Research Article

New Exact Solutions to the Lakshmanan–Porsezian–Daniel Equation with Kerr Law of Nonlinearity

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In this study, some new exact travelling wave solutions to the Lakshmanan–Porsezian–Daniel (LPD) equation with Kerr law of nonlinearity are retrieved by the complete discrimination system for the polynomial method. Under the travelling wave transformation, the LPD equation is reduced to an ordinary differential equation. The new exact travelling wave solutions including rational solutions, triangle function solutions, solitary wave solutions, and Jacobian elliptic function solutions are obtained and graphically illustrated using three-dimensional and two-dimensional graphs. Comparing with the previous results for LPD equation, some of new solutions in this work such as elliptical solution are not studied, which shows the complete discrimination system method is efficient.

1. Introduction

In this study, we consider the dimensionless form of Lakshmanan–Porsezian–Daniel (LPD) equation [1]:

\[ ip_t + lp_{xx} + mp_{xt} + nf(|p|^2)p = \tau p_{xxxx} + \varepsilon (p_x)^2 p^* + \rho |p|^2 p_x + \nu |p|^2 p_{xx} + \pi p^2 p_{xx} + \eta |p|^4 p. \] (1)

Here, the quantity governing the optical solitons is a complex-valued variable denoted as \( p(x, t) \); meanwhile, the independent variables \( x \) and \( t \) denote the space and time, respectively. Item \( ip_t \) represents the temporal evolution of the optical pulse. \( p^* \) is the complex conjugate of \( p \). Coefficients \( l \) and \( m \) are the group velocity dispersion (GVD) and spatiotemporal dispersion (STD) of the model. \( \tau \) is the fourth-order dispersion. \( \eta \) is two-photon absorption. The parameters \( \varepsilon, \rho, \nu, \) and \( \pi \) indicate the dispersion of other perturbation terms with nonlinear forms. The functional \( f \) is a real-valued function, and it intrinsically satisfies the smoothness \( f(|p|^2)p: \mathbb{C} \rightarrow \mathbb{C} \):

\[
f(|p|^2)p \in \bigcup_{i,j=1}^{\infty} C_k((-i, i) \times (-j, j): \mathbb{R}^3),
\] (2)

where we consider the complex plane \( \mathbb{C} \) as a two-dimensional linear space \( \mathbb{R}^2 \) and \( f(|p|^2)p \) is \( k \)-times differentiable.
The Lakshmanan–Porsezian–Daniel (LPD) equation was first proposed in [2, 3] to describe continuum isotropic biquadratic Heisenberg spin chain. The LPD model, among the infinite hierarchies of the nonlinear Schro dinger equation (NLSE) [4], similarly governs the dynamical behavior of optical solitons and transmission of solitons through a variety of waveguide [5, 6]. It is well known that nonlinear Schro dinger equations (NLSE) are essential to the research of various physical systems in the field of mathematical physics, nonlinear optics, optical fibres, and telecommunication engineering [7]. In this context, it is imperative to study exact optical solitons’ solutions of the LPD model, which made a revolutionary effect in modern electronic communication system. Moreover, it can accurately reflect the propagation of nonlinear systems and better understand nonlinear physical phenomena. For the last few years, the travelling wave equation solutions of LPD equation was widely studied by many authors. Biswas et al. [8] used two integration schemes to get the dark and singular solutions of the model. Rezazadeh et al. [9] studied the rational-type hyperbolic and trigonometric function solutions for the model by utilizing the Ba cklund transformation method. Akram et al. [10] proposed the improved tanΨ (η)/2-expansion method to give abundant optical solutions for the LPD equation.

Recently, Liu has developed the complete discrimination system for the polynomial method [11] for partial differential equation (PDE), which has been widely used to obtain the single travelling wave solutions from different models [12–15]. This method, comparing with the well-known (G'/G)-expansion method [16, 17], the B-spline collocation method [18–20], the integral bifurcations [21, 22], the Lie symmetry analysis method [23, 24], first integral method [25, 26], modified trial equation method [27], the exp-function method [28, 29], F-expansion method [30], the Kudryashov method [31], and so on, can be utilized to compute higher order polynomials deduced form multiple nonlinear PDE and enable researchers getting new exact travelling wave solutions to understand nonlinear physical phenomena more deeply.

The study consists of the following parts. In Section 2, we introduce the complete discrimination system for the polynomial method. In Section 3, we establish the travelling wave transformation and reduce (1) with the Kerr law to an ordinary differential equation. The main method is to use the complete discrimination system to classify the solutions and give the exact expressions. In Section 4, the focus is on the numerical illustration for the obtained solutions. We fix parameters to draw the numerical simulations using 3D-surface plots. Finally, the comparison of the obtained results is discussed in Section 5.

### 2. Mathematical Preliminaries

We commence with a brief introduction of the method which we will use to reduce the equation. Consider the nonlinear differential equation in the form

\[ G(p, p_t, p_x, p_{xt}, p_{xxx}, \ldots) = 0, \]

where \( p = p(t, x) \) is unknown function and \( G \) is a polynomial of all the derivatives of \( p = p(t, x) \).

By using the classical complex travelling wave transformation \( p(x, t) = U(\xi)e^{i\phi} \), we can easily convert PDE (3) into an ordinary differential equation, which can be written as

\[ P(U, U', U'', \ldots) = 0, \]

where \( P \) is a polynomial of \( U \) and its derivatives and notation ‘\( \xi \)’ denotes the derivative with respect to \( \xi \).

Next, the ordinary differential (4) can be reduced to

\[ \pm (\xi - \xi_0) = \int \frac{dU}{\sqrt{F(U)}}, \]

where \( \xi_0 \) is an integral constant and \( F(U) \) denotes a \( n \) degree polynomial.

In this study, the triple-order polynomial is considered:

\[ F(U) = U^3 + d_1U^2 + d_1U + d_0. \]

Its complete discrimination system is presented as follows:

\[
\begin{align*}
\Delta &= -27\left(\frac{2d_1^3}{27} + d_0 - \frac{d_1d_2}{3}\right)^2 - 4\left(d_1 - \frac{d_2}{3}\right)^3, \\
D_1 &= d_1 - \frac{d_2}{3}.
\end{align*}
\]

According to the complete discriminant system (7), the classification of the roots for polynomial (6) aims to retrieve all types of single-wave solutions of equation (1) which is obtained.

### 3. Applications

In this section, the main result of the exact single travelling wave solutions of (1) will be given. We use the following travelling wave transformation to investigate (1):

\[
p(x, t) = Q(\xi)e^{i\phi(x,t)}, \quad p^*(x, t) = Q(\xi)e^{-i\phi(x,t)},
\]

where \( c \) is the velocity of the soliton, \( k \) is the frequency of the soliton, \( \omega \) is the wavenumber, and \( \theta \) is the phase constant.

Inserting (8) into equation (1), we can deduce the real and imaginary parts from equation (1), respectively. The real part takes the form

\[
\tau Q^{(4)} + \left(mc - 6k^2 \tau \right)Q'' + \left(\omega + k^2 l - k \omega m + k^4 r\right)Q - nf(Q^2)Q
\]

\[-k^2 (\epsilon - \rho + \nu + \pi)Q^3 + \eta Q^2 + (\nu + \pi)Q^2 Q''
\]

\[+ (\epsilon + \rho)Q(Q')^2 = 0.
\]

Meanwhile, the imaginary part is
\[
\left(mkc - c + m\omega - 2lk - 4\tau k^3\right)Q' + 2k(\epsilon + \nu - \pi)Q^2(Q')^2 + 4\tau kQ'' = 0.
\] (10)

Since (9) and (10) are linearly independent, we take the coefficients equal to zero. Set
\[
t = 0, \nu + \pi = 0, \epsilon + \rho = 0,
\]
\[
\epsilon + \nu - \pi = 0, mkc - c + m\omega - 2lk - 4\tau k^3 = 0.
\] (11)

Then, (9) can be written as
\[
(mc - l)Q'' - (mk\omega - lk^2 - \omega)Q - n f(Q^2)Q + 4vk^3Q^3 + \eta Q^5 = 0.
\] (12)

In addition, the speed of solition is obtained from (11) as
\[
c = \frac{m\omega - 2lk}{1 - mk}, \text{ for } mk \neq 1.
\] (13)

The Kerr law of nonlinearity tells the fact that if there is an optical fibre with an external electric field, then the light wave in the fibre encounters nonlinear responses. In this case, \( f(Q) = Q \) so that (12) is reduced to
\[
(mc - l)Q'' - (mk\omega - lk^2 - \omega)Q - n(4vk^3)Q^3 + \eta Q^5 = 0.
\] (14)

Multiplying \( Q' \) on each side of (14) and integrating on \( \xi \), one can obtain that
\[
(mc - l)(Q')^2 - (mk\omega - lk^2 - \omega)Q^2 - \frac{n}{2}4vk^3Q^4 + \frac{\eta}{3}Q^6 = D,
\] (15)

where \( D \) is an arbitrary constant.

We then take the transformation
\[
U = \left(\frac{\epsilon}{3} \frac{\eta}{l - mc}\right)^{1/4} Q^2, \xi_1 = \left(\frac{\epsilon}{3} \frac{\eta}{l - mc}\right)^{1/4} \xi,
\] (16)

where \( \epsilon = 1 \) for \((\eta/l - mc > 0)\) and \( \epsilon = -1 \) for \((\eta/l - mc < 0)\); (15) becomes
\[
\left(U_{\xi_1}\right)^2 = eU(U^3 + d_2U^2 + d_1U + d_0),
\] (17)

where \( e = \pm 1, d_1 = 2c(n - 4vk^3/mc - l)(4\epsilon \eta /3l - 3mc)^{-3/4}, d_2 = 4emk\omega - lk^2 - \omega/mc - l(4\epsilon \eta /3l - 3mc)^{-1/2}, \) and \( d_0 \) is an arbitrary constant. Integrating (17), the final expression is given by
\[
\pm \left(\xi_1 - \xi_0\right) = \int \frac{dU}{\sqrt{eU(F(U)}} = \int \frac{dU}{\sqrt{eU(U^3 + d_2U^2 + d_1U + d_0)}},
\] (18)

where \( \xi_0 \) is the integration constant.

According to the discrimination system (7) for the triple order polynomial, the roots of \( F(U) \) will be classified, which implies each case for the solutions of equation (18).

Case 1. \( \Delta = 0 \) and \( D_1 = 0 \). \( F(U) \) has a real root of multiplicity three, namely, \( F(U) = (U + d_2)/3 \). Then, we have
\[
\pm \left(\xi_1 - \xi_0\right) = \int \frac{dU}{\sqrt{eU(U + d_2/3)}}.
\] (19)

If \( \epsilon = 1 \), then \( U > - (d_2/3) \). The solution of (1) is

\[
p_1(x, t) = \left(\frac{4}{3} \frac{\eta}{l - mc}\right)^{-1/8} \left(\frac{-12d_2}{d_2^2 \left(4/3\eta(l - mc)^{1/2}(x - ct) - \xi_0\right)^2 - 36} - \frac{d_2}{3}\right)^{1/2} e^{i(kx + ut + \theta)}.
\] (20)

If \( \epsilon = -1 \), then \( 0 < U < -(d_2/3) \). The solution of (1) is

\[
p_2(x, t) = \left(\frac{-4}{3} \frac{\eta}{l - mc}\right)^{-1/8} \left(\frac{12d_2}{d_2^2 \left(-4/3\eta(l - mc)^{1/2}(x - ct) - \xi_0\right)^2 - 36} - \frac{d_2}{3}\right)^{1/2} e^{i(kx + ut + \theta)}.
\] (21)
Case 2. \( \Delta = 0 \) and \( D_1 < 0 \). \( F(U) \) has a single real root and a real root of multiplicities two, namely, \( F(U) = (U - \lambda_1)^2 (U - \lambda_2) \). By equation (18), we deduce that

\[
\pm (\xi_1 - \xi_0) = \int \frac{dU}{[U - \lambda_1] \sqrt{\varepsilon U(U - \lambda_2)}}
\]

\[= \varepsilon \frac{2}{\lambda_2 - \lambda_1} \int \frac{d\Phi}{1 + \varepsilon(\lambda_1/\lambda_2 - \lambda_1)\Phi^2}, \tag{22}\]

where \( \Phi = \sqrt{\varepsilon U - \lambda_2}/U \).

When \( \epsilon = 1 \) and \( \lambda_1/\lambda_2 - \lambda_1 > 0 \) or \( \epsilon = -1 \) and \( \lambda_1/\lambda_2 - \lambda_1 < 0 \), (22) becomes

\[
\pm (\xi_1 - \xi_0) = 2\lambda_1^{-1} \varepsilon \frac{\lambda_1}{\lambda_2 - \lambda_1} \arctan \left( \frac{\lambda_1}{\lambda_2 - \lambda_1} \Phi \right). \tag{23}\]

For example, when \( l = 1, \ m = 0.5, \ n = 11, \ v = 0.25, \ k = 1, \ \omega = 16, \ \theta = 0, \) and \( d_0 = -4 \), we can get \( d_2 = -5, d_1 = 8 \), and \( c = 10 \). Then, we can deduce \( \epsilon = -1, \lambda_1 = 2, \) and \( \lambda_2 = 1 \).

Assuming \( \xi_0 = 0 \), the solution of (1) is

\[
p_3(x, t) = \frac{1}{1 + 1/2\tan^2 (x - 10t/\sqrt{2})} e^{i(x - \sqrt{116})}. \tag{24}\]

When \( \epsilon = 1 \) and \( \lambda_1/\lambda_2 - \lambda_1 < 0 \) or \( \epsilon = -1 \) and \( \lambda_1/\lambda_2 - \lambda_1 > 0 \), (22) becomes

\[
\pm (\xi_1 - \xi_0) = -\lambda_1^{-1} \varepsilon \frac{\lambda_1}{\lambda_1 - \lambda_2} \ln \left( \frac{1 + \sqrt{\varepsilon} \lambda_1/\lambda_2 - \lambda_2 \Phi}{1 - \sqrt{\varepsilon} \lambda_1/\lambda_2 - \lambda_2 \Phi} \right). \tag{25}\]

For example, when \( l = 1, \ m = 0.58258, n = 9, \ v = 0.25, \ k = 1, \ \omega = 9.58258, \ \theta = 0, \) and \( d_0 = -2 \), we can get \( d_2 = -4, d_1 = 5 \), and \( c = 8.58251 \). Then, we can deduce \( \epsilon = -1, \lambda_1 = 1, \) and \( \lambda_2 = 2 \), and (25) can be written as

\[
\pm (\xi_1 - \xi_0) = \begin{cases} \text{arcth} \Phi, & \text{for } |\Phi| < 1, \\ \text{arctanh} \Phi, & \text{for } |\Phi| > 1. \end{cases} \tag{26}\]

Assuming \( \xi_0 = 0 \), the solution of (1) is

\[
p_4(x, t) = \sqrt{1 + \frac{1}{\cosh (x - ct)}} e^{i(-kx + ut + \theta)}, \tag{27}\]

where \( c = 8.58251 \).

Case 3. \( \Delta > 0 \); then, \( D_1 < 0 \). \( F(U) \) can be decomposed into three distinct factors, namely \( F(U) = (U - \lambda_1)(U - \lambda_2)(U - \lambda_3) \). For convenience, we assume \( \lambda_1 > \lambda_2 > \lambda_3 \). Here, we denote equation (18) as

\[
\pm (\xi_1 - \xi_0) = \int \frac{dU}{\sqrt{\varepsilon(U - \lambda_1)(U - \lambda_2)(U - \lambda_3)}} \tag{28}
\]

\[
\int \frac{dU}{\sqrt{\varepsilon(U - l_1)(U - l_2)(U - l_3)(U - l_4)}}, \tag{29}\]

where any \( l_i \) could been 0 if we fix the parameters in (1).

Then, we take the transformation

\[
U = \begin{cases} 0, & \text{for } U > l_1 \text{ or } U < l_4, \\ \frac{l_2(l_1 - l_3)\sin^2 \phi - l_1(l_2 - l_4)}{(l_1 - l_4)\sin^2 \phi - (l_2 - l_4)}, & \text{for } l_3 < U < l_2, \\ \frac{l_1(l_2 - l_3)\sin^2 \phi - l_3(l_2 - l_4)}{(l_2 - l_4)\sin^2 \phi - (l_2 - l_4)}, & \text{for } l_2 < U < l_3. \end{cases}
\]

(29) to deduce the solutions of (1).

Firstly, we consider \( \epsilon = 1 \); hence, \( F(U) > 0 \).

IF \( U > 0 > \lambda_1 > \lambda_2 > \lambda_3 \), we can obtain the solution of (1) by using transition (29) as

\[
p_5(x, t) = \left( \frac{4}{3} \frac{\eta}{l - mc} \right)^{-1/8} \frac{\lambda_1 \lambda_3 \sin^2 \left( \sqrt{\lambda_2 \lambda_3} / 2 \left( (4/3 \eta l - mc)^{1/2} (x - ct) - \xi_0 \right), m \right)}{\lambda_3 \sin^2 \left( \sqrt{\lambda_2 \lambda_3} / 2 \left( (4/3 \eta l - mc)^{1/2} (x - ct) - \xi_0 \right), m \right) + (\lambda_1 - \lambda_3)} \tag{30}\]

\[
\times e^{i(-kx + ut + \theta)}, \]

where \( m^2 = \lambda_3(\lambda_1 - \lambda_2)/\lambda_2(\lambda_1 - \lambda_3) \).

Similarly, if \( U > \lambda_1 > 0 > \lambda_2 > \lambda_3 \), the solution is

\[
p_6(x, t) = \left( \frac{4}{3} \frac{\eta}{l - mc} \right)^{-1/8} \frac{\lambda_1 \lambda_3}{(\lambda_1 - \lambda_3) \sin^2 \left( \sqrt{\lambda_2 \lambda_3} / 2 \left( (4/3 \eta l - mc)^{1/2} (x - ct) - \xi_0 \right), m^{-1} \right) + \lambda_3} \tag{31}\]

\[
\times e^{i(-kx + ut + \theta)}, \]
where $m^2 = \lambda_3 (\lambda_1 - \lambda_2)/\lambda_2 (\lambda_1 - \lambda_3)$.

If $U > \lambda_1 > \lambda_2 > 0 > \lambda_3$, the solution is

$$p_1(x,t) = \left(\frac{4}{3} \frac{\eta}{1 - mc}\right)^{-1/8} \frac{\lambda_1 (\lambda_1 - \lambda_2) \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right)}{\lambda_2 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right)} - \lambda_1 (\lambda_2 - \lambda_3) \right) \lambda_2 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right) \right) \times e^{i(-kx+ut+\theta)},$$

where $n^2 = (\lambda_1 - \lambda_3) \lambda_2/\lambda_1 - \lambda_2 \lambda_3$.

If $\lambda_1 > \lambda_2 > U > 0 > \lambda_3$, the solution is

$$p_1(x,t) = \left(\frac{4}{3} \frac{\eta}{1 - mc}\right)^{-1/8} \frac{\lambda_3 \lambda_2 \lambda_3 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right)}{\lambda_3 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right) \right) \times e^{i(-kx+ut+\theta)},$$

where $n^2 = (\lambda_1 - \lambda_3) \lambda_2/\lambda_1 - \lambda_2 \lambda_3$.

If $\lambda_1 > \lambda_2 > \lambda_3 > 0$, the solution is

$$p_1(x,t) = \left(\frac{4}{3} \frac{\eta}{1 - mc}\right)^{-1/8} \frac{\lambda_1 \lambda_2 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right)}{\lambda_1 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right) \right) \times e^{i(-kx+ut+\theta)},$$

where $n^2 = (\lambda_1 - \lambda_3) \lambda_2/\lambda_1 - \lambda_2 \lambda_3$.

If $\lambda_1 > \lambda_2 > \lambda_3 > 0$, the solution is

$$p_1(x,t) = \left(\frac{4}{3} \frac{\eta}{1 - mc}\right)^{-1/8} \frac{-\lambda_2 \lambda_3 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right)}{\lambda_2 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right) \right) \times e^{i(-kx+ut+\theta)},$$

where $n^2 = (\lambda_1 - \lambda_3) \lambda_2/\lambda_1 - \lambda_2 \lambda_3$.

Next, consider $\varepsilon = -1$; then, $F(U) < 0$.

When $\lambda_1 > 0 > \lambda_2 > \lambda_3$, the solution is denoted as

$$p_1(x,t) = \left(\frac{4}{3} \frac{\eta}{1 - mc}\right)^{-1/8} \frac{\lambda_2 \lambda_1 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right)}{\lambda_1 \text{sn}^2\left(\sqrt{\lambda_1 (\lambda_2 - \lambda_3)/2 (4/3 \eta l - mc)^{1/2}} (x - ct) - \xi_0\right) \right) \times e^{i(-kx+ut+\theta)},$$

where $n^2 = (\lambda_1 - \lambda_3) \lambda_2/\lambda_1 - \lambda_2 \lambda_3$. 
where \( h^2 = \frac{(\lambda_1 (\lambda_2 - \lambda_3))}{(\lambda_1 - \lambda_2)} \frac{(\lambda_3 - \lambda_2)}{(\lambda_1 - \lambda_2)} \) (\(-\lambda_3\)).

When \( \lambda_1 > U > \lambda_2 > 0 > \lambda_3 \), the solution is denoted as

\[
p_{12}(x, t) = \left( \frac{4}{3} \frac{\eta}{l - mc} \right)^{-1/8} \sqrt{\frac{-\lambda_2 \lambda_1}{(\lambda_1 - \lambda_2) \left( \frac{(\lambda_2 - \lambda_1) / 2}{(4/3) \eta / l - mc} \right)^{1/2} (x - ct - \xi_0), h^{-1} - \lambda_1}} \\
\times e^{i(-kx + \omega t + \theta)},
\]

\[ (37) \]
where \( h^2 = (\lambda_1 (\lambda_2 - \lambda_3)/(\lambda_1 - \lambda_2) (-\lambda_3)) \).

When \( \lambda_1 > U > 0 > \lambda_2 > \lambda_3 \), the solution is denoted as

\[
p_{13}(x, t) = \left( \frac{4}{3} \eta/l - mc \right)^{-1/8} \frac{\lambda_3 (\lambda_1 - \lambda_2) \text{sn}^2 \left( \sqrt{\left(\lambda_1 - \lambda_3\right)\lambda_2 / 2 \left( (4/3)\eta/l - mc \right)^{1/2} (x - ct) - \xi_0}, m \right) - (\lambda_1 - \lambda_3)}{\lambda_3 \text{sn}^2 \left( \sqrt{\left(\lambda_1 - \lambda_3\right)\lambda_2 / 2 \left( (4/3)\eta/l - mc \right)^{1/2} (x - ct) - \xi_0}, m \right) - (\lambda_1 - \lambda_3)}
\times e^{i(-kx + wt + \theta)}.
\]

(Figure 3: The solitary wave solution of equation (1) by assuming \( l = 1, m = 0.58258, n = 9, \nu = 0.25, k = 1, \omega = 9.58258, \theta = 0, d_0 = -2 \), and \( \xi_0 = 0 \). (a) Three dimensional graph of (27). (b) Two dimensional shape of (27) along the \( z \)-axis.)

(Figure 4: The solitary wave solution of equation (1) by assuming \( l = 1, m = 0.58258, n = 9, \nu = 0.25, k = 1, \omega = 9.58258, \theta = 0, d_0 = -2 \), and \( \xi_0 = 0 \). (a) Three dimensional graph of (27). (b) Two dimensional shape of (27) along the \( z \)-axis.)
where \( m^2 = (\lambda_1 - \lambda_2)\lambda_3/(\lambda_1 - \lambda_3)\lambda_2). \)

When \( \lambda_1 > \lambda_2 > \lambda_3 > U > 0 \), the solution is denoted as

\[
p_{14}(x,t) = \left(\frac{4}{3} \frac{\eta}{l - mc}\right)^{-1/8} \sqrt{\frac{\lambda_1\lambda_3\sin^2\left(\sqrt{\frac{1}{2}(4/3\eta l - mc)}(x - ct) - \xi_0\right)}{\lambda_2\sin^2\left(\sqrt{\frac{1}{2}(4/3\eta l - mc)}(x - ct) - \xi_0\right)}} \cdot (\lambda_3 - \lambda_1) \tag{39}
\]

where \( m^2 = (\lambda_1 - \lambda_2)\lambda_3/(\lambda_1 - \lambda_3)\lambda_2). \)

Case 4. \( \Delta < 0, F(U) \) has a real root and a pair of conjugate complex roots, namely, \( F(U) = (U - \lambda)((U - l)^2 + s^2) \), where \( l \) and \( s > 0 \) are real numbers. Hence, equation (18) is

\[
\pm (\xi_1 - \xi_0) = \int_0^1 \frac{dU}{\sqrt{U(U - \lambda)((U - l)^2 + s^2)}} \tag{40}
\]

Suppose that \( \lambda > 0 \), and take the transformation

\[
U = \frac{a\cos \phi + b}{c\cos \phi + d} \tag{41}
\]

which is an elliptical solution.

\[
p_{15}(x,t) = \left(\frac{4}{3} \frac{\eta}{l - mc}\right)^{-1/8} \sqrt{\frac{\acn\sqrt{-2s}\frac{1}{\lambda}(4/3\eta l - mc)^{1/2}(x - ct) - \xi_0)m + b}{\ccn\sqrt{-2s}\frac{1}{\lambda}(4/3\eta l - mc)^{1/2}(x - ct) - \xi_0)m + d} \times e^{i(kx + wt + \theta)}, \tag{43}
\]

Similarly, when \( \lambda < 0 \), then \( \epsilon = 1 \). The solution of (1) is given by

\[
p_{16}(x,t) = \left(\frac{4}{3} \frac{\eta}{l - mc}\right)^{-1/8} \sqrt{\frac{\acn\sqrt{-2s}\frac{1}{\lambda}(4/3\eta l - mc)^{1/2}(x - ct) - \xi_0)m + \bar{b}}{\ccn\sqrt{-2s}\frac{1}{\lambda}(4/3\eta l - mc)^{1/2}(x - ct) - \xi_0)m + \bar{d}} \times e^{i(kx + wt + \theta)}, \tag{44}
\]

where \( \bar{a} = (\lambda/2)b + (\lambda/2)d, \bar{b} = (\lambda/2)d + (\lambda/2)b, \bar{c} = -l - (s/m_1), \) and \( \bar{d} = -l - sm_1. \)

4. Graphical Representation

In this section, the graphical description of exact solutions of the LPD (1) is shown in Figure 1. The presented solutions can be used to investigate the propagation of the origin (1) and understand the mechanism of the related physical phenomenon. Travelling wave solutions plotted by using Maple software include three types: rational, trigonometry, and solitary wave solutions. In Figure 1, rational function solution \( p_2(x,t) \) is obtained by taking suitable choice of parameters satisfying (11). In Figure 2, plots for triangle solution \( p_3(x,t) \) is presented. Particularly, the solitary wave solution \( p_4(x,t) \) taking \( \pm \) in equation (27) is given in Figure 3 and Figure 4, respectively.

5. Conclusion

In this work, the exact solutions of the LPD equation with Kerr law of nonlinearity are obtained by applying the complete discrimination system for the polynomial method. It is worthwhile to mention that this method is first applied on the LPD model to the best of our knowledge. Comparing the results published in other works [8, 9], the exact expressions of the solutions obtained in this study, including rational function solutions, triangle function solutions, the solitary wave solutions, and elliptic solutions have not been reported. Moreover, the Jacobian elliptic function solutions
listed in Case 3 and Case 4 cannot be obtained from other methods. In addition, the exact solutions of the LPD equation of importance for describing optical solitons help us deeply understand the dynamical behavior of the transmission of solitons through a variety of waveguides and provide a direct insight to the physical structure of waves.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**Authors’ Contributions**

Chen Peng investigated the study, developed methodology, reviewed and edited, curated data, and helped software. Zhao Li administrated project, developed methodology, collected resources, helped with software, supervised the study, validated the study, –and reviewed and edited the study. Hongwei Zhao helped with software, curated data, and helped carry out formal analysis.

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