A Bias-Corrected Method for Fractional Linear Parameter Varying Systems

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This paper proposes an identification algorithm for the fractional Linear Parameter Varying (LPV) system considering noisy scheduling and output measurements. A bias correction technique is provided in order to compensate for the bias caused by the least squares algorithm. This approach was created to estimate either coefficients or fractional-order differentiation, and it has been proven to produce unbiased and reliable results. The suggested method’s performance is assessed by the identification of two fractional models and was compared with Nelder–Mead Simplex method.

1. Introduction

Fractional-order systems may be utilized to more effectively represent the dynamics of many real-world systems due to the memory characteristic and nonlocality of the fractional-order calculus [1–4]. As a result, the study of fractional-order systems has become a hotspot for research (see [5–7] and the references therein), and it has evolved in a variety of fields, including system identification [2, 4, 8, 9], stability [10, 11], robust stability [12, 13], control [14–16], and diagnosis [17–19]. For more than two decades, fractional-order system identification has been a major research issue. As a result, various approaches in the literature have been created [20]. This paragraph provides a quick overview of fractional system identification methods. Le Lay [21] developed the first study that dealt with fractional systems identification in 1998. The identification of open-loop systems with fractional models is established in this article. Battaglia et al. [6] established a fractional model which produced the transient thermal behavior of a system in 2000. Nearby this year, authors in [22] have discussed new ways to model a system with the fractional model. In this work, the estimation of the fractional dynamics of a lead-acid battery has been investigated. Subsequently, in 2001, the use of state variable filters (svf) was first extended by Cois et al. to fractional systems, with an instrumental variable (IV) procedure [23]. Later, in [24], the fractional orthogonal bases Laguerre functions have been synthesized. In 2006, the identification of lead-acid battery state of charge estimation was developed by Sabatier et al. in [1]. In 2008, Malti et al. have developed the fractional optimal IV method for model identification [25]. Then, in 2011, fractional thermal systems identification was illustrated by Gabano et al. [26]. In [27], the genetic algorithm is used for the continuous-time identification. More significantly, in 2015, and prompted by the reality that many technologies can only run in fractional closed-loop configurations due to stability and sensitivity issues, many systems can only work in fractional closed-loop configurations. The authors in [28] have proposed a bias compensated method for system identification of the fractional closed-loop system. In 2018, the nonlinear system identification of fractional Wiener models has been established by Sersour et al. [29]. Very recently, in 2020–2021, the
fractional order system identification has attracted more and more attention: In [30] the authors have proposed an output error-based method to identify multi input single output (MISO) systems using fractional models. In paper [31], the authors consider the problem of identifying linear time invariant (LTI) fractional systems from noisy input and output measurements using the bias correction scheme. Moreover, in [32, 33], fractional derivatives based on the epidemic system have also been used to deal with some epidemic behaviors like the CORONA virus. In conclusion, the expansion of fractional calculus applications and other mathematical phenomena more closely reflects a real system than traditional integer models.

According to the abovementioned studies, despite the integer LPV modeling field has emerged rapidly over the recent years and it was used in many control applications like aircrafts [34–36], only few literatures have been presented on fractional-order LPV systems. Therefore, the fractional LPV system identification was developed firstly in [37]. The authors developed the simplified refined instrumental variable continuous-time (SRIVC) method to identify fractional LPV systems with output error (OE) technique and a nonlinear programming. More recently, the fractional closed-loop LPV system identification has been investigated in [38].

According to the abovementioned studies, the main contribution of this paper is to establish a parameter identification method for continuous-time fractional-order LPV systems under the Grünwald–Letnikov definition. This method is based on the bias compensated least squares algorithm known by its simplicity of concept and its power in the fields of identification and control theory. Firstly, the parameters vector estimation is obtained by the fractional-order least squares algorithm combined with the state variable filter approach. Then, to achieve the estimation consistency, the bias produced by the least squares is eliminated. Moreover, this algorithm combines with a nonlinear algorithm optimization to identify both fractional-order differentiations and linear coefficients in this work. In addition, this contribution seems interesting to deal with the more realistic case where the scheduling variables are affected by a measurement noise.

The paper has the following structure: the fractional calculus and systems description is given in section 2. In section 3, the identification problem is formulated. Section 4 establishes the creation of the fractional-order bias removed least squares technique for estimating both fractional-order and LPV coefficients. Section 5 contains two numerical examples as well as comparison simulations. In section 6, the closing comments are presented.

2. Preliminaries

In this section, we introduce the main notations, definitions, and theorems that will be used in the rest of this paper.

2.1. Notation. This section presents the used notations in this paper. Let \( \mathbb{R} \) denote the set of all real numbers. \( \mathbb{R}^n \) is defined as the set of real vectors of dimensions \( n \). \( \mathbb{R}^{n \times m} \) is the set of \( n \times m \) matrices. The inverse and the transpose of the matrix \( X \) are defined as follows \( X^{-1} \) and \( X^T \), respectively. \( \|x\|^2 = x^T x \) designates the squares of the 2-norm of \( x \). \( D = \frac{d}{dt} \) is the differential operator. The Laplace operator is symbolized by \( s \). The sampled period is denoted by \( T \), and \( N_d \) defines the number of data.

2.2. Fractional-Order Differentiation. Unlike integer calculus, there are several definitions of fractional derivative. The known three kinds of definitions for fractional calculus are Riemann–Liouville (R-L), Grünwald–Letnikov (G-L), and Caputo definitions [39]. In this paper, the Grünwald–Letnikov definition is used.

Definition 1 (see [40]). The Grünwald–Letnikov fractional derivative approximation of the order \( \alpha \) of function \( \chi(t) \) is described as

\[
\text{GLD}_{\rho}^{\chi}(t) = \frac{1}{T} \sum_{k=0}^{\nu T} (-1)^k \binom{\alpha}{k} (\chi(t) - kT),
\]

where \( \binom{\alpha}{k} \) is the generalized form of the Newton’s binomial to non-integer orders.

2.3. Fractional LPV System Description. Consider the single-input single-output (SISO) fractional-order LPV differential equation using the definition of Grünwald–Letnikov defined as follows:

\[
\sum_{n=0}^{N} a_n(\rho(t))D^\alpha y(t) = \sum_{m=0}^{M} b_m(\rho(t))D^{\beta}u(t),
\]

where the input and the output signals are presented by \( u(t) \) and \( y(t) \), respectively. The scheduling measurement is given by \( \rho(t) \). The linear coefficients \( (a_n, b_m) \in \mathbb{R}^+(0 < n \leq N; 0 \leq m \leq M) \) and the non-integer orders are \( (\alpha, \beta) \in \mathbb{R}^+ \) and they verify the following equation:

\[
\alpha_1 < \alpha_2 < \cdots < \alpha_N; \beta_0 < \beta_1 < \cdots < \beta_M.
\]

The commensurate order differential equation is given by

\[
\sum_{j=0}^{n_0} b_j(\rho(t))D^{\beta_j} y(t) = \sum_{j=0}^{m_0} b_j(\rho(t))D^{\beta_j} u(t),
\]

with \( \nu \) designates the commensurate order and where \( n_0 = \alpha/v \) and \( n_0 = \beta/v \) are integers and verify \( n_0 > n_0 \).

Assuming null initial conditions, the SISO fractional commensurate order LPV system representation in the Laplace domain is described by

\[
G(s) = \frac{B(s, \rho(t))}{A(s, \rho(t))} = \frac{\sum_{j=0}^{m_0} b_j(\rho(t))s^{\beta_j}}{\sum_{j=0}^{n_0} a_j(\rho(t))s^{\alpha_j}},
\]

where \( \rho(t) \) is the scheduling variable.
3. Problem Formulation

3.1. Data Generating Fractional-Order LPV System. In this subsection, a brief description of Continuous-Time (CT) fractional order LPV system \( \Theta_0 \) is presented. The generic form of the considered model is depicted in Figure 1 and governing mathematical relation is given by

\[
\Theta_0: \begin{cases}
\mathcal{A}_0(p, \rho_0(t))\chi_0(t) = \mathcal{B}_0(p, \rho_0(t))u(t), \\
y(t) = \chi_0(t) + e_0(t).
\end{cases}
\]

Hence, \( p = d/dt \) denotes the differentiation operator. \( u(t) \) denotes the input signal of the plant. Let \( \chi_0(t) \) designates the noise-free output. \( y(t) \) is the available observation of the output signal that is affected by a noise sequence \( e_0(t) \) which assumed to be a white noise.

Remark 1. The scheduling signal measurements are also contaminated by an additive measurement noise \( v_0(t) \)

\[
\rho_0(t) = \rho^0(t) + v_0(t),
\]

where \( \rho^0(t) \) is the noise-free scheduling signal and \( v_0(t) \) is an additive white noise with finite variance \( \sigma_v^2 \) and independent of the output noise \( e_0(t) \).

The vector of scheduling parameters \( P_0 \in \mathbb{R}^{n_1+1} \) is expressed by

\[
P_0 = \left[ 1, \rho_0(t), \rho_0^2(t), \ldots, \rho_0^{n_1}(t) \right]^T.
\]

\( \mathcal{A}_0(p, \rho_0(t)) \) and \( \mathcal{B}_0(p, \rho_0(t)) \) are polynomials in \( p \) of degree \( n_a, n_b \) expressed as follows:

\[
\mathcal{A}_0(p, \rho_0(t)) = 1 + \sum_{j=1}^{n_a} a_j^0(\rho_0(t)) p^j;
\]

\[
\mathcal{B}_0(p, \rho_0(t)) = \sum_{j=0}^{n_b} b_j^0(\rho_0(t)) p^j,
\]

where \( \nu \) is the commensurate order of the process.

The coefficients \( a_j^0 \) and \( b_j^0 \) are meromorphic and polynomial functions in \( \rho_0(t) \) of maximum degree \( n_p \) and defined as follows:

\[
a_j^0(\rho_0(t)) = a_{j,0} + \sum_{l=1}^{n_p} a_{j,l} p_l^j(\rho_0(t));
\]

\[
b_j^0(\rho_0(t)) = \sum_{l=0}^{n_p} b_{j,l} p_l^j(\rho_0(t)),
\]

where \( a_{j,l} \) and \( b_{j,l} \) are the unknown parameters.

3.2. System Model: Fractional-Order LPV System. Consider the parameterized model presented by the following equation:

\[
M: y(t) = -\sum_{l=1}^{n_1} a_l(\rho(t)) D^\nu y(t) + \sum_{l=0}^{n_p} b_j(\rho(t)) D^\nu u(t) + \varepsilon(t).
\]

For convenience, the following notations are introduced in this subsection.

The regression vector \( \varphi^T(t) \) is expressed by

\[
\varphi^T(t) = [-D^\nu y(t), \ldots, -D^{n_1+\gamma} y(t), u(t), \ldots, D^{n_p+\gamma} u(t)].
\]

The findings of direct fractional derivative of noisy output are not precise. It is suggested that the state variable filter (svf) be used to offer fractional-order differentiation of both input and output signals to overcome this problem [41].

The svf is presented by this equation as

\[
F_\alpha(p) = p^\alpha \left( \frac{\lambda_F}{\lambda_F + p} \right)^\kappa,
\]

where \( \alpha \in \mathbb{R}^*_+ \) is the order differentiation and \( \lambda_F \) is the cut-off frequency of the svf and \( \kappa \in \mathbb{N} \) selected so that \( \kappa > n_p \nu \).

We apply \( F_\alpha(p) \) to both sides of (11) and then

\[
\mathcal{A}(p, \rho(t)) y_j(t) = \mathcal{B}(p, \rho(t)) u_j(t) + \varepsilon_j(t).
\]

The filtered input/output signals \( u_j(t) \) and \( y_j(t) \) are written as

\[
L \text{ svf is presented by this equation as}
\]

\[
F_\alpha(p) = p^\alpha \left( \frac{\lambda_F}{\lambda_F + p} \right)^\kappa,
\]

where \( \alpha \in \mathbb{R}^*_+ \) is the order differentiation and \( \lambda_F \) is the cut-off frequency of the svf and \( \kappa \in \mathbb{N} \) selected so that \( \kappa > n_p \nu \).

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\]

The filtered input/output signals \( u_j(t) \) and \( y_j(t) \) are written as

\[
L \text{ svf is presented by this equation as}
\]
Consider \( e_f (t) \) represents the combined effects of the output observations noise \( e (t) \) and the error introduced by the operation filtering. Hence, the error equation \( e_f (t) \) may be rewritten as
\[
e_f (t) = J (p, \rho (t)) e_f (t).
\] (17)

The filtered regression vector is expressed as follows:
\[
\Phi_f (t) = \begin{bmatrix} D^\nu y_f (t), \ldots, -D^\nu y_f (t), \end{bmatrix}
\] (18)

The following linear regression form can be used to show the fractional system’s input and output behavior:
\[
y_f (t) = \Phi_f (t) \theta + \epsilon_f (t),
\] (19)

where
\[
\Phi_f (t) = \Phi_f^T (t) \otimes \rho (t),
\] (20)

where \( \otimes \) denotes the kronecker product.

The parameters vector \( \theta \) is defined as
\[
\theta = \left[ a_1^T, \ldots, a_{n_f}^T, b_0^T, \ldots, b_{n_f}^T \right] \in \mathbb{R}^{(n_f+1)(n_f+n_f+1)},
\] (21)

where
\[
b_j = \begin{bmatrix} b_{j,1} & b_{j,2} & \cdots & b_{j,n_f} \end{bmatrix}^T,
\] (22)

\[
a_i = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,n_f} \end{bmatrix}^T.
\]

Certainly, the novelty made in this study is the use of the bias compensated method to identify the fractional LPV system from \( N_d \) data points of the input and noisy output signals.

**4. Fractional LPV System Identification**

The fractional parameters vector and fractional-order differentiation are both identified in this section. To begin, the best coefficient is calculated using the fractional-order bias compensated least squares (fols) technique for initial values of the fractional-order differentiation. The fractional-order differentiation is then computed by combining the fols technique with a nonlinear algorithm.

**4.1. Fractional-Order Least Squares (fols) Estimates.**

Firstly, the fractional commensurate order \( \nu \) is meant to be known a priori, and the goal of this work is to identify only the coefficients.

In this section, in order to identify the continuous-time fractional order LPV systems, the fractional-order least squares (fols) method is detailed.

Let \( \hat{\theta}_{fols} (N_d) \), the estimation of \( \theta \), be found by minimizing the following criterion:
\[
C_{N_d} (\hat{\theta}_{fols} (N_d)) = \frac{1}{N_d} \sum_{i=1}^{N_d} \epsilon_i^2 (t),
\] (23)

where \( \epsilon_i (t) \) denotes the filtered equation error, which is provided by the equation as follows:
\[
\epsilon_i (t) = y_i (t) - \Phi_i^T (t) \theta.
\] (24)

As a result, the least squares estimated \( \theta \) is defined in the following way:
\[
\hat{\theta}_{fols} (N_d) = (E [\Phi_f \Phi_f^T])^{-1} (E [\Phi_f y_f]),
\] (25)

where \( E [\Phi_f \Phi_f^T] \) is the autocovariance matrix defined by
\[
E [\Phi_f \Phi_f^T] = \sum_{i=0}^{N_d} \Phi_f (t) \Phi_f^T (t).
\] (26)

And \( E [\Phi_f y_f] \) is the cross-covariance vector given by
\[
E [\Phi_f y_f] = \sum_{i=0}^{N_d} \Phi_f (t) y_f (t).
\] (27)

**Proposition 1.** Under the assumption that the matrix \( E [\Phi_f \Phi_f^T] \) is invertible, the difference between the fols estimates and the true parameters vector is given by
\[
\lim_{N_d \to \infty} \hat{\theta}_{fols} (N_d) - \theta = \hat{\Delta}_\theta,
\] (28)

where \( \hat{\Delta}_\theta \) is the bias yields by the fols method and given by the following expression:
\[
\hat{\Delta}_\theta = (E [\Phi_f \Phi_f^T])^{-1} (E [\Phi_f \epsilon_f]),
\] (29)

where the matrix \( \Lambda = \begin{bmatrix} I_{n_f \times (n_f+1)} \\
0_{(n_f+1) \times (n_f+1)} \end{bmatrix} \in \mathbb{R}^{(n_f \times (n_f+1)) \times (n_f \times (n_f+1))} \) and \( Y_f (t) = (y_f (t) \otimes \rho (t)) \).

**Proof.** Substituting the filtered output signal \( y_f (t) \) with equations (18) in (24) produced the following equation:
\[
\hat{\theta}_{fols} (N_d) = \left( \sum_{i=0}^{N_d} \Phi_f (t) \Phi_f^T (t) \right)^{-1} \left( \sum_{i=0}^{N_d} \Phi_f (t) \epsilon_f (t) \right) = \theta + \left( E [\Phi_f \Phi_f^T] \right)^{-1} (E [\Phi_f \epsilon_f]).
\] (30)
Substituting equation (20) in the above equation yields
\[ \hat{\theta}_{fcls}(N_d) = \theta + \left( E[\Phi_f \Phi_f^T] \right)^{-1} \left( E[\Phi_f(t) \otimes \rho(t) e_f(t)] \right), \]
(31)

Where
\[ E\left[(\varphi_f(t) \otimes \rho(t))e_f(t)\right] = E\left[\begin{bmatrix} -D^\gamma y_f(t) & -D^\gamma y_f(t)\rho(t) & \cdots & -D^\gamma y_f(t)\rho^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ -D^n y_f(t) & -D^n y_f(t)\rho(t) & \cdots & -D^n y_f(t)\rho^n(t) \\ u_f(t) & u_f(t)\rho(t) & \cdots & u_f(t)\rho^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ D^n u_f(t) & D^n u_f(t)\rho(t) & \cdots & D^n u_f(t)\rho^n(t) \end{bmatrix} e_f(t) \right]. \]
(32)

Since \( u(t) \) and \( v(t) \) are uncorrelated with the noise \( e(t) \), equation (32) can be written as follows:
\[ E\left[(\varphi_f(t) \otimes \rho(t))e_f(t)\right] = E\left[\begin{bmatrix} -D^\gamma y_f(t) & -D^\gamma y_f(t)\rho(t) & \cdots & -D^\gamma y_f(t)\rho^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ -D^n y_f(t) & -D^n y_f(t)\rho(t) & \cdots & -D^n y_f(t)\rho^n(t) \\ u_f(t) & u_f(t)\rho(t) & \cdots & u_f(t)\rho^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ D^n u_f(t) & D^n u_f(t)\rho(t) & \cdots & D^n u_f(t)\rho^n(t) \end{bmatrix} e_f(t) \right] = E \left[ \begin{bmatrix} -D^\gamma y_f(t) & -D^\gamma y_f(t)\rho(t) & \cdots & -D^\gamma y_f(t)\rho^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ -D^n y_f(t) & -D^n y_f(t)\rho(t) & \cdots & -D^n y_f(t)\rho^n(t) \\ u_f(t) & u_f(t)\rho(t) & \cdots & u_f(t)\rho^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ D^n u_f(t) & D^n u_f(t)\rho(t) & \cdots & D^n u_f(t)\rho^n(t) \end{bmatrix} e_f(t) \right], \]
(33)

Replacing equations (33) in (30), we get the following equation:
\[ \lim_{N_d \to \infty} \hat{\theta}_{fcls}(N_d) = \theta + \left( E[\Phi_f \Phi_f^T] \right)^{-1} \Lambda E[(\gamma_f(t) \otimes \rho(t)) e_f(t)] = \theta + \Delta_{\theta}. \]
(34)

Consequently, subtracting \( \theta \) from \( \lim_{N_d \to \infty} \hat{\theta}_{fcls}(N_d) \) in equation (34) makes us obtain (29).

Equation (34) shows that the fcls method fails to produce unbiased parameters and correct results because \( e(t) \) and \( \Phi(t) \) are mutually correlated. As a result, the fractional-order bias compensated least squares (fobcls) approach is detailed in the next section to address the bias problem.

4.2. Fractional-Order Bias Compensated Least Squares (fobcls) Algorithm. The bias compensated least squares (bcls) algorithm was introduced firstly by Zheng [42]. In this study, the aim is to formulate an update version of the bcls algorithm for parameter estimation of a fractional LPV system from noisy scheduling and output observations.

Thus, the fobcls estimator is expressed as
\[ \hat{\theta}_{fobcls}(N_d) = \hat{\theta}_{fcls}(N_d) - \Delta_{\theta} = \hat{\theta}_{fcls}(N_d) - \left( E[\Phi_f \Phi_f^T] \right)^{-1} \Lambda E[\gamma_f e_f]. \]
(35)

Here, the assessment of the bias is important to the actual use of the unbiased identification method. The fcls approach, on the other hand, can give a single formula for computing the estimations. Therefore, further equations connecting the noise to other known variables are necessary.

In order to achieve this aim, an enhanced system was used to raise the nominator parameters by \( n_a - n_b \) dimensions which is necessary, where the introduced parameter \( b_j = 0 \) \( \in [n_b + 1, \ldots, n_a + n_b] \). Therefore, the augmented transfer function is given by

\[ G(p, \rho(t)) = \frac{\bar{B}(p, \rho(t))}{A(p, \rho(t))} = \frac{b_0(p(t)) + b_1(p(t))\rho^n + \cdots + b_{n_b}(p(t))\rho^{n_b}}{1 + a_1(p(t))\rho^n + \cdots + a_{n_a}(p(t))\rho^{n_a}}, \]
(36)

and the augmented numerator \( \bar{B}(p) \) is written as
\[ \bar{B}(p) = b_0 + b_1\rho^n + \cdots + b_{n_b}\rho^{n_b} + \cdots + b_{n_a+n_b}\rho^{n_a+n_b}. \]
(37)

\( \bar{g} \) is presented by
\[ \bar{g} = \left[ \hat{\theta} \bar{B}^T(\rho(t)) \right], \]
(38)

where
\[ \bar{B}^T = \begin{bmatrix} b_{n_b+1}^T & \cdots & b_{n_a+n_b}^T \end{bmatrix} = [0 \ldots 0] \in \mathbb{R}^{n_{\gamma+1}}. \]

Similarly to (18), the filtered regression form is described by
\[ \bar{Y}_f(t) = \bar{B}^T(t) \bar{g} + e_f(t), \]
(40)
where

$$\overline{\Phi}_j(t) = \overline{\Psi}_j(t) \otimes \rho(t),$$ \hspace{1cm} (41)

$$\overline{\Psi}_j(t) = \left[ -D^*y_j(t) \ldots -D^{n_\nu}y_j(t) \right. \left. u_f(t) \ldots D^{(n_u+n_\nu)}u_f(t) \right], \hspace{1cm} (42)$$

$$\overline{\Psi}_j^T(t) = \left[ \Phi_j^T(t) \right],$$

where

$$\overline{\Phi}_j(t) = \left[ D^{(n_u+1)}u_f(t) \ldots D^{(n_u+n_\nu)}u_f(t) \right]. \hspace{1cm} (43)$$

The estimator $\tilde{\theta}_{fols}(N_d)$ using the fols method is expressed as

$$\tilde{\theta}_{fols}(N_d) = \left( E\left[ \overline{\Phi}_j \overline{\Phi}_j^T \right] \right)^{-1} \left( E\left[ \overline{\Phi}_j y_j(t) \right] \right)$$

$$= \overline{\theta} + \left( E\left[ \overline{\Phi}_j \overline{\Phi}_j^T \right] \right)^{-1} E\left[ \overline{\Phi}_j y_j(t) \right],$$ \hspace{1cm} (44)

where

$$E\left[ \overline{\Phi}_j \overline{\Phi}_j^T \right] = \sum_{t=0}^{N_d} \overline{\Phi}_j(t) \overline{\Phi}_j^T(t).$$ \hspace{1cm} (45)

Since $\nu(t)$, $e(t)$, and $u(t)$ are mutually independent, we obtain

$$\tilde{\theta}_{fols}(N_d) = \overline{\theta} + E\left[ \overline{\Phi}_j \overline{\Phi}_j^T \right]^{-1} \Lambda E\left[ \left( y_j(t) \otimes \rho(t) \right) \epsilon_j(t) \right].$$ \hspace{1cm} (46)

The array $\Lambda$ is given by the following equation:

$$\Lambda = \begin{bmatrix} I_{n_u(n_u+1)} \\ 0_{(n_u+n_\nu+1)^2 \times (n_u+1)} \end{bmatrix} \in \mathbb{R}^{((2n_u+n_\nu+1)(n_u+1) \times (n_u+1))},$$ \hspace{1cm} (47)

and the matrix $E[\overline{\Phi}_j \overline{\Phi}_j^T]$ may be described by

$$E[\overline{\Phi}_j \overline{\Phi}_j^T]^{-1} = \begin{bmatrix} E[\Phi_j \Phi_j]^T + X & -E[\Phi_j \Phi_j]^T E[\Phi_j \Phi_j] D^{-1} \\ -D^{-1} E[\Phi_j \Phi_j]^T E[\Phi_j \Phi_j] D^{-1} & D^{-1} \end{bmatrix},$$ \hspace{1cm} (48)

where $X = E[\Phi_j \Phi_j]^T E[\Phi_j \Phi_j] D^{-1} E[\Phi_j \Phi_j]^T E[\Phi_j \Phi_j]$ and $D = E[\Phi_j \Phi_j] - E[\Phi_j \Phi_j]^T E[\Phi_j \Phi_j] E[\Phi_j \Phi_j]^{-1} E[\Phi_j \Phi_j]$.

Combining (43) with (48),

$$\tilde{\theta} = -D^{-1} E[\Phi_j \Phi_j]^T E[\Phi_j \Phi_j]^{-1} E[\Phi_j \Phi_j] D^{-1} E[\Phi_j \Phi_j]^{-1} E[\Phi_j \Phi_j] + D^{-1} E[\Phi_j \Phi_j]^{-1} E[\Phi_j \Phi_j] D^{-1} E[\Phi_j \Phi_j]$$

$$= -D^{-1} E[\Phi_j \Phi_j]^T \tilde{\theta}_{fols}(N_d) - E[\Phi_j \Phi_j],$$ \hspace{1cm} (50)
On the other hand, it follows from (37), (38), (45), and (49) and since the augmented polynomial $\tilde{B}(p)$ has known parameters vector $\tilde{b} = 0$, it is evident that

$$\tilde{b} = b - D^{-1}E[\Phi_f \pi_f] E[\Phi_f \Phi_f^T]^{-1} \Lambda E[Y_f \epsilon_f],$$

$$= -D^{-1}E[\Phi_f \pi_f] E[\Phi_f \Phi_f^T]^{-1} \Lambda E[Y_f \epsilon_f]. \tag{51}$$

Thus, (50) and (52) give the following equality:

$$-D^{-1}E[\Phi_f \pi_f] E[\Phi_f \Phi_f^T]^{-1} \Lambda E[Y_f \epsilon_f]$$

$$= -D^{-1}(E[\Phi_f \pi_f]^T \tilde{\theta}_fobs(N_d) - E[\pi_f y_f]). \tag{52}$$

From (52), it is clear that

$$E[Y_f \epsilon_f] = \left(E[\Phi_f \pi_f]^T E[\Phi_f \Phi_f^T]^{-1} \Lambda \right)^{-1} \left(E[\Phi_f \pi_f]^T \tilde{\theta}_fobs(N_d) - E[\pi_f y_f] \right). \tag{53}$$

Replacing (53) in (28) leads to

$$\tilde{\Delta}_\theta = E[\Phi_f \Phi_f^T]^{-1} \Lambda \left(E[\Phi_f \pi_f]^T E[\Phi_f \Phi_f^T]^{-1} \Lambda \right)^{-1} \left(E[\Phi_f \pi_f]^T \tilde{\theta}_fobs(N_d) - E[\pi_f y_f] \right). \tag{54}$$

Finally, we obtain

$$\tilde{\theta}_fobs(N_d) = \tilde{\theta}_fobs(N_d) - \tilde{\Delta}_\theta,$$

$$= \tilde{\theta}_fobs(N_d) - E[\Phi_f \Phi_f^T]^{-1} \Lambda \left(E[\Phi_f \pi_f]^T E[\Phi_f \Phi_f^T]^{-1} \Lambda \right)^{-1} \times \left(E[\Phi_f \pi_f]^T \tilde{\theta}_fobs(N_d) - E[\pi_f y_f] \right). \tag{55}$$

To facilitate the use of this method, a detailed description is given in Algorithm 1.

4.2.1. Convergence Analysis. We begin this part by demonstrating that the fobcls method is highly reliable.

**Theorem 1.** The proposed fobcls estimator for fractional order system identification gives consistent estimates. The optimal augmented estimator $\tilde{\theta}_fobs(N_d)$ verifies the following equation:

$$\lim_{N_d \to \infty} \tilde{\theta}_fobs(N_d) = \theta. \tag{56}$$

**Proof.** Firstly, we recall that the filtered output signal is expressed as

$$y_f(t) = \Phi_f^T \theta + \epsilon_f(t). \tag{57}$$

Multiplying both side of equation (57) by $\pi_f(t)$, we obtain

$$\pi_f(t) y_f(t) = \pi_f(t) \Phi_f^T \theta \tag{58}$$

Because the noises $e(t)$ and $v(t)$ are mutually uncorrelated of each other and of $u(t)$, we get

$$E[\pi_f y_f] = E[\Phi_f \pi_f \theta]. \tag{59}$$

So, using equation (53),

$$\lim_{N_d \to \infty} E[Y_f \epsilon_f] = \left(E[\Phi_f \pi_f]^T E[\Phi_f \Phi_f^T]^{-1} \Lambda \right)^{-1} \times \left(E[\Phi_f \pi_f]^T \tilde{\theta}_fobs(N_d) - E[\pi_f y_f] \right). \tag{60}$$

Replacing $E[\pi_f y_f]$ in the above equation by its expression given in equation (57), we obtain

$$\lim_{N_d \to \infty} E[Y_f \epsilon_f] = \left(E[\Phi_f \pi_f]^T E[\Phi_f \Phi_f^T]^{-1} \Lambda \right)^{-1} \times \left(E[\Phi_f \pi_f]^T \tilde{\theta}_fobs(N_d) - E[\Phi_f \pi_f \theta] \right). \tag{61}$$

Thus, the result is shown as the following equation:

$$\lim_{N_d \to \infty} E[Y_f \epsilon_f] = \left(E[\Phi_f \pi_f]^T E[\Phi_f \Phi_f^T]^{-1} \Lambda \right)^{-1} \times \left(E[\Phi_f \pi_f]^T \left(\tilde{\theta}_fobs(N_d) - E[\pi_f y_f]\right)\right). \tag{62}$$

Now, replacing the term $(\tilde{\theta}_fobs(N_d) - \theta)$ by the expression of $\Delta_\theta$ defined in equation (54),

$$\lim_{N_d \to \infty} E[Y_f \epsilon_f] = \left(E[\Phi_f \pi_f]^T E[\Phi_f \Phi_f^T]^{-1} \Lambda \right)^{-1} \times \left(E[\Phi_f \pi_f]^T \tilde{\theta}_fobs(N_d) - E[\Phi_f \pi_f \theta] \right). \tag{63}$$

Consequently,
Using equation (65) and equation (25), we find
\[
\lim_{N_d \to \infty} \frac{\partial f_{bcls}}{\partial \theta} = \theta_{fobs}(N_d) - E[\Phi_f \Phi_f^T \lambda E[Y_f \xi_f]]
\]
\[
= \left( \theta + E[\Phi_f \Phi_f^T]^{-1} \lambda E[Y_f \xi_f]\right) - E[\Phi_f \Phi_f^T]^{-1} \lambda E[Y_f \xi_f] .
\]
Finally, letting $N_d \to \infty$, we obtain
\[
\lim_{N_d \to \infty} \frac{\partial f_{bcls}}{\partial \theta} = \theta.
\]

4.3. Fractional-Order Estimation. The $f_{bcls}$ technique can only be used in the scenario where the fractional orders are considered to be known a priori. The main contribution of this part is to design an algorithm extending the identification method studied previously to a more normal situation in which the fractional-order differentiation is supposed to be unknown and the coefficients are approximated. Fractional-order optimization-bias compensated least squares $f_{oobs}$ is the name of this algorithm. It is based on a combination of the $f_{bcls}$ coefficient estimate technique and a nonlinear differentiation order optimization algorithm. As an option to tackling the optimization problem, the Gauss–Newton method is offered.

For that, consider the parameter vector $\theta$ as follows:
\[
\theta = [a_1, \ldots, a_{n_a}, b_0, \ldots, b_{n_b}, \nu].
\]

The challenge of parameter identification is framed as a functional minimization. As a result, the primary purpose of this method is to reduce the error described by equation (69) with regard to $\nu$. As a result, the Gauss–Newton method is used to repeatedly update the commensurate order. As a result, the quadratic criteria are as follows:
\[
\mathcal{H}(\theta_{fobs}) = \frac{1}{2} \epsilon(t) \| \epsilon(t) \|^2,
\]
where the error $\epsilon(t)$ is expressed by
\[
\epsilon(t) = y(t) - \bar{y}(t).
\]

At each the iteration $i+1$, this algorithm calculates the fractional differentiation order $y^{i+1}$
\[
y_{j+1} = y_j - \left[ \frac{\partial \mathcal{W}}{\partial \nu} \right]_{v_j},
\]

The gradient $\frac{\partial \mathcal{W}}{\partial \nu}$ is described as follows:
\[
\frac{\partial \mathcal{W}}{\partial \nu} = \frac{\partial \epsilon^T(I)}{\partial \nu} \epsilon(t),
\]
and the approximated hessian $\mathcal{H}$ is given by
\[
\mathcal{H} = \frac{\partial \epsilon^T(I)}{\partial \nu} \frac{\partial \epsilon(I)}{\partial \nu}.
\]

The description of the $f_{oobs}$ method is given by Algorithm 2.

5. Numerical Example

The numerical examples provided in this section serve to verify the different aspects highlighted in the previous section of the paper.

In order to give a quantitative evaluation of the proposed method in this paper, we adopted the performances indexes: the Best Fit Rate (BFR), which is defined as follows:
\[
BFR = \max \left\{ 1 - \sum_{i=1}^{N_d} \frac{y(t) - \bar{y}(t)}{\bar{y}(t)}^2, 0 \right\},
\]
where $\bar{y}$ is the mean of the output signal and $\bar{y}$ is the estimated model output and also the normalized relative quadratic error (NRQE) is expressed by
\[
NRQE = \sqrt{\frac{1}{n_{mc}} \sum_{k=1}^{n_{mc}} \frac{||\bar{y} - \bar{y}||^2}{||\bar{y}||^2}}.
\]

where $n_{mc}$ designates the Monte Carlo realization number.

The fractional-order optimization-least squares ($f_{oobs}$) and $f_{oobs}$ techniques are used to estimate fractional commensurate order and fractional LPV model coefficients. Moreover, these algorithms are compared with Nelder–Mead (NM) method which has been widely used in many scientific and engineering applications [43]. It is based on the Matlab fminsearch optimization function without constrained. In fact, the Matlab fminsearch is a nonlinear function for solving mathematical problems available on the matlab platform. It is based on the Simplex algorithm and used to optimize the solution by minimizing a given cost function [44]. It starts with the initial vector and tries to find the local minimum. The main reason for the popularity of this algorithm in practice is its simplicity.
Step 1: Generate the data using the svf \( \{ u_f(t), y_f(t) \mid t = 1, \ldots, N_d \} \).
Step 2: Build the filtered regressor using equation (13)
Step 3: Apply the fols method to obtain \( \hat{\theta}_{fols} \) using equation (43)
Step 4: Compute the matrix \( \Lambda \) and the vector \( \varphi_f(t) \) and build the augmented regressor vector \( \varphi_f^T(t) \) using equations (46), (42), and (41)
Step 5: Compute the bias expression via equation (53)
Step 6: Calculate \( \hat{\theta}_{fols}(N_d) \) using equation (54)

Algorithm 1: Fractional-order bias compensated least squares algorithm.

5.1. Example 1. In this problem, the simulation studies are performed for the following system:

\[
G_1(p, \rho(t)) = \frac{b_0(\rho(t))}{(1 + a_1(\rho(t))p)},
\]

where \( \gamma = 0 \cdot 400 \) is the dependence of the coefficients.

The fractional LPV model input-output is given by

\[
\mathcal{A}(p, \rho(t))y(t) = \mathcal{B}(p, \rho(t))u(t) + \epsilon(t),
\]

where

\[
\begin{align*}
A(p, \rho(t)) &= 1 + a_1(\rho(t))p^n, \\
B(p, \rho(t)) &= b_0(\rho(t)), \\
a_1(\rho(t)) &= a_{1,0} + a_{1,1}(\rho(t)), \\
b_0(\rho(t)) &= b_{0,0} + b_{0,1}(\rho(t)),
\end{align*}
\]

with

\[
\begin{align*}
b_{0,0} &= 1, \\
b_{0,1} &= 1, \\
a_{1,0} &= 1, \\
a_{1,1} &= 1 \cdot 5
\end{align*}
\]

Figure 2 depicts the input signal \( u(t) \) as a pseudo random binary sequence (PRBS) with a uniform distribution between \([-1, 1]\). With a sample time of \( T = 0.05 \) seconds, the scheduling and output signals are monitored (see Figure 2). The sample count is \( N_d = 6000 \). The output measurement is affected by a white noise \( \epsilon(t) \) with varying signal-to-noise (SNR) ratios, which is represented as

\[
SNR_y = 10 \log \left( \frac{\text{var}(y(t))}{\text{var}(\epsilon(t))} \right).
\]

Figure 3 shows the observed scheduling signal defined by

\[
\rho(t) = 0.5 \sin(0.3 \frac{\pi}{12} t) + \nu(t),
\]

where \( \nu(t) \) is the white noise with a \( SNR_{\rho(t)} \) of 20 [dB].

5.1.1. The svf Parameters Selection. Our focus in this section is on the effects of the state variables parameters on the \( \text{foo-}ls \) and \( \text{foo-bcls} \) algorithms. For the \( nmc = 100 \) runs, a Monte Carlo simulation has been performed. In practice, finding the exact values of the svf parameters may be difficult, and iterative testing is typically required. NRQE will be used for this exam in this post. Accordingly, \( \kappa \) and \( \lambda_F \) can only be approximated based on the noise and order information.

(1) The Choice of \( \lambda_F \). The choice of \( \lambda_F \) is too difficult so when it is too high, the noise is not filtered properly, and when it is too low, the system dynamics are filtered with the noise. A study on the effect of \( \lambda_F \) on the \( \text{foo-}ls \) and \( \text{foo-bcls} \) methods is handled in this paragraph. Thus, the svf order \( \kappa = 1 \) and \( \lambda_F \) is chosen as follows:

\[
\lambda_F = \{0.01, 0.5, 1, 10\} \text{ rad/seconds}.
\]

The results are illustrated in Table 1. The low NRQE values confirm that the \( \text{foo-bcls} \) performs good results when \( \lambda_F = 0.5 \) rad/seconds.
(2) Choice of $\lambda_F$. At this level, $\lambda_F = 1$ rad/seconds and the svf order $\kappa$ is selected as

$$\kappa = \{1, 2, 3\}.$$  (82)

Table 2 shows that the simplicity of the implementation and the low value of NRQE is ensured by $\kappa = 1$.

5.1.2. Comparative Study. To show the relevance of the presented algorithm, a Monte Carlo simulation of $mnc = 200$ experiments is done. A Gaussian white noise is affected the output signal $y(t)$; the values $SNR_x = 12$ [dB] and $SNR_y = 6$ [dB] were considered.

The fractional LPV model parameters vector is expressed as

$$\theta = [a_{1,0}a_{1,1}b_{0,0}b_{0,1}y].$$  (83)

$\lambda_F = 0 \cdot 3$ rad/seconds and $\kappa = 1$.

The results are recapitulated in Tables 3 and 4 and illustrated graphically in Figure 4. It is evident from the values shown in Tables 3 and 4 that the $foo-ls$ gives biased parameters estimates. On the other side, the $foo-bcls$ method produces unbiased parameters estimates with a low NRQE, $BFR = 1$, and low values of standard deviation. The observed good performance of this method was noticed even if in the case of a lower value of $SNR_y$ ($SNR_y = 6$[dB]).

In the histograms below, we can clearly see that the obtained parameter estimates using $foo-bcls$ are centered around the true ones, which confirms the consistency of this method.

### Table 1: Effect of the parameter $\lambda_F$ on the $foo-ls$ and $foo-bcls$ methods ($SNR_x = SNR_y = 20$ [dB] and $mnc = 100$ runs).

<table>
<thead>
<tr>
<th>$\lambda_F$</th>
<th>$foo-ls$</th>
<th>$foo-bcls$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.569</td>
<td>0.1589</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1823</td>
<td>0.0085</td>
</tr>
<tr>
<td>1</td>
<td>0.2536</td>
<td>0.0256</td>
</tr>
<tr>
<td>10</td>
<td>0.9850</td>
<td>0.269</td>
</tr>
</tbody>
</table>

### Table 2: Effect of the parameter $\kappa$ on the $foo-ls$ and $foo-bcls$ methods ($SNR_x = SNR_y = 20$ [dB] and $mnc = 100$ runs).

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$foo-ls$</th>
<th>$foo-bcls$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1823</td>
<td>0.0085</td>
</tr>
<tr>
<td>2</td>
<td>0.1528</td>
<td>0.0045</td>
</tr>
<tr>
<td>3</td>
<td>0.1376</td>
<td>0.006</td>
</tr>
<tr>
<td>4</td>
<td>0.1090</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

### Table 3: Fractional system parameters estimation resulted by $foo-ls$ and the $foo-bcls$ methods ($SNR_x = 20$, $SNR_y = 12$, and $6$ [dB] $mnc = 200$ runs).

<table>
<thead>
<tr>
<th>True</th>
<th>$SNR = 12$ dB</th>
<th></th>
<th>$SNR = 6$ dB</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{1,0}=1$</td>
<td>$0.7740 \pm 0.0126$</td>
<td>$0.9989 \pm 0.0164$</td>
<td>$0.2764 \pm 0.0176$</td>
<td>$0.9960 \pm 0.0359$</td>
</tr>
<tr>
<td>$a_{1,1}=1.5$</td>
<td>$1.1256 \pm 0.0072$</td>
<td>$1.0007 \pm 0.0086$</td>
<td>$0.5842 \pm 0.0111$</td>
<td>$0.9980 \pm 0.0199$</td>
</tr>
<tr>
<td>$b_{0,0}=1$</td>
<td>$0.8706 \pm 0.0072$</td>
<td>$1.0007 \pm 0.0086$</td>
<td>$0.5842 \pm 0.0111$</td>
<td>$0.9980 \pm 0.0199$</td>
</tr>
<tr>
<td>$b_{0,1}=1$</td>
<td>$0.7683 \pm 0.0175$</td>
<td>$1.0001 \pm 0.0083$</td>
<td>$0.3315 \pm 0.0330$</td>
<td>$0.9979 \pm 0.0212$</td>
</tr>
<tr>
<td>$\gamma=0.4$</td>
<td>$0.3068 \pm 0.0125$</td>
<td>$0.3999 \pm 0.0040$</td>
<td>$0.1513 \pm 0.0265$</td>
<td>$0.4053 \pm 0.0094$</td>
</tr>
</tbody>
</table>

### Table 4: NRQE and BFR achieved by $foo-ls$ and the $foo-bcls$ methods ($SNR_x = 20$, $SNR_y = 12$ and $6$ [dB], and $mnc = 200$ runs).

<table>
<thead>
<tr>
<th>Methods</th>
<th>$SNR = 12$ dB</th>
<th></th>
<th>$SNR = 6$ dB</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$foo-ls$</td>
<td>0.2238</td>
<td>0.628</td>
<td>0.6580</td>
<td>0.473</td>
</tr>
<tr>
<td>$foo-bcls$</td>
<td>0.0097</td>
<td>0.9984</td>
<td>0.0221</td>
<td>0.9955</td>
</tr>
</tbody>
</table>

5.2. Example 2. The model structure in this example is given by

$$G_2(p, \rho(t)) = \frac{b_0(\rho(t))}{1 + a_1(\rho(t))p^\gamma + a_2(\rho(t))p^{2\gamma}}$$  (84)

The fractional LPV model input-output is given as follows:

$$\mathcal{A}(p, \rho(t))y(t) = \mathcal{B}(p, \rho(t))u(t) + \epsilon(t),$$  (85)

where

$$\begin{aligned}
\mathcal{A}(p, \rho(t)) &= 1 + a_1(\rho(t))p^\gamma + a_2(\rho(t))p^{2\gamma}, \\
\mathcal{B}(p, \rho(t)) &= b_0(\rho(t)),
\end{aligned}$$  (86)

and

$$\begin{aligned}
a_1 &= a_{1,0} + a_{1,1}(\rho(t)), \\
a_2 &= a_{2,0} + a_{2,1}(\rho(t)), \\
b_0 &= b_{0,0} + b_{0,1}(\rho(t)),
\end{aligned}$$  (87)

where $\gamma = 0 \cdot 6$ and the coefficients are described by the following equation:
Figure 4: The process estimates histograms obtained by the foo-ls and the foo–bcls methods (SNR$_y = 20$ [dB], SNR$_y = 12$ and 6 [dB], and nmc = 200 runs). (a) foo-ls. (b) foo-bcls.

Figure 5: The input and the output signal.

Figure 6: The scheduling signal.
Table 5: Fractional system parameters estimation obtained by the \textit{foo-bcls} method and NM algorithm \(\text{SNR}_p = 20\), \(\text{SNR}_y = 10\) [dB], and \(nmc = 200\) runs.

<table>
<thead>
<tr>
<th>True</th>
<th>\textit{foo-bcls}</th>
<th>NM algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{1,0} = 1.000)</td>
<td>1.0121 ± 0.0157</td>
<td>0.8763 ± 0.1124</td>
</tr>
<tr>
<td>(a_{1,1} = 0.500)</td>
<td>0.5057 ± 0.0274</td>
<td>0.4590 ± 0.0693</td>
</tr>
<tr>
<td>(a_{2,0} = 0.500)</td>
<td>0.5001 ± 0.0123</td>
<td>0.6495 ± 0.1173</td>
</tr>
<tr>
<td>(a_{2,1} = 0.250)</td>
<td>0.2514 ± 0.0328</td>
<td>0.3224 ± 0.0631</td>
</tr>
<tr>
<td>(b_{1,0} = 1.000)</td>
<td>1.0053 ± 0.0505</td>
<td>0.9942 ± 0.0125</td>
</tr>
<tr>
<td>(b_{1,1} = 0.200)</td>
<td>0.2023 ± 0.1195</td>
<td>0.2091 ± 0.0173</td>
</tr>
<tr>
<td>(\nu = 0.600)</td>
<td>0.5985 ± 0.0227</td>
<td>0.5688 ± 0.0286</td>
</tr>
</tbody>
</table>

Table 6: NRQE and BFR obtained by the \textit{foo-bcls} method and NM algorithm (\(\text{SNR}_p = 20\), \(\text{SNR}_y = 10\) [dB], and \(nmc = 200\) runs).

<table>
<thead>
<tr>
<th>Methods</th>
<th>NRQE</th>
<th>BFR</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textit{foo-bcls}</td>
<td>0.0586</td>
<td>0.9935</td>
</tr>
<tr>
<td>NM algorithm</td>
<td>0.0712</td>
<td>0.8986</td>
</tr>
</tbody>
</table>

Figure 7: The process estimates histograms of the estimates with the \textit{foo-bcls} method and NM algorithm (\(\text{SNR}_p = 20\) [dB], \(\text{SNR}_y = 10\) [dB], and \(nmc = 200\) runs). (a) NM algorithm. (b) \textit{foo-bcls}.
The output signal is observed with a sampling period $T = 0 \cdot 1$ seconds. The number of samples is $N_T = 400$. A Monte Carlo simulation of $\text{nnmc} = 200$ running with new noise realization is achieved at noise level $\text{SNR}_v = 10 \text{ [dB]}$.

In the upcoming example, Figure 5 shows that the input signal $u(t)$ is, also, a PRBS distributed between $[-1 1]$ and the output signal is observed with a sampling period $T = 0 \cdot 1$ seconds. The number of samples is $N_T = 400$. A Monte Carlo simulation of $\text{nnmc} = 200$ running with new noise realization is achieved at noise level $\text{SNR}_v = 10 \text{ [dB]}$.

Figure 6 presents the scheduling signal expressed as

$$\rho(t) = 1 - 1 \left( 0 \cdot 5 \sin \left( 0 \cdot 35 \left( \frac{\pi}{12} t \right) \right) + v(t), \right)$$

(90)

where $v(t)$ is a white noise with a $\text{SNR}_{\rho(t)}$ of 20 [dB].

We treated the same study developed in section (5.1.1), and the svf parameters are chosen according to the values of the NRQE, respectively, as follows: $\lambda_F = 0 \cdot 3 \text{ rad/seconds and } \kappa = 2$.

Table 5 and Table 6 summarize the results about means, standard deviation, BFR, and NRQE of the estimated LPV model parameters over 200 Monte Carlo simulations. Table 5 and Table 6 demonstrate that the $\text{foo-bcls}$ method offers satisfying results in terms of unbiased parameters estimates with low value of the normalized relative quadratic error and standard deviation values.

Figure 7 illustrates the histograms of the estimates obtained by $\text{foo-bcls}$ and NM algorithm. It checks that the estimates resulting using the $\text{foo-bcls}$ method are close to the true parameters. On the other hand and despite its faster convergence, the NM algorithm gives biased estimates far for the true ones. Moreover, this algorithm requires an initial condition of a starting simplex, which can slow down the algorithm if it is badly chosen. In addition, the NM algorithm failed to converge in some of the Monte Carlo runs.

These results show that even with a large number of parameters, it is still possible to detect the coefficients as well as the fractional-order differentiation (case of the second example: seven parameters). This approach has the benefit of strong and faster convergence of the estimates to the true values when compared with other such methods.

**6. Conclusion**

In this paper, a method on system identification using the least squares algorithm for fractional-order linear parameters varying systems is proposed. The estimates obtained by the least squares algorithm are biased. To compensate the measurement noise in the output signal, an effective method based on the well-known bias correction scheme is established. The calculation formula of the bias compensation is offered to obtain the satisfactory estimations of the model coefficients and fractional-order differentiation. The identification findings support the suggested method’s powerful capacity in managing with fractional-order LPV systems.

**Data Availability**

The present work is purely fundamental research based on theoretical analysis. It develops illustrative numerical models by means of simulations.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


