Application of Two Delay Differential Equations in the Evolutionary Game of Public Goods Supply

Simo Sun, Wang Man, and Hui Yang

School of Mathematics & Statistics, Guizhou University, Guiyang 550025, Guizhou, China

1 Correspondence should be addressed to Simo Sun; sunsimo@mail.gufe.edu.cn

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Starting from the leading role of public goods suppliers’ supply strategies, the government’s incentives and support to suppliers, and consumers’ influence on public opinion and other factors, the matrix game models between public goods suppliers and between the suppliers and the government are established, respectively. Considering the delay of government and supplier strategies, establish two delay differential equations between the two suppliers and between the supplier and the government, obtain the evolutionary game model of public goods supply, and solve the equilibrium point of the evolutionary game model with or without delay and its stability. Then, through model parameter analysis, verify the rationality of public goods supplier’s supply and government incentive strategy. Finally, through numerical simulation, it is verified that the delay does not affect the stability of the equilibrium point of the game party but changes the rate at which the game party’s strategy reaches a steady state, conducive to rapid, scientific, and reasonable decision-making between the suppliers of public goods and between the government and suppliers.

1. Introduction

In 1954, American economist Paul Samuelson pointed out in the Pure Theory of Public Expenditure that pure public goods are nonexclusive and noncompetitive. Public goods cannot be simply balanced through market allocation. The supply of public goods is the result of different stakeholders’ choice over a part of the social resource allocation strategy. Namely, public goods are a kind of product, where “public” means those products and services that can be shared by many people at the same time [1], such as science, education, culture, health, public facilities, environmental protection, diplomacy, national defense, etc. Public goods can be provided by the public sector, the private sector, or a combination of the two. The two have different orientations: public goods provided by the public sector focus on social benefits, while the private sector expects to obtain economic benefits.

It is precisely because of the different orientations of the two sectors that provide supply strategies, the social state will fall into a game dilemma. For example, when the government gives insufficient incentives to suppliers, the prisoner’s dilemma of insufficient public goods supply will arise. If the government encourages excessively, suppliers’ supply problems will arise again, like pig’s payoffs or chicken game dilemma. Literature [1–3] makes a comprehensive analysis. Literature [4, 5] uses population dynamics to study the two-predator-prey population equation, and it analyzes the application of delay differential equations in the public goods bidding game mechanism. Literature [6, 7] establishes a pair of evolutionary game between peasant entrepreneurs and village public goods providers and between the entrepreneurs and the government. Literature [8] analyzes the public goods game theory model of collective risk and adds growth factors to analyze the evolution of replication dynamics. Literature [9] uses the evolutionary game theory, introduces compensation coefficients based on the evolution of the government and social capital, builds a tripartite evolutionary game model for the government, enterprises,
and the public, and analyzes the evolutionary equilibrium of the tripartite under different strategies. Literature [10–15] discusses the evolutionary game strategy equilibrium and other issues regarding local government itself and the coordinated control of air pollution, industrial buildings, campus online loan supervision, public transportation, information sharing, and enterprise knowledge sharing from the perspective of evolutionary game. Literature [16] aims at the evolutionary trend of the time-lagged evolutionary game of switching topology enterprise innovation, constructs enterprise cooperative innovation game network through enterprise nodeization, establishes the time-lagged evolutionary game model of switching topology enterprise innovation, and verifies the feasibility of equilibrium with simulation examples. Literature [17, 18] applies dynamic-related knowledge, the steady state of the players’ strategies in the game, and analyzes the impact of delay on decision-making through numerical simulation.

It can be seen from the existing literature that most scholars do research on the supply of public goods based on the evolutionary game model, however, there are few articles that really consider supply and encourage delay. This article starts with the factors that affect the game party’s income and establishes a matrix game model between public goods suppliers and between suppliers and the government. Starting from the delay of the game party’s strategy, this article establishes two delay differential equations between the suppliers and between the suppliers and the government and attempts to solve the equilibrium point of evolutionary game model with or without a delay and its stability. Then, through model parameter analysis, verify the rationality of public goods supplier’s supply and government incentive strategy. Finally, through numerical simulation, it is verified that the delay does not affect the stability of the equilibrium point of the game party but changes the rate at which the game party’s strategy reaches a steady state, conducive to rapid, scientific, and reasonable decision-making between the suppliers of public goods, such as coal mine, water, electricity, and gas, and between the government and suppliers.

2. Analysis of Game Model between Two Suppliers of Public Goods

2.1. Establish a Matrix Game Model between Public Goods Suppliers. According to the introduction analysis and literature [6], the assumptions made to establish a matrix game model between public goods suppliers, as shown in Table 1, are as follows:

Assumption 1. when supplier A and B choose to supply public goods, the cost they need to pay is $C_A > 0, C_B > 0$. The tangible benefits that public goods suppliers choose to bring to each other are as follows: $\delta_A > 0, \delta_B > 0$. At the same time, the hidden market benefits of the supplier who chooses the supply strategy are $R_A > 0, R_B > 0$.

Assumption 2. the government’s incentives for suppliers to choose supply strategies, such as rewards, subsidies, and preferential policies, are as follows: $E_G > 0$. When two suppliers chooses the supply strategy at the same time, the government’s incentives are evenly distributed, that is $E_G/2 > 1$.

Assumption 3. when only one supplier chooses the supply strategy, the costs and benefits remain the same, however, the overall revenue is less than 0, and the nonsuppliers’ product costs and revenues are 0. However, the intangible benefits of them losing market or public support because of nonsupply are far greater than the tangible benefits that they can obtain from other suppliers.

Assumption 4. when all suppliers do not supply, the supplier does not incur supply costs and benefits, i.e., the costs and benefits are 0.

Assumption 5. the supply probabilities of public goods provided by supplier A and B are $S_A$ and $S_B$, respectively, where $0 \leq S_A \leq 1$, and $0 \leq S_B \leq 1$.

Based on the matrix game model, the expected return of supplier A is as follows:

$$S_{A_{supply}} = S_B \left( \delta_A + \delta_B - C_A + R_A + \frac{E_G}{2} \right) + (1 - S_B) \left( \delta_A - C_A + R_A + E_G \right),$$

$$S_{A_{non-supply}} = S_B (\delta_B - 1).$$

The replication dynamic equation of supplier A 18,19is obtained as follows:

$$\frac{dS_A}{dt} = S_A (1 - S_A) \left( E_{A_{supply}} - E_{A_{non-supply}} \right)$$

$$= S_A (1 - S_A) \left[ S_B \left( \delta_A + \delta_B - C_A + R_A + \frac{E_G}{2} \right) + (1 - S_B) \left( \delta_A - C_A + R_A + E_G \right) - S_B (\delta_B - 1) \right]$$

$$\frac{dS_A(t)}{dt} = S_A(t) (1 - S_A(t)) \left[ \delta_A - C_A + R_A + E_G + S_B(t) \left( 1 - \frac{E_G}{2} \right) \right].$$
Based on the matrix game model, the expected return of supplier B is as follows:

\[ SB = S_A \left( \delta_B + \delta_A - C_B + R_B + \frac{E_G}{2} \right) + (1 - S_A) \left( \delta_B - C_B + R_B + E_G \right), \quad (3) \]

The replication dynamic equation of supplier B is obtained as follows:

\[
\frac{dSB(t)}{dt} = S_B(t)(1 - S_B(t)) \left[ \delta_B - C_B + R_B + E_G + S_A(t) \left( I - \frac{E_G}{2} \right) \right].
\]

The evolutionary game model between suppliers is obtained from (1) and (2).

\[
\begin{align*}
\frac{dS_A}{dt} &= S_A(t)(1 - S_A(t)) \left[ \delta_A - C_A + R_A + E_G + S_B(t) \left( I - \frac{E_G}{2} \right) \right], \\
\frac{dS_B}{dt} &= S_B(t)(1 - S_B(t)) \left[ \delta_B - C_B + R_B + E_G + S_A(t) \left( I - \frac{E_G}{2} \right) \right].
\end{align*}
\]

2.2. Establish Delay Differential Equations. Since the supply strategy of the public goods supplier directly affects its own revenue and the expected revenue obtained in the current period is related to the game strategy of the previous competitors. The supplier will predict the current strategy based on the competitor’s previous game strategy, i.e., the delay of the competitor’s game strategy needs to be considered.

Here, suppose that the delay of public goods supplier B to A is \( \tau_1 \) and the delay of A to B is \( \tau_2 \). Suppliers A and B evolve and copy the dynamic (6) to obtain the delay differential equation.

\[
\begin{align*}
\frac{dS_A(t)}{dt} &= S_A(t)(1 - S_A(t)) \left[ \delta_A - C_A + R_A + E_G + S_B(t - \tau_1) \left( I - \frac{E_G}{2} \right) \right], \\
\frac{dS_B(t)}{dt} &= S_B(t)(1 - S_B(t)) \left[ \delta_B - C_B + R_B + E_G + S_A(t - \tau_2) \left( I - \frac{E_G}{2} \right) \right].
\end{align*}
\]
2.3. Analysis on the Stability of Equilibrium Points between Public Goods Providers

2.3.1. Stability of Equation (6) Equilibrium Points

(1) Equation (6) equilibrium points and Jacobian matrix

\[
J = \begin{bmatrix}
(1 - 2S_A)(\delta_A - C_A + R_A + E_G + S_B\left(I - \frac{E_G}{2}\right)) & S_A(1 - S_A)I \\
S_B(1 - S_B)I & (1 - 2S_B)\left[\delta_B - C_B + R_B + E_G + S_A\left(I - \frac{E_G}{2}\right)\right]
\end{bmatrix},
\]

As \(0 < -\delta_B + C_B - R_B - E_G < 1\), \(0 < -\delta_A + C_A - R_A - E_G < 1\),

So, \(0 > R_B + \frac{E_G}{2} - C_B > -I, 0 > R_A + \frac{E_G}{2} - C_A > -I\).

(2) Stability of Equation (6)

From the Jacobian matrix trace and determinant symbol of equation (6), the stability results of equation (6) equilibrium points are shown in Table 2. According to evolutionary game theory and as can be seen from Table 2, among the five equilibrium points, only the Nash equilibrium point O (0,0) and B (1,1) are the ESS points, and C (0,1) and A (1,0) are the unstable points, while D \((-\delta_B + C_B - R_B - E_G/I - (E_G/2), -\delta_A + C_A - R_A - E_G/I - (E_G/2))\) is a saddle point. According to Table 3, the replication dynamic relationship phase of the evolutionary game party can be obtained as shown in Figure 1.

(3) Parameter Analysis

Figure 1 shows that the equilibrium result depends on the initial state of the game. When the initial state is in the region ABCDA, the system converges to point B, i.e., evolutionary stability point (ESS). When the initial state is in the region OADCO, the system converges to point O, i.e., evolutionary stability point (ESS). Currently, there is an overall lack of public goods supply. Hence, it is clear that the (supply, supply) strategy combination is the best result. Consequently, to continuously expand the ABCDA area and shrink the OADCO area, the saddle point D must be moved downward.

Namely, the value of \(S_A^* = (-\delta_B + C_B - R_B - E_G/I - (E_G/2), S_B^* = -\delta_A + C_A - R_A - E_G/I - (E_G/2))\) becomes smaller. Hence, it is necessary for public goods supplier to increase the probability of the supply strategy. Suppliers need to reduce the supply cost of public goods, increase the supply of public goods, or reduce the loss \(I\) when public goods are not supplied.

2.3.2. Stability of Equation (7) Equilibrium Points

(1) Equation (9) equilibrium points

From replication dynamic (9), it is easy to find out that its equilibrium points are the same with (6), i.e., O(0,0), A(0,1), B(1,0), C(1,1), D \((-\delta_B + C_B - R_B - E_G/I - (E_G/2), -\delta_A + C_A - R_A - E_G/I - (E_G/2))\).

Since (9) is a replication dynamic equation of the mutual influence with two time delays, the sum of the two time delays, \(\tau_1\) and \(\tau_2\), will be used as a parameter in the following paragraphs to analyze the stability of the equilibrium points.
For the convenience of calculation, the matrix game model between the two suppliers is replaced, where
\[ A = \delta_A + \delta_B - C_A + R_A + E_G/2, \]
\[ B = \delta_A + \delta_B - C_B + R_B + E_G/2, \]
\[ C = \delta_B - I, \]
\[ D = \delta_B - C_B + R_B + E_G, \]
\[ E = \delta_A - C_A + \]

Table 2: Analysis on the stability of equilibrium points.

<table>
<thead>
<tr>
<th>Equilibrium points</th>
<th>Tr/J symbol</th>
<th>Det/J symbol</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>-</td>
<td>+</td>
<td>ESS</td>
</tr>
<tr>
<td>(0,1)</td>
<td>+</td>
<td>+</td>
<td>Unstable</td>
</tr>
<tr>
<td>(1,0)</td>
<td>+</td>
<td>+</td>
<td>Unstable</td>
</tr>
<tr>
<td>(1,1)</td>
<td>-</td>
<td>+</td>
<td>ESS</td>
</tr>
</tbody>
</table>

\[-\delta_B + C_B - R_B - E_G/2 - (E_G/2), -\delta_A + C_A - R_A - E_G/2 - (E_G/2)\]

Uncertain

- Saddle point

Table 3: Matrix game model between two suppliers.

<table>
<thead>
<tr>
<th>Supplier B</th>
<th>Supply (S_B)</th>
<th>Nonsupply (1-S_B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supplier A</td>
<td>Supply (S_A)</td>
<td>A,B</td>
</tr>
<tr>
<td></td>
<td>Nonsupply (1-S_A)</td>
<td>C,D</td>
</tr>
<tr>
<td></td>
<td></td>
<td>E,F</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0,0</td>
</tr>
</tbody>
</table>

Figure 1: Replication dynamic relationship phase of evolutionary game.

(2) Analysis on the stability of equation (9)

For the convenience of calculation, the matrix game model between the two suppliers is replaced, where
\[ A = \delta_A + \delta_B - C_A + R_A + E_G/2, \]
\[ B = \delta_A + \delta_B - C_B + R_B + E_G/2, \]
\[ C = \delta_B - I, \]
\[ D = \delta_B - C_B + R_B + E_G, \]
\[ E = \delta_A - C_A + \]

\[ \frac{dS_A(t)}{dt} = S_A(t)(1 - S_A(t))[E + S_B(t - \tau_3)(A - E - C)], \]
\[ \frac{dS_B(t)}{dt} = S_B(t)(1 - S_B(t))[D + S_A(t - \tau_2)(B - D - F)]. \]

The equilibrium points of equation 10 are as follows:
O(0,0), A(0, 1), B(1,0), C(1,1), and D(−D/B − D − F), (−E/A − E − C)).

Now, discuss the stability of equation (10) at the equilibrium point C (1,1).

At C (1,1), equation (10) is transformed into the following:
\[ R_A + E_G, F = \delta_A - I, M = B - D - F = I - E_G/2, \] and \[ N = A - E - C = I - E_G/2. \]

Then, equation (9) is transformed into the following:
\[ R_A + E_G, F = \delta_A - I, M = B - D - F = I - E_G/2, \] and \[ N = A - E - C = I - E_G/2. \]

Now, discuss the stability of equation (10) at the equilibrium point C (1,1).
A Jacobian matrix is obtained as follows:

\[
J = \begin{bmatrix}
-2R(t - \tau_2) + \left(1 + \frac{B-F}{M}\right)R(t - \tau_2) & S(t - \tau_1) & (R(t - \tau_2) + 1)\left(\frac{B-F}{M} - R(t - \tau_2)\right)N & (R(t - \tau_2) + 1)\left(\frac{B-F}{M} - R(t - \tau_2)\right)N \\
(S(t - \tau_1) + 1)\left(\frac{A-C}{N} - S(t - \tau_2)\right)M & -2S(t - \tau_2) + \left(1 + \frac{A-C}{N}\right)S & R(t - \tau_2)M
\end{bmatrix}
\]

(11)

Hence, the characteristic equation of (16) is as follows:

\[
\lambda^2 - \Delta e^{-\lambda t} = 0,
\]

(16)

where \(\Delta = H_{12}H_{21}, \tau = \tau_1 + \tau_2\).

Therefore, the stability of equation (10) at B (1,1) is obtained by discussing the roots of characteristic equation (18). Assuming \(\Delta \neq 0\), regarding the roots of equation (17), we have the following:

**Theorem 1.** When \(\tau = 1/\sqrt{|\Delta|} [\arccos(-|\Delta|/\Delta) + 2\pi_i](Z = 0, 1, 2, \ldots), equation (10) has a pair of pure imaginary root \(\lambda = \pm \sqrt{|\Delta|} i\).

Prove: let \(\lambda = \eta + \xi i\), and substituting into equation (10), get:

\[
(\eta+i\xi)^2 - \Delta e^{-(\eta+i\xi)i} = 0.
\]

(18)

\[
(\eta+i\xi)^2 - \Delta e^{-(\eta+i\xi)i} = \Delta e^{-(\eta+i\xi)i} = \Delta e^{-\eta}\cos\xi - i \sin\xi.
\]

At the same time, using Euler’s formula, there are,

\[
\eta^2 + 2\eta i\xi - \xi^2 - \Delta e^{-\eta t} \cdot [\cos\xi t - i \sin\xi t] = 0,
\]

\[
\eta^2 - \xi^2 - \Delta e^{-\eta t} \cos\xi t + 2\eta i\xi + \Delta e^{-\eta t} i \sin\xi t = 0.
\]

(19)

Separate the real part and the imaginary part of (19), and let both parts be 0.
Let \( \eta \) be the evolutionary stable point of ESS.

\[
\begin{aligned}
\eta^2 - \xi^2 - \Delta e^{-\eta \tau} \cos(\xi \tau) &= 0, \\
2\eta \xi + \Delta e^{-\eta \tau} \sin(\xi \tau) &= 0,
\end{aligned}
\]  

(20)

So, \( \eta = 0 \) and:

\[
\begin{aligned}
\xi^2 = -\Delta \cos(\xi \tau) \\
\Delta \sin(\xi \tau) &= 0
\end{aligned}
\]

(21)

Hence, equation (22) has a positive real solution, i.e., \( \xi_0 = \pm \sqrt{\Delta} \) i.

From (21), get the following:

\[
\xi_0 = -\cos(\xi_0 \tau)
\]

Hence,

\[
(\lambda^2 - \Delta e^{-\lambda \tau})' = 2\lambda \frac{d\lambda}{d\tau} - \Delta e^{-\lambda \tau} (\lambda - \tau) = 2\lambda \frac{d\lambda}{d\tau} - \Delta e^{-\lambda \tau} (\lambda - \tau \frac{d\lambda}{d\tau}) = 2\lambda \frac{d\lambda}{d\tau} + \Delta e^{-\lambda \tau} \left( \lambda + \tau \frac{d\lambda}{d\tau} \right).
\]

(23)

Let \( \lambda (d\lambda/d\tau) + \Delta e^{-\lambda \tau} (\lambda + \tau (d\lambda/d\tau)) = 0 \)

so, \( d\lambda/d\tau = -(\Delta e^{-\lambda \tau} \lambda/2 \Delta + \Lambda e^{-\lambda \tau} + \Delta \tau) \)

\[
\frac{d\lambda}{d\tau} = \sqrt{\Delta} \lambda = -\frac{\Delta \lambda}{2 \lambda e^{-\lambda \tau} + \Delta \tau} \lambda = \sqrt{\Delta} \lambda \\
\tau = \tau_Z \\
\lambda = \frac{-\Delta \sqrt{\Delta} \lambda}{2 \sqrt{\Delta} \lambda (\cos(\sqrt{\Delta} \tau_Z) + i \sin(\sqrt{\Delta} \tau_Z)) + \Delta \tau_Z} \]

(24)

Where, \( R = \Delta \tau_Z - 2 \sqrt{\Delta} \sin(\sqrt{\Delta} \tau_Z) \), and \( S = 2 \sqrt{\Delta} \cos(\sqrt{\Delta} \tau_Z) \).

From equation (2)–(10),

\[
dR_e^\lambda/d\tau |_{\lambda = \sqrt{\Delta} \lambda} = R_e \left( \frac{d\lambda}{d\tau} \lambda = \sqrt{\Delta} \lambda \right) = \frac{-2\Delta \lambda |\cos(\sqrt{\Delta} \tau_Z)|}{R^2 + S^2} > 0.
\]

(25)

Therefore, the conclusion is established. (27) at B(1,1) is the evolutionary stable point of ESS.
Now, discuss the stability of (27) at equilibrium point D 
\((-D/M, -E/N), \) where \(M = B - D - F, N = A - E - C, R(t) = S_\delta(t) - D/M, \) and \(S(t) = S_\delta(t) - E/N. \)

\[
\begin{align*}
\frac{dS_\delta(t)}{dt} &= S_\delta(t)(1 - S_\delta(t))[E + S_B(t - \tau_1)(A - E - C)] \\
\frac{dS_B(t)}{dt} &= S_B(t)(1 - S_B(t))[D + S_A(t - \tau_2)(B - D - F)].
\end{align*}
\]  
(26)

Then, (27) is transformed at \((-D/M, -E/N)\)s.

\[
\begin{align*}
\frac{dR(t)}{dt} &= \left(R(t - \tau_2) + \frac{D}{M} \frac{B - F}{M} - R(t - \tau_2)\right)(t - \tau_1)N \\
\frac{dS(t)}{dt} &= \left(S(t - \tau_1) + \frac{E}{N}\left(\frac{A - C}{N} - S(t - \tau_1)\right)\right)R(t - \tau_2)M.
\end{align*}
\]  
(27)

A Jacobian matrix is obtained as follows:

\[
J = \begin{bmatrix}
-2R(t - \tau_2) + \left(\frac{M}{D} + \frac{B - F}{M} \right)R(t - \tau_2) & S(t - \tau_1)N \\
\left(S(t - \tau_1) + \frac{E}{N}\left(\frac{A - C}{N} - S(t - \tau_1)\right)\right)M & \left(R(t - \tau_2) + \frac{D}{M} \frac{B - F}{M} - R\right)N
\end{bmatrix}
\]  
(28)

Similarly, study the stability of equilibrium point D 
\((-D/M, -E/N)\) by discussing the stability of equation
\((2)–(5)\) at origin (zero solution), i.e., \(R(t - \tau_2) = 0\) and \(S(t - \tau_1) = 0.\)

So, \(J = \begin{bmatrix}
0 & -D/MB - F/MN \\
-E/NA - C/NM & 0
\end{bmatrix}\)

The linear approximation of equation \((2)–(5)\) is obtained as follows:

\[
\begin{align*}
\frac{dR(t)}{dt} &= \frac{-D(B - F)N}{M^2}S(t - \tau_1), \\
\frac{dS(t)}{dt} &= \frac{-E(A - C)M}{N^2}R(t - \tau_2).
\end{align*}
\]  
(29)

As \(A = \delta_A + \delta_B - C_A + R_A + E_G/2, \) \(B = \delta_B + \delta_A - C_B + R_B + E_G/2, \) \(C = \delta_B - I, \) \(D = \delta_B - C_B + R_B + E_G, \) \(E = \delta_A - C_A + R_A + E_G, \)
\(F = \delta_A - I, \) \(M = B - D - F = I - E_G/2, \) and \(N = A - E - C = I - E_G/2, \) substitute equation \((2)–(7)\) to get the approximate equation of the original game model.

\[
\begin{align*}
\frac{dR(t)}{dt} &= \frac{(\delta_B - C_b + R_B + E_G)}{I - (E_G/2)} \frac{\left[\delta_A - I - (\delta_B + \delta_A - C_B + R_B + E_G/2)\right]}{S(t - \tau_1)}, \\
\frac{dS(t)}{dt} &= \frac{(\delta_A - C_A + R_A + E_G)}{I - (E_G/2)} \frac{\left[\delta_B - I - (\delta_A + \delta_B - C_A + R_A + E_G/2)\right]}{R(t - \tau_2)}.
\end{align*}
\]  
(30)
Let $H_{12} = ((\delta_B - C_B + R_E + E_G)(\delta_A - I - (\delta_B + \delta_A - C_B + R_E + E_G/2)/I - (E_G/2)) > 0, H_{21} = ((\delta_A - C_A + R_A + E_G)(\delta_B - I - (\delta_A + \delta_B - C_A + R_A + E_G/2)/I - (E_G/2)) > 0$. Then, (16) is transformed into the following:

$$\begin{align*}
\frac{dR(t)}{dt} &= H_{12}S_B(t - \tau_1), \\
\frac{dS(t)}{dt} &= H_{21}S_A(t - \tau_2).
\end{align*}$$

(31)

Hence, the characteristic equation of (32) is as follows:

$$\lambda^2 - \Delta e^{-\lambda \tau} = 0.$$  

(32)

where $\Delta = H_{12}H_{21} > 0$, and $\tau = \tau_1 + \tau_2$.

When $\tau = \tau_1 + \tau_2 = 0$, equation (33) is transformed into the following:

$$\lambda^2 - \Delta = 0.$$  

(33)

Therefore, the stability of (7) is obtained by discussing the roots of characteristic (33). Let $\Delta \neq 0$, regarding the roots of (33), we conclude that Theorem 1 and Theorem 2 are established.

Consequently, according to literature [19–23], there is an increasing sequence of real numbers $\tau_Z (Z = 1, 2, \ldots)$, and all the solutions of (33) on $\tau \in (0, \tau_Z)$ have negative real parts. When $\tau = \tau_Z$, there is a pair of pure imaginary solutions, and the others are solutions with strictly negative real parts. When $\tau > \tau_Z$, there is at least one solution with strictly positive real parts, i.e., when $\tau = \tau_Z$, the origin of equation (18) is at the Hopf branch, which is asymptotically stable on $\tau \in (0, \tau_Z)$ and unstable on $\tau > \tau_Z$.

The above is the analysis of equilibrium points $B(1, 1)$ and $D((-\delta_B + C_B - R_E - E_G/I - (E_G/2)), (-\delta_A + C_A - R_A - E_G/I - (E_G/2)))$. Other equilibrium points can be studied similarly, and it can be concluded that the stability of the equilibrium points of model (2-3) and model (2-4) are the same, except that there is a delay in the model (2-4). Moreover, the rate of equilibrium points evolving to a steady state is different from that of model (2-3), which provides a reference for public goods suppliers to provide rapid and reasonable supply strategies.

3. Game Analysis between Public Goods Suppliers and Government

3.1. Establish a Matrix Game Model between Public Goods Suppliers. To increase the supply of public goods, the government encourages suppliers to choose supply strategies through incentive policies. According to the introduction analysis and literature [6], the following assumptions are made to establish an evolutionary game model between public goods suppliers and the government, as shown in Table 4.

Assumption 1. when the government adopts incentive strategies, the social benefits, such as political achievements and public support, are as follows: $\delta_G > 0$. The cost paid by the government to encourage publicity and mobilization is as follows: $C_G > 0$.

Assumption 2. when suppliers adopt the supply strategy, the income obtained is divided into tangible income, $\delta_E > 0$, and invisible income, $R_E > 0$. At the same time, the government incentive is $S_G > 0$, however, suppliers have to bear the supply cost, i.e., $C_E > 0$.

Assumption 3. when the government adopts incentives and suppliers provide supplies, the total government revenue is as follows: $\delta_G - C_G - S_G < 0$. The total revenue of suppliers is as follows: $\delta_E + R_E - C_E + S_G > 0$.

Assumption 4. when the government adopts a nonincentive strategy and suppliers provide supplies, the government’s revenue and cost are 0. However, losing public the support and other social benefits is $\delta_G > 0$, i.e., the total government revenue is $-\delta_G$, and the total revenue of suppliers is divided into $\delta_E - C_E + R_E < 0$.

Assumption 5. when suppliers choose not to supply and the government chooses an incentive strategy, suppliers will lose the opportunity revenue: $I > 0$, and the government’s revenue is $\delta_G - C_G < 0$.

Assumption 6. when suppliers choose not to supply and the government chooses not to incentivize, suppliers will lose the opportunity revenue, i.e., $I > 0$. The government loses public support and other social benefits, i.e., $\delta_G/2$.

Assumption 7. the probability of the public goods supplier choosing the supply strategy is $P_E$, and the probability of the government providing incentives is $P_G$. Among them, $0 \leq P_E \leq 1$ and $0 \leq P_G \leq 1$.

According to the assumptions, the model has pure strategy Nash equilibrium points (1, 1) and (0, 0), as well as mixed strategy Nash equilibrium points.

$$\left(\frac{\delta_G - 2C_G}{2S_G - \delta_G}, \frac{-\delta_E + C_E - I - R_E}{S_G} \right)$$

(34)

Based on the payment matrix, suppliers’ expected return is as follows:

$$E_{\text{supply}} = P_G (\delta_E + R_E - C_E + S_G) + (1 - P_G)(\delta_E - C_E + R_E)$$

(35)

$$E_{\text{non-supply}} = -1.$$ 

Suppliers’ replication dynamic equation is obtained as follows:

$$\frac{dP_E}{dt} = P_E (1 - P_E)(\delta_E - C_E + R_E + I + P_G S_G).$$

(36)

Based on the payment matrix, the government’s expected return is as follows:
3.3. Analysis on the Stability of Equilibrium Points between Public Goods Suppliers and Government

3.3.1. Analysis on the Stability of Equations (42) and (43)

(1) Equilibrium points of (40) and (41)

Equation (40) has the following equilibrium points: O(0,0), B(1,0), D(1,1), F(0,1), and P((δ_G - 2C_G/2S_G - δ_G), (−δ_E + C_E - I - R_E/S_G)).

Meanwhile, it is known that equations (2)-(3) have the same equilibrium points, i.e., O(0,0), B(1,0), D(1,1), F(0,1), and P((δ_G - 2C_G/2S_G - δ_G), (−δ_E + C_E - I - R_E/S_G)).

3.2. Establish Two Delay Differential Equations. When the government provides incentive strategies, such as tax incentives and financial subsidies, to public goods suppliers, to make the incentive strategy more reasonable and achieve the expected benefits of social and economic development, the government needs to fully consider suppliers’ supply delay. At the same time, suppliers must consider their own development, and the delay of incentives given by the government should also be fully taken into account when selecting public goods supply strategies.

Here, suppose that the delay of public goods suppliers to the government’ incentives is τ_1, and the government’s supply delay to suppliers is τ_2. Based on the evolutionary game model (3-1) between suppliers and the government, a two delay differential equation is obtained as follows:

\[
\frac{dP_E(t)}{dt} = P_E(t)\left(1 - P_E(t)\right)\left[\frac{3}{2}\delta_G - C_G - P_E(t - \tau_1)S_G\right],
\]

\[
\frac{dP_G(t)}{dt} = P_G(t)\left(1 - P_G(t)\right)\left[\frac{3}{2}\delta_G - C_G - P_G(t - \tau_2)\left(\frac{S_G - \delta_G}{2}\right)\right].
\]
\((42)\) is replaced as shown, where
\[ M = \delta_E - C_E + R_E + I, \quad N = 3/2\delta_G - C_G. \]

\[
\begin{align*}
\frac{dP_E(t)}{dt} &= P_E(t)(1 - P_E(t))(\delta_E - C_E + R_E + I + P_G S_G), \\
\frac{dP_G(t)}{dt} &= P_G(t)(1 - P_G(t))\left[\frac{3}{2}\delta_G - C_G - P_G\left(S_G - \frac{\delta_G}{2}\right)\right].
\end{align*}
\]

\[
\begin{align*}
\frac{d\phi(t)}{dt} &= P_E(t)(1 - P_E(t))(M + P_G S_G), \\
\frac{d\psi(t)}{dt} &= P_G(t)(1 - P_G(t))\left(N - P_G\left(S_G - \frac{\delta_G}{2}\right)\right).
\end{align*}
\]

Let \( \phi(t) = P_E(t) - \delta_G - C_G/S_G = P_E(t) - N/S_G, \quad \psi(t) = P_G(t) - \delta_E - C_E + R_E + I/S_G = P_G(t) - M/S_G. \) Then, model \((44)\) is transformed into the following:

\[
\begin{align*}
\frac{d\phi(t)}{dt} &= \left(\phi(t) + \frac{N}{S_G}\right)\left[1 - \phi(t) - \frac{N}{S_G}\right]\left[M + \left(\psi(t) + \frac{M}{S_G}\right)S_G\right], \\
\frac{d\psi(t)}{dt} &= \left(\psi(t) + \frac{M}{S_G}\right)\left[1 - \psi(t) - \frac{M}{S_G}\right]\left[N - \left(\phi(t) + \frac{N}{S_G}\right)\left(S_G - \frac{\delta_G}{2}\right)\right].
\end{align*}
\]

A Jacobian matrix is obtained as follows:

\[
J = \begin{bmatrix}
\left(1 - 2\phi(t) - \frac{2N}{S_G}\right)
& \left[1 - \phi(t) - \frac{M}{S_G}\right]S_G
\
\left[1 - \psi(t) - \frac{M}{S_G}\right]S_G
& \left(\phi(t) + \frac{N}{S_G}\right)\left(1 - \psi(t) - \frac{M}{S_G}\right)S_G
\end{bmatrix}.
\]

The linear approximation of \((45)\) is obtained as follows:

\[
\begin{align*}
\frac{d\phi(t)}{dt} &= \frac{N}{S_G}(S_G - N)\psi(t), \\
\frac{d\psi(t)}{dt} &= \left[\frac{M}{S_G}(S_G - M) + \frac{M\delta_G}{2S_G}\right] \left(1 - \frac{M}{S_G}\right)\phi(t).
\end{align*}
\]

Substitute \( M = \delta_E - C_E + R_E + I, N = (3/2)\delta_G - C_G \) into equation \((49)\) to get the approximate equation of the original game model.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Equilibrium points & Tr\{ symbol & det\{ symbol & Equilibrium results \\
\hline
(0, 0) & Uncertain & – & Saddle point \\
(0, 1) & Uncertain & – & Saddle point \\
(1, 0) & Uncertain & – & Saddle point \\
(1, 1) & Uncertain & – & Saddle point \\
$(\delta_G - C_G/S_G, -\delta_E + C_E + I - R_E/S_G)$ & $= 0$ & + & Center point \\
\hline
\end{tabular}
\caption{Analysis on the stability of equilibrium points.}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Replication dynamic phase of evolutionary game.}
\end{figure}

The equilibrium points of (40) are as follows: O(0,0),
B(1,0), D(1,1), F(0,1), P(, $(\delta_G - C_G/S_G)$,
$(-\delta_E + C_E + I - R_E/S_G)$), and the replication dynamic phase of
the evolutionary game is illustrated by Figure 2.

According to Table 5 and Figure 2,

(1) When the initial state of the game system is in the
OAPGO area, saddle point O is stable along the OF
direction, i.e., when the suppliers’ supply probability
is $P_E < \delta_G - C_G/S_G$ and continues to decrease, the
government will provide more incentives to suppliers.

(2) When the initial state of the game system is in the
ABCPA area, saddle point B is stable along the BO
direction, i.e., the probability that the government
provides incentives is $P_G < -\delta_E + C_E + I - R_E/S_G$
and $P_G$ continues to decrease, the intensity of sup-
pliers to provide supply will reduce.

(3) When the initial state of the game system is in the
PCDEP area, saddle point D is stable along the DB
direction, i.e., when the supply probability of public
goods suppliers is $P_E > \delta_G - C_G/S_G$ and $P_E$ continues
to increase, the intensity of government to offer
incentives will reduce.
Figure 3: Time delay-free deterministic systems (a) $S_A = 0.2$, $S_B = 0.2$. (b) $S_A = 0.2$, $S_B = 0.5$.

Figure 4: Time delay system. (a) $\tau_1 = 2.5$, $\tau_2 = 2.5$. (b) $\tau_1 = 4$, $\tau_2 = 1$. 
3.3.2. Parameter Analysis. The following parts will use the replication dynamic phase of evolutionary game to analyze the parameters, illustrating the practical significance of public goods suppliers’ selection of supply strategies and government’s incentive strategies.

(1) From the government’s perspective, the most ideal evolutionary trend state is in the CDEP area, namely, public goods suppliers continue to increase the probability of supply, and the government continues to reduce support. At this time, suppliers are more proactive and enthusiastic, which may form an endogenous mechanism for suppliers to provide supply. Furthermore, it means the government expects that the central point P will gradually move downward, the value of \( P' \) will remain unchanged or decrease, and the value of \( P'' \) will also decrease, thereby expanding the CDEP area. On the premise of \( \delta_G - C_G/S_G < 1,0 < -\delta_E + C_E + I - R_E/S_E < 1 \), with all the parameters above \( \delta_G \), \( C_G \) and \( I \) increase, \( C_E, \delta_E, R_E, S_E \) and \( S_G (t - \tau) \) increase, making the value of \( P' \) bigger and that of \( P'' \) smaller. Center point \( P \) moves downward. Suppliers reduce the supply cost \( C_E \), the government increases incentives, and suppliers increase supply revenue, which can fuel the enthusiasm of suppliers. The government can gradually reduce the incentives to suppliers based on the delay of revenue \( \delta_G \).

(2) From suppliers’ perspective, the most ideal evolution trend state is in the EFGP area, i.e., while suppliers continue to increase the probability of supply, they can get incentives from the government. Therefore, public goods suppliers’ expectation \( P \) gradually moves to the lower right to continuously expand the EFGP area.

On the premise of \( 0 < \delta_G - C_G/S_G < 1 \) and \( 0 < -\delta_E + C_E + I - R_E/S_E < 1 \), with all the parameters above \( C_E \) and \( C_G \) decrease, while \( \delta_E, R_E, \delta_G \) and \( S_G (t - \tau) \) increase, making the value of \( P' \) bigger and that of \( P'' \) smaller. Center point \( P \) moves to the lower right, i.e., reducing suppliers’ supply cost \( C_E \) and the government cost \( C_G \), increasing suppliers’ revenue, or enhancing public support, for the government can mobilize the supply enthusiasm of public goods suppliers. At the same time, according to the delay of revenue \( \delta_G \), increase the probability of government incentives.

(3) For OAPG and ABCP area, its space will become smaller, which is a state that neither the government nor public goods suppliers want. It will not be discussed in the article.

4. Numerical Simulation

4.1. Numerical Simulation of Equations (2)–(3). \( A = 2.5, B = 1.5, C_A = 4.0, C_B = 4.5, I = 1.5, E_G = 4, \delta_A = 0, \) and \( \delta_B = 3 \). Let \( A = \delta_A + \delta_B - C_A + R_A + E_G/2, B = \delta_B + \delta_A - C_B + R_B + E_G/2, C = \delta_B - I, D = \delta_B - C_B + R_B + E_G, E = \delta_A - C_A + R_A + E_G, \) and \( F = \delta_B - I \). Let \( A = 6.5, B = 5, C = 1.5, D = 4, E = 5.5, F = 1.5, G = 0, \) and \( H = 0 \) are the initial values of the parameter, and the time length is 10. Systems with different supply probabilities are obtained (Figure 3(a), 3(b)).

4.2. Numerical Simulation of Equations (2)–(4). \( A = 6.5, B = 5, C = 1.5, D = 4, E = 5.5, F = 1.5, G = 0, \) and \( H = 0 \) are still the initial values of the parameter, and the time length is 10. Take \( S_A = 0.2 \) and \( S_B = 0.2 \), respectively (as shown in Figures 4(c), 4(d)), and when \( S_A = 0.2 \) and \( S_B = 0.5 \) (as shown in figures 4(e), 4(f)), the two time-delay systems are as follows:

4.3. Numerical Simulation of Equation (50)

\[
\begin{align*}
\frac{dP_E(t)}{dt} &= P_E(t)(1 - P_E(t)) (\delta_E - C_E + R_E + I + P_G S_G), \\
\frac{dP_G(t)}{dt} &= P_G(t)(1 - P_G(t)) (\delta_G - C_G - P_E S_G).
\end{align*}
\]

When \( P_G = P_E = 0.2 \) (Figure 5(a)), \( P_G = P_E = 0.4 \) (Figure 5(b)), \( P_G = 0.5, P_E = 0.2 \) (Figure 5(c)), and \( P_G = 0.2 \) and \( P_E = 0.8 \) (Figure 5(d)), it is a deterministic system.
Table 6: Payment matrix of the game between suppliers and government.

<table>
<thead>
<tr>
<th>Supplier</th>
<th>Government</th>
<th>Incentive ($P_G$)</th>
<th>Nonincentive ($1-P_G$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supply ($P_E$)</td>
<td>$\delta_E + R_E - C_E + S_E, \delta_G - C_G - S_G$</td>
<td>$\delta_E - C_E + R_E, 0$</td>
<td>$-1, \delta_G - C_G$</td>
</tr>
<tr>
<td>Nonsupply ($1-P_E$)</td>
<td>$-I, \delta_G - C_G$</td>
<td>$-I, 0$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5: Time delay-free deterministic systems (a) $P_E = 0.2, P_G = 0.2$, (b) $P_E = 0.4, P_G = 0.4$, (c) $P_E = 0.2, P_G = 0.5$, and (d) $P_E = 0.8, P_G = 0.2$. 
4.4. Numerical Simulation of Equation (50)

\[
\begin{align*}
\frac{dP_E(t)}{dt} &= P_E(t)(1 - P_E(t)) \left[ \delta_E - C_E + R_E + I + P_G(t - \tau_1)S_G \right], \\
\frac{dP_G(t)}{dt} &= P_G(t)(1 - P_G(t)) \left[ \delta_G - C_G - P_E(t - \tau_2)S_G \right].
\end{align*}
\]

(50)

\[\delta_E = a = 2.5, R_E = b = 0.5, C_E = c = 5,\]
\[\delta_G = \Gamma = 2.0, C_G = g = 3.0, S_G = e = 4 \text{ and } I = d = 1 \text{ are the initial values of the parameter, and the time length is 10.}
\]

When \(P_G = 0.2\) and \(P_E = 0.2\), it is a system with different time delays (Figure 6(a), 6(b)).

Based on numerical simulation figures,

(1) From the game process, regardless of whether there is a delay between suppliers, it does not affect the final stable state of the strategy [6] (see Figures 3 and 4). Finally, \(S_A\) and \(S_B\) are asymptotically stable at 1, which is the same as the previous conclusion. However, compared with deterministic systems, time delay systems have different
rates from decision-making to a stable state of suppliers because of delays. Both are slower, and the main reason is that delays lead to a decrease in the number of times players learn to adjust strategies during the game. However, when the suppliers are in a delay, the longer the delay, the faster the evolution rate. See Figure 3(a), Figure 4(a), 4(b), Figures 3(b), and Figure 4(e), 4(f). It shows that in the game, because of excessive consideration of the competitor’s previous strategy, although the frequency of the continuous adjustment of the strategy is less, the time consumed to reach a stable strategy will be shortened because the strategy selection is more conservative and prudent.

(2) In the game between public goods suppliers and the government, the cyclical evolutionary relationship between $P_E$ and $P_G$ is shown in Figure 5, however, it cannot reach a stable state. However, because of delay, the supply and incentive strategies between suppliers and the government have an uncertain probability of growth, as shown in Figure 6. It shows that as for the supply of public goods, if there is a delay between suppliers and the government, both sides will act cautiously, which can largely avoid losses caused by improper supply and incentive strategies.

5. Research Conclusions

5.1. Conclusion. The equilibrium results of the evolutionary game between public goods suppliers depend on the initial state of the game. In the absence of overall supply, the (supply, supply) strategy combination is the game result expected by suppliers. Therefore, suppliers should strive to reduce the supply cost of public goods, increase the supply revenue of public goods, and/or increase the loss of nonsupply, which can increase the probability of supplying public goods.

5.2. Conclusion

(1) Considering the evolutionary trend of the equation from the perspective of the government, as public goods suppliers continue to increase the probability of supply, and the government improves publicity, incentives, subsidies, and the supply revenue of public goods, the enthusiasm of public goods suppliers will be mobilized. Consequently, the government can gradually reduce incentives and other measures for public goods suppliers.

(2) Considering the evolutionary game from the perspective of public goods suppliers, suppliers can obtain government incentives and others while continuously increasing the probability of public goods supply. Increasing the revenue of public goods suppliers, enhancing public support for the government, and raising the probability of government incentives can mobilize the enthusiasm of public goods suppliers to supply.

5.3. Conclusion. The government and suppliers should view delay gains more rationally. Only when the government handles the supply incentive mechanism of public goods suppliers under the principle of satisfying public needs, can it handle the structural equilibrium of public goods supply. Otherwise, because of improper government incentives and other measures, there will be insufficient supply of public goods, resulting in a prisoner's dilemma in the supply of public goods. At the same time, it is necessary to prevent the failure of incentive policies because of excessive government incentives and guard against the problem of pig's payoffs or chicken game dilemma. In the government's incentive regulation, how to obtain more time-sensitive data is of great significance to the state of public goods supply.

It can be drawn from the above conclusions that in the game of public goods supply, the government focuses on social benefits, while suppliers focus on obtaining economic benefits. It is of great significance for the government to encourage suppliers to provide public goods to improve the government's social service level, enhance the city's brand and humanistic literacy, and foster a beautiful living environment.

6. Further Studies

Based on the existing research on the evolutionary game delay of public goods supply, the next step will be to consider the game delay between public goods multisuppliers and the government and the multigroup game model for research.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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