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Research Article

New Fractional Mercer–Ostrowski Type Inequalities with Respect to Monotone Function

Saad Ihsan Butt[1] **Ammara Nosheen**[2] **Jamshed Nasir**[3] **Khuram Ali Khan**[2] **and Rostin Matendo Mabela**[4]

[1]*COMSATS University Islamabad, Lahore Campus, Pakistan*
[2]*Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*
[3]*Virtual University of Pakistan, Lahore Campus 54000, Pakistan*
[4]*Department of Maths and Computer Science, Faculty of Science, University of Kinshasa, Kinshasa, Democratic Republic of the Congo*

Correspondence should be addressed to Rostin Matendo Mabela; rostin.mabela@unikin.ac.cd

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This research focuses on Ostrowski type inequality in the form of classical Mercer inequality via ψ-Riemann–Liouville fractional integral (F-I) operators. Using the ψ-Riemann–Liouville F-I operator, we first develop and demonstrate a new generalized lemma for differentiable functions. Based on this lemma, we derive some fractional Mercer–Ostrowski type inequalities by using the convexity theory. These new findings extend and recapture previous published results. Finally, we presented applications of our work via the known special functions of real numbers such as q-digamma functions and Bessel function.

1. Introduction and Preliminaries

The well-known Ostrowski inequality, developed in 1938, established the following helpful and noteworthy integral inequality (see [1], page 468).

Suppose a mapping $\lambda\colon [a,b] \longrightarrow \mathfrak{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $|\lambda'(z)| \leq M$, for all $z \in [a,b]$, then, the following inequality holds:

$$\left|\lambda(z) - \frac{1}{b-a}\int_a^b \lambda(v)\mathrm{d}v\right| \leq M(b-a)\left[\frac{1}{4} + \frac{(z-(a+b)/2)^2}{(b-a)^2}\right]. \qquad (1)$$

This finding is known as the Ostrowski inequality in the literature. Some generalizations, variations, and extensions of the Ostrowski inequality have been proposed in light of current findings and their related generalizations, variants, and extensions (see [2–4]).

This inequality yields an upper bound for the approximation of the integral average $1/(b-a)\int_a^b \lambda(v)\mathrm{d}v$ by the value of $\lambda(v)$ at the point $v \in [a,b]$.

In recent years, because of the widespread interest in the theory of inequalities, the theory of convex functions is now at the center of many studies. Convex functions are the topic of research in a number of disciplines due to their applicability in inequality theory [5–8] and defined as:

Definition 1. [5] A mapping $\lambda\colon \mathcal{I} \subset \mathfrak{R} \longrightarrow \mathfrak{R}$ is called to be convex on $\mathcal{I}$, if

$$\lambda((1-\zeta)y_1 + \zeta y_2) \leq (1-\zeta)\lambda(y_1) + \zeta\lambda(y_2), \qquad (2)$$

holds for every $y_1, y_2 \in \mathcal{I}$ and $\zeta \in [0,1]$.

Kian and Moslehian used the Jensen–Mercer inequality and demonstrated the Hermite–Hadamard Mercer inequality in [9] as:

$$\lambda\left(a+b-\frac{y_1+y_2}{2}\right)\leq\frac{1}{y_2-y_1}\int_{y_1}^{y_2}\lambda(a+b-\zeta)\mathrm{d}\zeta$$
$$\leq\frac{\lambda(a+b-y_1)+\lambda(a+b-y_2)}{2}\leq\lambda(a)+\lambda(b)-\frac{\lambda(y_1)+\lambda(y_2)}{2}, \tag{3}$$

where λ is convex function on $[a,b]$.

The famous Jensen inequality, (see [5], Ch. 1) in the literature states that, if λ is convex function on the interval $[y_1, y_2]$, then

$$\lambda\left(\sum_{i=1}^{n}\sigma_i\ell_i\right)\leq\left(\sum_{i=1}^{n}\sigma_i\lambda(\ell_i)\right), \tag{4}$$

holds for all $\ell_i \in [y_1, y_2]$ and all $\sigma_i \in [0,1], \sum_{i=1}^{n}\sigma_i = 1$.

Jensen's inequality was modified by Mercer (see [10]) as

$$\lambda\left(y_1+y_2-\sum_{i=1}^{n}\sigma_i\ell_i\right)\leq\lambda(y_1)+\lambda(y_2)-\sum_{i=1}^{n}\sigma_i\lambda(\ell_i), \tag{5}$$

where λ is a convex function on $[y_1, y_2]$ holds for all $\ell_i \in [y_1, y_2]$ and

$$\sigma_i \in [0,1], \sum_{i=1}^{n}\sigma_i = 1. \tag{6}$$

Jensen and Hermite–Hadamard type inequalities are the most dynamic inequalities pertaining convex functions. Jensen and its related inequalities are well-known and significant inequalities in mathematical analysis due to its diverse applications and useability in applied and information sciences. Some recent discoveries can be found in [11, 12].

Jensen-type Mercer's inequality is an effective inequality since it provides additional information with specific boundary constraints. The study of generalizations and improvements of Mercer's variants of Hermite–Hadamard type inequalities considering the variety of fractional integral (F-I) operators have been of great interest for researchers in recent years, as evidenced by a large amount of research on it (see [13–15]).

The fractional calculus has been extensively studied by many researchers from the last few decades to generalize, improve, and extend several classic inequalities in order to obtain new variants in different dimensions. There are not only global derivatives in so called fractional calculus (for example: Riemann–Liouville and Caputo), but also local fractional derivatives (Khalil and Almeida, among others) (see [16–18]).

Yue [19], in 2013, discovered new Ostrowski inequalities for fractional integral (F-I) operators along with its associated fractional inequalities. Later in 2014, Aljinović [20] first developed Montgomery identity for fractional integrals of one function with respect to another function and then derived generalized fractional Ostrowski inequality from it. In the same article, he also presented the associated Ostrowski fractional inequalities for fractional integrals of functions with first derivatives in L_p spaces and computed sharp bounds. In the same year, Yildirim and Kirtay [21] used the generalized Riemann–Liouville F-I to establish new variants for Ostrowski inequalities. Some recent development about weighted Ostrowski fractional inequalities can be observed in [22].

Vanterler da Costa Sousa and Capelas de Oliveira in [23] recently introduced the ψ-Hilfer fractional derivative with respect to another function. Also, they investigate some Gronwall inequalities and Cauchy–type problem using the newly introduced ψ-Hilfer operator.

Definition 2. ([24], p.3) Let $(y_1, y_2)\,(-\infty \leq y_1 < y_2 \leq \infty)$ be finite or infinite interval in $\mathfrak{R}$ and $\varrho > 0$. Also, let $\psi\colon (y_1, y_2) \longrightarrow \mathfrak{R}_+$ be positive strictly increasing function possessing continuous derivative ψ' on (y_1, y_2). Then, left- and right-sided ψ-Riemann–Liouville F-I of a function λ with respect to another function ψ on $[y_1, y_2]$ can be given as

$$\left(I_{y_1+}^{\varrho:\psi}\right)\lambda(\ell)=\frac{1}{\Gamma(\varrho)}\int_{y_1}^{\ell}\psi'(\zeta)(\psi(\ell)-\psi(\zeta))^{\varrho-1}\lambda(\zeta)\mathrm{d}\zeta,\quad y_1<\ell, \tag{7}$$

and

$$\left(I_{y_2-}^{\varrho:\psi}\right)\lambda(\ell)=\frac{1}{\Gamma(\varrho)}\int_{\ell}^{y_2}\psi'(\zeta)(\psi(\zeta)-\psi(\ell))^{\varrho-1}\lambda(\zeta)\mathrm{d}\zeta,\quad \ell<y_2. \tag{8}$$

Remark 1. F-I operators elaborated in (7) and (8) yield several known F-I operators corresponding to various suitable selections of function ψ (see [25]), that are independently introduced by several authors with related results.

(i) By taking $\psi(y) = y$ as an identity function in (7) and (8), we get Riemann–Liouville F-I operators [24].

(ii) For $\psi(y) = y^{\varrho}/\varrho$, $\varrho > 0$ in (7) and (8) produce Katugampola F-I operators defined by Chen and Katugampola in [26].

(iii) For $\psi(y) = y^{\tau+s}/\tau + s$, $\tau > 0, s > 0$ in (7) and (8) produce generalized conformable F-I operators defined by Khan and Khan in [27].

(iv) For $\psi(y) = (y - y_1)^s/s$, $s > 0$, in (7), and $\psi(y) = -(y_2 - y)^s/s$, $s > 0$, in (8), one can get conformable F-I operators presented by Jarad et al. in [28].

The main good articles about Hermite–Hadamard inequalities involving ψ-Riemann–Liouville F-I operators are in references [29–31]. Some recent results about Hermite–Jensen–Mercer inequalities for ψ-Riemann–Liouville F-I operators can be seen in [14, 15].

The striking motive of this study is to develop generalized fractional equality for ψ-Riemann–Liouville F-I operators, which has a unique place among fractional integral operators, and to use this identity to generate some new Mercer–Ostrowski type inequalities for convex functions. The study also included applications of the findings, taking into account several specific circumstances of the primary conclusions.

2. New Mercer–Ostrowski Type Inequalities

Throughout this portion, Mercer–Ostrowski inequalities for the ψ-Riemann–Liouville F-I operators are obtained for differentiable functions on (a,b). As a result, we present a novel identity pertaining ψ-Riemann–Liouville F-I operators, that will serve as an auxiliary equality to produce subsequent inequalities.

Lemma 1. *Consider λ: $[a,b] \subset (0,\infty) \longrightarrow \mathfrak{R}$ be a differentiable function and $\lambda' \in \mathscr{L}_1[a,b]$, with $b>a$. If ψ is a strictly increasing, positive monotone function on $(a,b]$ with continuous derivative ψ' on (a,b), then, for all $\varkappa, y_1, y_2, \nu \in [a,b]$ and $\varrho > 0$, the following identity holds*

$$\begin{aligned}\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,\nu) &= \frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+a-\nu)}^{\psi^{-1}(\varkappa+a-y_1)} (\psi(\zeta)-(\varkappa+a-\nu))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta \\ &+ \frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+b-y_2)}^{\psi^{-1}(\varkappa+b-\nu)} ((\varkappa+b-\nu)-\psi(\zeta))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,\nu) &= \frac{(\nu-y_1)^{\varrho}}{y_2-y_1}\lambda(\varkappa+a-y_1) + \frac{(y_2-\nu)^{\varrho}}{y_2-y_1}\lambda(\varkappa+b-y_2) \\ &- \frac{\Gamma(\varrho+1)}{y_2-y_1}\Big\{ I^{\varrho:\psi}_{\psi^{-1}(\varkappa+a-y_1)^-}(\lambda^{\circ}\psi)\big(\psi^{-1}(\varkappa+a-\nu)\big) \\ &+ I^{\varrho:\psi}_{\psi^{-1}(\varkappa+b-y_2)^+}(\lambda^{\circ}\psi)\big(\psi^{-1}(\varkappa+b-\nu)\big)\Big\}.\end{aligned} \tag{10}$$

Proof. Let us start with

$$I_1:\ = \frac{\Gamma(\varrho+1)}{y_2-y_1} I^{\varrho:\psi}_{\psi^{-1}(\varkappa+a-y_1)^-}(\lambda^{\circ}\psi)\big(\psi^{-1}(\varkappa+a-\nu)\big), \tag{11}$$

$$\begin{aligned} &= \frac{\varrho}{y_2-y_1}\int_{\psi^{-1}(\varkappa+a-\nu)}^{\psi^{-1}(\varkappa+a-y_1)} (\psi(\zeta)-(\varkappa+a-\nu))^{\varrho-1}\psi'(\zeta)(\lambda^{\circ}\psi)(\zeta)d\zeta \\ &= \frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+a-\nu)}^{\psi^{-1}(\varkappa+a-y_1)} (\lambda^{\circ}\psi)(\zeta)d(\psi(\zeta)-(\varkappa+a-\nu))^{\varrho}d\zeta \\ &= \frac{(\nu-y_1)^{\varrho}}{y_2-y_1}\lambda(\varkappa+a-y_1) \\ &- \frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+a-\nu)}^{\psi^{-1}(\varkappa+a-y_1)} (\psi(\zeta)-(\varkappa+a-\nu))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta,\end{aligned} \tag{12}$$

and similarly, we get

$$\begin{aligned} I_2:\ &= \frac{\Gamma(\varrho+1)}{y_2-y_1} I^{\varrho:\psi}_{\psi^{-1}(\varkappa+b-y_2)^+}(\lambda^{\circ}\psi)\big(\psi^{-1}(\varkappa+b-\nu)\big) \\ &= \frac{(y_2-\nu)^{\varrho}}{y_2-y_1}\lambda(\varkappa+b-y_2) \\ &- \frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+b-y_2)}^{\psi^{-1}(\varkappa+b-\nu)} \\ &((\varkappa+b-\nu)-\psi(\zeta))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta.\end{aligned} \tag{13}$$

It follows from (11) and (13) that

$$\begin{aligned} &\frac{(\nu-y_1)^{\varrho}}{y_2-y_1}\lambda(\varkappa+a-y_1) + \frac{(y_2-\nu)^{\varrho}}{y_2-y_1}\lambda(\varkappa+b-y_2) - I_1 - I_2 \\ &= \frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+a-\nu)}^{\psi^{-1}(\varkappa+a-y_1)} (\psi(\zeta)-(\varkappa+a-\nu))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta \\ &+ \frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+b-y_2)}^{\psi^{-1}(\varkappa+b-\nu)} ((\varkappa+b-\nu)-\psi(\zeta))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta.\end{aligned} \tag{14}$$

By simplifying, we get the required result. □

Remark 2. Placing identity function $\psi(y) = y$ in (9), then, we get Lemma 2.1 given in [32].

Remark 3. If we set $\psi(y) = y$, $y_1 = a$, $y_2 = b$, and $\nu = \varkappa$ in (9), it reduces to Lemma 2 in [33].

Remark 4. Setting $\psi(y) = y$, $y_1 = a$, $y_2 = b$, $\nu = \varkappa$, and $\varrho = 1$ in (9), it recaptures Lemma 1 proved in [2].

Theorem 1. *Under the assumptions of Lemma 1, if $|\lambda'|$ is convex function on $[a,b]$, then for all $\varrho > 0$, the following inequality is valid.*

$$\begin{aligned} &\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,\nu)\right| \\ &\leq \frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{\varrho+1}\left(|\lambda'(\varkappa)|+|\lambda'(a)|\right) - \left[\frac{1}{\varrho+2}|\lambda'(y_1)| + \frac{1}{(\varrho+1)(\varrho+2)}|\lambda'(\nu)|\right]\right\} \\ &+ \frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{\varrho+1}\left(|\lambda'(\varkappa)|+|\lambda'(b)|\right) - \left[\frac{1}{\varrho+2}|\lambda'(y_2)| + \frac{1}{(\varrho+1)(\varrho+2)}|\lambda'(\nu)|\right]\right\}.\end{aligned} \tag{15}$$

Proof. By means of (9),

$$\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)$$
$$=\frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+a-v)}^{\psi^{-1}(\varkappa+a-y_1)}(\psi(\zeta)-(\varkappa+a-v))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta$$
$$+\frac{1}{y_2-y_1}\int_{\psi^{-1}(\varkappa+b-y_2)}^{\psi^{-1}(\varkappa+b-v)}((\varkappa+b-v)-\psi(\zeta))^{\varrho}\psi'(\zeta)(\lambda'\circ\psi)(\zeta)d\zeta. \tag{16}$$

Change of variables $s_1=\psi(\zeta)-(\varkappa+a-v)/v-y_1$ and $s_2=(\varkappa+b-v)-\psi(\zeta)/y_2-v$ and then $\zeta=s_1=s_2$ into the resulting equality, we get

$$\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)$$
$$=\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])\right|d\zeta$$
$$+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])\right|d\zeta. \tag{17}$$

Since $|\lambda'|$ is convex function on $[a,b]$, we obtain

$$\begin{aligned}\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right|&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])\right|d\zeta\\&+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])\right|d\zeta\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left\{|\lambda'(\varkappa)|+|\lambda'(a)|-\left[\zeta|\lambda'(y_1)|+(1-\zeta)|\lambda'(v)|\right]\right\}d\zeta\\&+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left\{|\lambda'(\varkappa)|+|\lambda'(b)|-\left[\zeta|\lambda'(y_2)|+(1-\zeta)|\lambda'(v)|\right]\right\}d\zeta\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{\varrho+1}\left(|\lambda'(\varkappa)|+|\lambda'(a)|\right)-\left[\frac{1}{\varrho+2}|\lambda'(y_1)|+\frac{1}{(\varrho+1)(\varrho+2)}|\lambda'(v)|\right]\right\}\\&+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{\varrho+1}\left(|\lambda'(\varkappa)|+|\lambda'(b)|\right)-\left[\frac{1}{\varrho+2}|\lambda'(y_2)|+\frac{1}{(\varrho+1)(\varrho+2)}|\lambda'(v)|\right]\right\}.\end{aligned} \tag{18}$$

□

Remark 5. If we set $\psi(y)=y$ in Theorem 1, one can get above inequality for Riemann–Liouville F-I operators given in Theorem 2.1 [32].

Remark 6. If we set $\psi(y)=y$, $y_1=a$, $y_2=b$, and $v=\varkappa$ in Theorem 1, it reduces to Theorem 7 in [33] that yields the same results with $s=1$.

Corollary 1. *If we set $\psi(y)=y$, $y_1=a$, $y_2=b$, and $v=\varkappa$ with $\varrho=1$ in Theorem 1, we get the following inequality*

$$\left|\lambda(\varkappa)-\frac{1}{b-a}\int_a^b\lambda(\zeta)d\zeta\right|$$
$$\leq\frac{(\varkappa-a)^2}{3(b-a)}\left\{\frac{1}{2}|\lambda'(a)|+|\lambda'(\varkappa)|\right\}+\frac{(b-\varkappa)^2}{3(b-a)}\left\{\frac{1}{2}|\lambda'(b)|+|\lambda'(\varkappa)|\right\}. \tag{19}$$

Corollary 2. *If we set $\psi(y)=y$ and $\varrho=1$ in Theorem 1, we get the following Mercer–Ostrowski inequality:*

$$\begin{aligned}&\left|\left\{\frac{v-y_1}{y_2-y_1}\lambda(\varkappa+a-y_1)+\frac{y_2-v}{y_2-y_1}\lambda(\varkappa+b-y_2)\right\}-\frac{1}{y_2-y_1}\left\{\int_{\varkappa+a-v}^{\varkappa+a-y_1}\lambda(\zeta)d\zeta+\int_{\varkappa+b-y_2}^{\varkappa+b-v}\lambda(\zeta)d\zeta\right\}\right|\\&\leq\frac{(v-y_1)^2}{y_2-y_1}\left\{\frac{1}{2}\left(|\lambda'(\varkappa)|+|\lambda'(a)|\right)-\left[\frac{1}{3}|\lambda'(y_1)|+\frac{1}{6}|\lambda'(v)|\right]\right\}\\&+\frac{(y_2-v)^2}{y_2-y_1}\left\{\frac{1}{2}\left(|\lambda'(\varkappa)|+|\lambda'(b)|\right)-\left[\frac{1}{3}|\lambda'(y_2)|+\frac{1}{6}|\lambda'(v)|\right]\right\}.\end{aligned} \tag{20}$$

Corollary 3. *The following Mercer–Ostrowski inequality can be found in Theorem 1 with* $|\lambda'|\leq M$

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right|\\&\leq\frac{M}{(y_2-y_1)(\varrho+1)}\left\{(v-y_1)^{\varrho+1}+(y_2-v)^{\varrho+1}\right\}.\end{aligned}\tag{21}$$

Proof. The result can be obtained by using $|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])|\leq M$ and $|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])|\leq M$. □

Remark 7. If we set $\psi(y)=y$, $y_1=a$, $y_2=b$ and $v=\varkappa$ in Corollary 3, it reduces to Corollary 1 in [33].

Remark 8. If we set $\psi(y)=y$, $y_1=a$, $y_2=b$ and $v=\varkappa$ and $\varrho=1$ in Corollary 3, it reduces to Theorem 2 in [2] that yields the same result with $s=1$.

Theorem 2. *We assume that all the conditions of Lemma 1 hold. If* $|\lambda'|^q$ *is convex function on* $[a,b]$*, then, for all* $\varrho>0$*, the following inequality*

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right|\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho p+1}\right)^{(1/p)}\left\{\left(|\lambda'(\varkappa)|^q+|\lambda'(a)|^q\right)-\frac{1}{2}\left[|\lambda'(y_1)|^q+|\lambda'(v)|^q\right]\right\}^{(1/q)}\\&\quad+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho p+1}\right)^{(1/p)}\left\{\left(|\lambda'(\varkappa)|^q+|\lambda'(b)|^q\right)-\frac{1}{2}\left[|\lambda'(y_2)|^q+|\lambda'(v)|^q\right]\right\}^{(1/q)},\end{aligned}\tag{22}$$

holds, where $p,q>1$ *are conjugate exponents.*

Proof. Applying classical Hölder integral inequality and the convexity of $|\lambda'|^q$ on the right side of (9), we get

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right|\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])\right|d\zeta\\&\quad+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])\right|d\zeta.\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho p}d\zeta\right)^{(1/p)}\left(\int_0^1\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])\right|^q d\zeta\right)^{(1/q)}\\&\quad+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho p}d\zeta\right)^{(1/p)}\left(\int_0^1\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])\right|^q d\zeta\right)^{(1/q)}\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho p}d\zeta\right)^{(1/p)}\\&\quad\times\left(\int_0^1\left\{|\lambda'(\varkappa)|^q+|\lambda'(a)|^q-\left[\zeta|\lambda'(y_1)|^q+(1-\zeta)|\lambda'(v)|^q\right]\right\}d\zeta\right)^{(1/q)}\\&\quad+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho p}d\zeta\right)^{(1/p)}\\&\quad\times\left(\int_0^1\left\{|\lambda'(\varkappa)|^q+|\lambda'(b)|^q-\left[\zeta|\lambda'(y_2)|^q+(1-\zeta)|\lambda'(v)|^q\right]\right\}d\zeta\right)^{(1/q)}\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho p+1}\right)^{(1/p)}\left\{\left(|\lambda'(\varkappa)|^q+|\lambda'(a)|^q\right)-\frac{1}{2}\left[|\lambda'(y_1)|^q+|\lambda'(v)|^q\right]\right\}^{(1/q)}\\&\quad+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho p+1}\right)^{(1/p)}\left\{\left(|\lambda'(\varkappa)|^q+|\lambda'(b)|^q\right)-\frac{1}{2}\left[|\lambda'(y_2)|^q+|\lambda'(v)|^q\right]\right\}^{(1/q)}.\end{aligned}\tag{23}$$

That finish the proof. □

Remark 9. If we set $\psi(y) = y$ in Theorem 2, it reduces to Theorem 2.2 in [32].

Remark 10. If we set $\psi(y) = y$, $y_1 = a$, $y_2 = b$, and $v = \varkappa$ in Theorem 2, it reduces to Theorem 8 in [33] that yields the same results with $s = 1$.

Corollary 4. *If we set $\psi(y) = y$, $y_1 = a, y_2 = b$, and $v = \varkappa$ with $\varrho = 1$ in Theorem 2, we have the following inequality:*

$$\left|\lambda(\varkappa) - \frac{1}{b-a}\int_a^b \lambda(\zeta)d\zeta\right| \leq \frac{1}{2^{(1/q)}(b-a)}\left(\frac{1}{p+1}\right)^{(1/p)} \times\left[(\varkappa-a)^2\left\{|\lambda'(a)|^q + |\lambda'(\varkappa)|^q\right\}^{(1/q)} + (b-\varkappa)^2\left\{|\lambda'(b)|^q + |\lambda'(\varkappa)|^q\right\}^{(1/q)}\right]. \tag{24}$$

Corollary 5. *If we set $\psi(y) = y$ and $\varrho = 1$ in Theorem 2, we lead to following Mercer–Ostrowski inequality:*

$$\begin{aligned}&\left|\left\{\frac{v-y_1}{y_2-y_1}\lambda(\varkappa+a-y_1) + \frac{y_2-v}{y_2-y_1}\lambda(\varkappa+b-y_2)\right\} - \frac{1}{y_2-y_1}\left\{\int_{\varkappa+a-v}^{\varkappa+a-y_1}\lambda(\zeta)d\zeta + \int_{\varkappa+b-y_2}^{\varkappa+b-v}\lambda(\zeta)d\zeta\right\}\right|\\ &\leq \frac{(v-y_1)^2}{y_2-y_1}\left(\frac{1}{p+1}\right)^{(1/p)}\left\{\left(|\lambda'(\varkappa)|^q + |\lambda'(a)|^q\right) - \frac{1}{2}\left[|\lambda'(y_1)|^q + |\lambda'(v)|^q\right]\right\}^{(1/q)}\\ &+ \frac{(y_2-v)^2}{y_2-y_1}\left(\frac{1}{p+1}\right)^{(1/p)}\left\{\left(|\lambda'(\varkappa)|^q + |\lambda'(b)|^q\right) - \frac{1}{2}\left[|\lambda'(y_2)|^q + |\lambda'(v)|^q\right]\right\}^{(1/q)}.\end{aligned} \tag{25}$$

Corollary 6. *Let the function $|\lambda'|$ in Theorem 2 is assumed to be bounded that is $|\lambda'| \leq M$, then the following result holds:*

$$\left|\Xi_{\varrho,\psi}\lambda(y_1, y_2, \varkappa, v)\right| \leq \frac{M}{y_2-y_1}\left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}}\left\{(v-y_1)^{\varrho+1} + (y_2-v)^{\varrho+1}\right\}. \tag{26}$$

Proof. The result can be demonstrated by using $|\lambda'(\varkappa + a - [\zeta y_1 + (1-\zeta)v])| \leq M$ and $|\lambda'(\varkappa + b - [\zeta y_2 + (1-\zeta)v])| \leq M$. □

Remark 11. If we set $\psi(y) = y$, $y_1 = a$, $y_2 = b$, and $v = \varkappa$ in Corollary 6, it reduces to Corollary 2 in [33].

Remark 12. If we set $\psi(y) = y$, $y_1 = a$, $y_2 = b$, $v = \varkappa$ and $\varrho = 1$ in Corollary 6, it reduces to Theorem 3 in [2] that yields the same result with $s = 1$.

Theorem 3. *We assume that all the conditions of Lemma 1 hold. If $|\lambda'|^q$ is convex function on $[a, b], q \geq 1$, then for all $\varrho > 0$, the following inequality*

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1, y_2, \varkappa, v)\right|\\ &\leq \frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho+1}\right)^{1-(1/q)}\\ &\times\left\{\frac{1}{\varrho+1}\left(|\lambda'(\varkappa)|^q + |\lambda'(a)|^q\right) - \left[\frac{1}{\varrho+2}|\lambda'(y_1)|^q + \frac{1}{(\varrho+1)(\varrho+2)}|\lambda'(v)|^q\right]\right\}^{(1/q)}\\ &+ \frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho+1}\right)^{1-(1/q)}\\ &\times\left\{\frac{1}{\varrho+1}\left(|\lambda'(\varkappa)|^q + |\lambda'(b)|^q\right) - \left[\frac{1}{\varrho+2}|\lambda'(y_2)|^q + \frac{1}{(\varrho+1)(\varrho+2)}|\lambda'(v)|^q\right]\right\}^{(1/q)},\end{aligned} \tag{27}$$

is valid.

Proof. Applying power-mean integral inequality and the convexity of $|\lambda'|^q$ on the right side of (9), we get

$$\begin{aligned}
&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,\nu)\right|\\
&\leq \frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\int_0^1 \zeta^{\varrho}\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)\nu])\right|d\zeta\\
&\quad+\frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\int_0^1 \zeta^{\varrho}\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)\nu])\right|d\zeta\\
&\leq \frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho}d\zeta\right)^{1-(1/q)}\left(\int_0^1 \zeta^{\varrho}\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)\nu])\right|^q d\zeta\right)^{(1/q)}\\
&\quad+\frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho}d\zeta\right)^{1-(1/q)}\left(\int_0^1 \zeta^{\varrho}\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)\nu])\right|^q d\zeta\right)^{(1/q)}\\
&\leq \frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho}d\zeta\right)^{1-(1/q)}\times\left(\int_0^1 \zeta^{\varrho}\left\{\left|\lambda'(\varkappa)\right|^q+\left|\lambda'(a)\right|^q-\left[\zeta\left|\lambda'(y_1)\right|^q+(1-\zeta)\left|\lambda'(\nu)\right|\right]\right\}^q d\zeta\right)^{(1/q)}\\
&\quad+\frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\left(\int_0^1\zeta^{\varrho}d\zeta\right)^{1-(1/q)}\\
&\quad\times\left(\int_0^1 \zeta^{\varrho}\left\{\left|\lambda'(\varkappa)\right|^q+\left|\lambda'(b)\right|^q-\left[\zeta\left|\lambda'(y_2)\right|^q+(1-\zeta)\left|\lambda'(\nu)\right|^q\right]\right\}d\zeta\right)^{(1/q)}\\
&\leq \frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho+1}\right)^{1-(1/q)}\\
&\quad\times\left\{\frac{1}{\varrho+1}\left(\left|\lambda'(\varkappa)\right|^q+\left|\lambda'(a)\right|^q\right)-\left[\frac{1}{\varrho+2}\left|\lambda'(y_1)\right|^q+\frac{1}{(\varrho+1)(\varrho+2)}\left|\lambda'(\nu)\right|^q\right]\right\}^{(1/q)}\\
&\quad+\frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\left(\frac{1}{\varrho+1}\right)^{1-(1/q)}\\
&\quad\times\left\{\frac{1}{\varrho+1}\left(\left|\lambda'(\varkappa)\right|^q+\left|\lambda'(b)\right|^q\right)-\left[\frac{1}{\varrho+2}\left|\lambda'(y_2)\right|^q+\frac{1}{(\varrho+1)(\varrho+2)}\left|\lambda'(\nu)\right|^q\right]\right\}^{(1/q)}.
\end{aligned}\tag{28}$$

Which ends the proof. □

Remark 13. If we set $\psi(y)=y$ in Theorem 3, it reduces to Theorem 2.3 in [32].

Remark 14. If we set $\psi(y)=y$, $y_1=a$, $y_2=b$, and $\nu=\varkappa$ in Theorem 3, it reduces to Theorem 9 in [33] that yields the same results with $s=1$.

Corollary 7. *If we set $\psi(y)=y$, $y_1=a$, $y_2=b$, and $\nu=\varkappa$ with $\varrho=1$ in Theorem 3, we have the following inequality:*

$$\begin{aligned}
&\left|\lambda(\varkappa)-\frac{1}{b-a}\int_a^b\lambda(\zeta)d\zeta\right|\\
&\leq\frac{1}{(b-a)}\left(\frac{1}{2}\right)^{1-(1/q)}\\
&\quad\times\frac{1}{3}\left[(\varkappa-a)^2\left\{\frac{1}{2}\left|\lambda'(a)\right|^q+\left|\lambda'(\varkappa)\right|^q\right\}^{(1/q)}\right.\\
&\quad\left.+(b-\varkappa)^2\left\{\frac{1}{2}\left|\lambda'(b)\right|^q+\left|\lambda'(\varkappa)\right|^q\right\}^{(1/q)}\right].
\end{aligned}\tag{29}$$

Corollary 8. *If we choose* $\psi(y) = y$ *and* $\varrho = 1$ *in Theorem 3, we have the following Mercer–Ostrowski inequality:*

$$\begin{aligned}&\left|\left\{\frac{v-y_1}{y_2-y_1}\lambda(\varkappa+a-y_1)+\frac{y_2-v}{y_2-y_1}\lambda(\varkappa+b-y_2)\right\}-\frac{1}{y_2-y_1}\left\{\int_{\varkappa+a-v}^{\varkappa+a-y_1}\lambda(\zeta)d\zeta+\int_{\varkappa+b-y_2}^{\varkappa+b-v}\lambda(\zeta)d\zeta\right\}\right|\\&\leq\frac{(v-y_1)^2}{y_2-y_1}\left(\frac{1}{2}\right)^{1-(1/q)}\left\{\frac{1}{2}\left(|\lambda'(x)|^q+|\lambda'(a)|^q\right)-\left[\frac{1}{3}|\lambda'(y_1)|^q+\frac{1}{6}|\lambda'(v)|^q\right]\right\}^{(1/q)}\\&+\frac{(y_2-v)^2}{y_2-y_1}\left(\frac{1}{2}\right)^{1-(1/q)}\left\{\frac{1}{2}\left(|\lambda'(\varkappa)|^q+|\lambda'(b)|^q\right)-\left[\frac{1}{3}|\lambda'(y_2)|^q+\frac{1}{6}|\lambda'(v)|^q\right]\right\}^{(1/q)}.\end{aligned}\tag{30}$$

Corollary 9. *Assuming that* $|\lambda'|\leq M$, *in Theorem 3, the following Mercer–Ostrowski inequality holds:*

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right|\\&\leq\frac{M}{(y_2-y_1)(\varrho+1)}\left\{(v-y_1)^{\varrho+1}+(y_2-v)^{\varrho+1}\right\}.\end{aligned}\tag{31}$$

Proof. Under the assumed conditions, we have $|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])|\leq M$ and $|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])|\leq M$. Thus, inequality (27) in Theorem 3 leads to inequality (31). □

Remark 15. If we set $\psi(y) = y$, $y_1 = a$, $y_2 = b$, and $v = \varkappa$ in Corollary 9, it reduces to Corollary 3 in [33].

Remark 16. If we set $\psi(y) = y$, $y_1 = a$, $y_2 = b$, $v = \varkappa$, and $\varrho = 1$ in Corollary 9, it reduces to Theorem 4 in [2] that yields the same result with $s = 1$.

Theorem 4. *We assume that all the assumptions of Lemma 1 holds. If* $|\lambda'|^q$ *is convex function on* $[a,b]$, *for all* $\varrho>0$, *the following inequality*

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right|\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{(\varrho p+1)p}+\frac{1}{q}\left(|\lambda'(\varkappa)|^q+|\lambda'(a)|^q-\frac{1}{2}\left[|\lambda'(y_1)|^q+|\lambda'(v)|\right]^q\right)\right\}\\&+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{(\varrho p+1)p}+\frac{1}{q}\left(|\lambda'(\varkappa)|^q+|\lambda'(b)|^q-\frac{1}{2}\left[|\lambda'(y_2)|^q+|\lambda'(v)|\right]^q\right)\right\},\end{aligned}\tag{32}$$

holds where $p,q>1$ *are conjugate exponents.*

Proof. From (9), we obtain

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,x,v)\right|\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])|d\zeta\\&+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\int_0^1\zeta^{\varrho}|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])|d\zeta.\end{aligned}\tag{33}$$

Utilizing Young's inequality as

$$uv<\frac{1}{p}u^p+\frac{1}{q}v^q.$$

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right|\\&\leq\frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{p}\int_0^1\zeta^{\varrho p}d\zeta\right.\\&\left.+\frac{1}{q}\int_0^1|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])|^q d\zeta\right\}\\&+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{p}\int_0^1\zeta^{\varrho p}d\zeta\right.\\&\left.+\frac{1}{q}\int_0^1|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])|^q d\zeta\right\}.\end{aligned}\tag{34}$$

By the convexity of $|\lambda'|^q$, we have

$$\begin{aligned}\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,\nu)\right| &\le \frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\\ &\times\left\{\frac{1}{p}\int_0^1 \zeta^{\varrho p}d\zeta+\frac{1}{q}\int_0^1\left\{|\lambda'(\varkappa)|^q+|\lambda'(a)|^q-\left[\zeta|\lambda'(y_1)|^q+(1-\zeta)|\lambda'(\nu)|^q\right]\right\}d\zeta\right\}\\ &+\frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\\ &\times\left\{\frac{1}{p}\int_0^1 \zeta^{\varrho p}d\zeta+\frac{1}{q}\int_0^1\left\{|\lambda'(\varkappa)|^q+|\lambda'(b)|^q-\left[\zeta|\lambda'(y_2)|^q+(1-\zeta)|\lambda'(\nu)|^q\right]\right\}d\zeta\right\}\\ &\le\frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{(\varrho p+1)p}+\frac{1}{q}\left(|\lambda'(\varkappa)|^q+|\lambda'(a)|^q-\frac{1}{2}\left[|\lambda'(y_1)|^q+|\lambda'(\nu)|^q\right]\right)\right\}\\ &+\frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\left\{\frac{1}{(\varrho p+1)p}+\frac{1}{q}\left(|\lambda'(\varkappa)|^q+|\lambda'(b)|^q-\frac{1}{2}\left[|\lambda'(y_2)|^q+|\lambda'(\nu)|^q\right]\right)\right\},\end{aligned} \tag{35}$$

and the proof is done. □

Remark 17. In Theorem 4, putting $\psi(y)=y$, then, one can get Mercer–Ostrowski inequality pertaining Riemann–Liouville F-I operators given in [32].

Corollary 10. *If we choose $\psi(y)=y$ and $\varrho=1$ in Theorem 4, the following Mercer–Ostrowski inequality holds:*

$$\begin{aligned}&\left|\left\{\frac{\nu-y_1}{y_2-y_1}\lambda(\varkappa+a-y_1)+\frac{y_2-\nu}{y_2-y_1}\lambda(\varkappa+b-y_2)\right\}-\frac{1}{y_2-y_1}\left\{\int_{\varkappa+a-\nu}^{\varkappa+a-y_1}\lambda(\zeta)d\zeta+\int_{\varkappa+b-y_2}^{\varkappa+b-\nu}\lambda(\zeta)d\zeta\right\}\right|\\ &\le\frac{(\nu-y_1)^2}{y_2-y_1}\left\{\frac{1}{(p+1)p}+\frac{1}{q}\left(|\lambda'(\varkappa)|^q+|\lambda'(a)|^q-\frac{1}{2}\left[|\lambda'(y_1)|^q+|\lambda'(\nu)|\right]^q\right)\right\}\\ &+\frac{(y_2-\nu)^2}{y_2-y_1}\left\{\frac{1}{(p+1)p}+\frac{1}{q}\left(|\lambda'(\varkappa)|^q+|\lambda'(b)|^q-\frac{1}{2}\left[|\lambda'(y_2)|^q+|\lambda'(\nu)|^q\right]\right)\right\}.\end{aligned} \tag{36}$$

Corollary 11. *The following Mercer–Ostrowski inequality can be obtained from Theorem 4 by assuming $|\lambda'|\le M$:*

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,\nu)\right|\\ &\le\frac{1}{y_2-y_1}\left\{\frac{1}{(\varrho p+1)p}+\frac{1}{q}M^q\right\}\left[(\nu-y_1)^{\varrho+1}+(y_2-\nu)^{\varrho+1}\right].\end{aligned} \tag{37}$$

Theorem 5. *We assume that all the assumptions of Lemma 1 hold. If $|\lambda'|^q$ is concave function on $[a,b], q>1$, for all $\varrho>0$, the following inequality holds:*

$$\begin{aligned}&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,\nu)\right|\\ &\le\left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}}\\ &\times\left\{\frac{(\nu-y_1)^{\varrho+1}}{y_2-y_1}\left|\lambda'\left(\varkappa+a-\frac{y_1+\nu}{2}\right)\right|\right.\\ &\left.+\frac{(y_2-\nu)^{\varrho+1}}{y_2-y_1}\left|\lambda'\left(\varkappa+b-\frac{y_2+\nu}{2}\right)\right|\right\},\end{aligned} \tag{38}$$

where $p,q>1$ are conjugate exponents.

Proof. From (9), by using the Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\Xi_{\varrho,\psi}\lambda(y_1,y_2,\varkappa,v)\right| \\
&\leq \frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\int_0^1 \zeta^{\varrho}\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])\right|d\zeta \\
&\quad+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\int_0^1 \zeta^{\varrho}\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])\right|d\zeta \\
&\leq \frac{(v-y_1)^{\varrho+1}}{y_2-y_1}\left(\int_0^1 \zeta^{\varrho p}d\zeta\right)^{(1/p)}\left(\int_0^1\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])\right|^q d\zeta\right)^{(1/q)} \\
&\quad+\frac{(y_2-v)^{\varrho+1}}{y_2-y_1}\left(\int_0^1 \zeta^{\varrho p}d\zeta\right)^{(1/p)}\left(\int_0^1\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])\right|^q d\zeta\right)^{(1/q)}.
\end{aligned} \tag{39}$$

Since $|\lambda'|^q$ is concave mapping, therefore from (3), we obtain

$$\int_0^1\left|\lambda'(\varkappa+a-[\zeta y_1+(1-\zeta)v])\right|^q d\zeta\leq\left|\lambda'\left(\varkappa+a-\frac{y_1+v}{2}\right)\right|^q, \tag{40}$$

and

$$\int_0^1\left|\lambda'(\varkappa+b-[\zeta y_2+(1-\zeta)v])\right|^q d\zeta\leq\left|\lambda'\left(\varkappa+b-\frac{y_2+v}{2}\right)\right|^q. \tag{41}$$

By placing the inequalities (40) and (41) in (39), leads to (38). □

Remark 18. If we set $\psi(y)=y$ in Theorem 5, it reduces to Theorem 2.5 in [32].

Remark 19. If we set $\psi(y)=y$, $y_1=a$, $y_2=b$, $v=\varkappa$, and $\varrho=1$ in Theorem 5, it reduces to Theorem 5 in [2] that yields the same result with $s=1$.

Corollary 12. *If we choose $\psi(y)=y$ and $\varrho=1$ in Theorem 5, we deduce the Mercer–Ostrowski inequality as*

$$\begin{aligned}
&\left|\left\{\frac{v-y_1}{y_2-y_1}\lambda(\varkappa+a-y_1)+\frac{y_2-v}{y_2-y_1}\lambda(\varkappa+b-y_2)\right\}-\frac{1}{y_2-y_1}\left\{\int_{\varkappa+a-v}^{\varkappa+a-y_1}\lambda(\zeta)d\zeta+\int_{\varkappa+b-y_2}^{\varkappa+b-v}\lambda(\zeta)d\zeta\right\}\right| \\
&\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\frac{(v-y_1)^2}{y_2-y_1}\left|\lambda'\left(\varkappa+a-\frac{y_1+v}{2}\right)\right|+\frac{(y_2-v)^2}{y_2-y_1}\left|\lambda'\left(\varkappa+b-\frac{y_2+v}{2}\right)\right|\right\}.
\end{aligned} \tag{42}$$

3. Some Applications

3.1. Applications to Means. For two real numbers $0<\kappa_1<\kappa_2$, consider the following two important means:

The arithmetic mean:

$$A(\kappa_1,\kappa_2)=\frac{\kappa_1+\kappa_2}{2}. \tag{43}$$

The generalized logarithmic-mean:

$$L_m(\kappa_1,\kappa_2)=\left[\frac{\kappa_2^{m+1}-\kappa_1^{m+1}}{(m+1)(\kappa_2-\kappa_1)}\right]^{(1/m)};\quad m\in\Re\smallsetminus\{-1,0\}. \tag{44}$$

Proposition 1. *Suppose $a,b>0$, then, we have the following inequality*

$$\begin{aligned}&\left|\left\{\frac{v-y_1}{y_2-y_1}(2A(\varkappa,a)-y_1)^n+\frac{y_2-v}{y_2-y_1}(2A(\varkappa,b)-y_2)^n\right\}\right.\\&\left.-\frac{1}{y_2-y_1}\left\{(v-y_1)L_m^m(\varkappa+a-y_1,\varkappa+a-v)+(v-y_2)L_m^m(\varkappa+b-v,\varkappa+a-y_2)\right\}\right|\\&\le\frac{n(v-y_1)^2}{y_2-y_1}\left(\frac{1}{2}\right)^{1-(1/q)}\left\{A\left(|\varkappa|^{(n-1)q},|a|^{(n-1)q}\right)-\left(\frac{1}{3}|y_1|^{(n-1)q}+\frac{1}{6}|v|^{(n-1)q}\right)\right\}^{(1/q)}\\&+\frac{n(y_2-v)^2}{y_2-y_1}\left(\frac{1}{2}\right)^{1-(1/q)}\left\{A\left(|\varkappa|^{(n-1)q},|b|^{(n-1)q}\right)-\left(\frac{1}{3}|y_2|^{(n-1)q}+\frac{1}{6}|v|^{(n-1)q}\right)\right\}^{(1/q)}.\end{aligned} \tag{45}$$

Proof. The result can be obtained immediately by taking into account Corollary 8 along with the convex function $\lambda(\varkappa)=\varkappa^n,\varkappa>0$. □

Proposition 2. *Suppose $a,b>0$, then, we have the following inequality:*

$$\begin{aligned}&\left|\left\{\frac{v-y_1}{y_2-y_1}(2A(\varkappa,a)-y_1)^n+\frac{y_2-v}{y_2-y_1}(2A(\varkappa,b)-y_2)^n\right\}\right.\\&\left.-\frac{1}{y_2-y_1}\left\{(v-y_1)L_m^m(\varkappa+a-y_1,\varkappa+a-v)+(v-y_2)L_m^m(\varkappa+b-v,\varkappa+a-y_2)\right\}\right|\\&\le\frac{(v-y_1)^2}{y_2-y_1}\left[\frac{1}{(p+1)p}+\frac{n}{q}\left\{2A\left(|\varkappa|^{(n-1)q},|a|^{(n-1)q}\right)-A\left(|y_1|^{(n-1)q},|v|^{(n-1)q}\right)\right\}\right]\\&+\frac{(y_2-v)^2}{y_2-y_1}\left[\frac{1}{(p+1)p}+\frac{n}{q}\left\{2A\left(|\varkappa|^{(n-1)q},|b|^{(n-1)q}\right)-A\left(|y_2|^{(n-1)q},|v|^{(n-1)q}\right)\right\}\right].\end{aligned} \tag{46}$$

Proof. The proof is the direct consequence of Corollary 10 by considering the convex function $\lambda(\varkappa)=\varkappa^n,\varkappa>0$. □

3.2. q-Digamma Function. The φ_{q}-digamma function, which is described as the logarithmic derivative of the q-gamma function, is an essential function related to the q-gamma function (see [34]) given as

$$\begin{aligned}\varphi_{\mathrm{q}}&=-\ln(1-\mathrm{q})+\ln\mathrm{q}\sum_{k=0}^{\infty}\frac{\mathrm{q}^{k+\zeta}}{1-\mathrm{q}^{k+\zeta}}\\&=-\ln(1-\mathrm{q})+\ln\mathrm{q}\sum_{k=0}^{\infty}\frac{\mathrm{q}^{k\zeta}}{1-\mathrm{q}^{k\zeta}}.\end{aligned} \tag{47}$$

For $\mathrm{q}>1$ and $\zeta>0$, q-digamma function φ_{q} can be given as

$$\begin{aligned}\varphi_{\mathrm{q}}&=-\ln(\mathrm{q}-1)+\ln\mathrm{q}\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathrm{q}^{-(k+\zeta)}}{1-\mathrm{q}^{-(k+\zeta)}}\right]\\&=-\ln(\mathrm{q}-1)+\ln\mathrm{q}\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathrm{q}^{-k\zeta}}{1-\mathrm{q}^{-k\zeta}}\right].\end{aligned} \tag{48}$$

Proposition 3. *Assume that a and b are the real numbers such that $0<a<b$, $q>1$, $0<q<1$, and $q^{-1}=1-p^{-1}$. Then, the following inequality is valid:*

$$\begin{aligned}&\left|\left\{\frac{v-y_1}{y_2-y_1}\varphi_{\mathbf{q}}(\varkappa+a-y_1)+\frac{y_2-v}{y_2-y_1}\varphi_{\mathbf{q}}(\varkappa+b-y_2)\right\}\right.\\&\left.-\frac{1}{y_2-y_1}\left\{\int_{\varkappa+a-v}^{\varkappa+a-y_1}\varphi_{\mathbf{q}}(\zeta)\mathrm{d}\zeta+\int_{\varkappa+b-y_2}^{\varkappa+b-v}\varphi_{\mathbf{q}}(\zeta)\mathrm{d}\zeta\right\}\right|\\&\leq\frac{(v-y_1)^2}{y_2-y_1}\left(\frac{1}{2}\right)^{1-(1/q)}\left\{A\left(\left|\varphi_{\mathbf{q}}'(\varkappa)\right|^q,\left|\varphi_{q}'(a)\right|^q\right)-\left[\frac{1}{3}\left|\varphi_{\mathbf{q}}'(y_1)\right|^q+\frac{1}{6}\left|\varphi_{\mathbf{q}}'(v)\right|^q\right]\right\}^{(1/q)}\\&+\frac{(y_2-v)^2}{y_2-y_1}\left(\frac{1}{2}\right)^{1-(1/q)}\left\{A\left(\left|\varphi_{\mathbf{q}}'(\varkappa)\right|^q,\left|\varphi_{\mathbf{q}}'(b)\right|^q\right)-\left[\frac{1}{3}\left|\varphi_{\mathbf{q}}'(y_2)\right|^q+\frac{1}{6}\left|\varphi_{\mathbf{q}}'(v)\right|^q\right]\right\}^{(1/q)}.\end{aligned}\tag{49}$$

Proof. The statement can be obtained by using Corollary 8 by considering $\lambda(\zeta)\longrightarrow\varphi_{\mathbf{q}}(\zeta)$. Since, $\varphi_{\mathbf{q}}'(\zeta)$ is a completely monotone function on $(0,\infty)$ for all $\zeta>0$, consequently, $\lambda'(\zeta):=\varphi_{\mathbf{q}}'(\zeta)$ is convex on the same interval $(0,\infty)$, (see [34]). □

3.3. Bounds Involving Modified Bessel Function. We know the first type of modified Bessel function $\mathfrak{B}_{\tau_1}$, which has the series interpretation (see [35], p.77)

$$\mathfrak{B}_{\tau_1}(\varkappa)=\sum_{n\geq 0}\frac{(\varkappa/2)^{\tau_1+2n}}{n!\Gamma(\tau_1+n+1)},\tag{50}$$

where $\varkappa\in\mathfrak{R}$ and $\tau_1>-1$, while the second kind modified Bessel function $\hbar_{\tau_1}$ (see [35], p.78) is usually defined as

$$\phi_{\tau_1}(\varkappa)=\frac{\pi}{2}\frac{\mathfrak{B}_{-\tau_1}(\varkappa)-\mathfrak{B}_{\tau_1}(\varkappa)}{\sin\tau_1\pi}.\tag{51}$$

Consider the function $\Psi_{\tau_1}:\mathfrak{R}\longrightarrow[1,\infty)$ defined by

$$\Psi_{\tau_1}(\varkappa)=2^{\tau_1}\Gamma(\tau_1+1)\varkappa^{-\tau_1}\mathfrak{B}_{\tau_1}(\varkappa).\tag{52}$$

The first- and second-order derivative formula of $\Psi_{\tau_1}(\varkappa)$ is given as [35]:

$$\Psi_{\tau_1}'(\varkappa)=\frac{\varkappa}{2(\tau_1+1)}\Psi_{\tau_1+1}(\varkappa),\tag{53}$$

$$\Psi_{\tau_1}''(\varkappa)=\frac{\varkappa^2\Psi_{\tau_1+2}(\varkappa)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(\varkappa)}{2(\tau_1+1)}.\tag{54}$$

Proposition 4. *Suppose that $\tau_1>-1$ and $0<a<b$, $q>1$. Then, we have*

$$\begin{aligned}&\left|\left\{\frac{v-y_1}{y_2-y_1}.\frac{\varkappa+a-y_1}{2(\tau_1+1)}.\Psi_{\tau_1+1}(\varkappa+a-y_1)+\frac{y_2-v}{y_2-y_1}.\frac{\varkappa+b-y_2}{2(\tau_1+1)}.\Psi_{\tau_1+1}(\varkappa+b-y_2)\right\}\right.\\&\left.-\frac{1}{y_2-y_1}\left\{\left(\Psi_{\tau_1}(\varkappa+a-y_1)-\Psi_{\tau_1}(\varkappa+a-v)\right)+\left(\Psi_{\tau_1}(\varkappa+b-v)-\Psi_{\tau_1}(\varkappa+b-y_2)\right)\right\}\right|\\&\leq\frac{(v-y_1)^2}{y_2-y_1}\left\{\frac{1}{(p+1)p}\right.\\&+\frac{1}{q}\left(\left(\frac{\varkappa^2\Psi_{\tau_1+2}(\varkappa)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(\varkappa)}{2(\tau_1+1)}\right)^q+\left(\frac{a^2\Psi_{\tau_1+2}(a)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(a)}{2(\tau_1+1)}\right)\right)^q\\&\left.-\frac{1}{2}\left[\left(\frac{y_1^2\Psi_{\tau_1+2}(y_1)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(y_1)}{2(\tau_1+1)}\right)^q+\left(\frac{v^2\Psi_{\tau_1+2}(v)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(v)}{2(\tau_1+1)}\right)^q\right]\right\}\\&+\frac{(y_2-v)^2}{y_2-y_1}\left\{\frac{1}{(p+1)p}\right.\\&+\frac{1}{q}\left(\left(\frac{\varkappa^2\Psi_{\tau_1+2}(\varkappa)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(\varkappa)}{2(\tau_1+1)}\right)^q+\left(\frac{b^2\Psi_{\tau_1+2}(b)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(b)}{2(\tau_1+1)}\right)^q\right)\\&\left.-\frac{1}{2}\left[\left(\frac{y_2^2\Psi_{\tau_1+2}(y_2)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(y_2)}{2(\tau_1+1)}\right)^q+\left(\frac{v^2\Psi_{\tau_1+2}(v)}{4(\tau_1+1)(\tau_1+2)}+\frac{\Psi_{\tau_1+1}(v)}{2(\tau_1+1)}\right)^q\right]\right\}.\end{aligned}\tag{55}$$

Proof. Substituting the mapping $\lambda \longrightarrow \Psi_{\tau_1}{}'$ to the inequality in Corollary 10. Note that all assumptions of Corollary 10 are satisfied (see [34]). Therefore, using the identities (53) and (54) gives required result. □

4. Conclusions

The objective of this study is to introduce the idea of new generalized fractional variants of Ostrowski inequality by employing Jensen–Mercer inequality for differentiable convex functions. The obtained results are intersting and generalized in a sense that by substituting identity function $\psi(y) = y$ and special value of $\varrho = 1$, we get connected to previously established results in the literature. Also, one can get variety of fractional Mercer–Ostrowski inequalities for different F-I operators by considering particular values of function ψ as mentioned in Remark 1. In addition, another motivating aspect of the study is that we try to give applications of means, q-digamma function, and Bessel function for Mercer–Ostrowski inequality in the similar passion as considered for Hermite–Hadamard type inequalities given in [8, 30]. Based on this study, researchers may contribute to the development of such results for twice differentiable functions.

Data Availability

No data are available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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