by the ARA-RPS method for different values of α, $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, and $\alpha = 0.9$ are plotted in Figure 1. Figure 2 portrays a very precise agreement of the exact solutions $(y_1(t), y_2(t))$ and the ARA-RPS solutions $(y_1^8(t), y_2^8(t))$ of Example 1 at different time levels with fixed α. The obtained results show that the ARA-RPS method is efficient in obtaining approximate solutions of systems of FDEs. This is obvious in the following tables, which illustrate the small errors between the ARA-RPS solutions and the exact ones.

Example 2. Consider the nonlinear system of FDEs

$$\begin{cases} D^\alpha y_1(t) = -1002 y_1(t) + 1000 y_2^2(t), \\ D^\alpha y_2(t) = y_1(t) - y_2(t) - y_2^2(t), \end{cases} 0 \le t \le 1, 0 < \alpha \le 1, \tag{47}$$

with the ICs

$$y_1(0) = 1, y_2(0) = 1. \tag{48}$$

Solution: Applying the ARA transform of order two $\mathscr{G}_2$, on both sides of the equations in system (47)

$$\begin{cases} \mathscr{G}_2[D^\alpha y_1(t)](s) = -1002\mathscr{G}_2[y_1(t)](s) \\ \quad +1000\mathscr{G}_2[y_2^2(t)](s), \\ \mathscr{G}_2[D^\alpha y_2(t)](s) = \mathscr{G}_2[y_1(t)](s) \\ \quad -\mathscr{G}_2[y_2(t)](s) - \mathscr{G}_2[y_2^2(t)](s). \end{cases} \tag{49}$$

Running the ARA transform on equations in system (49), we get

$$\begin{cases} s^\alpha \mathscr{G}_2[y_1(t)](s) - \alpha s^{\alpha-1}\mathscr{G}_1[y_1(t)](s) + (\alpha-1)s^{\alpha-1}y_1(0) \\ = -1002\mathscr{G}_2[y_1(t)](s) + 1000\mathscr{G}_2[y_2^2(t)](s), \\ s^\alpha \mathscr{G}_2[y_2(t)](s) - \alpha s^{\alpha-1}\mathscr{G}_1[y_2(t)](s) + (\alpha-1)s^{\alpha-1}y_2(0) \\ = \mathscr{G}_2[y_1(t)](s) - \mathscr{G}_2[y_2(t)](s) - \mathscr{G}_2[y_2^2(t)](s). \end{cases} \tag{50}$$

Substituting the ICs (48) and simplifying system (50), to get

$$\begin{cases} \mathscr{G}_2[y_1(t)](s) - \frac{\alpha}{s}\mathscr{G}_1[y_1(t)](s) \\ +\frac{\alpha-1}{s} + \frac{1002}{s^\alpha}\mathscr{G}_2[y_1(t)](s) \\ -\frac{1000}{s^\alpha}\mathscr{G}_2\left[\left(\mathscr{G}_2^{-1}[\mathscr{G}_2[y_2(t)]]\right)^2\right](s) = 0, \\ \mathscr{G}_2[y_2(t)](s) - \frac{\alpha}{s}\mathscr{G}_1[y_2(t)](s) + \frac{\alpha-1}{s} \\ -\frac{\mathscr{G}_2[y_1(t)](s)}{s^\alpha} + \frac{\mathscr{G}_2[y_2(t)](s)}{s^\alpha} \\ +\frac{\mathscr{G}_2\left[\left(\mathscr{G}_2^{-1}[\mathscr{G}_2[y_2(t)]]\right)^2\right](s)}{s^\alpha} = 0. \end{cases} \tag{51}$$

That the ARA-RPS solutions of system (51) have the series representations

$$\begin{aligned} \mathscr{G}_1[y_1(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{1,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathscr{G}_2[y_1(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{1,n}}{s^{n\alpha+1}}, \\ \mathscr{G}_1[y_2(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{2,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathscr{G}_2[y_2(t)](s) &= \sum_{n=0}^{\infty} \frac{h_{2,n}}{s^{n\alpha+1}}. \end{aligned} \tag{52}$$

Using the fact that

$$\lim_{s\longrightarrow\infty} s\mathscr{G}_2[y_i(t)](s) = y_i(0), i = 1, 2, \tag{53}$$

we get the coefficients $h_{1,0} = 1, h_{2,0} = 1$, and so the k^{th} ARA-RPS solutions of system (51) have the form

Table 1: Numerical results of $y_1(t)$ and $y_2(t)$ for Example 1 at $\alpha = 1$ and $k = 8$.

t_i	Numerical results of $y_1(t)$ Exact solution	ARA-RPS	$\lvert y_1(t) - y_1^8(t)\rvert$
0.16	0.1869616474087370	0.18696164740560864	$3.1283586832 \times 10^{-12}$
0.32	0.4331983443561256	0.4331983427026164	$1.65350916292 \times 10^{-9}$
0.48	0.7462695050583291	0.7462694394452845	$6.56130446641 \times 10^{-8}$
0.64	1.132569735751 8580	1.1325688339164080	$9.01835450096 \times 10^{-7}$
0.80	1.5965053406002512	1.5964984076190478	$6.93298120336 \times 10^{-6}$
0.96	2.1394797299896706	2.1394428275308255	$3.69024588451 \times 10^{-5}$
t_i	Numerical results of $y_2(t)$ Exact solution	ARA-RPS	$\lvert y_2(t) - y_2^8(t)\rvert$
0.16	1.1585219491810106	1.1585219491779821	$3.0284663666 \times 10^{-12}$
0.32	1.3072184491323793	1.3072184475840865	$1.54829282728 \times 10^{-9}$
0.48	1.4334497895786633	1.4334497302105291	$5.93681341865 \times 10^{-8}$
0.64	1.5211592681991748	1.5211584804906808	$7.87708493988 \times 10^{-7}$
0.8 0	1.5505492968074224	1.5505434575238095	$5.83928361286 \times 10^{-6}$
0.96	1.4978601250631105	1.4978301895607207	$2.99355023898 \times 10^{-5}$

Table 2: Approximate solutions of $y_1(t)$ and $y_2(t)$ at $k = 8$, with different values of α, for Example 1.

t_i	Approximate solution of $y_1(t)$ $\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.75$	$\alpha = 0.55$
0.25	0.317672972	0.359660542	0.605763904	1.046207110
0.50	0.790438988	0.874865261	1.318003799	1.834029727
0.75	1.443025426	1.569624175	2.131354212	2.293549329
1.00	2.287301587	2.442844705	2.939834465	2.158261999
t_i	Approximate solution of $y_2(t)$ $\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.75$	$\alpha = 0.55$
0.25	1.244108176	1.264502635	1.338303344	1.292780399
0.50	1.446888951	1.454674061	1.382264212	0.891643391
0.75	1.548982075	1.512267129	1.116486985	0.075214909
1.00	1.468650794	1.348487278	0.461189999	−1.010101627867

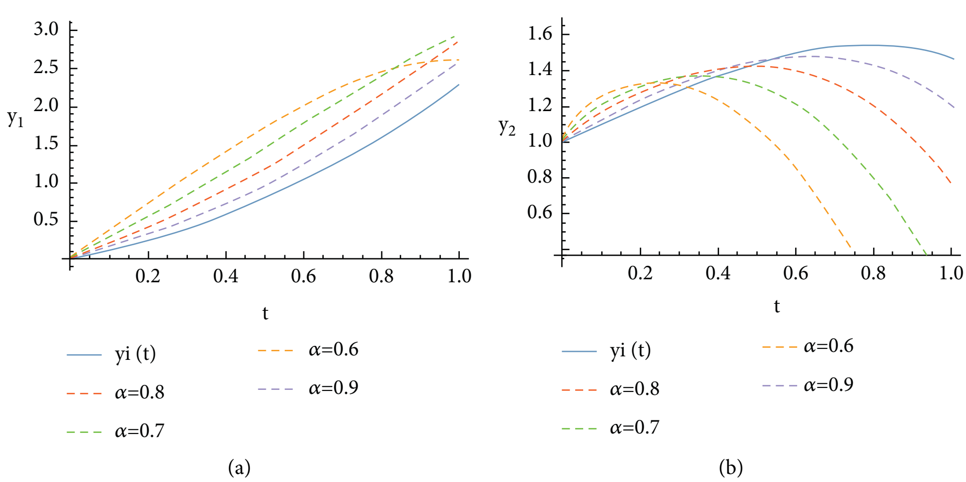

Figure 1: (a) Plots of the exact solution $y_1(t)$ and the approximate solutions $y_1^8(t)$ at various α values and (b) plots of the exact solution $y_2(t)$ and the approximate solutions $y_2^8(t)$ at various α values, for Example 1.

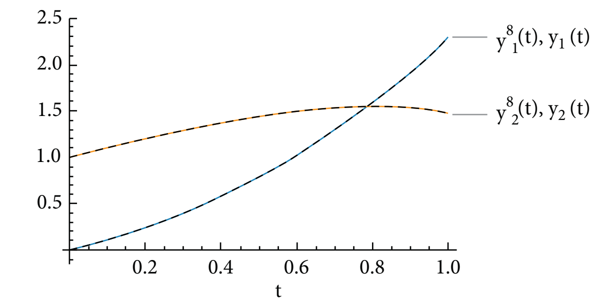

Figure 2: Plots of the exact solutions $(y_1(t), y_2(t))$ and the ARA-RPS solutions $(y_1^8(t), y_2^8(t))$ at $\alpha = 1$ of Example 1.

$$\begin{aligned}
&\mathcal{G}_1[y_1(t)]_k(s) = 1 + \sum_{n=1}^{k} \frac{h_{1,n}}{(n\alpha+1)s^{n\alpha}},\\
&\mathcal{G}_2[y_1(t)]_k(s) = \frac{1}{s} + \sum_{n=1}^{k} \frac{h_{1,n}}{s^{n\alpha+1}},\\
&\mathcal{G}_1[y_2(t)]_k(s) = 1 + \sum_{n=1}^{k} \frac{h_{2,n}}{(n\alpha+1)s^{n\alpha}},\\
&\mathcal{G}_2[y_2(t)]_k(s) = \frac{1}{s} + \sum_{n=1}^{k} \frac{h_{2,n}}{s^{n\alpha+1}}.
\end{aligned} \tag{54}$$

Define the ARA-residual functions of the equations in system (51) as follows:\openup5

$$\begin{cases}
\mathcal{G}_2Res^1(s) = \mathcal{G}_2[y_1(t)](s) - \frac{\alpha}{s}\mathcal{G}_1[y_1(t)](s)\\
+\frac{\alpha-1}{s} + \frac{1002}{s^\alpha}\mathcal{G}_2[y_1(t)](s)\\
-\frac{1000}{s^\alpha}\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}[\mathcal{G}_2[y_2(t)](s)]\right)^2\right](s),\\
\mathcal{G}_2Res^2(s) = \mathcal{G}_2[y_2(t)](s) - \frac{\alpha}{s}\mathcal{G}_1[y_2(t)](s)\\
+\frac{\alpha-1}{s} - \frac{\mathcal{G}_2[y_1(t)](s)}{s^\alpha}\\
+\frac{\mathcal{G}_2[y_2(t)](s)}{s^\alpha} + \frac{\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}[\mathcal{G}_2[y_2(t)](s)]\right)^2\right](s)}{s^\alpha},
\end{cases} \tag{55}$$

and the k^{th} ARA-residual functions of the equations in system (55)

$$\begin{cases}
\mathcal{G}_2Res_k^1(s) = \mathcal{G}_2[y_1(t)]_k(s) - \frac{\alpha}{s}\mathcal{G}_1[y_1(t)]_k(s) + \frac{\alpha-1}{s}\\
+\frac{1002}{s^\alpha}\mathcal{G}_2[y_1(t)]_k(s)\\
-\frac{1000}{s^\alpha}\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}[\mathcal{G}_2[y_2(t)]_k(s)]\right)^2\right]_k(s),\\
\mathcal{G}_2Res_k^2(s) = \mathcal{G}_2[y_2(t)]_k(s) - \frac{\alpha}{s}\mathcal{G}_1[y_2(t)]_k(s)\\
+\frac{\alpha-1}{s} - \frac{\mathcal{G}_2[y_1(t)]_k(s)}{s^\alpha}\\
+\frac{\mathcal{G}_2[y_2(t)]_k(s)}{s^\alpha} + \frac{\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}[\mathcal{G}_2[y_2(t)]_k(s)]\right)^2\right]_k(s)}{s^\alpha}.
\end{cases} \tag{56}$$

To find the first unknown coefficients $h_{1,1}$ and $h_{2,1}$ of the series expansions (54), we substitute the first truncated expansions $\mathcal{G}_1[y_1(t)]_1(s)$, $\mathcal{G}_1[y_2(t)]_1(s)$, $\mathcal{G}_2[y_1(t)]_1(s)$, and $\mathcal{G}_2[y_2(t)]_1(s)$ into the first ARA-residual functions $\mathcal{G}_2Res_1^1(s)$ and $\mathcal{G}_2Res_1^2(s)$ in system (56) to get

$$\begin{cases}
\mathcal{G}_2Res_1^1(s) = \frac{1}{s} + \frac{h_{1,1}}{s^{\alpha+1}} - \frac{\alpha}{s}\left(1 + \frac{h_{1,1}}{(\alpha+1)s^\alpha}\right) + \frac{\alpha-1}{s}\\
+\frac{1002}{s^\alpha}\left(\frac{1}{s} + \frac{h_{1,1}}{s^{\alpha+1}}\right) - \frac{1000}{s^\alpha}\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}\left[\frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}}\right]\right)^2\right]_1(s),\\
\mathcal{G}_2Res_1^2(s) = \frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}} - \frac{\alpha}{s}\left(1 + \frac{h_{2,1}}{(\alpha+1)s^\alpha}\right) + \frac{\alpha-1}{s} - \frac{1}{s^\alpha}\left(\frac{1}{s} + \frac{h_{1,1}}{s^{\alpha+1}}\right)\\
+\frac{1}{s^\alpha}\left(\frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}}\right) + \frac{1}{s^\alpha}\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}\left[\frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}}\right]\right)^2\right]_1(s),
\end{cases} \tag{57}$$

which can be written as

$$\begin{cases}
\mathcal{G}_2Res_1^1(s) = \frac{h_{1,1}}{s^{\alpha+1}} - \frac{\alpha h_{1,1}}{(\alpha+1)s^{\alpha+1}} + \frac{1002}{s^{\alpha+1}} + \frac{1002h_{1,1}}{s^{2\alpha+1}}\\
-\frac{1000}{s^\alpha}\mathcal{G}_2\left[1 + \frac{2t^\alpha h_{2,1}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}h_{2,1}^2}{\Gamma^2(\alpha+1)}\right]_1(s),\\
\mathcal{G}_2Res_1^2(s) = \frac{h_{2,1}}{s^{\alpha+1}} - \frac{\alpha h_{2,1}}{(\alpha+1)s^{\alpha+1}} - \frac{h_{1,1}}{s^{2\alpha+1}} + \frac{h_{2,1}}{s^{2\alpha+1}}\\
+\frac{1}{s^\alpha}\mathcal{G}_2\left[t + \frac{2t^\alpha h_{2,1}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}h_{2,1}^2}{\Gamma^2(\alpha+1)}\right]_1(s).
\end{cases} \tag{58}$$

Multiplying both sides of equations in system (58) by $s^{\alpha+1}$ and taking the limit as $s \longrightarrow \infty$, one can get

$$\begin{aligned} h_{1,1} &= -2(\alpha+1), \\ h_{2,1} &= -1(\alpha+1). \end{aligned} \tag{59}$$

To find out the second unknown coefficients $h_{1,2}$ and $h_{2,2}$ of the series expansions (54), we substitute the second truncated expansions $\mathcal{G}_1[y_1(t)]_2(s)$, $\mathcal{G}_1[y_2(t)]_2(s)$, $\mathcal{G}_2[y_1(t)]_2(s)$, and $\mathcal{G}_2[y_2(t)]_2(s)$ into the second ARA-residual functions $\mathcal{G}_2 Res_2^1(s)$ and $\mathcal{G}_2 Res_2^2(s)$ in system (56) to get

$$\begin{aligned} \mathcal{G}_2 Res_2^1(s) &= \frac{1}{s} + \frac{h_{1,1}}{s^{\alpha+1}} + \frac{h_{1,2}}{s^{2\alpha+1}} - \frac{\alpha}{s}\left(1 + \frac{h_{1,1}}{(\alpha+1)s^{\alpha}} + \frac{h_{1,2}}{(2\alpha+1)s^{2\alpha}}\right) + \frac{\alpha-1}{s} \\ &\quad + \frac{1002}{s^{\alpha}}\left(\frac{1}{s} + \frac{h_{1,1}}{s^{\alpha+1}} + \frac{h_{1,2}}{s^{2\alpha+1}}\right) - \frac{1000}{s^{\alpha}}\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}\left[\frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}} + \frac{h_{2,2}}{s^{2\alpha+1}}\right]\right)^2\right]_1(s), \\ \mathcal{G}_2 Res_2^2(s) &= \frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}} + \frac{h_{2,2}}{s^{2\alpha+1}} - \frac{\alpha}{s}\left(1 + \frac{h_{2,1}}{(\alpha+1)s^{\alpha}} + \frac{h_{2,2}}{(2\alpha+1)s^{2\alpha}}\right) + \frac{\alpha-1}{s} - \frac{1}{s^{\alpha}}\left(\frac{1}{s} + \frac{h_{1,1}}{s^{\alpha+1}} + \frac{h_{1,2}}{s^{2\alpha+1}}\right) \\ &\quad + \frac{1}{s^{\alpha}}\left(\frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}} + \frac{h_{2,2}}{s^{2\alpha+1}}\right) + \frac{1}{s^{\alpha}}\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}\left[\frac{1}{s} + \frac{h_{2,1}}{s^{\alpha+1}} + \frac{h_{2,2}}{s^{2\alpha+1}}\right]\right)^2\right]_1(s). \end{aligned} \tag{60}$$

Simplifying equations in system (60), we have

$$\begin{aligned} \mathcal{G}_2 Res_2^1(s) &= \frac{1}{s^{\alpha+1}}\left(h_{1,1} - \frac{h_{1,1}}{\alpha+1} + 2\right) + \frac{h_{1,2}}{s^{2\alpha+1}} - \frac{\alpha h_{1,2}}{(2\alpha+1)s^{2\alpha+1}} + \frac{1002 h_{1,1}}{s^{2\alpha+1}} + \frac{1002 h_{1,2}}{s^{3\alpha+1}} \\ &\quad - \frac{1000}{s^{\alpha}}\mathcal{G}_2\left[t^2 + \frac{2t^{1+\alpha}h_{2,1}}{\Gamma(1+\alpha)} + \frac{2t^{1+2\alpha}h_{2,1}}{\Gamma(1+2\alpha)} + \frac{t^{2\alpha}h_{2,1}^2}{\Gamma^2(1+\alpha)} + \frac{t^{4\alpha}h_{2,1}^2}{\Gamma^2(1+2\alpha)} + \frac{2t^{3\alpha}h_{2,1}^2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right]_1(s), \\ \mathcal{G}_2 Res_2^2(s) &= \frac{1}{s^{\alpha+1}}\left(h_{2,1} - \frac{\alpha h_{2,1}}{\alpha+1} + 1\right) + \frac{1}{s^{2\alpha+1}}\left(h_{2,2} - \frac{\alpha h_{2,2}}{2\alpha+1} - h_{1,1} + h_{2,1}\right) + \frac{1}{s^{3\alpha+1}}\left(h_{1,2} + h_{2,2}\right) \\ &\quad + \frac{1}{s^{\alpha}}\mathcal{G}_2\left[t^2 + \frac{2t^{1+\alpha}h_{2,1}}{\Gamma(1+\alpha)} + \frac{2t^{1+2\alpha}h_{2,1}}{\Gamma(1+2\alpha)} + \frac{t^{2\alpha}h_{2,1}^2}{\Gamma^2(1+\alpha)} + \frac{t^{4\alpha}h_{2,1}^2}{\Gamma^2(1+2\alpha)} + \frac{2t^{3\alpha}h_{2,1}^2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right]_1(s). \end{aligned} \tag{61}$$

Multiplying both sides of equations in system (61) by $s^{2\alpha+1}$ and taking the limit as $s \longrightarrow \infty$, we have

$$\begin{aligned} h_{1,2} &= 4(2\alpha+1), \\ h_{2,2} &= 2\alpha+1. \end{aligned} \tag{62}$$

We obtain the coefficients $h_{1,n}$ and $h_{2,n}$, $n = 3, 4, \ldots$, in the expansions (54), by solving the system

$$\lim_{s\longrightarrow\infty} s^{k\alpha+1}\mathcal{G}_2 Res_k^1(s) = 0, \tag{63}$$

and

$\lim_{s\longrightarrow\infty} s^{k\alpha+1}\mathcal{G}_2 Res_k^2(s) = 0$, for $k = 3, 4, \ldots$, which lead to

$$
\begin{aligned}
h_{1,3} &= -8(3\alpha+1)\left(251 - \frac{125\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right),\\
h_{2,3} &= (3\alpha+1)\left(1 - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right),\\
h_{1,4} &= (4\alpha+1)\left(2014016 - \frac{1004000\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - \frac{2000\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right),\\
h_{2,4} &= (4\alpha+1)\left(2011 + \frac{1003\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right),\\
h_{1,5} &= (5\alpha+1)\left(-2022066032 + \frac{1008014000\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2008000\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{1000\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)}\right.\\
&\quad \left. - \frac{2000\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)} + \frac{2000\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)}\right),\\
h_{2,5} &= (5\alpha+1)\left(2020049 - \frac{1007009\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - \frac{2006\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} - \frac{\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)}\right.\\
&\quad \left. + \frac{2\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)} - \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)}\right).
\end{aligned} \tag{64}
$$

Consequently, the ARA-RPS solutions of system (51) can be expressed as

$$
\begin{cases}
\mathcal{G}_2[y_1(t)](s) = \dfrac{1}{s} - \dfrac{2(\alpha+1)}{s^{\alpha+1}} + \dfrac{4(2\alpha+1)}{s^{2\alpha+1}} \\
\quad - \dfrac{8(3\alpha+1)}{s^{3\alpha+1}}\left(251 - \dfrac{125\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) + \dots, \\
\mathcal{G}_2[y_2(t)](s) = \dfrac{1}{s} - \dfrac{\alpha+1}{s^{\alpha+1}} - \dfrac{2\alpha+1}{s^{2\alpha+1}} \\
\quad + \dfrac{3\alpha+1}{s^{3\alpha+1}}\left(1 - \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) + \dots.
\end{cases} \tag{65}
$$

To get the series solutions in the original space of system (47), we apply the inverse ARA transform of order two $\mathcal{G}_2^{-1}$ to both sides of equations in system (65) to get

$$
\begin{cases}
y_1(t) = 1 - \dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} \\
\quad + \left(\dfrac{1000\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} - \dfrac{2008}{\Gamma(3\alpha+1)}\right)t^{3\alpha} + \cdots, \\
y_2(t) = 1 - \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
\quad + \left(\dfrac{1}{\Gamma(3\alpha+1)} + \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right)t^{3\alpha} + \dots.
\end{cases} \tag{66}
$$

For $\alpha = 1$, the series expansions (66) have the forms, $y_1(t) = e^{-2t}$, $y_2(t) = e^{-t}$ which are coincide with the exact solutions for the ordinary system (47) and (48).

Numerical results of Example 2 with $k = 6$, and different values of α are listed in Table 3. In addition, the comparison of the results obtained for the exact solutions corresponding to $\alpha = 1$ and the numerical solutions given by the ARA-RPS method for different values of α: $\alpha = 0.6$, $\alpha = 0.7$ and $\alpha = 0.8$ are plotted in Figure 3. Figure 4 portrays a very precise agreement of the exact solutions $(y_1(t), y_2(t))$ and the ARA-RPS solutions $(y_1^6(t), y_2^6(t))$ of Example 2 at different time levels with fixed α. The obtained results show that the ARA-RPS method is efficient in obtaining approximate solutions of systems of FDEs. This is obvious in the following tables, which illustrate the small errors between the ARA-RPS solutions and the exact ones.

Example 3. Consider the following nonlinear system of ordinary FDEs

$$
\begin{cases}
D^{\alpha} y_1(t) = y_1(t), \\
D^{\alpha} y_2(t) = 2y_1^2(t), \\
D^{\alpha} y_3(t) = 3y_1(t)y_2(t),
\end{cases} \tag{67}
$$

with the ICs

$$
y_1(0) = y_2(0) = y_3(0) = 1. \tag{68}
$$

Solution. For each $t \in [0,1]$, $0 < \alpha \le 1$. The exact solution of the nonlinear system (67) and (68), when $\alpha = 1$, is $y_1(t) = e^t$, $y_2(t) = e^{2t}$ and $y_3(t) = e^{3t}$.

Applying the ARA transform of order two $\mathcal{G}_2$ on both sides of equations in system (67)

Table 3: Numerical results of $y_1(t)$ and $y_2(t)$ in Example 2 at $\alpha = 1$ and $k = 6$.

t_i	Numerical results of $y_1(t)$		
	Exact solution	ARA-RPS	$\lvert y_1(t) - y_1^6(t)\rvert$
0.15	0.7408182206817179	0.74081826249999990	$4.18182820594 \times 10^{-8}$
0.30	0.5488116360940265	0.54881680000000000	$5.16390597349 \times 10^{-6}$
0.45	0.4065696597405990	0.40665486250000005	$8.52027594009 \times 10^{-5}$
0.60	0.3011942119122020	0.30181120000000006	$6.16988087798 \times 10^{-4}$
0.75	0.2231301601484298	0.22597656249999998	$2.84640235157 \times 10^{-3}$
0.90	0.1652988882215860	0.17517520000000003	$9.87631177841 \times 10^{-3}$
t_i	Numerical results of $y_2(t)$		
	Exact solution	ARA-RPS	$\lvert y_2(t) - y_2^6(t)\rvert$
0.15	0.8607079764250578	0.8607079767578124	$3.3275460165 \times 10^{-10}$
0.30	0.7408182206817179	0.7408182624999999	$4.18182820594 \times 10^{-8}$
0.45	0.6376281516217733	0.6376288533203125	$7.01698539163 \times 10^{-7}$
0.60	0.5488116360940265	0.5488168000000000	$5.16390597349 \times 10^{-6}$
0.75	0.4723665527410147	0.4723907470703125	$2.41943292978 \times 10^{-5}$
0.90	0.4065696597405991	0.4066548625000000	$8.52027594009 \times 10^{-5}$

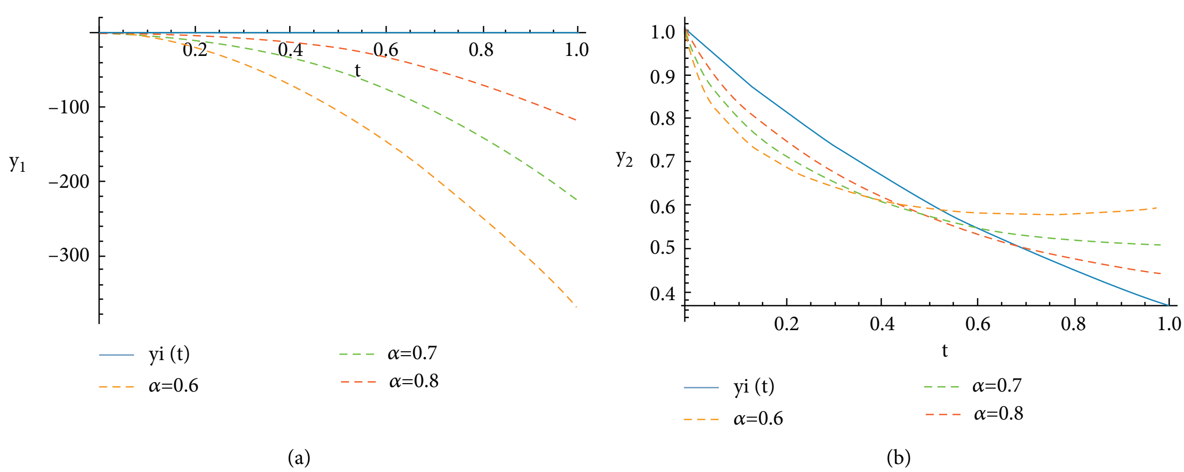

Figure 3: (a) Plots of the exact solution $y_1(t)$ and the approximate solution $y_1^6(t)$ at various α values and (b) plots of the exact solution $y_2(t)$ and the approximate solution $y_2^6(t)$ at various α values, for Example 2.

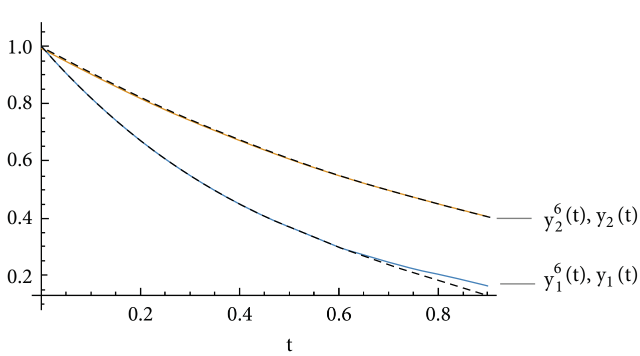

Figure 4: Plots of the exact solutions $(y_1(t), y_2(t))$ and the ARA-RPS solutions $(y_1^6(t), y_2^6(t))$ at $\alpha = 1$ of Example 2.

$$\begin{cases} \mathcal{G}_2[D^\alpha y_1(t)](s) = \mathcal{G}_2[y_1(t)](s), \\ \mathcal{G}_2[D^\alpha y_2(t)](s) = 2\mathcal{G}_2\left[y_1^2(t)\right](s), \\ \mathcal{G}_2[D^\alpha y_3(t)](s) = 3\mathcal{G}_2[y_1(t)y_2(t)](s). \end{cases} \tag{69}$$

Running the ARA transform on the equations in system (69), we have

$$\begin{cases} s^\alpha\mathcal{G}_2[y_1(t)](s) - \alpha s^{\alpha-1}\mathcal{G}_1[y_1(t)](s) + (\alpha-1)s^{\alpha-1}y_1(0) \\ = \mathcal{G}_2[y_1(t)](s), \\ s^\alpha\mathcal{G}_2[y_2(t)](s) - \alpha s^{\alpha-1}\mathcal{G}_1[y_2(t)](s) + (\alpha-1)s^{\alpha-1}y_2(0) \\ = 2\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}[\mathcal{G}_2[y_1(t)]]\right)^2\right](s), \\ s^\alpha\mathcal{G}_2[y_3(t)](s) - \alpha s^{\alpha-1}\mathcal{G}_1[y_3(t)](s) + (\alpha-1)s^{\alpha-1}y_3(0)(s) \\ = 2\mathcal{G}_3\left[\mathcal{G}_2^{-1}[\mathcal{G}_2[y_1(t)]]\,\mathcal{G}_2^{-1}[\mathcal{G}_2[y_2(t)]]\right]. \end{cases} \tag{70}$$

Substituting the ICs (67) and simplifying system (69), we get\

$$\begin{cases} \mathcal{G}_2[y_1(t)](s) - \frac{\alpha}{s}\mathcal{G}_1[y_1(t)](s) + \frac{\alpha-1}{s} + \frac{1}{s^\alpha}\mathcal{G}_2[y_1(t)](s) = 0, \\ \mathcal{G}_2[y_2(t)](s) - \frac{\alpha}{s}\mathcal{G}_1[y_2(t)](s) + \frac{\alpha-1}{s} + 2\frac{\mathcal{G}_2\left[\left(\mathcal{G}_2^{-1}[\mathcal{G}_2[y_1(t)]]\right)^2\right](s)}{s^\alpha} = 0, \\ \mathcal{G}_2[y_3(t)](s) - \frac{\alpha}{s}\mathcal{G}_1[y_3(t)](s) + \frac{\alpha-1}{s} + 3\frac{\mathcal{G}_2\left[\mathcal{G}_2^{-1}[\mathcal{G}_2[y_1(t)]]\,\mathcal{G}_2^{-1}[\mathcal{G}_2[y_2(t)]]\right](s)}{s^\alpha} = 0. \end{cases} \tag{71}$$

\

Assuming that the ARA-RPS solutions of system (71) have the series representations

$$\begin{aligned} \mathcal{G}_1[y_1(t)](s) &= \sum_{n=0}^{\infty}\frac{h_{1,n}}{(n\alpha+1)s^{n\alpha}}, \quad \mathcal{G}_2[y_1(t)](s) = \sum_{n=0}^{\infty}\frac{h_{1,n}}{s^{n\alpha+1}}, \\ \mathcal{G}_1[y_2(t)](s) &= \sum_{n=0}^{\infty}\frac{h_{2,n}}{(n\alpha+1)s^{n\alpha}}, \quad \mathcal{G}_2[y_2(t)](s) = \sum_{n=0}^{\infty}\frac{h_{2,n}}{s^{n\alpha+1}}, \\ \mathcal{G}_1[y_3(t)](s) &= \sum_{n=0}^{\infty}\frac{h_{3,n}}{(n\alpha+1)s^{n\alpha}}, \quad \mathcal{G}_2[y_3(t)](s) = \sum_{n=0}^{\infty}\frac{h_{2,n}}{s^{n\alpha+1}}. \end{aligned} \tag{72}$$

Using the fact that

$$\lim_{s\longrightarrow\infty} s\mathcal{G}_2[y_i(t)](s) = y_i(0), \quad i = 1,2,3, \tag{73}$$

we get the coefficients $h_{1,0} = 1,\ h_{2,0} = 1,\ h_{3,0} = 1$, and so the k^{th} ARA-RPS solutions of system (71) have the form

$$\begin{aligned} \mathcal{G}_1[y_1(t)]_k(s) &= 1 + \sum_{n=1}^{k}\frac{h_{1,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathcal{G}_2[y_1(t)]_k(s) &= \frac{1}{s} + \sum_{n=1}^{k}\frac{h_{1,n}}{s^{n\alpha+1}}, \\ \mathcal{G}_1[y_2(t)]_k(s) &= 1 + \sum_{n=1}^{k}\frac{h_{2,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathcal{G}_2[y_2(t)]_k(s) &= \frac{1}{s} + \sum_{n=1}^{k}\frac{h_{2,n}}{s^{n\alpha+1}}, \\ \mathcal{G}_1[y_3(t)](s) &= 1 + \sum_{n=1}^{k}\frac{h_{3,n}}{(n\alpha+1)s^{n\alpha}}, \\ \mathcal{G}_2[y_3(t)](s) &= \frac{1}{s} + \sum_{n=1}^{k}\frac{h_{3,n}}{s^{n\alpha+1}}, \end{aligned} \tag{74}$$

where the coefficients $h_{1,n}$, $h_{2,n}$, and $h_{3,n}$ could be obtained by considering the following ARA-residual functions of the equations in system (71), we get

$$\begin{cases} \varsigma_2\mathrm{Res}^1(s) = \varsigma_2[y_1(t)](s) - \frac{\alpha}{s}\varsigma_1[y_1(t)](s) \\ +\frac{\alpha-1}{s} - \frac{1}{s^\alpha}\varsigma_2[y_1(t)](s), \\ \varsigma_2\mathrm{Res}^2(s) = \varsigma_2[y_1(t)](s) - \frac{\alpha}{s}\varsigma_1[y_2(t)](s) \\ +\frac{\alpha-1}{s} - 2\varsigma_2\frac{\varsigma_2\left[\left(\varsigma_2^{-1}[\varsigma_2[y_1(t)](s)]\right)^2(s)\right]}{s^\alpha}, \\ \varsigma_2\mathrm{Res}^3(s) = \varsigma_3[y_1(t)](s) - \frac{\alpha}{s}\varsigma_1[y_2(t)](s) \\ +\frac{\alpha-1}{s} - 3\frac{\varsigma_2\left[\left(\varsigma_2^{-1}[\varsigma_2[y_1(t)](s)]\right)\varsigma_2^{-1}[\varsigma_2[y_1(t)](s)](s)\right]}{s^\alpha}, \end{cases} \tag{75}$$

and the k^{th} ARA-residual functions of the equations in system (75)

$$\begin{cases} \varsigma_2\mathrm{Res}_k^1(s) = \varsigma_2[y_1(t)]_k(s) - \frac{\alpha}{s}\varsigma_1[y_1(t)]_k(s) \\ +\frac{\alpha-1}{s} - \frac{1}{s^\alpha}\varsigma_2[y_1(t)]_k(s), \\ \varsigma_2\mathrm{Res}_k^1(s) = \varsigma_2[y_1(t)]_k(s) - \frac{\alpha}{s}\varsigma_1[y_2(t)]_k(s) \\ +\frac{\alpha-1}{s} - 2\frac{\varsigma_2\left[\left[\left(\varsigma_2^{-1}[\varsigma_2[y_1(t)]_k(s)]\right)^2\right]_k(s)\right]}{s^\alpha}, \\ \varsigma_2\mathrm{Res}_k^3(s) = \varsigma_3[y_1(t)]_k(s) - \frac{\alpha}{s}\varsigma_1[y_2(t)]_k(s) + \frac{\alpha-1}{s} \\ -3\frac{\varsigma_2\left[\left(\varsigma_2^{-1}[\varsigma_2[y_1(t)]_k(s)]\right)\varsigma_2^{-1}[\varsigma_2[y_1(t)]_k(s)]_k(s)\right]}{s^\alpha}. \end{cases} \tag{76}$$

Table 4: Numerical results of $y_1(t)$, $y_2(t)$ and $y_3(t)$ in Example 3 at $\alpha = 1$ and $k = 6$.

t_i	Numerical results of $y_1(t)$		
	Exact solution	ARA-RPS	$\lvert y_1(t) - y_1^6(t)\rvert$
0.1	1.105170918075648	1.105170918055556	$2.0091928121 \times 10^{-11}$
0.2	1.221402758160169	1.221402755555556	$2.60461430202 \times 10^{-9}$
0.3	1.349858807576003	1.349858762500000	$4.50760029302 \times 10^{-8}$
0.4	1.491824697641270	1.491824355555556	$3.42085714866 \times 10^{-7}$
0.5	1.648721270700128	1.648719618055555	$1.65264457275 \times 10^{-6}$
0.6	1.822118800390509	1.822112800000000	$6.00039050899 \times 10^{-6}$
t_i	Numerical results of $y_2(t)$		
	Exact solution	ARA-RPS	$\lvert y_2(t) - y_2^6(t)\rvert$
0.1	1.221402758160169	1.221402755555556	$2.60461430202 \times 10^{-9}$
0.2	1.491824697641270	1.491824355555556	$3.42085714866 \times 10^{-7}$
0.3	1.822118800390509	1.822112800000000	$6.00039050913 \times 10^{-6}$
0.4	2.225540928492468	2.225494755555556	$4.61729369121 \times 10^{-5}$
0.5	2.718281828459045	2.718055555555555	$2.26272903489 \times 10^{-4}$
0.6	3.320116922736547	3.319283200000000	$8.33722736547 \times 10^{-4}$
t_i	Numerical results of $y_3(t)$		
	Exact solution	ARA-RPS	$\lvert y_3(t) - y_3^6(t)\rvert$
0.1	1.349858807576003	1.349858762500000	$4.50760029302 \times 10^{-7}$
0.2	1.822118800390509	1.822112800000000	$6.00039050913 \times 10^{-6}$
0.3	2.459603111156950	2.459496362500000	$1.06748656949 \times 10^{-5}$
0.4	3.320116922736548	3.319283200000000	$8.33722736548 \times 10^{-4}$
0.5	4.481689070338065	4.477539062500000	$4.15000783806 \times 10^{-3}$
0.6	6.049647464412945	6.034103200000000	$1.55442644129 \times 10^{-2}$

Using the fact that $\lim_{s\longrightarrow\infty} s^{k\alpha+1}\mathcal{G}_2 Res_k^i(s) = 0$ for $k = 1, 2, \ldots$ and $i = 1, 2, 3$.

We can find out the coefficients, $h_{1,n}$, $h_{2,n}$, and $h_{3,n}$ as follows:

$$\begin{cases} h_{1,1} = \alpha + 1, \\ h_{2,1} = 2(\alpha + 1), \\ h_{3,1} = 3(\alpha + 1), \end{cases}$$

$$\begin{cases} h_{1,2} = 2\alpha + 1, \\ h_{2,2} = 4(2\alpha + 1), \\ h_{3,2} = 9(2\alpha + 1), \end{cases}$$

$$\begin{cases} h_{1,3} = 3\alpha + 1, \\ h_{2,3} = 4(3\alpha + 1), \\ h_{3,3} = 4(3\alpha + 1)\left(15 + \dfrac{6\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)}\right), \end{cases}$$

$$\begin{cases} h_{1,4} = 4\alpha + 1, \\ h_{2,4} = (4\alpha + 1)\left(4 + \dfrac{4\Gamma(3\alpha + 1)}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)}\right), \\ h_{3,4} = (4\alpha + 1)\left(15 + \dfrac{6\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)} + \dfrac{18\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)}\right), \end{cases}$$

$$\begin{cases} h_{1,5} = 5\alpha + 1, \\ h_{2,5} = (5\alpha + 1)\left(4 + \dfrac{2\Gamma(3\alpha + 1)}{\Gamma^2(2\alpha + 1)} + \dfrac{4\Gamma(3\alpha + 1)}{\Gamma(\alpha + 1)\Gamma(3\alpha + 1)}\right), \\ h_{3,5} = (5\alpha + 1)\left(15 + \dfrac{12\Gamma(3\alpha + 1)}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} + \dfrac{12\Gamma(4\alpha + 1)}{\Gamma^2(2\alpha + 1)}, \right. \\ \left. + \dfrac{18\Gamma(4\alpha + 1)}{\Gamma(\alpha + 1)\Gamma(3\alpha + 1)} + \dfrac{6\Gamma(2\alpha + 1)\Gamma(4\alpha + 1)}{\Gamma^2(\alpha + 1)\Gamma(3\alpha + 1)}\right). \end{cases} \tag{77}$$

Consequently, the ARA-RPS solutions of system (71) can be expressed as

$$\begin{cases} \mathcal{G}_2[y_1(t)](s) = \dfrac{1}{s} + \dfrac{\alpha + 1}{s^{\alpha+1}} + \dfrac{2\alpha + 1}{s^{2\alpha+1}} + \dfrac{3\alpha + 1}{s^{3\alpha+1}} + \ldots, \\ \mathcal{G}_2[y_2(t)](s) = \dfrac{1}{s} + \dfrac{2(\alpha + 1)}{s^{\alpha+1}} + \dfrac{4(2\alpha + 1)}{s^{2\alpha+1}} \\ + \dfrac{3\alpha + 1}{s^{3\alpha+1}}\left(4 + \dfrac{2\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)}\right) + \ldots, \\ \mathcal{G}_2[y_3(t)](s) = \dfrac{1}{s} + \dfrac{2(\alpha + 1)}{s^{\alpha+1}} + \dfrac{4(2\alpha + 1)}{s^{2\alpha+1}} \\ + \dfrac{3\alpha + 1}{s^{3\alpha+1}}\left(15 + \dfrac{6\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)}\right) + \ldots. \end{cases} \tag{78}$$

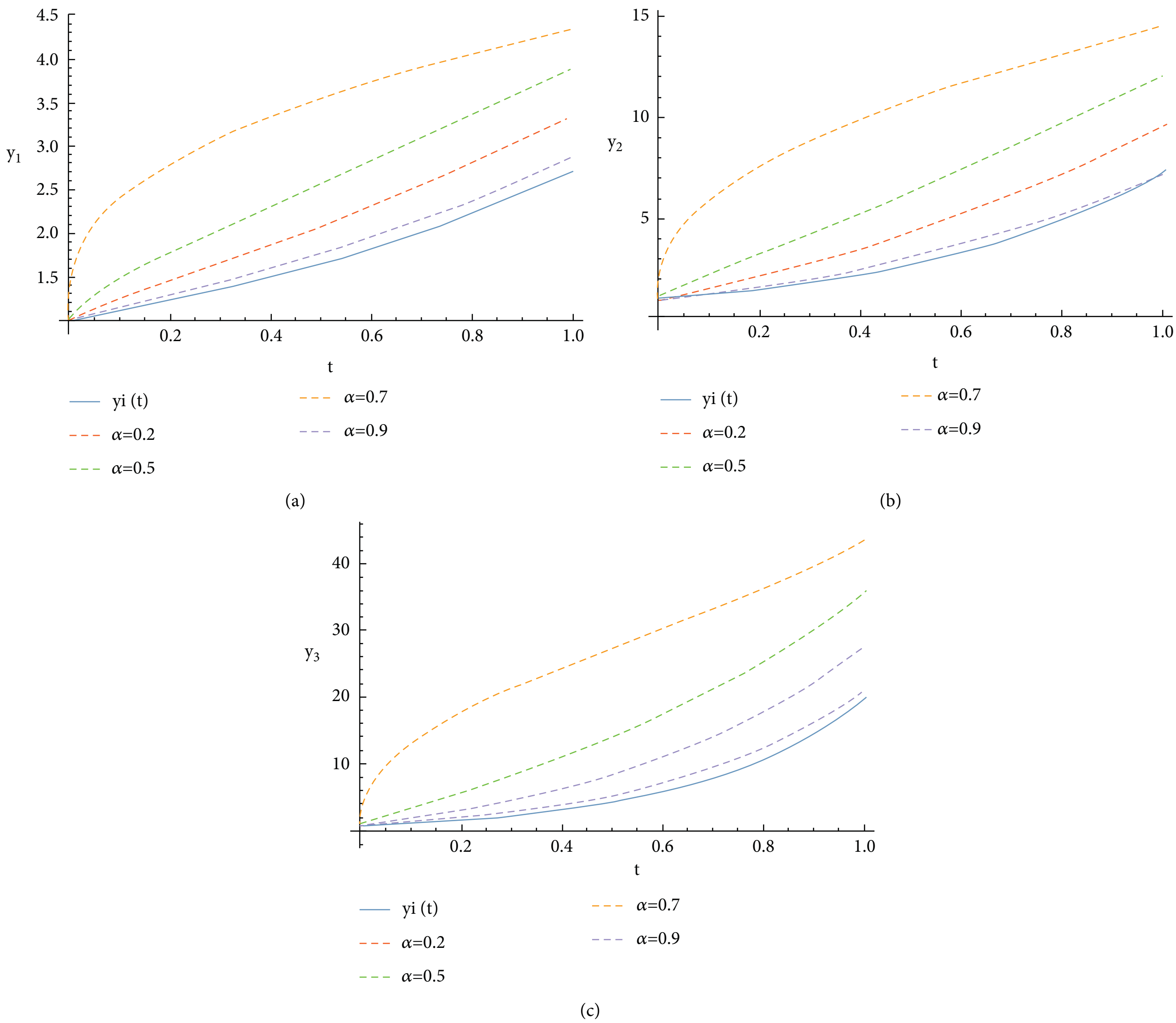

FIGURE 5: (a) Plots of the exact solution $y_1(t)$ and the approximate solution $y_1^6(t)$ at various α values; (b) plots of the exact solution $y_2(t)$ and the approximate solution $y_2^6(t)$ at various α values; and (c) plots of the exact solution $y_3(t)$ and approximate solution $y_3^6(t)$ at various α values, for Example 3.

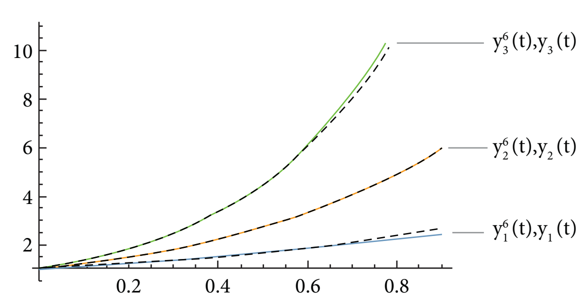

FIGURE 6: Plots of the exact solutions $(y_1(t), y_2(t), y_3(t))$ and the ARA-RPS solutions $(y_1^6(t), y_2^6(t), y_2^6(t))$ at $\alpha = 1$ of Example 3.

To get the series solutions in the original space of system (67), we apply the inverse ARA transform of order two $\mathcal{G}_2^{-1}$ to both sides of equations in system (78) to get

$$\begin{cases} y_1(t) = 1 + \dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots, \\ y_2(t) = 1 + \dfrac{2t^\alpha}{\Gamma(\alpha+1)} + \dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} \\ +\left(\dfrac{4}{\Gamma(3\alpha+1)} + \dfrac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right)t^{3\alpha} + \ldots, \\ y_2(t) = 1 + \dfrac{3t^\alpha}{\Gamma(\alpha+1)} + \dfrac{9t^{2\alpha}}{\Gamma(2\alpha+1)} \\ +\left(\dfrac{15}{\Gamma(3\alpha+1)} + \dfrac{6\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right)t^{3\alpha} + \ldots. \end{cases} \quad (79)$$

For $\alpha = 1$, the series expansions (79) have the form, $y_1(t) = e^t$, $y_2(t) = e^{2t}$, and $y_3(t) = e^{3t}$ which are coincide with the exact solutions for the ordinary system (67) and (68).

Numerical results of Example 3 with $k = 6$, and different values of α are listed in Table 4. In addition, the comparison of the results obtained for the exact solution corresponding to $\alpha = 1$ and the numerical solutions given by the ARA-RPS method for different values of α: $\alpha = 0.2$, $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$ are plotted in Figure 5. Figure 6 portrays a very precise agreement of the exact solutions $(y_1(t), y_2(t), y_3(t))$ and the ARA-RPS solutions $(y_1^6(t), y_2^6(t), y_2^6(t))$ of Example 3 at different time levels with fixed α. The obtained results show that the ARA-RPS method is efficient in obtaining approximate solutions of systems of FDEs. This is obvious in the following tables which illustrate the small errors between the ARA-RPS solutions and the exact ones.

5. Conclusion

Many numerical techniques have been established to investigate numerical solutions of FDEs. Some methods may not provide exact solutions. This research emphasized that the proposed technique, ARA-RPS, is an effective method to construct series solutions that may match with the exact for some systems of FDEs with suitable ICs. The proposed scheme provides us the solutions in a new space "ARA transform space" in which it is easy to determine the unknown constants of the suggested series representations using the concept of the limit. The ARA-RPS solutions are easy to compute with fewer terms. We mention that our results are obtained by using the Mathematica software package 12. As future work, we plan to solve fractional integral equations and systems of fractional partial differential equations with different orders of α using the proposed method [48–50].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no known conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper. The authors declare that there are no conflicts of interest.

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