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Research Article

The Rayleigh–Stokes Problem for a Heated Generalized Second-Grade Fluid with Fractional Derivative: An Implicit Scheme via Riemann–Liouville Integral

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The goal of this study is to use the fast algorithm to solve the Rayleigh–Stokes problem for heated generalized second-grade fluid (RSP-HGSGF) with Riemann–Liouville time fractional derivative using the fast algorithm. The modified implicit scheme, which is formulated by the Riemann–Liouville integral formula and applied to the fractional RSP-HGSGF, is proposed. Numerical experiments will be carried out to demonstrate that the scheme is simple to implement, and the results will reveal the best way to implement the suggested technique. The proposed scheme's stability and convergence will be examined using the Fourier series. The method is stable, and the approximation solution approaches the exact solution. A numerical demonstration will be provided to demonstrate the applicability and viability of the suggested strategy.

1. Introduction

The study and application of arbitrary-order derivatives and integrals are associated with fractional calculus. The use of fractional-order calculus in a variety of fields of science and engineering, including geometric phenomena, has sparked a lot of interest in this area [1]. The first discussion of fractional calculus took place between Leibniz and L'Hospital at the end of the seventeenth century [2]. The great mathematicians Erdelyi, Abel, Riemann, Laplace, Heaviside, Levy, Liouville, Riesz, Gunwald, Letnikov, and Fourier worked on it and had contributed [3]. Fractional-order integrals and derivatives play an important role in solving some chemical problems, and this field has been paid much attention since 1968. The most well-known book in the field of fractional calculus, originally written by Ross and Miller and Ross [4], Spanier and Oldham [5], Podlubny [6], and Samko et al. [7], explains the underlying theory of fractional calculus as well as its applications and solutions.

Many researchers have solved fractional-order problems using various methods. For example, Shivanian and Jafferabadi [8] used fractional derivatives to find the numerical solution for the RSP-HGSGF using spectral meshless radial point interpolation. The time-fractional derivative has been defined in the Riemann–Liouville sense. The Shape functions are created by using a point interpolation method and radial basis functions as basic functions. An efficient numerical approach for approximating RSP-HGSGF in a bounded domain is described by Liu et al. [9]. They investigated the proposed scheme's stability and convergence. To solve SFP-HGSGF, Wu [10] used a numerical approach. The stability, convergence, and consistency of the INAS for the SFP HGSGF have been investigated. RSPHGSGF was studied in a flow on a heated flat plate and within a heated edge by Shen et al. [11]. A viscoelastic fluid was described using the fractional calculus technique in the constitutive relationship model. For the exact solution of the velocity and temperature fields, the Fourier transform on fractional-order Laplace operator is used. Yu et al. [12] used the Adomian decomposition method to solve the RSP-HGSGF. In general, without discretizing the problem, such series solutions converge quickly and the Adomian decomposition approach yields very precise numerical solutions. In this study, Chen et al. [13] presented two numerical methods for solving a two-dimensional variable-order subdiffusion anomalous problem. Their stability, convergence, and solvability were investigated using Fourier analysis. The numerical approximation for the Riemann–Liouville fractional-order derivative for the fractional SFP-HGSGF was studied by Yu et al. [14]. They used the implicit scheme with Riemann–Liouville fractional derivative to solve the direct and inverse problems. Lin and Jiang [15] devised a straightforward method for calculating the fractional derivative of an RSP-HGSG. They created the series of the exact solution to the problem using kernel theory and established the approximate solution of its fractional derivative using truncating series, which are uniformly convergent. Meanwhile, their method includes error estimation and stability analysis. Chen et al. [16] proposed the implicit and explicit techniques for solving the RSP-HGSGF of fractional order. The convergence, stability, and solvability of the problem have all been determined. In recent years, Chen et al. [17] discussed Stokes' initial challenge attention. The variable-order nonlinear RSP-HGSGF is investigated, and the fourth-order numerical technique is discussed. The Fourier approach is used to investigate the numerical scheme's theoretical analysis. Dehghan and Abbas Zadeh [18] developed a numerical solution for 2D fractional-order RSP-HGSGF on rectangular domains such as circular, L-shaped, and a unit square with circular holes. The RL principle is used to calculate the fractional derivatives. They used the Galerkin FEM to obtain a fully discrete scheme for the space direction by integrating the equation for the time variable. Finally, we compare the results of Galerkin FEM to those of other numerical techniques. The Rayleigh–Stokes problem for an edge in a generalized Oldroyd-B fluid was solved by Nikan and Avazzadeh et al. [19] using the radial basis function and fractional derivatives. The temporal derivative terms are discretized using the finite difference technique, while the spatial derivative terms are discretized using the local RBF-FD.

To maintain a constant number of nodes, they evaluate the distribution of data nodes within the local support area. The stability and convergence of the proposed method are also investigated. The RBF-FD results are compared to those of previous approaches on irregular domains, demonstrating the novel methodology's viability and efficiency. RSP-HGSGF flow was investigated by Zhai et al. [20] on a heated flat plate and within a heated edge. To describe such a viscoelastic fluid, a fractional calculus methodology was used in the constitutive relationship model. The velocity and temperature fields were solved in closed form using the Fourier transform and the fractional Laplace operator. Another study looked at the same model to describe a viscoelastic fluid [21, 22]. For the finite difference/finite element technique, Guan et al. [23] provided an enhanced version of a nonlinear source term with a fractional RSP. The backward difference formula and second-order Grünwald–Letnikov derivative are used to discretize the first-order time derivative. They use the Galerkin finite element approach to define a fully discrete strategy for the fractional RSP-HGSGF with a nonlinear source term in the space direction. A novel analytical technique is used to calculate the level of accuracy in the L2 norm in great detail. For the 2D modified anomalous fractional subdiffusion equation, Ali et al. [24] used a modified implicit difference approximation. The proposed scheme's convergence and stability are investigated using the Fourier series approach. It is shown that the scheme is unconditionally stable, and that the approximate solution converges to the exact solution. Bazhlekova et al. [25] investigated the RSP-HGSGF in time using the RL fractional derivative, and the problem was analysed in space using semidiscrete, continuous, and completely discrete formulations. Mohebbi et al. [26] compared the meshless approach to a fourth-order approximation for 2D fractional RSP and generated a completely discrete implicit scheme. Sun et al [27] contributed a review article on important fractional calculus information. They talked about the most important real-world applications as well as powerful mathematical tools. The numerical solution of a nonlinear fractional-order reaction-subdiffusion model was investigated by Nikan et al. [28]. For spatial discretization, they used the radial base function-finite difference method, and for time discretization, they used a weighted discrete scheme. They discussed theoretical analysis and tested two numerical examples for the computational efficiency of the proposed scheme, which yielded accurate results. In a separate study [29], the author proposed a meshless scheme for the fractional-order diffusion model. They eliminated the time derivative by integrating both sides of the proposed model and used local hybridization of cubic and radial basis functions for space derivatives. Nikan et al. [30] investigated the local hybrid kernel meshless approach for fractional-order model approximation. To approximate the time and space directions, they used the central difference approximation and Gaussian kernels, respectively. They verified the validity of the proposed method using numerical examples that are both accurate and efficient. Liu et al. [31] discussed the fractional dynamics modelled from the fractional-order PDEs. Fractional-order systems have importance in the field of electrochemistry, chaotic systems, biology etc. Ahmad et al. [32] formulated a new methodology named as variational iteration method I and successfully applied to a nonlinear model. They explained the compactness of the method and compared their results with the existed literature and found that the proposed method is more productive and reliable than others. Khan et al. [33] considered the numerical approach based on the collocation method for the inverse heat source problem and tasted the method both on regular and irregular domain. Different researchers discussed various numerical approaches for time and space fractional-order models in the research [27, 30, 34–38]. The goal of this research is to propose a new scheme for this model modified implicit scheme for fractional-order RSP-HGSGF. It lowers the computational cost and allows for easy theoretical analysis using any method for the final scheme. In the procedure, the discretized form of the Riemann–Liouville integral operator is used to replace the Riemann–Liouville derivative with the first-order time derivative. The partial derivative with respect to time is then eliminated using backward difference approximation. Additionally, we use the Fourier series method to investigate the established method's stability and convergence criterion. Finally, numerical examples are presented and solved using the proposed method to verify the method's accuracy and feasibility. Maple 15 is used to code the numerical examples.

The following is how the rest of the paper is organized: The methodology of the proposed scheme is discussed in Section 2, followed by stability and convergence analysis in Sections 2.1 and 2.2. The numerical experiments and results are presented in Section 3 and discussed in Section 4. The conclusion is discussed in Section 5 of the report.

The aim of this study is to propose a modified implicit scheme for fractional RSP-HGSGF based on the formulated Riemann–Liouville integral operator. The partial derivative w.r.t. time is eliminated by backward difference approximation. Additionally, we investigate the stability and convergence criterion of the established method by the Fourier series method.

Here, we consider the following two-dimensional RSP-HGSGF with fractional derivative [22].

$$\frac{\partial Y(x,y,t)}{\partial t} = D_t^{1-\beta}\left(\frac{\partial^2 Y(x,y,t)}{\partial x^2} + \frac{\partial^2 Y(x,y,t)}{\partial y^2}\right) + \frac{\partial^2 Y(x,y,t)}{\partial x^2} + \frac{\partial^2 Y(x,y,t)}{\partial y^2} + H(x,y,t). \tag{1}$$

Initial and boundary conditions are as follows:

$$Y(x,y,t) = \varphi(x,y), \tag{2}$$

$$\begin{aligned} Y(0,y,t) &= \Omega_1(y,t), \mathrm{Y}(\mathrm{L},\mathrm{y},\mathrm{t}) \\ &= \Omega_2(y,t), \\ Y(x,0,t) &= \Omega_3(y,t) Y(x,L,t) \\ &= \Omega_4(x,t), \\ 0 \le x,\, y &\le L, 0 \le t \le T, \end{aligned} \tag{3}$$

where ${}_0D_t^{1-\beta} Y(x,y,t)$ represents the fractional-order Riemann–Liouville derivative of order $1-\beta$.

Lemma 1. *The β $(0\langle\beta\langle 1)$-order Riemann–Liouville fractional integral of the function $Y(x,y,t)$ on [0,T] can be defined in discretized form as*

$$I_0^{\beta} Y(x,y,t_m) = \frac{\tau^{\beta}}{\Gamma(\beta+1)} \sum_{j=0}^{m-1} d_j^{(\beta)} Y(x,y,t_{m-j}). \tag{4}$$

Lemma 2. *The coefficients constant $d_m^{(\beta)}$ $(m = 0, 1, 2, \ldots.)$ fulfils the following properties [29]:*

(i) $d_0^{\beta} = 1, \quad d_m^{\beta}\rangle 0,\ m = 0,1,2\ldots$

(ii) $d_{m-1}^{\beta}\rangle d_m^{\beta}, \quad m = 1,2\ldots$

(iii) There exists a positive constant $C\rangle 0$, such that $\tau \le C d_m^{\beta}\tau^{\beta}, \quad m = 1,2,\ldots$

(iv) $\sum_{j=0}^{m} d_j^{(\beta)}\tau^{\beta} = (m+1)^{\beta} \le T^{\beta}$

2. Methodology of the Proposed Scheme

The 2D RSP-HGSGF in equations (1)–(3) is solved by the modified implicit scheme. We utilized the Riemann–Liouville approximation for time-fractional and central difference for space derivative and partitioned the bounded domain into subintervals of lengths Δx and Δy. The space steps are $x_i = i\Delta x$, in the x-direction with $i = 1,\ldots,M_1 - 1$, $\Delta x = L/M_1$, and $y_j = j\Delta y$, in the y-direction with $j = 1,\ldots, M_2 - 1$, $\Delta y = L/M_2$. The time step is $t_m = m\tau$, $m = 1,\ldots,N$ where $\tau = T/N$. Let $Y_{i,j}^m$ be the numerical approximation to $Y(x_i, y_j, t_m)$; by applying (2) to (1), we obtain

$$\frac{\partial Y(x,y,t)}{\partial t} = \frac{\partial}{\partial t} I_0^{\beta}\left(\frac{\partial^2 Y(x,y,t)}{\partial x^2} + \frac{\partial^2 Y(x,y,t)}{\partial y^2}\right) + \frac{\partial^2 Y(x,y,t)}{\partial x^2} + \frac{\partial^2 Y(x,y,t)}{\partial y^2} + H(x,y,t). \tag{5}$$

Applying Lemma 1 and backward difference approximation w.r.t. time, we obtain

$$Y_{i,j}^m - Y_{i,j}^{m-1} = R_1 \sum_{j=o}^{m-1} d_j^{(\beta)}\left(\delta x^2 Y_{I,j}^{m-j} - \delta x^2 Y_{i,j}^{m-j-i}\right) + R_2 \sum_{j=o}^{m-1} d_j^{(\beta)}\left(\delta x^2 Y_{I,j}^{m-j} - \delta x^2 Y_{i,j}^{m-j-i}\right) + R_3\left(\frac{Y_{i,j}^m}{\Delta x^2}\right) + R_4\left(\frac{Y_{i,j}^m}{\Delta y^2}\right) + H(x,\ y,t), \tag{6}$$

where

$$R_1 = \frac{\tau^\beta}{\Gamma(\beta+1)\Delta x^2}, \quad R_2 = \frac{\tau^\beta}{\Gamma(\beta+1)\Delta y^2}, \quad R_3 = \frac{\tau}{\Delta x^2}, \quad R_4 = \frac{\tau}{\Delta y^2}. \tag{7}$$

$$\delta x^2 Y_{i,j}^m = Y_{i+1,j}^m - 2Y_{i,j}^m + Y_{i-1,j}^m. \tag{8}$$

The simplified form of the proposed scheme for 2D RSP-HGSGF (1)-(3) and the conditions are as follows:

$$Y_{i,j}^m - Y_{i,j}^{m-1} = R_1 \delta x^2 Y_{I,j}^m - R_1 d_{m-1}^{(\beta)} \delta x^2 Y_{I,j}^0 - R_1 \sum_{s=1}^{m-1}\left(d_{s-\beta}^{(\beta)} - d_s^{(\beta)}\right)\delta x^2 Y_{I,j}^{m-s} + R_2 \delta y^2 Y_{I,j}^m + R_2 \sum_{s=1}^{m-1}\left(d_{s-\beta}^{(\beta)} - d_s^{(\beta)}\right)\delta y^2 Y_{I,j}^{m-s} + R_3 \delta x^2 Y_{I,j}^m + R_4 \delta y^2 Y_{I,j}^m + \tau H_{i,j}^m, \tag{9}$$

where $i = 1, 2, \ldots, M_1 - 1$, $j = 1, 2, \ldots, M_2 - 1$, and $m = 1, 2, \ldots, N-1$.

$$\begin{aligned} &Y_{I,j}^0 = \Omega(x_i, y_j),\\ &Y_{0,j}^m = \Omega_1(y_j, t_m),\ Y_{i,0}^m = \Omega_2(x_i, t_m),\\ &Y_{M_1,j}^m = \Omega_3(y_j, t_m),\ Y_{i,M_2}^m = \Omega_4(x_i, t_m),\\ &0 \le x, y \le L,\ 0 \le t \le T. \end{aligned} \tag{10}$$

2.1. Stability. We find the stability of the proposed scheme by Fourier technique. Let the approximate solution be $\Psi_{i,j}^m$ for (9); we have

$$\begin{aligned} \Psi_{i,j}^m - \Psi_{i,j}^{m-1} = {} & R_1\left(\Psi_{i+1,j}^m - 2\Psi_{i,j}^m + \Psi_{i-1,j}^m\right) - R_1 d_{m-1}^{(\beta)}\left(\Psi_{i+1,j}^0 - 2\Psi_{i,j}^0 + \Psi_{i-1,j}^0\right) - R_1 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\left(\Psi_{i+1,j}^{m-s} - 2\Psi_{i,j}^{m-s} + \Psi_{i,j+1}^{m-s}\right) + \\ & R_2\left(\Psi_{i,j}^{m-s} - 2\Psi_{i,j}^{m-s} + \Psi_{i,j-1}^{m-s}\right) + R_2 d_{m-1}^{(\beta)}\left(\Psi_{i,j+1}^0 - 2\Psi_{i,j}^0 + \Psi_{i,j-1}^0\right) - R_2 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\left(\Psi_{i,j+1}^{m-s} - 2\Psi_{i,j}^{m-s} + \Psi_{i,j-1}^{m-s}\right) \\ & + R_3\left(\Psi_{i+1,j}^m - 2\Psi_{i,j}^m + \Psi_{i-1,j}^m\right) \\ & + R_4\left(\Psi_{i,j+1}^m - 2\Psi_{i,j}^m + \Psi_{i,j-1}^m\right). \end{aligned} \tag{11}$$

Next, the error is defined as

$$\varphi_{i,j}^m = Y_{i,j}^m - \Psi_{i,j}^m. \tag{12}$$

where $\varphi_{i,j}^m$ satisfies (11) and

$$\begin{aligned} \varphi_{i,j}^m - \varphi_{i,j}^{m-1} = {} & R_1\left(\varphi_{i+1,j}^m - 2\varphi_{i,j}^m + \varphi_{i-1,j}^m\right) - R_1 d_{m-1}^{(\beta)}\left(\varphi_{i+1,j}^0 - 2\varphi_{i,j}^0 + \varphi_{i-1,j}^0\right) - R_1 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\left(\varphi_{i+1,j}^{m-s} - 2\varphi_{i,j}^{m-s} + \varphi_{i,j+1}^{m-s}\right) \\ & + R_2\left(\varphi_{i,j+1}^{m-s} - 2\varphi_{i,j}^{m-s} + \varphi_{i,j-1}^{m-s}\right) + R_2 d_{m-1}^{(\beta)}\left(\varphi_{i,j+1}^0 - 2\varphi_{i,j}^0 + \varphi_{i,j-1}^0\right) - R_2 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\left(\varphi_{i,j+1}^{m-s} - 2\varphi_{i,j}^{m-s} + \varphi_{i,j-1}^{m-s}\right) + \\ & R_3\left(\varphi_{i+1,j}^m - 2\varphi_{i,j}^m + \varphi_{i-1,j}^m\right) + R_4\left(\varphi_{i,j+1}^m - 2\varphi_{i,j}^m + \varphi_{i,j-1}^m\right). \end{aligned} \tag{13}$$

The error initial and boundary conditions are given as

$$\begin{aligned}\varphi_{0,j}^{m} &= \varphi_{M_1,j}^{m}\\ &= \varphi_{i,0}^{m}\\ &= \varphi_{i,M_2}^{m}\\ &= \varphi_{i,j}^{0}\\ &= 0.\end{aligned} \tag{14}$$

Define the following grid functions for $m = 1, 2 \ldots, N$:

$$\varphi^{m}(x,y) = \left\{ \begin{array}{c} \varphi_{i,j}^{m}, \text{ when } x_{i-\frac{\Delta x}{2}} < x \le x_{i+\frac{\Delta x}{2}},\ y_{j-\frac{\Delta y}{2}} < y \le y_{i+\frac{\Delta y}{2}}, \\ 0, \text{ when } 0 \le x \le \dfrac{\Delta x}{2} \text{ or } L - \dfrac{\Delta x}{2} \le x \le L, \\ 0, \text{ when } 0 \le y \le \dfrac{\Delta y}{2} \text{ or } L - \dfrac{\Delta y}{2} \le y \le L. \end{array} \right\}. \tag{15}$$

Then, $\varphi^{m}(x,y)$ can be expanded in Fourier series such as

$$\varphi^{m}(x,y) = \sum_{l_1,l_2=-\alpha}^{\alpha} X^{m}(l_1,l_2) e^{\sqrt[2]{-1}\pi\left(\frac{l_1 x}{L} + \frac{l_2 y}{L}\right)}, \tag{16}$$

where

$$X^{m}(l_1,l_2) = \frac{1}{L}\int_0^L\int_0^L \varphi^{m}(x,y) e^{-\sqrt[2]{-1}\pi\left(\frac{l_1 x}{L} + \frac{l_2 y}{L}\right)} \mathrm{d}x\mathrm{d}y. \tag{17}$$

From the definition of l^2 norm and Parseval equality, we have

$$\|\varphi^{m}\|_{\alpha}^{2} = \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \Delta x \Delta y \left|\varphi_{i,j}^{m}\right|^{2} = \sum_{l_1,l_2=-\alpha}^{\alpha} \left|X^{m}(l_1,l_2)\right|^{2}. \tag{18}$$

Suppose that

$$\varphi_{i,j}^{m} = X^{m} e^{\sqrt{-1}\left(\alpha_1 i\Delta x + \alpha_2 j \Delta y\right)}, \tag{19}$$

where $\alpha_1 = 2\pi l_1/L$ and $\alpha_2 = 2\pi l_2/L$, and substituting (19) in (13), we get $X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} - X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} = R_1 (X^{m}e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)} - 2X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{m} e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)}) - R_1 d_{m-1}^{(\beta)}(X^{0}e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)} - 2X^{0}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{0}e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)}) - R_1 \sum_{s=1}^{m-1}(d_{s-1}^{(\beta)} - d_{s}^{(\beta)})(X^{m-s}e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)} - 2X^{m-s}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{m-s}e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)}) + R_2(X^{m} e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j+1)\Delta y)} - 2X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{m} e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j-1)\Delta y)} + R_2 d_{m-1}^{(\beta)}(X^{0}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} - 2X^{0} e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{0} e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}) - R_2\sum_{s=1}^{m-1}(d_{s-1}^{(\beta)} - d_{s}^{(\beta)})(X^{m-s}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j+1)\Delta y)} - 2X^{m-s}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{m-s}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j-1)\Delta y)}) + R_3(X^{m}e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)} - 2X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{m}e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)}) + R_4(X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j+1)\Delta y)} - 2X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)} + X^{m}e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j-1)\Delta y)})$.

After simplifying, we get

$$\begin{aligned} X^{m}[1+\nu_1+\nu_2] &= X^{m-1} + X^{0}d_{m-1}^{(\beta)}\nu_1 + \nu_1\sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_{s}^{(\beta)}\right)X^{m-s},\\ X^{m} &= \frac{X^{m-1} + X^{0}d_{m-1}^{(\beta)}\nu_1 + \nu_1\sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_{s}^{(\beta)}\right)X^{m-s}}{[1+\nu_1+\nu_2]}, \end{aligned} \tag{20}$$

where $\nu_1 = [4R_1 \sin \alpha_1\Delta x/2 + 4R_2 \sin \alpha_2\Delta y/2]$ and $\nu_2 = [4R_3 \sin \alpha_1\Delta x/2 + 4R_4 \sin \alpha_2\Delta y/2]$.

Proposition 1. *If* X^{m} $(m = 1, 2, \ldots, N)$ *satisfies (20), then* $|X^{m+1}| \le |X^{0}|$.

Proof: By using mathematical induction, we take $m = 1$ in (20).

$$X^{1} = \frac{\left(1 + d_0^{(\beta)}\nu_1\right)X^{0}}{\left(1+\nu_1+\nu_2\right)}, \tag{21}$$

and as $\nu_1, \nu_2 \geq 0, \quad b_0^{(\beta)} = 1$, then

$$\left|X^1\right| \leq \left|X^0\right|. \tag{22}$$

Now, assume that

$$\left|X^n\right| \leq \left|X^0\right|;\ n = 1, 2, \ldots, m-1, \tag{23}$$

and as $0 < \beta < 1$, from (20) and Lemma 2, we obtain

$$\begin{aligned}
\left|X^m\right| &\leq \frac{\left|X^{m-1}\right| + b_{m-1}^{(\beta)}\nu_1\left|X^0\right| + \nu_1 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)X^{m-s}}{1 + \nu_1 + \nu_2}, \leq \frac{1 + d_{m-1}^{(\beta)}\nu_1 + \nu_1 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)}{\left(1 + \nu_1 + \nu_2\right)}\left|X^0\right|,\\
&= \frac{1 + d_{m-1}^{(\beta)}\nu_1 + \nu_2\left(1 - d_{m-1}^{(\beta)}\right)}{\left(1 + \nu_1 + \nu_2\right)}\left|X^0\right|,\\
&= \frac{1 + \nu_1}{1 + \nu_1 + \nu_2}X^0,\\
\left|X^m\right| &\leq \left|X^0\right|.
\end{aligned} \tag{24}$$

This completes the proof.

Based on the above proof, it can be summarized that the solution of (5) satisfies the following inequality:

$X^m{}_2 \leq X^0{}_2$.

And, we demonstrated that the proposed scheme is unconditionally stable. □

2.2. Convergence. Here, we use a similar method to examine the convergence of the scheme. Let $Y(x_i, y_j, t_m)$ represent the exact solution; then, the truncation error of the scheme is obtained as follows: from (3),

$$\begin{aligned}
T_{i,j}^m = {} & Y\left(x_i, y_j, t_m\right) - Y\left(x_i, y_j, t_{m-1}\right) - R_1 \sum_{j=o}^{m-1} d_s^{(\beta)} \delta x^2\left(Y\left(x_i, y_j, t_{m-s}\right) - Y\left(x_i, y_j, t_{m-s-1}\right)\right),\\
& + R_2 \sum_{j=o}^{k-1} d_s^{(\beta)} \delta y^2\left(Y\left(x_i, y_j, t_{m-s}\right) - Y\left(x_i, y_j, t_{m-s-1}\right)\right) + R_3 \delta x^2 Y\left(x_i, y_j, t_m\right) + R_4 \delta y^2 Y\left(x_i, y_j, t_m\right) - \tau h\left(x_i, y_j, t_m\right).
\end{aligned} \tag{25}$$

From (1), we have

$$\begin{aligned}
T_{i,j}^m = {} & \frac{Y_{i,j}^m - Y_{i,j}^{m-1}}{\tau} - \frac{\partial Y\left(x_i, y_j, t_m\right)}{\partial t} + \left(\frac{\partial^2 Y\left(x_i, y_j, t_m\right)}{\partial x^2}\right) - R_1 \sum_{s=0}^{m-1} d_s^{(\beta)} \delta x^2\left(Y_{i,j}^{m-s} - Y_{i,j}^{m-s-1}\right) + \left(\frac{\partial^2 Y\left(x_i, y_j, t_m\right)}{\partial y^2}\right)\\
& - R_2 \sum_{s=0}^{m-1} b_s^{(\beta)} \delta y^2\left(Y_{i,j}^{m-s} - Y_{i,j}^{m-s-1}\right) + \left(\frac{\partial^2 Y\left(x_i, y_j, t_m\right)}{\partial x^2}\right) - R_3 \delta x^2\left(Y_{i,j}^m\right) + \left(\frac{\partial^2 Y\left(x_i, y_j, t_m\right)}{\partial x^2}\right) - R_4 \delta y^2\left(Y_{i,j}^m\right)\\
= {} & O(\tau + (\tau(\Delta x)) + \tau(\Delta y)).
\end{aligned} \tag{26}$$

Since $i, j, an\ d m$ are finite, there is a positive constant C_1, for all $i, j, an\ d m$, which then have

$$\left|T_{i,j}^m\right| \leq C_1(\tau + \tau(\Delta x)) + \tau(\Delta y)). \tag{27}$$

The error is defined as

$$\phi_{i,j}^m = Y\left(x_i, y_j, t_m\right) - Y_{i,j}^m. \tag{28}$$

From (25), we have

$$
\begin{aligned}
Y(x_i, y_j, t_m) = {} & Y(x_i, y_j, t_{m-1}) + R_1\big(Y(x_i, y_j, t_m) - 2Y(x_i, y_j, t_m) + Y(x_i, y_j, t_m)\big) - \\
& R_1 d_{m-1}^{(\beta)}\big(\big(Y(x_{i+1}, y_j, t_0) - 2Y(x_i, y_j, t_0) + Y(x_{i-1}, y_j, t_0)\big) - \\
& R_1 \sum_{s=1}^{m-1} \left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\big(Y(x_{i+1}, y_j, t_{m-s}) - 2Y(x_i, y_j, t_{m-s}) - Y(x_{i-1}, y_j, t_{m-s})\big) \\
& + R_2\big(Y(x_i, y_{j+1}, t_m) - 2Y(x_i, y_j, t_m) + Y(x_i, y_{j-1}, t_m)\big) - \\
& R_2 d_{m-1}^{(\beta)}\big(\big(Y(x_i, y_{j+1}, t_m) - 2Y(x_i, y_j, t_m) + Y(x_i, y_{j-1}, t_m)\big) \\
& - R_2 \sum_{s=1}^{m-1} \left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\big(Y(x_i, y_{j+1}, t_{m-s}) - 2Y(x_i, y_j, t_{m-s}) + Y(x_i, y_{j-1}, t_{m-s})\big) \\
& + R_3\big(Y(x_{i+1}, y_{j+1}, t_m) - 2Y(x_i, y_{j+1}, t_m) + Y(x_{i-1}, y_{j+1}, t_m)\big) \\
& + R_4\big(Y(x_i, y_{j+1}, t_m) - 2Y(x_i, y_{j+1}, t_m) + Y(x_i, y_{j-1}, t_m)\big) + \tau h(x_i, y_j, t_m).
\end{aligned} \tag{29}
$$

To obtain the error equation, subtract (29) from (5) to obtain

$$
\begin{aligned}
\phi_{i,j}^{m} - \phi_{i,j}^{m-1} = {} & R_1\left(\phi_{i+1,j}^{m} - 2\phi_{i,j}^{m-1} + \phi_{i-1,j}^{m}\right) - R_1 d_{m-1}^{(\beta)}\left(\phi_{i+1,j}^{m} - 2\phi_{i,j}^{m-1} + \phi_{i-1,j}^{m} - R_1 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\left(\phi_{i+1,j}^{m} - 2\phi_{i,j}^{m-1} + \phi_{i-1,j}^{m}\right)\right. \\
& + R_2\left(\phi_{i,j}^{m} - 2\phi_{i,j}^{m-1} + \phi_{i,j}^{m}\right) - \\
& R_2 d_{m-1}^{(\beta)}\left(\phi_{i,j+1}^{0} - 2\phi_{i,j}^{0} + \phi_{i,j-1}^{0}\right) - R_2 \sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)} - d_s^{(\beta)}\right)\left(\phi_{i,j+1}^{m-s}2 - \phi_{i,j}^{m-s} + \phi_{i,j-1}^{m-s}\right) + R_3\left(\phi_{i+1,j}^{m} - 2\phi_{i,j}^{m-1} + \phi_{i-1,j}^{m}\right) \\
& + R_4\left(\phi_{i,j+1}^{m-s} - 2\phi_{i,j}^{m-s} + \phi_{i,j-1}^{m-s}\right) + \tau T_{i,j}^{m}.
\end{aligned} \tag{30}
$$

With error boundary conditions,

$$
\phi_{0,j}^{m} = \phi_{M_1,j}^{m} = \phi_{0,j}^{m} = \phi_{i,M_2}^{m} = 0,\ m = 1, 2, \ldots, N. \tag{31}
$$

And, the initial condition

$$
\phi_{i,j}^{0} = 0,\ i = 1, 2, \ldots, M_1 - 1,\ j = 1, 2, \ldots, M_2 - 1. \tag{32}
$$

Next, we define the following grid functions for $m = 1, 2, \ldots, N$:

$$
\phi^{m}(x, y) = \left\{
\begin{array}{c}
\phi_{i,j}^{m},\ \text{when}\ x_{i-\frac{\Delta x}{2}} < x \le x_{i+\frac{\Delta x}{2}},\ y_{j-\frac{\Delta y}{2}} \langle\, y \le y_{j+\frac{\Delta y}{2}}, \\
\\
0,\ \text{when}\ 0 \le x \le \dfrac{\Delta x}{2}\ or\ L - \dfrac{\Delta x}{2} \le x \le L, \\
\\
0,\ \text{when}0\ \le y \le \dfrac{\Delta y}{2}\ or\ L - \dfrac{\Delta y}{2} \le y \le L.
\end{array}
\right\}. \tag{33}
$$

$$T^m(x,y)=\begin{cases} T_{i,j}^m, \text{ when } x_{i-\frac{\Delta x}{2}}<x\le x_{i+\frac{\Delta x}{2}},\ y_{j-\frac{\Delta y}{2}}<y\le y_{j+\frac{\Delta y}{2}}, \\ 0, \text{ when } 0\le x\le \frac{\Delta x}{2} \text{ or } L-\frac{\Delta x}{2}\le x\le L, \\ 0, \text{ when } 0\le y\le \frac{\Delta y}{2} \text{ or } L-\frac{\Delta y}{2}\le y\le L. \end{cases}. \quad (34)$$

Here, the $\phi^m(x,y)$ *an d* $T^m(x,y)$ can be expanded in Fourier series such as

$$\phi^m(x,y)=\sum_{l_1,l_2=-\alpha}^{\alpha}\zeta^m(l_1,l_2)e^{2\sqrt{-1}\pi(l_1x/L+l_2y/L)},\ m=1,2,\ldots,N,$$
$$T^m(x,y)=\sum_{l_1,l_2=-\alpha}^{\alpha}\varphi^m(l_1,l_2)e^{2\sqrt{-1}\pi(l_1x/L+l_2y/L)},\ m=1,2,\ldots,N, \quad (35)$$

where

$$\zeta^m(l_1,l_2)=\frac{1}{L}\int_0^L\int_0^L\phi^m(x,y)e^{2\sqrt{-1}\pi\left(\frac{l_1x}{L}+\frac{l_2y}{L}\right)}\mathrm{d}x\mathrm{d}y, \quad (36)$$

$$\varphi^m(l_1,l_2)=\frac{1}{L}\int_0^L\int_0^L\phi^m(x,y)e^{2\sqrt{-1}\pi(l_1x/L+l_2y/L)}\mathrm{d}x\mathrm{d}y. \quad (37)$$

From the definition of l^2 norm and the Parseval equality, we have

$$\|\phi^m\|_{l^2}^2=\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\Delta x\Delta y\left|e_{i,j}^m\right|=\sum_{l_1,l_2=-\alpha}^{\alpha}\left|\rho^m(l_1,l_2)\right|^2, \quad (38)$$

$$T^{m2}_{l^2}=\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\Delta x\Delta y\left|e_{i,j}^m\right|=\sum_{l_1,l_2=-\alpha}^{\alpha}\left|\varphi^m(l_1,l_2)\right|^2. \quad (39)$$

Based on the previous equations, suppose that

$$\varphi_i^m=\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 i\Delta y)}, \quad (40)$$

$$T_i^m=\varphi^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 i\Delta y)}. \quad (41)$$

Respectively, we have $\alpha_1=2\pi l_1/L$ and $\alpha_2=2\pi l_2/L$; substitute (40) and (41) into (30) and we get $\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}-\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}=R_1(\zeta^m e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)})-R_1 d_{m-1}^{(\beta)}(\zeta^m e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)})-R_1\sum_{s=1}^{m-1}(d_{s-1}^{(\beta)}-d_s^{(\beta)})(\zeta^m e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)})+R_2(\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j+1)\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j-1)\Delta y)}+R_2 d_{m-1}^{(\beta)}(\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)})-R_2\sum_{s=1}^{m-1}(d_{s-1}^{(\beta)}-d_s^{(\beta)})(\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j+1)\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j-1)\Delta y)})+R_3(\zeta^m e^{\sqrt{-1}(\alpha_1(i+1)\Delta x+\alpha_2 j\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1(i-1)\Delta x+\alpha_2 j\Delta y)})+R_4(\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j+1)\Delta y)}-2\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)}+\zeta^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2(j-1)\Delta y)})+\tau(\varphi^m e^{\sqrt{-1}(\alpha_1 i\Delta x+\alpha_2 j\Delta y)})$,

Simplifying the previous equation, we obtain

$$\zeta^m=\frac{\zeta^{m-1}+\zeta^0 b_{m-1}^{(\beta)}\nu_1+\nu_1\sum_{s=1}^{m-1}\left(d_{s-1}^{(\beta)}-d_s^{(\beta)}\right)\zeta^{m-s}}{[1+\nu_1+\nu_2]}, \quad (42)$$

where

$$\nu_1=4R_1\sin\frac{\alpha_1\Delta x}{2}+4R_2\sin\frac{\alpha_2\Delta y}{2}\Bigg],$$
$$\nu_2=4R_3\sin\frac{\alpha_1\Delta x}{2}+4R_4\sin\frac{\alpha_2\Delta y}{2}\Bigg]. \quad (43)$$

Proposition 2. *Let ζ^m $(m=1,2,\ldots,N)$ be the solution of (42); then, there is a positive constant C_2 so that*

$$|\zeta^m|\le C_2 m\tau|\varphi^1|.$$

Proof: From $\phi^0=0$ and (36), we have

$$\zeta^o=\zeta^o(l_1,l_2)=0. \quad (44)$$

From (37) and (39), then there is a positive constant C_2, such that

$$\left|\varphi^m\right|\le C_2\left|\varphi^1(l_1,l_2)\right|. \quad (45)$$

Using mathematical induction, for $m=1$, then from (42) and (44), we obtain

$$\zeta^1=1/1+\nu_1+\nu_2\left(\tau\varphi^1\right). \quad (46)$$

Since $\nu_1,\nu_2\ge 0$, from (45), we get

$$\left|\zeta^1\right|\le\tau\left|\varphi^1\right|\le C_2\tau\left|\varphi^1\right|. \quad (47)$$

Now, suppose that

$$\left|\zeta^m\right|\le C_2 m\tau\left|\varphi^1\right|,\ n=1,2,\ldots,m-1. \quad (48)$$

As $0<\beta<1$, $\nu_1,\nu_2\ge 0$.

From (41) and (44) and Lemma 2, we have

$$\begin{aligned}
|\zeta^m| &= \frac{|\zeta|^{m-1} + \nu_1 \sum_{s=1}^{m-1}\left(d_{s-1}^{\beta} - d_s^{\beta}\right)|\zeta|^{m-s} + \tau|\varphi|^m}{\left(1 + \nu_1 + \nu_2\right)},\\
|\zeta^m| &= \frac{C_2\,(m-1)\tau|\varphi|^1 + \nu_1 \sum_{s=1}^{m-1}\left(d_{s-1}^{\beta} - d_s^{\beta}\right)C_2\,(m-s)\tau|\varphi|^1 + C_2\tau|\varphi|^1}{\left(1 + \nu_1 + \nu_2\right)},\\
&\leq \left[\frac{(m-1) + \nu_1\,(m-1)\sum_{s=1}^{m-1}\left(b_{s-1}^{\beta} - b_s^{\beta}\right) + 1}{\left(1 + \nu_1 + \nu_2\right)}\right]C_2\tau|\varphi|^1,\\
&= \left[\frac{m + \nu_1\,(m-1)\sum_{s=1}^{m-1}\left(b_{s-1}^{\beta} - b_s^{\beta}\right) + 1}{\left(1 + \nu_1 + \nu_2\right)}\right]C_2\tau|\varphi|^1,\\
&= \left[\frac{m + \nu_1\,(m-1) + \left(1 - b_{m-1}^{(\beta)}\right)}{\left(1 + \nu_1 + \nu_2\right)}\right]C_2\tau|\varphi|^1,\\
&\leq mC_2\tau\left|\varphi^1\right|.
\end{aligned} \tag{49}$$

The proof is completed via the induction method. □

Theorem 1. *The modified implicit difference scheme l^2 is convergent, and the order of convergence is $O(\tau + \tau(\Delta x)^2 + \tau(\Delta y)^2)$.*

Proof: From (27) and (39), we obtain

$$\begin{aligned}
T^k &\leq \sqrt{M_x\Delta x}\sqrt{M_y\Delta y}\,C_1\left(\tau + \tau(\Delta x)^2 + \tau(\Delta y)^2\right) = LC_1\left(\tau + \tau(\Delta x)^2 + \tau(\Delta y)^2\right)\\
\varphi^k{}_{l^2} &\leq kC_2\tau T^1 \leq C_1C_2k\tau L\left(\tau + \tau\left(\Delta x^2\right) + \tau\left(\Delta y^2\right)\right).
\end{aligned} \tag{50}$$

As $k\tau \leq T$, thus

$$\varphi^k{}_{l^2} \leq C_1C_2k\tau L\left(\tau + \tau\left(\Delta x^2\right) + \tau\left(\Delta y^2\right)\right), \tag{51}$$

where $C = C_1C_2TL$. □

3. Numerical Experiment

Example 1. Consider the following two-dimensional Rayleigh–Stokes problem for heated generalized second-grade fluid with the fractional derivative [22]:

$$\frac{\partial Y(x,y,t)}{\partial t} = {}_0D_t^{1-\beta}\left(\frac{\partial^2 Y(x,y,t)}{\partial x^2} + \frac{\partial Y(x,y,t)}{\partial y^2}\right) + \frac{\partial^2 Y(x,y,t)}{\partial x^2} + \frac{\partial^2 Y(x,y,t)}{\partial y^2} + h(x,y,t), 0 \leq \beta \leq 1,\ 0 \leq t \leq T, \tag{52}$$

with initial and boundary conditions

$$\begin{aligned}
&Y(x,y,0) = 0,\ 0 \leq x,\ y \leq 1,\\
&Y(0,y,t) = e^y t^{1+\beta},\ Y(1,y,t)\\
&\quad = e^{1+y}t^{1+\beta},\\
&Y(x,0,t)) = e^x t^{1+\beta},\ Y(x,1,t)\\
&\quad = e^{1+x}t^{1+\beta},\ 0 \leq t \leq T.
\end{aligned} \tag{53}$$

Here, $h(x,y,t) = ((1+\beta)t^{\beta} - 2\Gamma(2+\beta)/\Gamma(1+2\beta)t^{2\beta} - 2t^{1+\beta})e^{x+y}$ and the exact solution of (52) is given by

$$Y(x,y,t) = e^{x+y}t^{1+\beta}. \tag{54}$$

The error between the numerical solution and exact solution is defined as follows:

$$E_{\infty} = \max_{0\leq i,j\leq M, 0\leq m\leq N}\left|Y\left(x_i, y_j, t_m\right) - Y_{i,j}^m\right|. \tag{55}$$

And, the rate of convergence for space variable can be defined as

$$= -\text{order} = \log_2\left(\frac{\left\|E_{\infty}(16\tau, 2\Delta x)\right\|}{\left\|E_{\infty}(\tau, \Delta x)\right\|}\right). \tag{56}$$

Table 1: The error table for different values at $\tau, \Delta x, \Delta y$, and γ.

τ	$\Delta x = \Delta y$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
1/4	1/2	$2.180E-2$	$1.316E-2$	$1.601E-2$	$1.889E-2$	$2.481E-2$
1/16	1/4	$4.573E-3$	$5.267E-3$	$5.934E-3$	$6.606E-2$	$7.312E-2$
1/64	1/8	$1.330E-3$	$1.484E-3$	$1.634E-3$	$1.789E-3$	$1.958E-3$
1/128	1/10	$7.701E-4$	$8.426E-4$	$9.141E-4$	$9.898E-4$	$1.073E-4$

Table 2: The error table for different values at $\tau, \Delta x, \Delta y$, and γ.

$\tau = \Delta x = \Delta y$	$\gamma = 0.35$	$\gamma = 0.65$	$\gamma = 0.85$
1/2	$7.9899E-3$	$2.4671E-2$	$3.8181E-2$
¼	$4.5383E-3$	$1.4258E-2$	$2.1050E-2$
1/6	$3.3873E-3$	$9.9350E-3$	$1.4272E-2$
1/8	$2.8140E-3$	$7.7703E-3$	$1.0965E-2$
1/10	$2.4289E-3$	$6.3977E-3$	$8.9159E-3$

Table 3: The error table for different values at $\tau, \Delta x, \Delta y$, and γ.

τ	$\Delta x = \Delta y$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
1/16	1/4	$5.0794E-3$	$5.9521E-3$	$6.7049E-3$	0.00746352	$8.2610E-3$
1/8	1/8	$1.4043E-3$	$1.6387E-3$	$8.5596E-3$	$1.0150E-3$	$1.8575E-3$
1/144	1/12	$6.2458E-4$	$6.8873E-4$	$7.5239E-4$	$8.1996E-4$	$8.9478E-4$

Table 4: The error table for different values at $\tau, \Delta x$, and Δy and at a fixed value of $\gamma = 0.25$.

N	$\Delta x = \Delta y = 1/5$	$\Delta x = \Delta y = 1/10$	$\Delta x = \Delta y = 1/15$	$\Delta x = \Delta y = 1/20$
20	$1.9274E-3$	$9.5392E-4$	$7.6704E-4$	$7.0147E-4$
40	$1.7467E-3$	$1.6060E-3$	$5.7985E-4$	$4.3020E-4$
60	$1.6504E-3$	$6.6854E-4$	$5.4537E-4$	$3.2953E-4$

The developed modified implicit scheme is applied to problems (52) to (54).

Tables 1–4 show the errors E_∞ for values of space step size $(\Delta x, \Delta y)$ and τ. Here, time step τ is defined by $\tau = T/N$.

Tables 1–4 indicate that, as we reduce the time and space step size τ and $(\Delta x, \Delta y)$, the error decreases for a fixed value of γ.This shows that the method converges to the exact solution.

Example 2. Consider the following Cable equation:

$$\frac{\partial Y(x,t)}{\partial t} = {}_0D_t^{1-\rho_1}\left(K\frac{\partial^2 Y(x,t)}{\partial x^2}\right) - \mu^2\,{}_0D_t^{1-\rho_2}Y(x,t) + 2\left(t + \frac{\pi^2 t^{1+\rho_1}}{\Gamma(2+\rho_1)} + \frac{t^{1+\rho_2}}{\Gamma(2+\rho_2)}\right)\sin(\pi x), \tag{57}$$

with initial and boundary conditions

$$\begin{aligned} &Y(x,0) = 0,\ 0 \le x \le L, \\ &Y(0,t) = t^2\sin(\pi x), \quad Y(L,t) = \beta_2(t),\ 0 \langle t \le T. \end{aligned} \tag{58}$$

The exact solution is $Y(x,t) = t^2\sin(\pi x)$.

4. Results and Discussion

A modified implicit scheme is developed and applied on RSP-HGSGF. Numerical example is given to support theoretical study. The error between the exact and numerical solution is calculated using different values of N and M. Also, at different values of γ, Tables 1–4 are created to show the comparison of the numerical scheme with the exact solution in terms of maximum error. In example 2, we solved the fractional-order Cable equation, and the numerical results are shown in Table 5 for various values of space and time step size. The values of ρ_1 and ρ_2 are also changed, and the obtained results are converging with reduced step sizes. Here, the error is calculated using Maple15 software with the increase in the number of space and time steps. Figures 1–3 are plotted for different values of M and N, and fractional order γ shows good agreement with the exact solution.

Table 5: Numerical results of example 2 of the proposed scheme for various values of ρ_1, ρ_2, N, and Δx.

Δx	N	$\rho_1, \rho_2 = 0.25$	$\rho_1, \rho_2 = 0.5$	$\rho_1, \rho_2 = 0.95$
1/10	40	$1.173E-2$	$8.131E-2$	$6.552E-3$
	80	$8.032E-3$	$6.452E-3$	$5.762E-3$
	110	$5.912E-3$	$5.528E-3$	$5.171E-3$
1/20	40	$8.146E-3$	$6.312E-3$	$3.272E-3$
	80	$4.537E-3$	$2.989E-3$	$2.409E-3$
	110	$2.428E-3$	$2.015E-3$	$9.146E-4$
1/40	40	$7.294E-3$	$3.781E-3$	$2.491E-3$
	80	$3.639E-3$	$2.173E-3$	$1.591E-3$
	110	$1.578E-3$	$1.257E-3$	$9.015E-4$

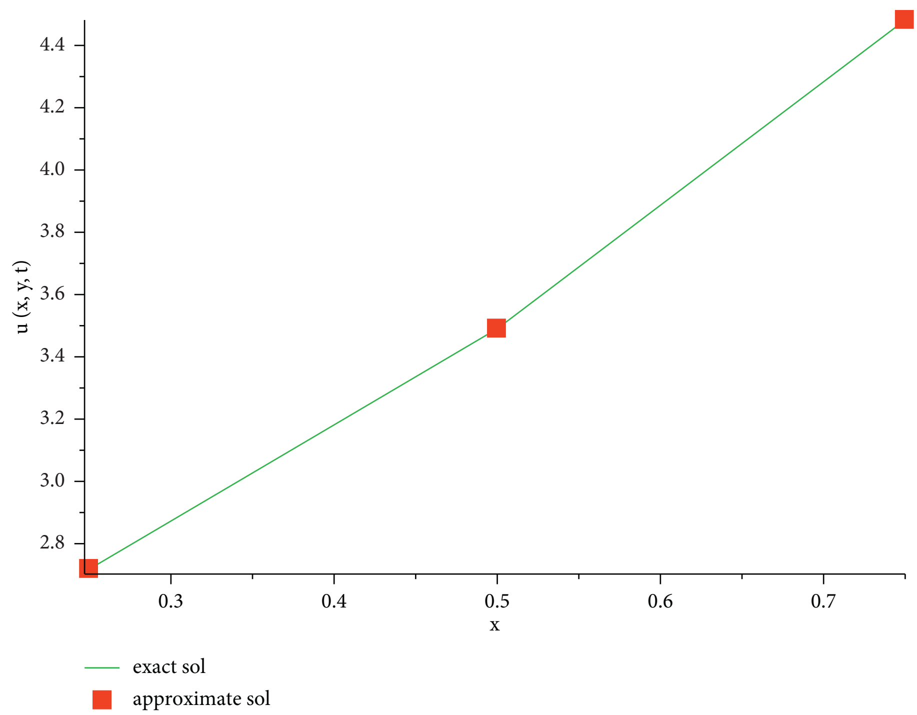

Figure 1: Comparing equations (52) and (54) at $M = 4$, $N = 2$, and $\gamma = 0.25$.

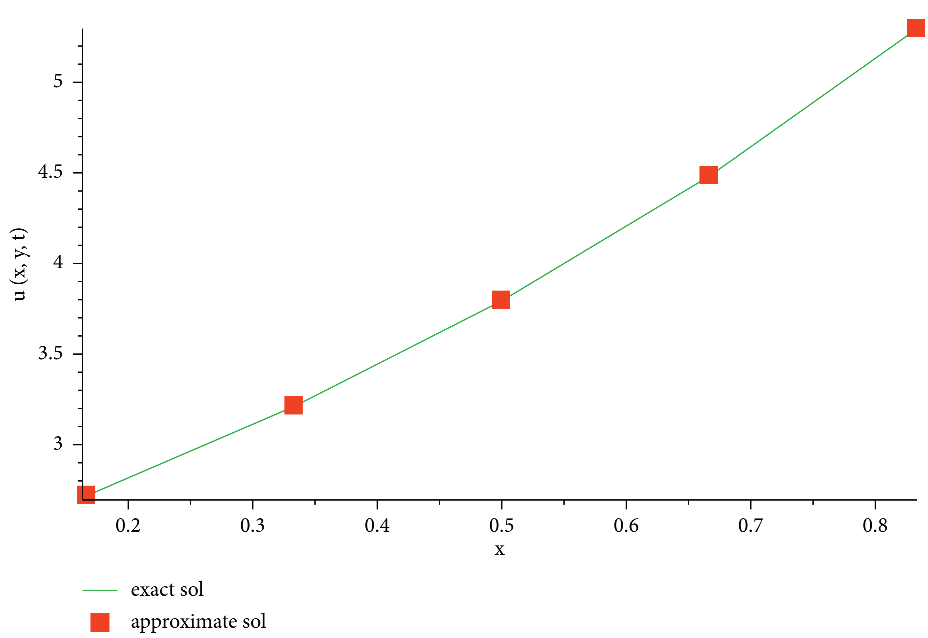

Figure 2: Comparing equations (52) and (54) at $M = 6$, $N = 6$, and $\gamma = 0.35$.

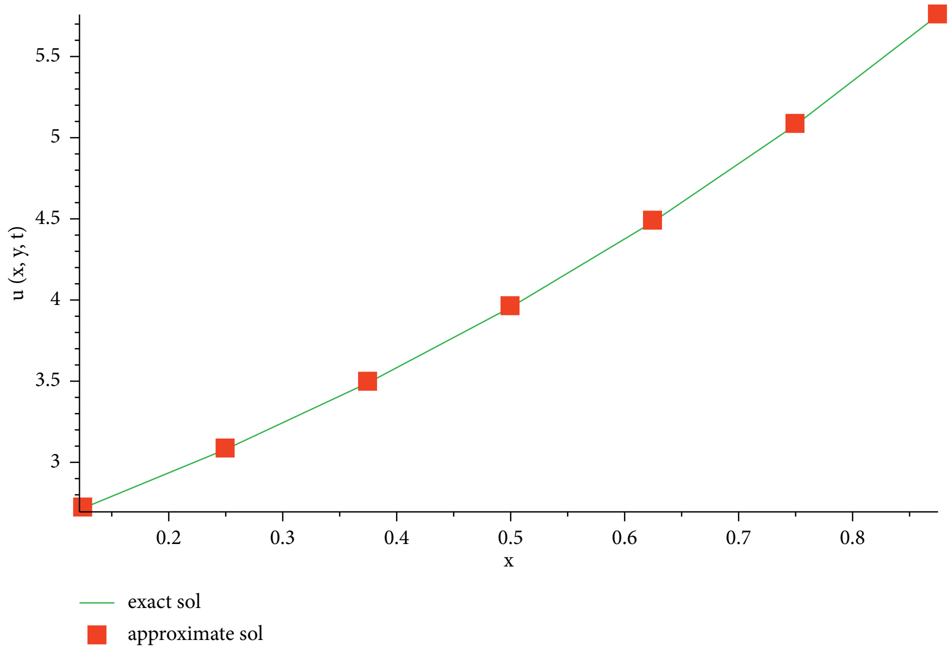

Figure 3: Comparing equations (52) and (54) at $M = 8$, $N = 8$, and $\gamma = 0.85$.

5. Conclusion

A modified implicit difference scheme is formulated for 2D RSP-HGSGF, and a derivative of fractional order has been described in this paper. The modified scheme has the improvement of low computational cost and can be easily applied. The Fourier technique has been used for the theoretical analysis stability, and convergence with order $(\tau + (\Delta x) + (\Delta y))$ is unconditionally stable and convergent. The numerical experiment for the 2D RSP-HGSGF and 1D Cable equation is conducted, which shows that the modified implicit scheme is easy to implement, and the results show good performance of the proposed schemes [39].

Abbreviations

RSP-HGSGF:	Rayleigh–Stokes problem for heated generalized second-grade fluid
SFP:	Stokes first problem
INAS:	Implicit numerical approximation scheme
RBF-FD:	Radial basis function finite difference
FEM:	Finite element method
2D:	Two-dimensional
RL:	Riemann–Liouville.

Data Availability

All the related materials have been cited in the paper.

Conflicts of Interest

The authors declare no conflicts of interest.

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