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Research Article

Investigation of the Generalized Proportional Langevin and Sturm–Liouville Fractional Differential Equations via Variable Coefficients and Antiperiodic Boundary Conditions with a Control Theory Application Arising from Complex Networks

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This article studies the existence theory of an innovation type of generalized proportional fractional differential equations with the assistance of the technique of Kuratowski measure on noncompactness combined with the fixed-point theorem of Mönch. Also, we use Lebesgue's dominated convergence theorem and Arzelá–Ascoli fixed point theorem on existence and uniqueness results. An application is also presented by employing two illustrative iks, which enrich our outcomes.

1. Introduction

The idea of a measure of noncompactness ($\mathbb{MNC}$) was presented by Kuratowski in 1958 for the first time. In fact, for any bounded subset $\mathfrak{M}$ of a metric space $\mathscr{C}$, the $\mathbb{MNC}$ denoted is defined to be the infimum of numbers $\omega > 0$ such that $\mathscr{C}$ can be covered by a finite number of sets of diameter less than ω.

The first who used the index α to the nonlinear analysis was Darbo [1]. The technique is a very useful tool for the existence solution of integral equations, which is well presented by Banaś et al. [2]. In 2008, Benchohra et al. considered the $\mathbb{IVP}$ for a nonlinear fractional differential equation

$$\mathbb{G}_c^{\sigma} q(\iota) = \wp(\iota, q(\iota)), \forall 0 \le \iota \le \tau^*, \tag{1}$$

where $\mathbb{G}_c^{\sigma}$ is the Caputo fractional derivative of order $1 < \sigma < 2$ and $\wp$ is a given function [3]. Some authors have been utilized comparative strategies through K $\mathbb{MNC}$ technique to diverse sorts of $\mathbb{FBVP}s$ [4,5]. In 2016, Kiataramkul et al studied the generalized Langevin and Sturm–Liouville fractional differential equations ($\mathbb{GL}$-SL$\mathbb{FDE}$s) of Hadamard type, with antiperiodic boundary conditions ($\mathbb{APBC}$s) of the form

$$\begin{cases} {}^H\mathbb{G}_c^{\sigma}\left(\left[y(\iota){}^H\mathbb{G}_c^{\sigma} + w(\iota)\right]\right)q(\iota) = \wp(\iota, q(\iota)), & (\iota \in [1, \tau^*]), \\ q(1) = -q(\tau^*), \quad {}^H\mathbb{G}_c^{\sigma} q(1) = -{}^H\mathbb{G}_c^{\sigma} q(\tau^*), \end{cases} \tag{2}$$

where ${}^H\mathbb{G}_c^{\eta}$ denotes the Caputo-type Hadamard fractional derivative of order $\eta \in (0, 1]$, $\wp$: $[1, \tau^*] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $y \in C([1, \tau^*], \mathbb{R})$ with $|y(\iota)| \ge K > 0$, and $w \in \mathscr{C} = C([1, \tau^*], \mathbb{R})$ [6].

In this manuscript, we consider an innovation class of $\mathbb{BVP}s$ by combining Langevin and Sturm–Liouville fractional differential equations (L-SL $\mathbb{FDE}$s). This work discusses the existence theory of the generalized proportional fractional differential equations with the help of the technique of K $\mathbb{MNC}$ combined with the Mönch fixed-point theorem. To be more precise, we initiate the study of the existence, uniqueness, and different types of Ulam stability ($\mathbb{US}$) for the following generalized proportional Langevin and Sturm–Liouville fractional differential problems ($\mathbb{GP}$ L-SL $\mathbb{FDP}$ s) with variable coefficients ($\mathbb{VC}$s) and $\mathbb{APBC}$s

$$\begin{cases} \mathbb{G}_c^{\sigma;\hbar}\left(\left[y(\iota)\mathbb{G}_c^{\acute{\sigma};\hbar}+w(\iota)\right]\right)q(\iota)=\wp(\iota,q(\iota)),\ (\iota\in\mathcal{J}=:[0,\tau^*]),\\ q(0)=-q(\tau^*),\mathbb{G}_c^{\acute{\sigma};\hbar}q(0)=-\mathbb{G}_c^{\acute{\sigma};\hbar}q(\tau^*). \end{cases} \tag{3}$$

So that $\mathbb{G}_c^{\eta;\hbar}$ denotes the generalized proportional Caputo fractional derivatives ($\mathbb{GPCFD}$) of order $0<\eta\leq 1$, $1<\sigma+\acute{\sigma}\leq 2$, $\hbar>0$, $\wp\colon\mathcal{J}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous, $y\in C(\mathcal{J},\mathbb{R}/\{0\})$ with $|y(\iota)|\geq\breve{K}>0$, and $w\in\mathcal{C}=C(\mathcal{J},\mathbb{R})$. Our proposed method is essentially based on the result given by Banaś et al. [2]. Therefore, it is emphasized that the K $\mathbb{MNC}$ technique is implemented for the first time on the $\mathbb{GP}$ L-SL $\mathbb{FDP}$s.

This article is arranged as follows: in Section 2, we provide the definitions and preliminary lemmas that we will use to verify our main theorems. In Section 3, we establish the existence and uniqueness of solutions to problem (3) by the K $\mathbb{MNC}$ -method. In Section 4, we discuss some types of fractional $\mathbb{US}$. We give an application by presenting two examples to illustrate numerical findings in Section 5. Finally, we conclude our exposition.

2. Preliminary Notions

We consider the space $\mathcal{C}$ with the norm

$$\|q\|_\infty=\sup\{|q(\iota)|\colon \iota\in\mathcal{J}\}. \tag{4}$$

And denote by $\mathcal{C}_B$ the class of all bounded mappings in $\mathcal{C}$. Regard $\mathcal{L}=L^1(\mathcal{J},\mathbb{R})$ as a Banach space of all measurable Bochner integrable mappings like as $q\colon\mathcal{J}\longrightarrow\mathbb{R}$ which is endowed with the norm

$$\|q\|_{\mathcal{L}}=\int_{\mathcal{J}}|q(\tau)|d\tau. \tag{5}$$

Now we look at the specifications of the concept of K $\mathbb{MNC}$.

Definition 1 (see [2, 7]). The mapping $\Upsilon\colon\mathcal{C}_B\longrightarrow[0,\infty)$ denoted by $\Upsilon(\mathfrak{M})$ for $\mathfrak{M}\in\mathcal{C}_B$, is named as the K$\mathbb{MNC}$, if

$$\Upsilon(\mathfrak{M}) := \inf\left\{\omega>0\colon \exists \text{ finitely many sets } \mathfrak{M}_i \ni \mathfrak{M}=\bigcup_{i=1}^{m}\mathfrak{M}_i \text{ and } D(\mathfrak{M}_i)\leq\omega\right\}, \tag{6}$$

where

$$D(\mathfrak{M}_i)=\sup\{|q-\widehat{q}|\colon q,\widehat{q}\in\mathfrak{M}_i\}. \tag{7}$$

Proposition 1 (see [2, 7]). *The K$\mathbb{MNC}$ fulfills the following:*

(1) $\Upsilon(\mathfrak{M})\leq\Upsilon(\acute{\mathfrak{M}})$ whenever $\mathfrak{M}\subset\acute{\mathfrak{M}}$, $\forall\mathfrak{M},\acute{\mathfrak{M}}\in\mathcal{C}_B$;

(2) $\Upsilon(\mathfrak{M})=0$ iff $\mathfrak{M}$ is relatively compact;

(3) $\Upsilon(\mathfrak{M})=\Upsilon(\overline{\mathfrak{M}})=\Upsilon(conv(\mathfrak{M}))$, where $\overline{\mathfrak{M}}$ and $conv(\mathfrak{M})$ represent the closure and the convex hull of $\mathfrak{M}$, respectively;

(4) $\Upsilon(\mathfrak{M}_1+\mathfrak{M}_2)\leq\Upsilon(\mathfrak{M}_1)+\Upsilon(\mathfrak{M}_2)$;

(5) $\Upsilon(y\mathfrak{M})=|y|\Upsilon(\mathfrak{M})$, $\forall y\in\mathbb{R}$.

Let Σ stands for the set of functions $q\colon\mathcal{J}\longrightarrow\mathbb{R}$, $\forall\iota\in\mathcal{J}$, set $\Sigma_\iota=\{q(\iota)\colon q\in\Sigma\}$, and

$$\Sigma(\mathcal{J})=\{q(\iota)\colon q\in\Sigma,\iota\in\mathcal{J}\}. \tag{8}$$

Theorem 1 (see [8–10]). *The continuous self function ϱ on convex bounded $0\in\mathfrak{X}\subseteq\mathcal{C}$ admits a fixed-point whenever*

$$\begin{aligned} &\Sigma=\overline{\text{conv}}\varrho(\Sigma),\\ &\text{or }\Sigma=\varrho(\Sigma)\cup\{0\}\}\Rightarrow\Upsilon(\Sigma)=0,\quad\forall\Sigma\subset\mathfrak{X}. \end{aligned} \tag{9}$$

Lemma 1 (see [10]). *Let $\mathfrak{M}\subset\mathcal{C}$ be convex bounded and closed, $\wp\colon\mathcal{J}\times\mathfrak{X}\longrightarrow\mathcal{C}$ be Carathéodory, and $\exists y\in L^1(\mathcal{J},\mathbb{R}^+)$ so that $\forall\iota\in\mathcal{J}$ and every bounded set $E\subset\mathfrak{M}$,*

$$\lim_{\delta\longrightarrow 0^+}\Upsilon\left(\wp\left(\mathcal{J}_{\iota,\delta}\times E\right)\right)\leq y(\iota)\Upsilon(E);\mathcal{J}_{\iota,\delta}=[\iota-\delta,\iota]\cap\mathcal{J}. \tag{10}$$

From equicontinuous of $\mathcal{J}\longrightarrow\mathfrak{X}$, we have

$$\begin{aligned} &\Upsilon\left(\left\{\int_{\mathcal{J}}\widehat{G}(\iota,\xi)\wp(\xi,q(\xi))d\xi\colon q\in\Sigma\right\}\right)\\ &\leq\int_{\mathcal{J}}\|\widehat{G}(\iota,\xi)\|y(\xi)\Upsilon(\Sigma(\xi))d\xi. \end{aligned} \tag{11}$$

Here $\mathcal{G}\in C(\mathcal{J}\times\mathcal{J})$.

Definition 2 (see [11]) The $\mathbb{GPF}$ integral of q of order $\sigma\in\mathbb{C}$, $(\text{Re}(\sigma)>0)$ is

$$\begin{aligned} \left({}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}q\right)(\iota)&=\int_a^\iota\frac{(\iota-s)^{\sigma-1}}{\hbar^\sigma\Gamma(\sigma)}\exp\left(\frac{\hbar-1}{\hbar}(\iota-\xi)\right)q(\xi)d\xi\\ &=\hbar^{-\sigma}\exp\left(\frac{\hbar-1}{\hbar}\iota\right)\mathbb{I}_a^\sigma\left(\exp\left(\frac{1-\hbar}{\hbar}\iota\right)q(\iota)\right). \end{aligned} \tag{12}$$

For $0<\hbar\leq 1$, where ${}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}$ is Riemann–Liouville fractional integral and the $\mathbb{GPF}$ Caputo type derivative of order σ is

$$\left(\mathbb{G}_c^{\sigma;\hbar}q\right)(\iota)=\int_a^{\iota}\frac{(\iota-\xi)^{n-\sigma-1}}{\hbar^{n-\sigma}\Gamma(n-\sigma)}\exp\left(\frac{\hbar-1}{\hbar}(\iota-\xi)\right)\left(\mathbb{D}^{n,\hbar}q\right)(\xi)\mathrm{d}\xi, \tag{13}$$

where $n=[\operatorname{Re}(\sigma)]+1$.

Lemma 2 (see [11]). *For the map q, we have*

$$\exp\left(\frac{\hbar-1}{\hbar}(\iota-\xi)\right)\left(\mathbb{D}^{n,\hbar}q\right)(\xi)\mathrm{d}\xi, \tag{14}$$

$$\left({}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\mathbb{G}_c^{\sigma;\hbar}q\right)(\iota)=q(\iota)-\sum_{k=0}^{n-1}\frac{\left(\mathbb{D}^{n,\hbar}q\right)(a)}{\hbar^k k!}(\iota-a)^k\exp\left(\frac{\hbar-1}{\hbar}(\iota-\xi)\right). \tag{15}$$

Proposition 2 (see [11]). *Let $\sigma,\acute{\sigma}\in\mathbb{C}$ be such that* $\operatorname{Re}(\acute{\sigma})>0$ *and $n=[\operatorname{Re}(\sigma)]+1$. Then, for any $0<\hbar\leq 1$, we have*

(i) $\left({}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\exp(\hbar-1/\hbar\iota)(\iota-a)^{\acute{\sigma}-1}\right)(\iota)=\Gamma(\acute{\sigma})/\hbar^{\sigma}\Gamma(\sigma+\acute{\sigma})\exp(\hbar-1/\hbar\iota)(\iota-a)^{\sigma+\acute{\sigma}-1}$, $\operatorname{Re}(\sigma)>0$;

(ii) $\left(\mathbb{G}_c^{\sigma;\hbar}\exp(\hbar-1/\hbar\iota)(\iota-a)^{\acute{\sigma}-1}\right)(\iota)=\hbar^{\sigma}\Gamma(\acute{\sigma})/\Gamma(\acute{\sigma}-\sigma)\exp(\hbar-1/\hbar)(\iota-a)^{\acute{\sigma}-\sigma-1}$, $\operatorname{Re}(\sigma)>n$;

(iii) $\left(\mathbb{G}_c^{\sigma;\hbar}\exp(\hbar-1/\hbar\iota)(\iota-a)^{k}\right)(\iota)=0$, $\operatorname{Re}(\sigma)>n$, $k=0,1,\ldots,n-1$.

3. Main Results

We investigate the given $\mathbb{GP}$ L-SL $\mathbb{FDE}$s (3) and present the solution's characterization in relation to it.

Definition 3. By a solution of the $\mathbb{GP}$ L-SL $\mathbb{FDE}$s with $\mathbb{VC}$s and $\mathbb{APBC}$s (3), we mean a measurable function $q\in\mathscr{C}$ such that $q(0)=-q(\tau^*)$,

$$\mathbb{G}_c^{\sigma;\hbar}q(0)=-\mathbb{G}_c^{\sigma;\hbar}q(\tau^*). \tag{16}$$

And $\mathbb{FDE}$s

$$\mathbb{G}_c^{\sigma;\hbar}\left(\left[y(\iota)\mathbb{G}_c^{\acute{\sigma};\hbar}+w(\iota)\right]\right)q(\iota)=\wp_q(\iota),\ (\iota\in\mathscr{J}), \tag{17}$$

are fulfilled on $\mathscr{J}$.

Lemma 3. *Let $\wp_q(\iota)\in\mathscr{C}$ and $0<\sigma,\acute{\sigma}\leq 1$ with $1<\sigma+\acute{\sigma}\leq 2$. Then, the solution of the linear $\mathbb{GP}$ L-SL $\mathbb{FDE}$s with $\mathbb{VC}$s and $\mathbb{APBC}$s*

$$\begin{cases}\mathbb{G}_c^{\sigma;\hbar}\left(\left[y(\iota)\mathbb{G}_c^{\acute{\sigma};\hbar}+w(\iota)\right]\right)q(\iota)=\wp_q(\iota),\ (\iota\in\mathscr{J}),\\ q(0)=-q(\tau^*),\mathbb{G}_c^{\acute{\sigma};\hbar}q(0)=-\mathbb{G}_c^{\acute{\sigma};\hbar}q(\tau^*).\end{cases} \tag{18}$$

$\forall\hbar\in C(\mathscr{J},\mathbb{R}/\{0\})$ *and $w\in\mathscr{C}$ is given by*

$$\begin{aligned}q(\iota)={}&{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{y_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q\right)(\iota)-\frac{w(\iota)}{y(\iota)_{\mathbb{GPF}}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\iota)\\&+\left\{-\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)-\frac{\Delta_2}{\Delta_1}q(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{w(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\&+\left\{-{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{\Delta_3 y_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q\right)(\tau^*)-\frac{\Delta_2}{\Delta_{3\mathbb{GPF}}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\tau^*)-\frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)-\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}q(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right),\end{aligned} \tag{19}$$

where

$$\Delta_1 = \frac{1}{y(0)} + \frac{1}{y(\tau^*)}\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right),$$
$$\Delta_2 = \frac{w(\tau^*)}{y(\tau^*)} - \frac{w(0)}{y(0)} \neq 0,$$
$$\Delta_3 = 1 + \exp\left(\frac{\hbar-1}{\hbar}\tau^*\right), \tag{20}$$
$$\Delta_4 = \frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar - 1/\hbar\tau^*)}{y(\tau^*)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}.$$

Proof. Taking the σ^{th}-$\mathbb{GPF}$ integral to $\mathbb{FDE}$ q of (3.1), we get

$$\mathbb{G}_c^{\acute{\sigma};\hbar} q(\iota) = \frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\iota) - w(\iota)q(\iota) + c_0\exp(\hbar - 1/\hbar\iota)}{y(\iota)}, \ (c_0 \in \mathbb{R}). \tag{21}$$

From the $\mathbb{BC}$s of system (3.1), we obtain

$$c_0 = -\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)}{\Delta_1 y(\tau^*)} - \frac{\Delta_2}{\Delta_1}q(\tau^*). \tag{22}$$

Taking the $\acute{\sigma}^{th}$-$\mathbb{GPF}$ integral to (3.4), we obtain

$$q(\iota) = {}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{y_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}q(\iota)\right)(\iota) - \frac{w(\iota)}{y(\iota)_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}q(\iota) + c_0\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} + c_1\exp\left(\frac{\hbar-1}{\hbar}\iota\right), \ (c_1 \in \mathbb{R}). \tag{23}$$

Using the condition $q(0) = -q(\tau^*)$ of (18), we have

$$c_1 = -{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}c_0 = -\frac{\wp_q(\tau^*)}{\Delta_1 y(\tau^*)} - \frac{\Delta_2}{\Delta_1}q(\tau^*).\left(\frac{1}{\Delta_3 y}{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q\right)(\tau^*) - \frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)}{\Delta_1 y(\tau^*)} - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_4}q(\tau^*). \tag{24}$$

So, we derive (19) by substituting (24), and the proof is ended.

Clearly, by using Lemma 2.7, the system (18) is immediately established whenever we apply the σ^{th}-$\mathbb{GPF}$-derivative and $\acute{\sigma}^{th}$-$\mathbb{GPF}$-derivative to both sides of (19). On the basis of Lemma 3, the solutions of (3) are corresponding to $\mathbb{GPF}$-integral equation in the following format:

$$\begin{aligned} q(\iota) &= {}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{y_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q\right)(\iota) - \frac{w(\iota)}{y(\iota)_{\mathbb{GPF}}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\iota) \\ &+ \left\{-\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*) - \frac{\Delta_2}{\Delta_1}q(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{w(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\ &+ \left\{-{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{\Delta_3 y_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q\right)(\tau^*) - \frac{\Delta_2}{\Delta_{3\mathbb{GPF}}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}q(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right). \end{aligned} \tag{25}$$

□

3.1. Existence Result via $\mathbb{KMNC}$-Method. We consider the following hypotheses:

(H1) $\wp_q: \mathcal{J} \times \mathcal{C} \longrightarrow \mathcal{C}$ fulfills the Carathéodory criterion;

(H2) $\exists \mathfrak{p} \in C(\mathcal{J}, \mathbb{R}^+)$ s.t.

$$\|\wp(\iota, q(\iota))\| \leq \mathfrak{p}(\iota)\|q\|, \quad \forall \iota \in \mathcal{J}, \ \forall q \in \mathcal{C}. \tag{26}$$

(H3) For any $\iota \in \mathcal{J}$ and each bounded measurable set $E \subset \mathcal{C}$,

$$\lim_{\hbar \longrightarrow 0^+} \Upsilon\left(\wp\left(\mathcal{J}_{\iota,\hbar} \times E\right), 0\right) \leq \mathfrak{p}(\iota)(\iota)\Upsilon(B), \tag{27}$$

where Υ is the $\mathbb{KMNC}$ and $\mathcal{J}_{\iota,\hbar} = [\iota - \hbar, \iota] \cap \mathcal{J}$.

Then, let us set

$$\begin{aligned} y^* &= \inf_{\iota\in\mathcal{J}} y(\iota),\\ w^* &= \sup_{\iota\in\mathcal{J}} w(\iota),\\ \mathfrak{p}^* &= \sup_{\iota\in\mathcal{J}} \mathfrak{p}(\iota). \end{aligned} \tag{28}$$

Theorem 2. *Let us (H1)-(H3) holds. If*

$$\vartheta\mathfrak{p}^* + \nu := \Lambda_1 < 1, \tag{29}$$

where

$$\begin{aligned} \vartheta = \Bigg\{ &\frac{1}{y^*}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{w^*}{y^*\Delta_1}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+1)}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y^*\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\ &+\left(\frac{1}{y^*\Delta_3}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{\Delta_4}{y^*\Delta_1\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)}\right)\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right)\Bigg\}. \end{aligned} \tag{30}$$

$$\begin{aligned} \nu = \Bigg\{ &\frac{w^*}{y^*}\frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} - \frac{\Delta_2}{\Delta_1}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y(\tau^*)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\ &+\left(-\frac{\Delta_2}{\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)} - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\right)\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right)\Bigg\}. \end{aligned} \tag{31}$$

Then the system of GPFDEs (3) possesses a solution on $\mathcal{J}$.

Proof. We define the self-operator $\mathcal{N}_*$ on $\mathscr{C}$ by

$$\begin{aligned} (\mathcal{N}_*q)(\iota) = {}& {}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{y}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q\right)(\iota) - \frac{w(\iota)}{y(\iota)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\iota)\\ &+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)}{\Delta_1 y(\tau^*)} - \frac{\Delta_2}{\Delta_1}\wp(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{y(\iota)\iota^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\ &+\left\{-{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{\Delta_3 y}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q\right)(\tau^*) - \frac{\Delta_2}{\Delta_3}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\tau^*) - \frac{\Delta_4}{\Delta_3}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)\Delta_1 y(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}q(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right). \end{aligned} \tag{32}$$

$\forall q\in\mathscr{C}$. Evidently, the fixed-point $\mathcal{N}_*$ is a solution of the $\mathbb{GP}$ L-SL $\mathbb{FDP}$s (3). We define

$$\Omega_M = \{q\in\mathscr{C}: \|q\|\le M\}. \tag{33}$$

We shall follow the proof in three steps. □

Step 1. Let $\{q_n\}_n$ be a sequence with $q_n \longrightarrow q$ in $\mathscr{C}$. Then $\forall\iota\in\mathcal{J}$, one may write

$$\begin{aligned}
\left\|(\mathcal{N}_* q_n)(\iota) - (\mathcal{N}_* q)(\iota)\right\| \le {}& {}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\left(\frac{1}{y}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left\|\wp(\xi, q_n(\xi)) - \wp(\xi, q(\xi))\right\|\right)(\iota) \\
& - \frac{w(\iota)}{p(\iota)}\left({}_{\mathbb{GPF}}\mathbb{I}^{\alpha,\rho}\left\|w_n(s) - w(s)\right\|\right)(\iota) \\
& + \left\{-\frac{1}{\Delta_1 y(\tau^*)}\left({}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left\|\wp(\xi, q_n(\xi)) - \wp(\xi, q(\xi))\right\|\right)(\tau^*) - \frac{\Delta_2}{\Delta_1}\left(\left\|q_n(\xi) - q(\xi)\right\|\right)(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\
& + \left\{\begin{array}{l} -\dfrac{1}{\Delta_3 y(\tau^*)}\left({}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\left\|\wp(\xi, q_n(\xi)) - \wp(\xi, q(\xi))\right\|\right)(\tau^*) - \dfrac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\left(\left\|q_n(\xi) - q(\xi)\right\|\right)(\tau^*) \\ -\dfrac{\Delta_4}{\Delta_3}\dfrac{1}{\Delta_1 y(\tau^*)}\left({}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left\|\wp(\xi, q_n(\xi)) - \wp(\xi, q(\xi))\right\|\right)(\tau^*) - \dfrac{\Delta_4}{\Delta_3}\dfrac{\Delta_2}{\Delta_1}\left(\left\|q_n(\xi) - q(\xi)\right\|\right)(\tau^*) \end{array}\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right).
\end{aligned} \tag{34}$$

By continuity q and from (H1), the function $\wp(\iota, q_n(\iota))$ tends uniformly to $\wp(\iota, q(\iota))$. In accordance with Lebesgue's dominated convergence theorem, $(\mathcal{N}_* q_n)(\iota)$ tends uniformly to $\mathcal{N}_*(q)(\iota)$, that is $\mathcal{N}_* q_n \longrightarrow \mathcal{N}_* q$. Hence $\mathcal{N}_*$ is sequentially continuous.

Step 2. Let is take $\overline{q} \in \Omega_M$, by (H2), and assume that $(\mathcal{N}_*\overline{q})(\iota) \neq 0$, we obtain

$$\begin{aligned}
\left|(\mathcal{N}_*\overline{q})(\iota)\right| \le {}& \left|{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{y}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}\right)(\iota)\right| + \left|\frac{w(\iota)}{y(\iota)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\iota)\right| \\
& + \left\{\left|\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}\right| + \left|\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right|\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\
& + \left\{\left|{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{\Delta_3 y}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}\right)(\tau^*)\right| + \left|\frac{\Delta_2}{\Delta_3}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\tau^*)\right| + \left|\frac{\Delta_4}{\Delta_3}\,\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}\right| + \left|\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right|\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right) \\
\le {}& {}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\left(\frac{1}{y}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left|\wp_{\overline{q}}\right|(\iota)\right) + \frac{w(\iota)}{y(\iota)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}|\overline{q}(\iota)| \\
& + \left\{\frac{1}{\Delta_1 y(\tau^*)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left|\wp_{\overline{q}}(\tau^*)\right| + \frac{\Delta_2}{\Delta_1}|\overline{q}(\tau^*)|\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\
& + \left\{\frac{1}{\Delta_3 y(\tau^*)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\left|\wp_{\overline{q}}(\tau^*)\right| + \frac{\Delta_2}{\Delta_3}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}|\overline{q}(\tau^*)| + \frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left|\wp_{\overline{q}}(\tau^*)\right| + \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}|\overline{q}(\tau^*)|\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right) \\
\le {}& \frac{1}{y(\iota)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\left[\|\overline{q}\|\mathfrak{p}(\xi)\right](\iota) + \frac{w(\iota)}{y(\iota)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}|\overline{q}(\xi)|(\iota) \\
& + \left\{\frac{1}{\Delta_1 y(\tau^*)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left[\|\overline{q}\|\mathfrak{p}(\xi)\right](\tau^*) + \frac{\Delta_2}{\Delta_1}|\overline{q}(\xi)|(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\
& + \left\{\frac{1}{\Delta_3 y(\tau^*)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\left[\|\overline{q}\|\mathfrak{p}(\xi)\right](\tau^*) + \frac{\Delta_2}{\Delta_3}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}|\overline{q}(\xi)|(\tau^*) + \frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)}\,{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left[\|\overline{q}\|\mathfrak{p}(\xi)\right](\tau^*) + \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}|\overline{q}(\xi)|(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right).
\end{aligned} \tag{35}$$

$\forall \iota \in \mathcal{J}$. By using the property

$$0 < \exp\left(\frac{\acute{\sigma}-1}{\hbar}(s-\xi)\right) \le 1. \tag{36}$$

For $a \le \xi < s < \iota \le \tau^*$, we have

$$\begin{aligned}
\left|\left(\mathcal{N}_*\overline{q}\right)(\iota)\right| &\le \frac{\mathfrak{p}^* M}{y(\iota)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}(1)(\iota) + M\frac{w(\iota)}{y(\iota)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}(1)(\iota)\\
&+\left\{\frac{\mathfrak{p}^* M}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}(1)(\tau^*) + M\frac{\Delta_2}{\Delta_1}\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\
&+\left\{\frac{\mathfrak{p}^* M}{\Delta_3 y(\tau^*)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}(1)(\tau^*) + M\frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}(1)(\tau^*) + \frac{\Delta_4}{\Delta_3}\frac{\mathfrak{p}^* M}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}(1)(\tau^*) + M\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right)\\
&\le M\left\{\begin{matrix}\dfrac{\mathfrak{p}^*}{y^*}\dfrac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \dfrac{w^*}{y^*}\dfrac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} + \left[\dfrac{\mathfrak{p}^*}{\Delta_1 y(\tau^*)}\dfrac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)} + \dfrac{\Delta_2}{\Delta_1}\right]\dfrac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y^*\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\[2ex]
+\left[\dfrac{\mathfrak{p}^*}{y(\tau^*)\Delta_3}\dfrac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \dfrac{\Delta_2}{\Delta_3}\dfrac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} + \dfrac{\Delta_4}{\Delta_3}\dfrac{\mathfrak{p}^*}{y(\tau^*)\Delta_1}\dfrac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)} + \dfrac{\Delta_4}{\Delta_3}\dfrac{\Delta_2}{\Delta_1}\right]\end{matrix}\right\}\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right) = M\left(\vartheta\mathfrak{p}^*+\nu\right).
\end{aligned} \tag{37}$$

Hence, we get

$$\left\|\mathcal{N}_* q\right\|_{\mathscr{C}} \le M\left(\vartheta\mathfrak{p}^*+\nu\right) \le M. \tag{38}$$

Indeed $\mathcal{N}_*(\Omega_M) \subseteq \Omega_M$.

Step 3. By Step 2, the $\mathcal{N}_*(\Omega_M)$ is bounded. Let $\iota_1, \iota_2 \in \mathcal{J}$, $\iota_1 < \iota_2$ and $\overline{q} \in \Omega_M$, so

$$\left(\mathcal{N}_*\overline{q}\right)(\iota_2) - \left(\mathcal{N}_*\overline{q}\right)(\iota_1) \ne 0. \tag{39}$$

Then

$$\begin{aligned}
&\left\|\left(=\left(\mathcal{N}_*\overline{q}\right)(\iota_2) - \left(\mathcal{N}_*\overline{q}\right)(\iota_1)\right)\right\|\\
&\le {}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left(\frac{1}{\hbar^*}{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\left|\wp(\xi,\overline{q}(\xi))(\iota_2) - \wp(\xi,\overline{q}(\xi))(\iota_1)\right|\right)\\
&\quad + {}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left(\frac{w^*}{\hbar^*}\left|\overline{q}(\xi)\right|(\iota_2) - \left|\overline{q}(\xi)\right|(\iota_1)\right)\\
&\quad + \left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)} - \frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\frac{\left(\iota_2^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota_2) - \iota_1^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota_2)\right)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\
&\quad + \left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_3 y(\tau^*)} - \frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\iota) - \frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)} - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\left(\exp\left(\frac{\hbar-1}{\hbar}\iota_2\right) - \exp\left(\frac{\hbar-1}{\hbar}\iota_1\right)\right)\\
&\le \mathfrak{p}^* M\left\{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\left(\frac{1}{y^*}\left|{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}(1)(\iota_2) - {}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}(1)(\iota_1)\right|\right)\right\}
\end{aligned}$$

$$+M\left(\frac{w^*}{y^*}\left({}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}(1)(\iota_2)-{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}(1)(\iota_1)\right)\right)$$
$$+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}-\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\frac{\left(\iota_2^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota_2)-\iota_1^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota_1)\right)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \tag{40}$$
$$+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_3 y(\tau^*)}-\frac{\Delta_2}{\Delta_{3\mathbb{GPF}}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\tau^*)-\frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}-\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\left(\exp\left(\frac{\hbar-1}{\hbar}\iota_2\right)-\exp\left(\frac{\hbar-1}{\hbar}\iota_1\right)\right).$$

By using the property

$$0<\exp\left(\frac{\acute{\sigma}-1}{\hbar}(s-\xi)\right)\leq 1. \tag{41}$$

For $a\leq\xi<s<\iota\leq\tau^*$, we get

$$\|(\mathcal{N}_*\overline{q})(\iota_2)-(\mathcal{N}_*\overline{q})(\iota_1)\|$$
$$\leq\frac{\mathfrak{p}^*M}{\hbar^{\sigma+\acute{\sigma}}y^*\Gamma(\sigma+\acute{\sigma})}\int_a^{\iota_1}\left|\exp\left(\frac{\hbar-1}{\hbar}(\iota_2-\xi)\right)(\iota_2-\xi)^{\sigma+\acute{\sigma}-1}-\exp\left(\frac{\hbar-1}{\hbar}(\iota_1-\xi)\right)(\iota_1-\xi)^{\sigma+\acute{\sigma}-1}\right|d\xi$$
$$+\frac{\mathfrak{p}^*M}{\hbar^{\sigma+\acute{\sigma}}y^*\Gamma(\sigma+\acute{\sigma})}\int_{\iota_1}^{\iota_2}\exp\left(\frac{y-1}{\hbar}(\iota_2-\xi)\right)(\iota_2-\xi)^{\sigma+\acute{\sigma}-1}d\xi$$
$$+\frac{w^*}{y^*}\frac{M}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma})}\int_a^{\iota_1}\left|\exp\left(\frac{\hbar-1}{\hbar}(\iota_2-\xi)\right)(\iota_2-\xi)^{\acute{\sigma}-1}-\exp\left(\frac{y-1}{\hbar}(\iota_1-\xi)\right)(\iota_1-\xi)^{\acute{\sigma}-1}\right|d\xi \tag{42}$$
$$+\frac{w^*}{y^*}\frac{M}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma})}\int_{\iota_1}^{\iota_2}\exp\left(\frac{y-1}{\hbar}(\iota_2-\xi)\right)(\iota_2-\xi)^{\acute{\sigma}-1}|\overline{q}(\xi)|d\xi$$
$$+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 w^*}-\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\frac{\iota_2^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota_2)-\iota_1^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota_1)}{y^*\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}$$
$$+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_3 y^*}-\frac{\Delta_2}{\Delta_{3\mathbb{GPF}}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\tau^*)-\frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y^*}-\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\left(\exp\left(\frac{\hbar-1}{\hbar}\iota_2\right)-\exp\left(\frac{\hbar-1}{\hbar}\iota_1\right)\right).$$

As $\mathfrak{t}_1\longrightarrow\mathfrak{t}_2$, the right-hand side of inequality (42) tends to zero. This means that $\mathcal{N}_*(\Omega_M)$ is equicontinuous. Now, let $\widehat{\Omega}\subset\Omega_M$ with

$$\widehat{\Omega}=\overline{conv}\left(\mathcal{N}_*(\widehat{\Omega})\cup\{0\}\right). \tag{43}$$

Since $\widehat{\Omega}$ is bounded and equicontinuous, the function $\iota\longrightarrow\omega(\iota)=\Upsilon(\widehat{\Omega}(\iota))$ is continuous on $\mathcal{J}$. By assumption (H2) and using the property

$$0<\exp\left(\frac{\hbar-1}{\hbar}(s-\xi)\right)\leq 1. \tag{44}$$

For $a \le \xi < s < \iota \le \tau^*$, we get.

$$\omega(\iota) \le \Upsilon\Big(\mathcal{N}_*(\widehat{\Omega})(\iota) \cup \{0\}\Big) \le \Upsilon\Big(\big(\mathcal{N}_*\widehat{\Omega}\big)(\iota)\Big) \le {}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{1}{\hbar}\, {}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}\mathfrak{p}(\xi)\Upsilon(\widehat{\Omega}(\xi))\Big)$$

$$(\iota) + {}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{w}{\hbar}\Upsilon(\widehat{\Omega}(\xi))\Big)(\iota) + \Big\{-{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{w}{\Delta_1 y}\Upsilon(\widehat{\Omega}(\xi))\Big)(\tau^*) - \frac{\Delta_2}{\Delta_1}\Upsilon(\widehat{\Omega}(\xi))(\tau^*)\Big\}$$

$$\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\iota^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} + \Big\{-{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{1}{\Delta_3 y}\, {}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}\mathfrak{p}(\xi)\Upsilon(\widehat{\Omega}(\xi))\Big)(\tau^*) - {}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{\Delta_2}{\Delta_3}\Upsilon(\widehat{\Omega}(\xi))\Big)(\tau^*)$$

$$-\frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\mathfrak{p}(\xi)\Upsilon(\widehat{\Omega}(\xi))(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\Upsilon(\widehat{\Omega}(\xi))(\tau^*)\Big\}$$

$$\exp\Big(\frac{\hbar-1}{\hbar}\iota\Big) \le \mathfrak{p}^*\|\omega\|\Bigg({}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{1}{\hbar}\,{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}(1)\Big)(\iota) - {}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{w}{\Delta_1 y}(1)\Big)(\tau^*)\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}$$

$$+\Big\{-{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{1}{\Delta_3 y}\,{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}(1)\Big)(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}(1)(\tau^*)\Big\}\exp\Big(\frac{\hbar-1}{\hbar}\iota\Big)\Bigg) \tag{45}$$

$$+\|\omega\|\Bigg({}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{w}{\hbar}(1)\Big)(\iota) - \frac{\Delta_2}{\Delta_1}\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} + \Big\{-{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\Big(\frac{\Delta_2}{\Delta_3}(1)\Big)(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\Big\}\exp\Big(\frac{\hbar-1}{\hbar}\iota\Big)\Bigg)$$

$$\le \mathfrak{p}^*\|\omega\|\Bigg\{\frac{1}{y^*}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{w^*}{y^*\Delta_1}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+1)}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y^*\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}$$

$$+\Bigg(\frac{1}{y^*\Delta_3}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{\Delta_4}{y^*\Delta_1\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)}\Bigg)\exp\Big(\frac{\hbar-1}{\hbar}\tau^*\Big)\Bigg\}$$

$$+\|\omega\|\Bigg\{\frac{w^*}{y^*}\frac{(\tau^*)^{\sigma}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} - \frac{\Delta_2}{\Delta_1}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y(\tau^*)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} + \Big\{-\frac{\Delta_2}{\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)} - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\Big\}\exp\Big(\frac{\hbar-1}{\hbar}\tau^*\Big)\Bigg\}$$

$$\le \mathfrak{p}^*\|\omega\|\vartheta + \|\omega\|\nu.$$

Indeed,

$$\|\omega\|(1 - \mathfrak{p}^*\vartheta - \nu) \le 0. \tag{46}$$

Equation (29) implies that $\|\omega\| = 0$ or $\omega(\iota) = 0$, $\forall \iota \in \mathcal{J}$. Hence $\widehat{\Omega}(\iota)$ is relatively compact in $\mathbb{R}$ and so Arzelá–Ascoli theorem implies that $\widehat{\Omega}$ is relatively compact in Ω_M. Now, from Theorem 3, we conclude that the problem (3) has a solution which is a fixed point of operator $\mathcal{N}_*$.

3.2. Uniqueness Criterion

Theorem 3. *Let the following assumptions hold:*

(H4) $\wp: \mathcal{J} \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function.*

(H5) there exists a nonnegative constant $M > 0$, *such that*

$$|\wp(\iota, q) - \wp(\iota, \acute{q})| \le M\|q - \acute{q}\|, \forall \iota \in \mathcal{J}, \forall q, \acute{q} \in \mathbb{R}. \tag{47}$$

Then, problem (3) has a unique solution on $\mathcal{J}$, provided that

$$\Lambda_2 = \vartheta M + \nu < 1, \tag{48}$$

where ϑ and ν are given by (30) and (31), respectively.

Proof. In the first step, we show that $\mathcal{N}_*(\Omega_M) \subset \Omega_M$, where the operator $\mathcal{N}_*: \mathcal{C} \longrightarrow \mathcal{C}(\mathcal{J})$ is defined by Equation (32), $\Omega_M = \{q \in \mathcal{C}: \|q\| \le M\}$, and we choose a real number $M >$, such that

$$M \geq \frac{\vartheta \wp_0}{1 - \Lambda_2},$$
$$\wp_0 = \sup_{1 \leq \iota \leq \tau^*} |\wp(\iota, 0)|. \tag{49}$$

For any $\overline{q} \in \Omega_M$, using (H5), we occur

$$\begin{aligned}
\left|(\mathcal{N}_* q)(\iota)\right| \leq & \left|\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_q(\iota)}{y(\iota)}\right| + \left|\frac{w(\iota)}{y(\iota)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\iota)\right| \\
& + \left\{\left|\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}\right| + \left|\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right|\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\
& + \left\{\left|\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_{\overline{q}}}{\Delta_3 y(\tau^*)}\right| + \left|\frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\tau^*)\right| + \left|\frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}\right| + \left|\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right|\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right).
\end{aligned} \tag{50}$$

By using the property

$$0 < \exp\left(\frac{\hbar - 1}{\hbar}(s - \xi)\right) \leq 1. \tag{51}$$

For $a \leq \xi < s < \iota \leq \tau^*$ and (H5) implies that

$$\begin{aligned}
\left|(\mathcal{N}_*)q(\iota)\right| \leq & {}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{1}{y^*}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}|\wp(\xi,\overline{q}(\xi)) - \wp(\xi,0)| + |\wp(\xi,0)|\right)(\iota) \\
& + {}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{w^*}{y^*}|\overline{q}(\xi)|\right)(\iota) + \left\{\frac{1}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}(|\wp(\xi,\overline{q}(\xi)) - \wp(\xi,0)| + |\wp(\xi,0)|)(\tau^*)\right. \\
& \left. + \frac{\Delta_2}{\Delta_1}|\overline{q}(\xi)|(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\
& + \left\{\frac{1}{\Delta_3 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma+\acute{\sigma},\hbar}(|\wp(\xi,\overline{q}(\xi)) - \wp(\xi,0)| + |\wp(\xi,0)|)(\tau^*)\right. \\
& + \frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}|\overline{q}(\xi)|(\tau^*) + \frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}(|\wp(\xi,\overline{q}(\xi)) - \wp(\xi,0)| + |\wp(\xi,0)|)(\tau^*) \\
& \left. + \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}|\overline{q}(\xi)|(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right) \leq {}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{1}{y^*}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}(M\|\overline{q}\| + \wp_0)\right)(\iota) + \left(\frac{w^*}{y^*}\right){}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\|\overline{q}\|(\iota) \\
& + \left\{\frac{1}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}(M\|\overline{q}\| + \wp_0)(\tau^*) + \frac{\Delta_2}{\Delta_1}\|\overline{q}\|(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}
\end{aligned}$$

$$
\begin{aligned}
&+\left\{\frac{1}{\Delta_3 y(\tau^*)_{\mathbb{GPF}}} I_a^{\sigma+\acute{\sigma},\hbar}\left(M\|\overline{q}\|+\wp_0\right)(\tau^*)+\frac{\Delta_2}{\Delta_{3\mathbb{GPF}}} I_a^{\acute{\sigma},\hbar}\|\overline{q}\|(\tau^*)+\frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}} I_a^{\sigma,\hbar}\left(M\|\overline{q}\|+\wp_0\right)(\tau^*)+\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\|\overline{q}\|\right\}\\
&\exp\left(\frac{\hbar-1}{\hbar}\iota\right)\leq\frac{\left(M\|\overline{q}\|+\wp_0\right)}{y^*}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)}+\|\overline{q}\|\frac{w^*}{y^*}\frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\
&+\left\{\frac{\left(M\|\overline{q}\|+\wp_0\right)}{\Delta_1 y(\tau^*)}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)}+M\frac{\Delta_2}{\Delta_1}\right\}\frac{(\tau^*)^{\acute{\sigma}}\exp\left(\hbar-1/\hbar\tau^*\right)}{y^*\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\
&+\left\{\frac{\left(M\|\overline{q}\|+\wp_0\right)}{y(\tau^*)\Delta_3}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)}+\|\overline{q}\|\frac{\Delta_2}{\Delta_3}\frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}+\frac{\Delta_4}{\Delta_3}\frac{\left(M\|\overline{q}\|+\wp_0\right)}{y(\tau^*)\Delta_1}\frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}+\|\overline{q}\|\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\right\}\\
&\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right)\leq\vartheta\left(M\left(\|\overline{q}\|\right)+\wp_0\right)+\nu\|\overline{q}\|\leq M.
\end{aligned}
\tag{52}
$$

Which implies $\|\mathcal{N}_*\overline{q}\|\leq M$ after taking the norm on $\mathcal{J}$. Thus, $\mathcal{N}_*$ maps Ω_M into itself. For $q,\acute{q}\in\mathcal{C}$, we obtain by using the notations in (30) and (31) that

$$
\begin{aligned}
&\left|(\mathcal{N}_*q)(\iota)-(\mathcal{N}_*\acute{q})(\iota)\right|\leq{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{1}{y^*}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}\left|\wp(\xi,q(\xi))-\wp(\xi,\acute{q}(\xi))\right|\right)(\iota)\\
&+{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{w^*}{y^*}\left|q(\xi)-\acute{q}(\xi)\right|\right)(\iota)+\left\{\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}I_a^{\sigma,\hbar}\left|\wp(\xi,q(\xi))-\wp(\xi,\acute{q}(\xi))\right|(\tau^*)+\frac{\Delta_2}{\Delta_1}\left|q(\xi)-\acute{q}(\xi)\right|(\tau^*)\right\}\\
&\frac{\iota^{\acute{\sigma}}\exp\left(\hbar-1/\hbar\iota\right)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}+\left\{\frac{1}{\Delta_3 y(\tau^*)_{\mathbb{GPF}}}I_a^{\sigma+\acute{\sigma},\hbar}\left|\wp(\xi,q(\xi))-\wp(\xi,\acute{q}(\xi))\right|(\tau^*)+\frac{\Delta_2}{\Delta_{3\mathbb{GPF}}}I_a^{\acute{\sigma},\hbar}\left|q(\xi)-\acute{q}(\xi)\right|(\tau^*)\right.\\
&\left.+\frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}I_a^{\sigma,\hbar}\left|\wp(\xi,q(\xi))-\wp(\xi,\acute{q}(\xi))\right|(\tau^*)+\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\left|q(\xi)-\acute{q}(\xi)\right|(\tau^*)\right\}\\
&\exp\left(\frac{\hbar-1}{\hbar}\iota\right)\leq{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{1}{y^*}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}\left(M\left|q(\xi)-\acute{q}(\xi)\right|\right)\right)(\iota)\\
&+{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{w^*}{y^*}\left|q(\xi)-\acute{q}(\xi)\right|\right)(\iota)+\left\{\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}I_a^{\sigma,\hbar}\left(M\left|q(\xi)-\acute{q}(\xi)\right|\right)(\tau^*)+\frac{\Delta_2}{\Delta_1}\left|q(\xi)-\acute{q}(\xi)\right|(\tau^*)\right\}\\
&\frac{\iota^{\acute{\sigma}}\exp\left(\hbar-1/\hbar\iota\right)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}+\left\{\frac{1}{\Delta_3 y(\tau^*)_{\mathbb{GPF}}}I_a^{\sigma+\acute{\sigma},\hbar}\left(M\left|q(\xi)-\acute{q}(\xi)\right|\right)(\tau^*)+\frac{\Delta_2}{\Delta_{3\mathbb{GPF}}}I_a^{\acute{\sigma},\hbar}\left|q(\xi)-\acute{q}(\xi)\right|(\tau^*)\right.\\
&\left.+\frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}}I_a^{\sigma,\hbar}\left(M\left|q(\xi)-\acute{q}(\xi)\right|\right)(\tau^*)+\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\left|q(\xi)-\acute{q}(\xi)\right|(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right).
\end{aligned}
\tag{53}
$$

By using the property

$$0 < \exp\left(\frac{\acute{\hbar}-1}{\hbar}(s-\xi)\right) \le 1. \tag{54}$$

For $a \le \xi < s < \iota \le \tau^*$, we have

$$\begin{aligned} \left|(\mathcal{N}_* q)(\iota) - (\mathcal{N}_* \acute{q})(\iota)\right| \le & \left\{ \frac{M}{y^*} \frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{w^*}{y^*} \frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \right. \\ & + \left\{ \frac{M}{\Delta_1 y(\tau^*)} \frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)} + \frac{\Delta_2}{\Delta_1} \right\} \frac{(\tau^*)^{\acute{\sigma}} \exp(\hbar - 1/\hbar\tau^*)}{y^* \hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\ & + \left\{ \frac{M}{y(\tau^*)\Delta_3} \frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{\Delta_2}{\Delta_3} \frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \right. \\ & \left.\left. + \frac{\Delta_4}{\Delta_3} \frac{M}{y(\tau^*)\Delta_1} \frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)} + \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1} \right\} \exp\left(\frac{\hbar-1}{\hbar}\tau^*\right) \right\} |q(\xi) - \acute{q}(\xi)| \le (\vartheta M + \nu)\|q - \acute{q}\|. \end{aligned} \tag{55}$$

Consequently, we get

$$\|\mathcal{N}_* q - \mathcal{N}_* \acute{q}\| \le (\vartheta M + \nu)\|q - \acute{q}\|. \tag{56}$$

Therefore, $\mathcal{N}_*$ is a contraction. Hence, the operator $\mathcal{N}_*$ has a unique fixed point, which corresponds to a unique solution of the (3) on $\mathcal{J}$. □

4. UH and UHR Stability of System

In the recent section, we are interested to study UH and UHR stability of the system (3). The system (3) is UH stable if $\exists c_q > 0$ such that, for each $\epsilon \in \mathbb{R}^+$ and for each $q \in \mathscr{C}$ satisfying

$$\begin{cases} \left| \mathbb{G}_c^{\sigma;\hbar}\left(\left[y(\iota)\mathbb{G}_c^{\acute{\sigma};\hbar} + w(\iota)\right]\right)q(\iota) - \wp(\iota, q(\iota)) \right| \le \varepsilon, \ (\iota \in \mathcal{J}), \\ q(0) = -q(\tau^*), \mathbb{G}_c^{\acute{\sigma};\hbar} q(0) = -\mathbb{G}_c^{\acute{\sigma};\hbar} q(\tau^*). \end{cases} \tag{57}$$

There exists a unique solution $\overline{q} \in \mathscr{C}$ of (3) with

$$\|q - \overline{q}\| \le c_q \varepsilon. \tag{58}$$

The system (3) is generalized UH stable (GUH) if there exists $c_q \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $c_q(0) = 0$ such that for each $\epsilon \in \mathbb{R}^+$ and for each $q \in \mathscr{C}$ satisfying (57), there exists a unique solution $\overline{q} \in \mathscr{C}$ of (3) with $\|q - \overline{q}\| \le c_q(\epsilon)$.

Remark 1. A function $\overline{q} \in \mathscr{C}$ is a solution of inequality (57) if $\exists q_1 \in \mathscr{C}$ (which depends on solution $\overline{q}$) such that, i) $|q_1(\iota)| \le \varepsilon$, $(\iota \in \mathcal{J})$; ii)

$$\mathbb{G}_c^{\sigma;\hbar}\left(\left[y(\iota)\mathbb{G}_c^{\sigma;\hbar} + q(\iota)\right]\right)q(\iota) = \wp(\iota, q(\iota)) + q_1(\iota), \iota \in \mathcal{J}. \tag{59}$$

Theorem 4. *Suppose that the condition (H5) and inequality (48) are fulfilled. Then, the solution of (3) is UH and GUH stable.*

Proof. Let $\epsilon > 0$ and let $\overline{q} \in \mathscr{C}$ be a function that satisfies the inequality (57) and let $q \in \mathscr{C}$ the unique solution of the problem

$$\begin{cases} \mathbb{G}_c^{\sigma;\hbar}\left(\left[y(\iota)\mathbb{G}_c^{\acute{\sigma};\hbar} + w(\iota)\right]\right)q(\iota) = \wp(\iota, q(\iota)), \ (\iota \in \mathcal{J}), \\ q(0) = -q(\tau^*), \mathbb{G}_c^{\acute{\sigma};\hbar} q(0) = -\mathbb{G}_c^{\acute{\sigma};\hbar} q(\tau^*). \end{cases} \tag{60}$$

Lemma 3 implies that

$$\begin{aligned} q(\iota) = & \frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_q(\iota)}{y(\iota)} - \frac{w(\iota)}{y(\iota)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar} q(\iota) \\ & + \left\{ -\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)}{\Delta_1 y(\tau^*)} - \frac{\Delta_2}{\Delta_1} q(\tau^*) \right\} \frac{\iota^{\acute{\sigma}} \exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \end{aligned}$$

$$+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_q(\tau^*)}{\Delta_3 y(\tau^*)}-\frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}q(\tau^*)-\frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_q(\tau^*)}{\Delta_1 y(\tau^*)}-\frac{\Delta_4}{\Delta_3}\frac{\Delta_3}{\Delta_1}q(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right). \tag{61}$$

Hence by Remark 1, we have

$$\begin{cases}\mathbb{G}_c^{\acute{\sigma};\hbar}\left(\left[y(\iota)\mathbb{G}_c^{\sigma;\hbar}+w(\iota)\right]\right)\overline{q}(\iota)=\wp(\iota,\overline{q}(\iota))+q_1(\iota),\ (\iota\in\mathcal{J}),\\ \overline{q}(0)=-\overline{q}(\tau^*),\mathbb{G}_c^{\acute{\sigma};\hbar}\overline{q}(0)=-\mathbb{G}_c^{\acute{\sigma};\hbar}\overline{q}(\tau^*).\end{cases} \tag{62}$$

Again Lemma 3 implies that

$$\begin{aligned}\overline{q}(\iota)&=\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_{\overline{q}}(\iota)}{y(\iota)}-\frac{w(\iota)}{y(\iota)}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\iota)\\&+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}-\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\&+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}\wp_{\overline{q}}(\overline{q})}{\Delta_3 y(\tau^*)}-\frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}\mathbb{I}_a^{\acute{\sigma},\hbar}\overline{q}(\tau^*)-\frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}\wp_{\overline{q}}(\tau^*)}{\Delta_1 y(\tau^*)}-\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\overline{q}(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right)\\&+\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}q_1(\iota)}{y(\iota)}+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}q_1(\tau^*)}{\Delta_1 y(\tau^*)}\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}+\left\{-\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma+\acute{\sigma},\hbar}q_1(\tau^*)}{\Delta_3 y(\tau^*)}-\frac{\Delta_4}{\Delta_3}\frac{{}_{\mathbb{GPF}}\mathbb{I}_a^{\sigma,\hbar}q_1(\tau^*)}{\Delta_1 y(\tau^*)}\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right).\end{aligned} \tag{63}$$

On the other hand, we have, for each $\iota\in\mathcal{J}$,

$$\begin{aligned}\overline{q}(\iota)&=\frac{1}{y(\iota)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{1}{y^*}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}|\wp(\xi,\overline{q}(\xi))-\wp(\xi,q(\xi))|\right)(\iota)+{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{w^*}{y^*}|\overline{q}(\xi)-q(\xi)|\right)(\iota)\\&+\left\{-\frac{1}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}\left|\wp\left(\xi,{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}(\xi)\right)-\wp(\xi,q(\xi))\right|(\tau^*)-\frac{\Delta_2}{\Delta_1}\left|\tau^*(\xi)-q(\xi)\right|(\tau^*)\right\}\frac{\iota^{\acute{\sigma}}\exp(\hbar-1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\&+\left\{-\frac{1}{\Delta_3 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{1}{y^*}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}|\wp(\xi,\overline{q}(\xi))-\wp(\xi,q(\xi))|\right)(\tau^*)-\frac{\Delta_2}{\Delta_3}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}|\overline{q}(\xi)-q(\xi)|(\tau^*)\right.\\&-\frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}|\wp(\xi,\overline{q}(\xi))-\wp(\xi,q(\xi))|(\tau^*)\\&\left.-\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}|\overline{q}(\xi)-q(\xi)|(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right)+\frac{1}{y(\iota)}{}_{\mathbb{GPF}}I_a^{\sigma,\hbar}\left(\frac{1}{y^*}{}_{\mathbb{GPF}}I_a^{\acute{\sigma},\hbar}|q_1(\xi)|\right)(\iota)\end{aligned}$$

$$+\left\{-\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}} I_a^{\sigma,\hbar} q_1(\tau^*)\right\} \frac{\iota^{\acute{\sigma}} \exp(\hbar - 1/\hbar\iota)}{y(\iota)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}$$
$$+\left\{-\frac{1}{\Delta_3 y(\tau^*)_{\mathbb{GPF}}} I_a^{\sigma,\hbar}\left(\frac{1}{y^*{}_{\mathbb{GPF}}} I_a^{\acute{\sigma},\hbar}|q_1(\xi)|\right)(\tau^*) - \frac{\Delta_4}{\Delta_3}\frac{1}{\Delta_1 y(\tau^*)_{\mathbb{GPF}}} I_a^{\sigma,\hbar} q_1(\tau^*)\right\}\exp\left(\frac{\hbar-1}{\hbar}\iota\right). \tag{64}$$

Hence using part 1 of Remark 1 and (H5), we can get

$$|\overline{q} - q| \leq \vartheta\varepsilon + \Lambda_2\|\overline{q} - q\|. \tag{65}$$

In consequence, it follows that

$$\|\overline{q} - q| \leq \frac{\vartheta}{1-\Lambda_2}\varepsilon. \tag{66}$$

If we let $c_q = \vartheta/1 - \Lambda_2$, then, the UH stability condition is satisfied. More generally, for

$$c_q(\varepsilon) = \frac{\vartheta}{1-\Lambda_2}\varepsilon. \tag{67}$$

$c_q(0) = 0$ the generalized UH stability condition is also satisfied. This completes the proof. □

5. Application and Numerical Examples

The notations and terminologies in this section are adopted from [11–13]. In control theory, a proportional derivative controller ($\mathbb{PDC}$) for controller output q at time ι has the following equation:

$$q(\iota) = \Upsilon_p \mathfrak{E}(\iota) + \Upsilon_d \frac{d}{d\iota}\mathfrak{E}(\iota), \tag{68}$$

where Υ_p, Υ_d, and $\mathfrak{E}$ are the proportional gain, the derivative gain, and the input deviation (or the error between the state variable and the process variable), respectively. Recent investigations have shown that $\mathbb{PDC}$ has direct incorporation in the control of complex networks models (see [14] for more details).

Let us consider the continuous functions $\Upsilon_0, \Upsilon_1\colon [0,1]\times\mathbb{R} \longrightarrow [0,\infty)$ such that, $\forall\iota \in \mathbb{R}$

$$\begin{aligned} &\lim_{\hbar\longrightarrow 0^+}\Upsilon_0(\hbar,\iota) = \lim_{\hbar\longrightarrow 0^-}\Upsilon_1(\hbar,\iota) = 0,\\ &\lim_{\hbar\longrightarrow 0^-}\Upsilon_0(\hbar,\iota) = \lim_{\hbar\longrightarrow 0^+}\Upsilon_1(\hbar,\iota) = 1. \end{aligned} \tag{69}$$

For $\hbar \in [0,1]$ and $\Upsilon_0(\hbar,\mathfrak{t}) \neq 0$, $\Upsilon_1(\hbar,\iota)\neq 0$ for $\hbar \in (0,1]$, $\hbar \in [0,1)$, respectively. Then, Anderson et al. [21] defined the proportional derivative of order $\hbar$ by

$$\mathbb{D}^{\hbar}q(\iota) = \Upsilon_0(\hbar,\iota)q'(\iota) + \Upsilon_1(\hbar,\iota)q(\iota), \quad (\forall\iota\in\mathbb{R}). \tag{70}$$

Here $q\prime := d/d\iota q$, provided that the right-hand side exists and Υ_1 is a type of proportional gain Υ_p, Υ_0 is a type of derivative gain Υ_d, q is the error, and $\mathbb{D}^{\hbar}q$ is the controller output (for more details refer to [15]). We next restrict ourselves to the case that

$$\begin{aligned} \Upsilon_0(\hbar,\iota) &= \hbar,\\ \Upsilon_1(\hbar,\iota) &= 1-\hbar. \end{aligned} \tag{71}$$

Therefore,

$$\mathbb{D}^{\hbar}q(\iota) = \hbar q'(\iota) + (1-\hbar)q(\iota). \tag{72}$$

Clearly,

$$\begin{aligned} &\lim_{\hbar\longrightarrow 0^+}\mathbb{D}^{\hbar}q(\iota) = q(\iota),\\ &\lim_{\hbar\longrightarrow 1^-}\mathbb{D}^{\hbar}q(\iota) = q'(\iota). \end{aligned} \tag{73}$$

Thus, equation (72) is considered to be more general than the conformable derivative, which evidently does not tend to the original functions as $\hbar$ tends to 0.

Now, we present some examples to illustrate our results.

Example 1. Based on (3), we consider an $\mathbb{GP}$ L-SL $\mathbb{FDP}$s as

$$\begin{cases} \mathbb{G}_c^{0.92;2.5}\left(\left[\dfrac{15\left(3+\sin^2(2\pi\iota)\right)}{2+\sin^2(2\pi\iota)}\mathbb{G}_c^{0.43;2.5} + \dfrac{1}{10(1+\exp(\iota/2))}\right]\right)q(\iota)\\ = \dfrac{10+\iota^2}{5+\iota^2}\tan^{-1}\dfrac{|q(\iota)|}{1+|q(\iota)|},\\ q(0)+q(1.85) = 0, \mathbb{G}_c^{0.43;2.5}q(0) + \mathbb{G}_c^{0.43;2.5}q(1.85) = 0. \end{cases} \tag{74}$$

For $\iota \in \mathcal{J} =: [0, 1.85]$. Clearly

$$\begin{aligned} \tau^* &= 1.85 > 0,\\ \sigma &= 0.92 \in (0,1],\\ \acute{\sigma} &= 0.43 \in (0,1],\\ \hbar &= 2.5. \end{aligned} \tag{75}$$

By taking

$$\begin{aligned} y(\iota) &= \frac{15\left(3+\sin^2(2\pi\iota)\right)}{2+\sin^2(2\pi\iota)},\\ w(\iota) &= \frac{1}{10(1+\exp(\iota/2))}. \end{aligned} \tag{76}$$

$$\wp(\iota, q(\iota)) = \frac{10+\iota^2}{5+\iota^2}\tan^{-1}\frac{|q(\iota)|}{1+|q(\iota)|}. \tag{77}$$

Hypothesis (H1) is held and we have

$$\begin{aligned}&\left\|\frac{10+\iota^2}{5+\iota^2}\tan^{-1}\frac{|q(\iota)|}{1+|q(\iota)|}\right\|\\&\leq\frac{10+\iota^2}{5+\iota^2}\left\|\tan^{-1}\frac{|q(\iota)|}{1+|q(\iota)|}\right\|\leq\mathfrak{p}(\iota)\|q(\iota)\|,\end{aligned}\tag{78}$$

where

$$\mathfrak{p}(\iota)=\frac{10+\iota^2}{5+\iota^2}.\tag{79}$$

By using Equations (20), (30), and (31), we got

$$\begin{aligned}\Delta_1&=\frac{1}{y(0)}+\frac{1}{y(\tau^*)}\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right)\simeq3.91977,\\
\Delta_2&=\frac{w(\tau^*)}{y(\tau^*)}-\frac{w(0)}{y(0)}\simeq-0.001704\neq0,\\
\Delta_3&=1+\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right)\simeq3.944679,\\
\Delta_4&=\frac{(\tau^*)^{\acute{\sigma}}\ \exp(\hbar-1/\hbar\tau^*)}{y(\tau^*)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\simeq0.05168,\\
\vartheta&=\frac{1}{y^*}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)}+\frac{w^*}{y^*\Delta_1}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+1)}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y^*\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}\\
&\quad+\left(\frac{1}{y^*\Delta_3}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)}+\frac{\Delta_4}{y^*\Delta_1\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)}\right)\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right)\\
&=\frac{1}{15}\frac{1.85^{1.35}}{2.5^{1.35}\Gamma(2.35)}+\frac{0.05}{15\times3.919779}\frac{1.85^{0.92}}{2.5^{1.35}\Gamma(2.35)}\frac{1.85^{0.43}\exp(1.5\times1.85/2.5)}{15\times2.5^{0.43}\Gamma(2.43)}\\
&\quad+\left(\frac{1}{15\times3.94467}\frac{1.85^{1.35}}{2.5^{1.35}\Gamma(2.35)}+\frac{0.0516813}{15\times3.919779\times3.944679}\frac{1.85^{0.92}}{2.5^{0.92}\Gamma(1.92)}\right)\\
&\quad\times\ \exp\left(\frac{1.5\times1.85}{2.5}\right)\simeq0.062698,\end{aligned}\tag{80}$$

here $y^*=15$, $w^*=0.05$, and $\mathfrak{p}^*=1.3155$, and

$$\begin{aligned}\nu&=\frac{w^*}{y^*}\frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}-\frac{\Delta_2}{\Delta_1}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y(\tau^*)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)}+\left(-\frac{\Delta_2}{\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)}-\frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\right)\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right)\\
&=\frac{0.05}{15}\frac{(1.85)^{0.43}}{2.5^{0.43}\Gamma(1.43)}-\left(\frac{-0.001704}{3.919779}\right)\frac{1.85^{0.43}\exp(1.5/2.51.85)}{15\times2.5^{0.43}\Gamma(1.43)}+\left(-\frac{-0.001704}{3.944679}\frac{1.85^{0.92}}{2.5^{0.92}\Gamma(1.92)}-\frac{0.0516813}{3.944679}\frac{-0.001704}{3.91977}\right)\\
&\quad\times\ \exp\left(\frac{1.5}{2.5}1.85\right)\simeq0.003619.\end{aligned}\tag{81}$$

```
clear;
format long;
syms v e;
sigma = 0.92; accutesigma = 0.43; hslash = 2.5;
a = 0; tauast = 1.85;
y = 15*(3sin(2 * pi*v)^2)/(2sin(2 * pi*v));
yast = 15;
w = 1/(10*(1exp(v/2)));
wast = 0.05;
wp = log(v);
mathfrakp= (10v^2)/(5v^2);
mathfrakpast=(10+tauast^2)/(5a^2);

n = floor(sigma)+1;
iota = a;
column = 1;
nn = 1;
while iota ≤ tauast
      MI(nn,column) = nn;
      MI(nn,column+1) = iota;
      Delta_1 = 1/(eval(subs(y, {v}, {a}))) ...
            + 1/(eval(subs(y, {v}, {iota}))) ...
            + exp ((hslash-1)/2 * iota);
      MI(nn,column+2) = Delta_1;
      Delta_2 = eval(subs(w, {v}, {iota}))/eval(subs(y, {v}, {iota})) ...
            - eval(subs(w, {v}, {a}))/eval(subs(y, {v}, {a}));
      MI(nn,column+3) = Delta_2;
      Delta_3 = 1 + exp((hslash-1)/hslash * iota);
      MI(nn,column+4) = Delta_3;
      Delta_4 = (iota^accutesigma * exp((hslash-1)/hslash * iota))...
            /(eval(subs(y, {v}, {iota})) * hslash^accutesigma * gamma(accutesigma+1));
      MI(nn,column+5) = Delta_4;
   vartheta = iota^(sigma + accutesigma)/(yast * hslash^(sigma + accutesigma)...
            * gamma(sigma + accutesigma+1)) + wast * iota^(sigma)...
            * iota^(accutesigma)*exp((hslash-1)/hslash * iota)...
            /(yast × Delta_1 * hslash^(sigma + accutesigma)...
            * gamma(sigma+1)* yast * hslash^(accutesigma)...
            * gamma (accutesigma+1)) + (iota^(sigma + accutesigma)...
            /(yast * Delta_3 * hslash^(sigma + accutesigma) ...
            * gamma(sigma + accutesigma+1)) + Delta_4 * iota^(sigma)...
            /(yast * Delta_1 * Delta_3 * hslash^(sigma) * gamma(sigma+1)))...
            * exp((hslash-1)/hslash * iota);
      MI(nn,column+6) = vartheta;
      nu = wast * iota^(accutesigma)/(yast * hslash^(accutesigma)...
            * gamma(accutesigma+1)) - Delta_2 * iota^(accutesigma)...
            * exp((hslash-1)/hslash * ...iota)/(Delta_1 * yast * hslash^(accutesigma+1))...
            + (-Delta_2 * iota^(sigma)/(Delta_3 * hslash * gamma(sigma+1))...
            - Delta_4 * Delta_2/(Delta_3 * Delta_1) * exp((hslash-1)/hslash * iota));
      MI(nn,column+7) = nu;
      MI(nn,column+8) = vartheta * mathfrakpast + nu;
      iota = iota + 0.1;
      nn = nn + 1;
end;
```

Algorithm 1: MATLAB lines for calculating values of Δ_i, ϑ, ν, and Λ_1 in Example 1 for $\iota \in \mathcal{J} = [0, \tau^*]$ and $\tau^* = 1.85$.

TABLE 1: Numerical results of ϑ, ν, and Λ_1 for $t \in \mathcal{J} = [0, \tau^*]$ in Example 1.

ι	Δ_1	Δ_2	Δ_3	Δ_4	ϑ	ν	$\Lambda_1 < 1$
0.00	1.088 89	0.000 00	2.000 00	0.000 00	0.000 00	0.000 00	0.000 00
0.10	1.173 90	0.000 29	2.061 84	0.015 48	0.001 12	0.000 93	0.003 92
0.20	1.256 67	0.000 17	2.127 50	0.021 64	0.002 87	0.001 26	0.008 96
0.30	1.347 15	0.000 11	2.197 22	0.027 36	0.005 01	0.001 50	0.014 95
0.40	1.445 87	0.000 10	2.271 25	0.033 65	0.007 46	0.001 70	0.021 73
0.50	1.543 88	−0.00028	2.349 86	0.033 89	0.010 15	0.001 92	0.029 16
0.60	1.640 90	−0.00103	2.433 33	0.024 65	0.013 01	0.002 17	0.037 09
0.70	1.752 81	−0.00148	2.521 96	0.017 80	0.016 09	0.002 38	0.045 58
0.80	1.884 47	−0.00150	2.616 07	0.020 01	0.019 44	0.002 53	0.054 72
0.90	2.03662	−0.00113	2.716 01	0.035 13	0.023 10	0.002 61	0.064 61
1.00	2.20589	−0.00054	2.822 12	0.061 64	0.027 05	0.002 63	0.075 24
1.10	2.377 89	−0.00034	2.934 79	0.079 11	0.031 11	0.002 71	0.086 22
1.20	2.554 44	−0.00044	3.054 43	0.085 21	0.035 25	0.002 83	0.097 46
1.30	2.746 00	−0.00049	3.181 47	0.093 65	0.039 58	0.002 94	0.109 18
1.40	2.953 66	−0.00051	3.316 37	0.105 06	0.044 09	0.003 04	0.121 40
1.50	3.169 11	−0.00080	3.459 60	0.099 05	0.048 60	0.003 19	0.133 65
1.60	3.392 70	−0.00135	3.611 70	0.068 47	0.053 01	0.003 39	0.145 70
1.70	3.641 06	−0.00169	3.773 20	0.047 49	0.057 66	0.003 53	0.158 33
1.80	3.919 78	−0.00171	3.944 68	0.051 68	0.062 70	0.003 62	0.171 93

We can see the results of ϑ, ν, and Λ_1 for $\iota \in [0, \tau^*]$ in Table 1. These results are plotted in Figure 1. By using the MATLAB program (Algorithm 1) according to (29), we find

$$\Lambda_1 = \vartheta \mathfrak{p}^* + \nu \simeq 0.171934. \tag{82}$$

Thus from Theorem 2, the system of $\mathbb{GP}$ L-SL $\mathbb{FDE}$s (74) possesses a solution on $\mathcal{J} = [0, 1.85]$.

Example 2. By taking $\sigma = 0.51 \in (0,1]$, $\acute{\sigma} = 0.86 \in (0,1]$, and $\hbar = 1.25$, let us have the following system of $\mathbb{GP}$ L-SL $\mathbb{FDP}$s:

$$\begin{cases} \mathbb{G}_c^{0.51;1.25}\left(\left[(\sqrt{\iota}+2)\mathbb{G}_c^{0.86;1.25}+\dfrac{\iota^2}{25}\right]\right)q(\iota) = \dfrac{3}{16}+\dfrac{\iota|q(\iota)|}{12(8+|q(\iota)|)}, \\ q(0)+q(0.95)=0, \quad \mathbb{G}_c^{0.86;2.5}q(0)+\mathbb{G}_c^{0.86;2.5}q(0.95)=0. \end{cases} \tag{83}$$

For $\iota \in \mathcal{J} =: [0, 0.95]$. We choose

$$\begin{aligned} y(\iota) &= (\sqrt{\iota}+2), \\ w(\iota) &= \frac{\iota^2}{25}. \end{aligned} \tag{84}$$

And

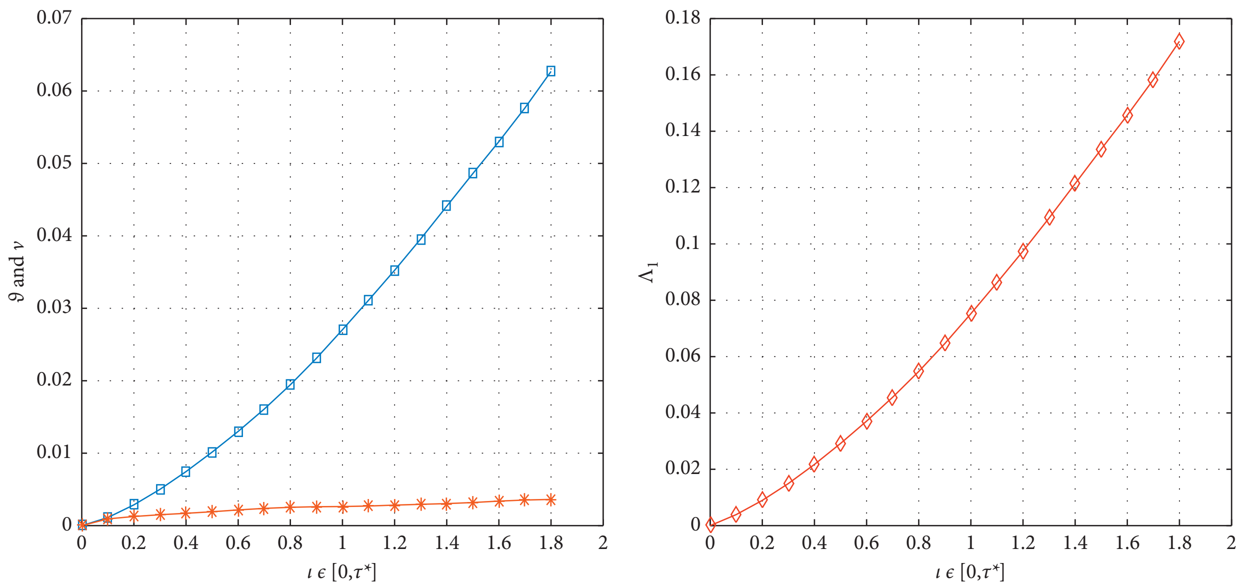
$$\wp(\iota, q(\iota)) = \frac{3}{16}+\frac{\iota|q(\iota)|}{12(8+|q(\iota)|)}. \tag{85}$$

So, for $\iota \in \mathcal{J}$ and $q, \acute{q}$, we have

$$\begin{aligned} &\left\|\frac{3}{16}+\frac{\iota|q(\iota)|}{12(3+|q(\iota)|)}-\left(\frac{3}{16}+\frac{\iota|\acute{q}(\iota)|}{12(3+|\acute{q}(\iota)|)}\right)\right\| \\ &\quad \le M\|q(\iota)-\acute{q}(\iota)\|, \end{aligned} \tag{86}$$

where $M = 1/8$. By using Equations (20), (30), and (31), we got

$$\begin{aligned} \Delta_1 &= \frac{1}{y(0)}+\frac{1}{y(\tau^*)}\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right) \simeq 1.96225, \\ \Delta_2 &= \frac{w(\tau^*)}{y(\tau^*)}-\frac{w(0)}{y(0)} \simeq 0.012135 \neq 0, \\ \Delta_3 &= 1+\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right) \simeq 2.20924, \\ \Delta_4 &= \frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y(\tau^*)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \simeq 0.33841, \end{aligned}$$



(a) (b)

FIGURE 1: Graphical representation of ϑ, ν, and Λ for $\iota \in [0, \tau^*]$ in Example 1. (a) ϑ and ν, (b) θ.

```
clear;
format long;
syms v e;
sigma = 0.51; accutesigma = 0.86; hslash = 1.25;
a = 0; tauast = 1.0;
y = sqrt(v)+2;
yast = 2;
w = vˆ2/25;
wast = 0.0361;
wp = 3/16 + v ∗ abs(e)/(12 ∗ (3abs(e)));
wpzero = 3/16;
M = 1/8;
n = floor(sigma)+1;
iota = a;
column = 1;
nn = 1;
while iota ≤ tauast
      MI(nn,column) = nn;
      MI(nn,column+1) = iota;
      Delta_1 = 1/(eval(subs(y, {v}, {a}))) ...
            + 1/(eval(subs(y, {v}, {iota}))) ...
            + exp ((hslash-1)/2 ∗ iota);
      MI(nn, column+2) = Delta_1;
      Delta_2 = eval(subs(w, {v}, {iota}))/eval(subs(y, {v}, {iota})) ...
            - eval(subs(w, {v}, {a}))/eval(subs(y, {v}, {a}));
      MI(nn, column+3) = Delta_2;
      Delta_3 = 1+ exp((hslash-1)/hslash ∗ iota);
      MI(nn, column+4) = Delta_3;
      Delta_4 = (iota^accutesigma ∗ exp((hslash-1)/hslash ∗ iota))...
            /(eval(subs(y, {v}, {iota})) ∗ hslash^accutesigma ∗ gamma(accutesigma+1));
      MI(nn, column+5) = Delta_4;
   vartheta = iota^(sigma + accutesigma)/(yast ∗ hslash^(sigma + accutesigma)...
            ∗ gamma(sigma + accutesigma+1)) + wast ∗ iota^(sigma)...
            ∗ iota^(accutesigma)∗exp((hslash-1)/hslash ∗ iota)...
```

ALGORITHM 2: Continued.

```
        /(yast * Delta_1 * hslash^(sigma + accutesigma)...
        * gamma(sigma+1) *  yast * hslash^(accutesigma)...
        * gamma  (accutesigma+1)) + (iota^(sigma + accutesigma)...
        /(yast * Delta_3 * hslash^(sigma + accutesigma)  ...
        *  gamma(sigma + accutesigma+1)) + Delta_4 * iota^(sigma)...
        /(yast *  Delta_1  *  Delta_3 * hslash^(sigma) *  gamma(sigma+1)))...
        *  exp((hslash-1)/hslash  *  iota);
    MI(nn, column+6) = vartheta;
    nu = wast * iota^(accutesigma)/(yast * hslash^(accutesigma)...
        *  gamma(accutesigma+1))  -  Delta_2 *  iota^(accutesigma)...
        *  exp((hslash-1)/hslash  *  iota)/(Delta_1 * yast * hslash^(accutesigma+1))...
        +  (-Delta_2  *  iota^(sigma)/(Delta_3 * hslash * gamma(sigma+1))...
        -  Delta_4 * Delta_2/(Delta_3 * Delta_1) * exp((hslash-1)/hslash * iota));
    MI(nn, column+7) = nu;
    Lamda_2 = vartheta  *  M + nu;
    MI(nn, column+8) = Lamda_2;
    MI(nn, column+9) = vartheta  *  wpzero/(1-Lamda_2);
    iota = iota+0.05;
    nn = nn+1;
end;
```

ALGORITHM 2: MATLAB lines for calculating values of Δ_i, ϑ, ν, and Λ_2 in Example 2 for $\iota \in \mathcal{J} = [0, \tau^*]$ and $\tau^* = 0.95$.

TABLE 2: Numerical results of ϑ, ν, and Λ_2 for $t \in \mathcal{J} = [0, \tau^*]$ in Example 2.

ι	Δ_1	Δ_2	Δ_3	Δ_4	ϑ	ν	$\Lambda_2 < 1$	$M \geq \vartheta\wp_0/1 - \Lambda_2$
0.00	2.000 00	0.000 00	2.000 00	0.000 00	0.000 00	0.000 00	0.000 00	0.000 00
0.05	1.955 99	0.000 05	2.010 05	0.030 06	0.008 40	0.001 19	0.002 24	0.001 58
0.10	1.944 32	0.000 17	2.020 20	0.052 90	0.021 69	0.002 14	0.004 85	0.004 09
0.15	1.937 81	0.000 38	2.030 46	0.073 47	0.037 82	0.002 99	0.007 72	0.007 15
0.20	1.933 94	0.000 65	2.040 81	0.092 71	0.056 13	0.003 76	0.010 78	0.010 64
0.25	1.931 74	0.001 00	2.051 27	0.111 06	0.076 28	0.004 47	0.014 00	0.014 51
0.30	1.930 72	0.001 41	2.061 84	0.128 76	0.098 05	0.005 10	0.017 36	0.018 71
0.35	1.930 58	0.001 89	2.072 51	0.145 97	0.121 26	0.005 67	0.020 83	0.023 22
0.40	1.931 15	0.002 43	2.083 29	0.162 81	0.145 82	0.006 17	0.024 40	0.028 02
0.45	1.932 28	0.003 03	2.094 17	0.179 37	0.171 61	0.006 60	0.028 05	0.033 11
0.50	1.933 89	0.003 69	2.105 17	0.195 69	0.198 57	0.006 96	0.031 78	0.038 45
0.55	1.935 92	0.004 41	2.116 28	0.211 84	0.226 64	0.007 25	0.035 58	0.044 06
0.60	1.938 30	0.005 19	2.127 50	0.227 86	0.255 76	0.007 46	0.039 43	0.049 92
0.65	1.940 99	0.006 02	2.138 83	0.243 77	0.285 89	0.007 59	0.043 33	0.056 03
0.70	1.943 97	0.006 91	2.150 27	0.259 61	0.316 99	0.007 65	0.047 28	0.062 39
0.75	1.947 20	0.007 85	2.161 83	0.275 39	0.349 03	0.007 63	0.051 26	0.068 98
0.80	1.950 66	0.008 85	2.173 51	0.291 15	0.381 98	0.007 52	0.055 27	0.075 81
0.85	1.954 34	0.009 89	2.185 31	0.306 90	0.415 81	0.007 33	0.059 31	0.082 88
0.90	1.958 21	0.010 99	2.197 22	0.322 65	0.450 51	0.007 06	0.063 37	0.090 19
0.950 00	1.962 26	0.012 14	2.209 25	0.338 42	0.486 05	0.006 69	0.067 45	0.097 72

$$\begin{aligned}\vartheta = {} & \frac{1}{y^*}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{w^*}{y^*\Delta_1}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+1)}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar-1/\hbar\tau^*)}{y^*\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} \\ & + \left(\frac{1}{y^*\Delta_3}\frac{(\tau^*)^{\sigma+\acute{\sigma}}}{\hbar^{\sigma+\acute{\sigma}}\Gamma(\sigma+\acute{\sigma}+1)} + \frac{\Delta_4}{y^*\Delta_1\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)}\right)\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right) \simeq 0.486044.\end{aligned} \tag{87}$$

Here $y^* = 2$, $w^* = 0.0361$ and

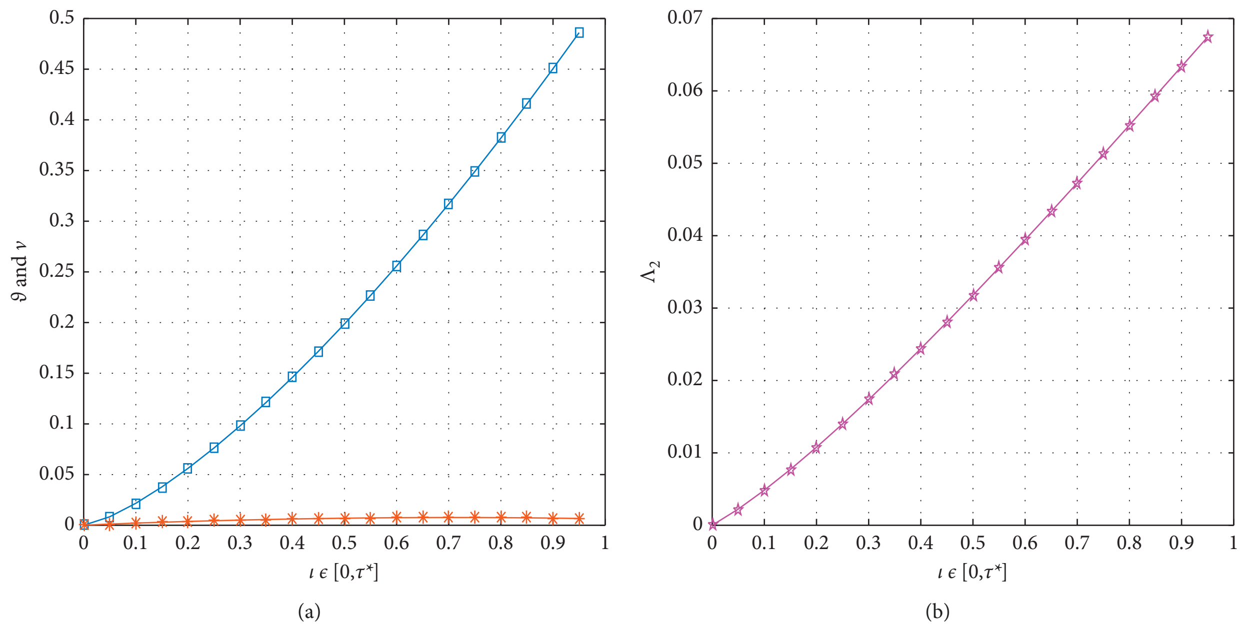


FIGURE 2: Graphical representation of ϑ, ν, and Λ_2 for $\iota \in [0, \tau^*]$ in Example 2. (a) ϑ and ν, (b) θ.

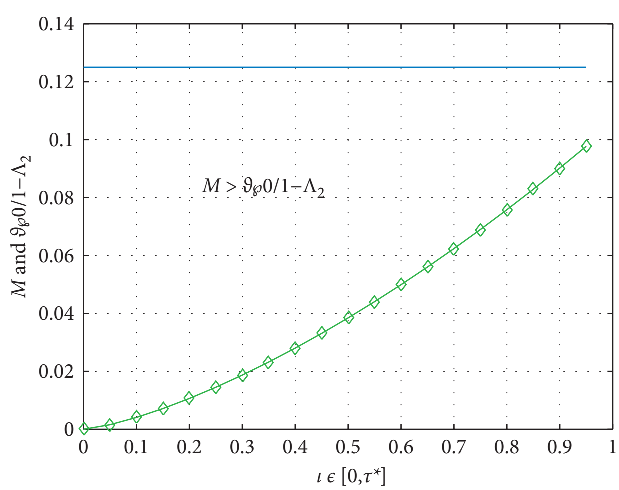


FIGURE 3: Numerical results of M, ν, and $\vartheta\wp_0/1-\Lambda_2$ for $t \in \mathcal{J} = [0, \tau^*]$ in Example 2.

$$\nu = \frac{w^*}{y^*}\frac{(\tau^*)^{\acute{\sigma}}}{\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} - \frac{\Delta_2}{\Delta_1}\frac{(\tau^*)^{\acute{\sigma}}\exp(\hbar - 1/\hbar\tau^*)}{y(\tau^*)\hbar^{\acute{\sigma}}\Gamma(\acute{\sigma}+1)} + \left(-\frac{\Delta_2}{\Delta_3}\frac{(\tau^*)^{\sigma}}{\hbar^{\sigma}\Gamma(\sigma+1)} - \frac{\Delta_4}{\Delta_3}\frac{\Delta_2}{\Delta_1}\right)\exp\left(\frac{\hbar-1}{\hbar}\tau^*\right) \simeq 0.006689. \tag{88}$$

Table 2 shows these results. Also, we can see from the graphical representation of ϑ, ν, and Λ_1 in Figures 2 and 3 whenever $\iota \in [0, 0.95]$. By using the MATLAB program (Algorithm 2) according to (48), we find

$$\Lambda_2 = \vartheta M + \nu \simeq 0.06744. \tag{89}$$

Thus, from Theorem 3 we conclude that the system of $\mathbb{GP}$ L-SL $\mathbb{FDP}$s (83) has a unique solution on $\mathcal{J}$.

6. Conclusion

With respect to the oddity of the display composition, no commitments exist, as far as we know, concerning the existence theory of the $\mathbb{GPFDE}$s (3) with the assistance of the strategy of K $\mathbb{MNC}$ combined with the Mönch fixed-point theorem. As a result, the objective of this paper is to enhance this scholastic zone by means of modern procedures based on an uncommon idea of Kuratowski measures. Subsequently, the $\mathbb{UH}$ and $\mathbb{UHR}$ stability were established for the proposed nonlinear $\mathbb{GP}$ L-SL $\mathbb{FDP}$*s* (3). We initiate the study of the existence, uniqueness, and different types of $\mathbb{US}$ for the $\mathbb{GP}$ L-SL $\mathbb{FDP}$s (3) with $\mathbb{VC}$s and $\mathbb{APBC}$s. Moreover, two examples were presented as an illustration of the obtained theory. With respect to another research venture, we are attending to proceed the investigation of such combined structures of physical and mathematical models by utilizing nonsingular fractional operators which provide more exact numerical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest with this study.

Authors' Contributions

Abdellatif Boutiara contributed to actualization, methodology, formal analysis, validation, investigation, and initial draft. Mohammed K. A. Kaabar contributed to actualization, methodology, formal analysis, validation, investigation, initial draft, supervision of the original draft, and editing.

Zailan Siri contributed to actualization, validation, methodology, formal analysis, investigation, and initial draft. Mohammad Esmael Samei contributed to actualization, methodology, formal analysis, validation, investigation, software, simulation, and the initial draft. Xiao-Guang Yue contributed to actualization, validation, methodology, formal analysis, investigation, and initial draft. All authors read and approved the final version.

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