Research Article

A Comparative Study of the Fractional-Order Nonlinear System of Physical Models via Analytical Methods

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This article is related to the fractional-order analysis of one- and two-dimensional nonlinear systems of third-order KdV equations and coupled Burgers equations, applying modified analytical methods. The proposed problems will be solved with the Caputo–Fabrizio fractional derivative operator and the Yang transform. The results we obtained by implementing the suggested methods are compared with the exact solution. The convergence of the method is successfully presented and mathematically proved. To show the effectiveness of the proposed methods, we compared exact and analytical results with the help of graphs and tables, which are in strong agreement with each other. Also, the results obtained by implementing the suggested methods at various fractional orders are compared, confirming that solution gets closer to exact solution as the value tends from fractional order towards integer order. Moreover, the proposed methods are attractive, easy, and highly accurate, which confirms that these methods are suitable methods for solving partial differential equations or systems of partial differential equations.

1. Introduction

Fractional calculus has overtaken ordinary calculus as a popular subject in recent years. Ordinary calculus has reached its peak of achievement in terms of discovery. As a result, fractional calculus is needed by mathematicians and researchers. This allows us to more properly describe real-world phenomena than by classic “integer” order. Many mathematicians, including Fourier, Laplace, Riesz, and others, were active and made significant contributions to the topic. The Atangana–Baleanu fractional integral [1], the Caputo fractional derivative [2], and the Caputo–Fabrizio fractional derivative [3] are modern examples of modern definitions of fractional-order derivatives and integrals that have brought in a new era in the history of fractional derivatives. Models based on fractional calculus can accurately represent many processes in engineering, physics, chemistry, and other fields [4]. Furthermore, fractional calculus is used to simulate the frequency-dependent damping behavior of several viscoelastic materials [5], economics [6], and dynamics of interfaces between nanoparticles and substrates [7].

Many physical and engineering phenomena are mathematically expressed using fractional differential equations. They have obtained a lot of attention in the natural and social sciences because they can accurately model phenomena dominated by memory effects. Because FDEs represent values on each point and differentiate the gaps between the two integers, they are a generalization of ordinary differential equations (ODEs). This is why, with the discovery of fractional calculus, it was discovered that FDEs have more real-world applications than ODEs [8, 9]. In many mathematical and scientific fields, FDEs in fractional calculus are one of the most popular subjects, such as biophysics, blood flow phenomena, aerodynamics, viscoelasticity, electrical circuits, electro-analytical chemistry, biology, control theory, finance, hydrology, and control systems [10–15].

In all of these fields of study, finding the actual or approximate solutions of FDEs is essential, but as we do not have a technique for finding the exact solution of these types
of FDEs, we must focus on approximation to exact solution [16–19]. In mathematics, determining the exact solution of such FDEs and other scientific applications is a challenging task. In comparison to the approximate solution [20], the exact solution helps us understand the mechanism and sophistication of the problem. Dealing with the difficulties of computations in these equations makes obtaining exact analytic solutions of FDEs extremely difficult, if not impossible. As a result, it is necessary to seek out some useful approximations and numerical techniques, such as the homotopy perturbation method [21], variation iteration method [22], residual power series method [23], approximate-analytical method [24], Elzaki transform decomposition method [25], iterative Laplace transform method [26], Adomian decomposition method [27], reduced differential transform method [28], and others.

In this paper, we proposed two analytical techniques with the aid of Laplace transform and the Atangana–Baleanu fractional derivative operator for solving fractional-order systems. The first technique is the mixture of Laplace transform (LT) and the variational iteration method known as the variational iteration transform method (VITM), which was first introduced by He [24] and is an effective approach for a wide range of problems in applied sciences [29–31]. The second important technique is the combination of Laplace transform and the Adomian decomposition method, first introduced by George Adomian (1923–1996) in the 1980s for solving nonlinear functional equations. The method is based on the decomposition of nonlinear functional equation into a series of functions. A polynomial created by a power series expansion of analytic function yields each series term. This method for solving various nonlinear fractional-order differential equations is exciting, simple, and accurate.

Harry Bateman initially introduced Burgers equation in 1915 [32], and it was later termed the Burgers equation. The Burgers equation has a wide range of applications in science and engineering, particularly for situations involving nonlinear equations. The importance and interest of Burgers equation applications by mathematical scientists and researchers have grown. This equation is recognized to represent a variety of phenomena, including dynamics modeling, heat conduction, acoustic waves, turbulence, and many others [33–35]. Korteweg and Vries derived the Korteweg–De Vries (KDV) equation for the first time in 1895. Long waves, tides, solitary waves, and wave propagation in a shallow canal are all modeled using KDV equation. Fluid mechanics, signal processing, hydrology, viscoelasticity, and fractional kinetics are just a few fields where KDV equation is used.

2. Preliminary Concepts

We provide fundamental definitions that will be used throughout the article. For the purpose of simplification, we write the exponential decay kernel as, \( K(\tau, \varrho) = e^{-\varrho^{s}(\tau-\varrho^{1-s})} \).

**Definition 1.** The Caputo–Fabrizio derivative is given as follows [36]:

\[
CFD^\delta_{\tau}[\mathcal{P}(\tau)] = \frac{N(\delta)}{1-\delta} \int_{0}^{\tau} \mathcal{P}(\tau) K(\tau, \varrho)d\varrho, \quad n - 1 < \delta \leq n. \tag{1}
\]

\( N(\delta) \) is the normalization function with \( N(0) = N(1) = 1 \).

\[
CFD^\delta_{\tau}[\mathcal{P}(\tau)] = \frac{N(\delta)}{1-\delta} \int_{0}^{\tau} \mathcal{P}[\mathcal{P}(\tau) - \mathcal{P}(\varrho)] K(\tau, \varrho)d\varrho. \tag{2}
\]

**Definition 2.** The fractional integral Caputo–Fabrizio is given as [36]

\[
CFI^\delta_{\tau}[\mathcal{P}(\tau)] = \frac{N(\delta)}{1-\delta} \mathcal{P}(\tau) + \frac{\delta}{N(\delta)} \int_{0}^{\tau} \mathcal{P}(\varrho)d\varrho, \quad \tau \geq 0, \delta \in (0, 1]. \tag{3}
\]

**Definition 3.** For \( \mathcal{P}(\tau) \) is expressed as [37]

\[
\forall \mathcal{P}(\tau) = \chi(s) = \int_{0}^{\infty} \mathcal{P}(\tau)e^{-\tau/s}d\tau, \quad \tau > 0. \tag{4}
\]

**Remark 1.** Yang transformation of few useful functions is defined as follows:

\[
\forall 1 = s, \quad \forall \varrho = s^2, \quad \forall [\tau^n] = \Gamma(i + 1)s^{i+1}. \tag{5}
\]

**Lemma 1** (Laplace–Yang duality). Let the Laplace transformation of \( \mathcal{P}(\tau) \) be \( F(s) \); then, \( \chi(s) = F(1/s) \) [38].

**Proof.** From equation (6), we can achieve another type of Yang transformation by putting \( \tau/s = \zeta \) as

\[
L[\mathcal{P}(\tau)] = \chi(s) = s\int_{0}^{\infty} \mathcal{P}(\tau) e^{\tau/s}d\zeta, \quad \zeta > 0. \tag{6}
\]

Since \( L[\mathcal{P}(\tau)] = F(s) \), this implies that

\[
F(s) = L[\mathcal{P}(\tau)] = \int_{0}^{\infty} \mathcal{P}(\tau)e^{-\tau/s}d\tau. \tag{7}
\]

Putting \( \tau = \zeta/s \) in (17), we have

\[
F(s) = \frac{1}{s} \int_{0}^{\infty} \mathcal{P}(\zeta) e^{\zeta/s}d\zeta. \tag{8}
\]

Thus, from equation (17), we achieve

\[
F(s) = \chi\left(\frac{1}{s}\right). \tag{9}
\]

Also, from equations (6) and (17), we achieve
The proof is completed. □

3. Idea of YDM

Here, we discuss the methodology of YDM for solving fractional-order partial differential equations.

\[ CF \frac{D^\varphi_s}{\partial t^\varphi} \mu(\xi, 3) + \mathcal{F}_1(\xi, 3) + N_1(\xi, 3) = \mathcal{F}(\xi, 3), 0 < \varphi \leq 1, \]

having initial terms

\[ \mu(\xi, 0) = \xi(\xi), \frac{\partial}{\partial \xi^\varphi} \mu(\xi, 0) = \zeta(\xi). \]

Here, the fractional-order CF operator is indicated from \( D^\varphi_s = (\partial^\varphi t^\varphi)^\varphi \) having orders \( \varphi, \mathcal{F}_1, \) and \( N_1 \) are linear and nonlinear operators and \( \mathcal{F}(\xi, 3) \) represents the source term.

On taking the Yang transform of (15), we get

\[ \mathcal{Y} \left[ D^\varphi_s \mu(\xi, 3) + \mathcal{F}_1(\xi, 3) + N_1(\xi, 3) \right] = \mathcal{Y} [\mathcal{F}(\xi, 3)]. \]

By the property of YT differentiation, we obtain

\[ \mathcal{Y} [\mu(\xi, 3)] = \Theta(\xi, \omega) \sum_{0}^{\infty} \mu_m(\xi, 3), \]

where

\[ \Theta(\xi, \omega) = 1/\omega^\varphi \sum_{\infty} g_0(\xi) + \omega^\varphi - 1 g_1(\xi) + \ldots + g_j(\xi). \]

Now, using the inverse Yang transform, we get

\[ \mu(\xi, 3) = \Theta(\xi, \omega) \sum_{0}^{\infty} \mu_m(\xi, 3), \]

where \( \Theta(\xi, \omega) \) shows the term that comes from the source factor. YDM generates the result of the infinite series of \( \mu(\xi, 3) \):

\[ \mu(\xi, 3) = \sum_{m=0}^{\infty} \mu_m(\xi, 3), \]

and we decompose the nonlinear operator \( N_1 \) as

\[ N_1(\xi, 3) = \sum_{m=0}^{\infty} \mathcal{A}_m, \]

where \( \mathcal{A}_m \) are Adomian polynomials given as

\[ \mathcal{A}_m = \frac{1}{m!} \left[ \frac{\partial^m}{\partial t^m} \mathcal{N}_1 \left( \sum_{k=0}^{\infty} \mathcal{F}_k(\xi, 3) \right) \right] \].

Putting equations (20) and (22) into (19) gives

\[ \sum_{m=0}^{\infty} \mu_m(\xi, 3) = \Theta(\xi, \omega) \sum_{m=0}^{\infty} \mu_m(\xi, 3) + \mathcal{A}_m \]

The terms listed are defined as follows:

\[ \mu_0(\xi, 3) = \Theta(\xi, \omega), \]

\[ \mu_1(\xi, 3) = \mathcal{Y}^{-1} \left[ 1 + \varphi(\omega - 1) \mathcal{Y} \left[ \mathcal{F}_1(\xi, 3) + \mathcal{A}_0 \right] \right]. \]
As a result, all of the components for \( m \geq 1 \) are calculated as
\[
\mu_{m+1}(\xi, \mathfrak{F}_m) = \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left[ \mathcal{T}_1(\mathfrak{F}_m, \mathfrak{F}_m) + \mathcal{D}_m \right] \right].
\] (25)

4. VITM Formulation

Here, we discuss the methodology of VITM for solving fractional-order partial differential equations.

\[
\mathcal{C} \mathcal{F} \mathcal{D}_m^\mu(\xi, \mathfrak{F}) + \mathcal{M}_\mu(\xi, \mathfrak{F}) + \mathcal{N}_\mu(\xi, \mathfrak{F}) - \mathcal{P}(\xi, \mathfrak{F}) = 0, m - 1 < \varphi \leq m.
\] (26)

having initial term
\[
\mu(\xi, 0) = g_1(\xi).
\] (27)

\[
\mathcal{Y} \left[ \mathcal{M}_\mu(\xi, \mathfrak{F}) + \mathcal{N}_\mu(\xi, \mathfrak{F}) - \mathcal{P}(\xi, \mathfrak{F}) \right].
\] (30)

\( \mathcal{Y} \) is the Lagrange multiplier, and
\[
\varphi(s) = -(1 + \varphi(\omega - 1)).
\] (31)

\[
\frac{\partial^\mu}{\partial \mathfrak{F}^\mu} = \frac{\partial^\mu}{\partial \xi^\mu} + \mu \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi}
\] (33)

5. Applications

To show the validity and capability of suggested techniques, we implemented proposed methods for solving four nonlinear systems.

**Problem 1.** Consider the system of homogeneous KdV equation having order three.

\[
\mathcal{C} \mathcal{F} \mathcal{D}_m^\mu(\xi, \mathfrak{F}) + \mathcal{M}_\mu(\xi, \mathfrak{F}) + \mathcal{N}_\mu(\xi, \mathfrak{F}) - \mathcal{P}(\xi, \mathfrak{F}) = 0, m - 1 < \varphi \leq m.
\] (26)

having initial term
\[
\mu(\xi, 0) = g_1(\xi).
\] (27)

\[\begin{align*}
\mu_0(\xi, \mathfrak{F}) &= \mu(0) + \mathcal{Y}^{-1} \left[ \varphi(s) \mathcal{Y} \left[ -\mathcal{P}(\xi, \mathfrak{F}) \right] \right], \\
\mu_1(\xi, \mathfrak{F}) &= \mathcal{Y}^{-1} \left[ \varphi(s) \mathcal{Y} \left[ \mathcal{M}_\mu + \mathcal{N}_\mu - \mathcal{P}(\xi, \mathfrak{F}) \right] \right], \\
&\vdots \\
\mu_{m+1}(\xi, \mathfrak{F}) &= \mathcal{Y}^{-1} \left[ \varphi(s) \mathcal{Y} \left[ \mathcal{M}_\mu + \mu_1(\xi, \mathfrak{F}) + \cdots + \mu_n(\xi, \mathfrak{F}) \right] \right] + \mathcal{N}[\mu_\mu(\xi, \mathfrak{F}) + \mu_1(\xi, \mathfrak{F}) + \cdots + \mu_n(\xi, \mathfrak{F})].
\end{align*}\]

with an initial source
\[
\mu(\xi, 0) = \left(3 - 6 \tanh^2 \xi/2\right) \nu(\xi, 0).
\] (34)

On taking the Yang transform of (34), we get
\[
\frac{1}{(1 + \varphi(\omega - 1))} \mathcal{Y}[\nu(\xi, \mathfrak{F})] = \mathcal{Y} \left[ -\mathcal{F}^\mu + \mu \frac{\partial \mu}{\partial \xi} + \nu \frac{\partial \nu}{\partial \xi} \right].
\] (35)

When we use the Yang inverse transform, we obtain
\[
\mu(\xi, \mathfrak{J}) = (3 - 6 \tanh^2 \frac{\xi}{2}) + \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left[ \frac{\partial^3 \mu}{\partial \xi^3} + \mu \frac{\partial \mu}{\partial \xi} + \varphi \frac{\partial \varphi}{\partial \xi} \right] \right],
\]

\[
\nu(\xi, \mathfrak{J}) = \left(3l\sqrt{2} \tanh^2 \frac{\xi}{2}\right) + \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left[ -2 \frac{\partial^3 \nu}{\partial \xi^3} + \mu \frac{\partial \nu}{\partial \xi} \right] \right].
\]

(36)

Assume that the solutions \(\mu(\xi, \mathfrak{J})\) and \(\nu(\xi, \mathfrak{J})\) in the form of infinite series are given by

\[
\mu(\xi, \mathfrak{J}) = \sum_{m=0}^{\infty} \mu_m(\xi, \mathfrak{J}), \quad \nu(\xi, \mathfrak{J}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{J}).
\]

(37)

Here, \(\mu_m = \sum_{m=0}^{\infty} \mathcal{A}_m\), \(\nu_m = \sum_{m=0}^{\infty} \mathcal{B}_m\), and \(\mu_m = \sum_{m=0}^{\infty} \mathcal{C}_m\) are the so-called Adomian polynomials that represent nonlinear terms, and so equation (36) is rewritten as follows:

\[
\sum_{m=0}^{\infty} \mu_m(\xi, \mathfrak{J}) = \left(3 - 6 \tanh^2 \frac{\xi}{2}\right) + \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left[ \frac{\partial^3 \mu}{\partial \xi^3} + \mu \frac{\partial \mu}{\partial \xi} + \varphi \frac{\partial \varphi}{\partial \xi} \right] \right],
\]

\[
\sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{J}) = \left(3l\sqrt{2} \tanh^2 \frac{\xi}{2}\right) + \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left[ -2 \frac{\partial^3 \nu}{\partial \xi^3} + \mu \frac{\partial \nu}{\partial \xi} \right] \right].
\]

(38)

The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (22).

\[
\mathcal{A}_0 = \mu_0 \mu_0',
\]

\[
\mathcal{A}_1 = \mu_1 \mu_0 + \mu_0 \mu_1',
\]

\[
\mathcal{A}_2 = \mu_2 \mu_0 + \mu_1 \mu_1 + \mu_0 \mu_2',
\]

\[
\mathcal{B}_0 = \nu_0 \nu_0',
\]

\[
\mathcal{B}_1 = \nu_1 \nu_0 + \nu_0 \nu_1',
\]

\[
\mathcal{B}_2 = \nu_2 \nu_0 + \nu_1 \nu_1 + \nu_0 \nu_2',
\]

\[
\mathcal{C}_0 = \mu_0 \nu_0,
\]

\[
\mathcal{C}_1 = \mu_1 \nu_0 + \mu_0 \nu_1,
\]

\[
\mathcal{C}_2 = \mu_2 \nu_0 + \mu_1 \nu_1 + \mu_0 \nu_2.
\]

(39)

As a result, when comparing the two sides of equation (38), we get

\[
\mu_0(\xi, \mathfrak{J}) = (3 - 6 \tanh^2 \frac{\xi}{2}),
\]

\[
\nu_0(\xi, \mathfrak{J}) = \left(3l\sqrt{2} \tanh^2 \frac{\xi}{2}\right).
\]

(40)

For \(m = 0\),

\[
\mu_1(\xi, \mathfrak{J}) = -6 \text{sech}^4 \frac{\xi}{2} \tanh \frac{\xi}{2} \left[1 + \varphi \mathfrak{J} - \varphi\right],
\]

\[
\nu_1(\xi, \mathfrak{J}) = 3l\sqrt{2} \text{sech}^4 \frac{\xi}{2} \tanh \frac{\xi}{2} \left[1 + \varphi \mathfrak{J} - \varphi\right].
\]

(41)

For \(m = 1\),

\[
\mu_2(\xi, \mathfrak{J}) = \frac{3}{2} \left(2 \text{sech}^2 \frac{\xi}{2} + 7 \text{sech}^4 \frac{\xi}{2} - 15 \text{sech}^6 \frac{\xi}{2}\right) \left(1 - \varphi\right) \left(2\varphi \mathfrak{J} + \left(1 - \varphi\right)^2 + \varphi^2 \mathfrak{J}^2\right),
\]

\[
\nu_2(\xi, \mathfrak{J}) = \frac{3l\sqrt{2}}{4} \left(2 \text{sech}^2 \frac{\xi}{2} + 21 \text{sech}^4 \frac{\xi}{2} - 24 \text{sech}^6 \frac{\xi}{2}\right) \left(1 - \varphi\right) \left(2\varphi \mathfrak{J} + \left(1 - \varphi\right)^2 + \varphi^2 \mathfrak{J}^2\right).
\]

(42)

The approximate solution to the series is written as follows:
\[
\mu(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \mu_m(\xi, \mathfrak{F}) = \mu_0(\xi, \mathfrak{F}) + \mu_1(\xi, \mathfrak{F}) + \mu_2(\xi, \mathfrak{F}) + \cdots, \\
\nu(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{F}) = \nu_0(\xi, \mathfrak{F}) + \nu_1(\xi, \mathfrak{F}) + \nu_2(\xi, \mathfrak{F}) + \cdots,
\]

\[
\mu(\xi, \mathfrak{F}) = 3 - 6\tanh^2(\mathfrak{F}/2) - 6\text{sech}^2\xi/2\tanh\xi/2\{1 + \varphi\mathfrak{F} - \varphi\} + \frac{3}{2}\left(2\text{sech}^2\xi/2 + 7\text{sech}^3\xi/2 - 15\text{sech}^6\xi/2\right) \tag{43}
\]

\[
\nu(\xi, \mathfrak{F}) = -\left(3l\sqrt{2} \tanh^2(\mathfrak{F}/2) + 3l\sqrt{2} \text{sech}^2\xi/2\tanh\xi/2\{1 + \varphi\mathfrak{F} - \varphi\} + \frac{3l\sqrt{2}}{4}\left(2\text{sech}^2\xi/2 + 21\text{sech}^4\xi/2 - \right)24\text{sech}^6(\xi/2)\{1 + \varphi\mathfrak{F} - \varphi\} + \frac{\varphi^2\mathfrak{F}^2}{2}\right) + \cdots.
\]

We achieve the exact solution by putting \(\varphi = 1\).

\[
\mu(\xi, \mathfrak{F}) = 3 - 6\tanh^2(\mathfrak{F}/2), \\
\nu(\xi, \mathfrak{F}) = -3l\sqrt{2}\tanh^2(\mathfrak{F}/2). \tag{44}
\]

\[
\mu_{m+1}(\xi, \mathfrak{F}) = \mu_m(\xi, \mathfrak{F}) - \mathcal{Y'}^{-1}\left[\{1 + \varphi(\omega - 1)\}\mathcal{Y'}\left\{\frac{\partial^3 \mu_m}{\partial \mathfrak{F}^3} + \frac{\partial \mu_m}{\partial \xi} + \frac{\partial \nu_m}{\partial \xi}\right\}\right], \tag{45}
\]

\[
\nu_{m+1}(\xi, \mathfrak{F}) = \nu_m(\xi, \mathfrak{F}) - \mathcal{Y'}^{-1}\left[\{1 + \varphi(\omega - 1)\}\mathcal{Y'}\left\{-2\frac{\partial^3 \nu_m}{\partial \mathfrak{F}^3} + \frac{\partial \mu_m}{\partial \xi} \frac{\partial \nu_m}{\partial \xi}\right\}\right],
\]

where

\[
\mu_0(\xi, \mathfrak{F}) = 3 - 6\tanh^2(\mathfrak{F}/2), \\
\nu_0(\xi, \mathfrak{F}) = -3l\sqrt{2}\tanh^2(\mathfrak{F}/2). \tag{46}
\]

For \(m = 0, 1, 2, \cdots\).
\[
\begin{align*}
\mu_1 (\xi, \mathfrak{F}) &= \mu_0 (\xi, \mathfrak{F}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^3 \mu_0}{\partial \xi^3} + \mu_0 \frac{\partial \mu_0}{\partial \xi} + \nu_0 \frac{\partial \nu_0}{\partial \xi} \right\} \right], \\
\mu_1 (\xi, \mathfrak{F}) &= -6 \text{sech}^2 \frac{\xi}{2} \tanh \frac{\xi}{2} \{ 1 + \varphi \mathfrak{F} - \varphi \}, \\
\nu_1 (\xi, \mathfrak{F}) &= \nu_0 (\xi, \mathfrak{F}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ -2 \frac{\partial^3 \nu_0}{\partial \xi^3} \mu_0 \frac{\partial \nu_0}{\partial \xi} \right\} \right], \\
\nu_1 (\xi, \mathfrak{F}) &= 3 l \sqrt{2} \text{sech}^2 \frac{\xi}{2} \tanh \frac{\xi}{2} \{ 1 + \varphi \mathfrak{F} - \varphi \}, \\
\mu_2 (\xi, \mathfrak{F}) &= \mu_1 (\xi, \mathfrak{F}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^3 \mu_1}{\partial \xi^3} + \mu_1 \frac{\partial \mu_1}{\partial \xi} + \nu_1 \frac{\partial \nu_1}{\partial \xi} \right\} \right], \\
\mu_2 (\xi, \mathfrak{F}) &= \frac{3}{2} \left\{ 2 \text{sech}^2 \xi/2 + 7 \text{sech}^4 \xi/2 - 15 \text{sech}^6 \xi/2 \right\} \{ 1 - \varphi \} \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \}, \\
\nu_2 (\xi, \mathfrak{F}) &= \nu_1 (\xi, \mathfrak{F}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ -2 \frac{\partial^3 \nu_1}{\partial \xi^3} \mu_1 \frac{\partial \nu_1}{\partial \xi} \right\} \right], \\
\nu_2 (\xi, \mathfrak{F}) &= \frac{3 l \sqrt{2}}{4} \left\{ 2 \text{sech}^2 \xi/2 + 21 \text{sech}^4 \xi/2 - 24 \text{sech}^6 \xi/2 \right\} \{ 1 - \varphi \} \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \), \\
\mu (\xi, \mathfrak{F}) &= \sum_{m=0}^{\infty} \mu_m (\xi, \mathfrak{F}) = \left( 3 - 6 \text{tanh}^2 \xi/2 \right) - 6 \text{sech}^2 \frac{\xi}{2} \tanh \frac{\xi}{2} \{ 1 + \varphi \mathfrak{F} - \varphi \} + \frac{3}{2} \left\{ 2 \text{sech}^2 \xi/2 + 7 \text{sech}^4 \xi/2 \right\} \{ 1 - \varphi \} \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} + \cdots, \\
\nu (\xi, \mathfrak{F}) &= \sum_{m=0}^{\infty} \nu_m (\xi, \mathfrak{F}) = - \left( 3 l \sqrt{2} \text{tanh}^2 \xi/2 \right) + 3 l \sqrt{2} \text{sech}^2 \frac{\xi}{2} \tanh \frac{\xi}{2} \{ 1 + \varphi \mathfrak{F} - \varphi \} + \frac{3 l \sqrt{2}}{4} \left\{ 2 \text{sech}^2 \xi/2 + 7 \text{sech}^4 \xi/2 \right\} \{ 1 - \varphi \} \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} + \cdots.
\end{align*}
\]

We achieve the exact solution by putting \( \varphi = 1 \).
\[
\begin{align*}
\mu (\xi, \mathfrak{F}) &= 3 - 6 \text{tanh}^2 (\mathfrak{F} + \xi/2), \\
\nu (\xi, \mathfrak{F}) &= -3 l \sqrt{2} \text{tanh}^2 (\mathfrak{F} + \xi/2).
\end{align*}
\]

Analytical solution and exact solution are shown in Figures 1(a) and 1(b) at \( \varphi = 1 \) and \( -5 \leq \xi \leq 5 \). Figure 1(c) shows the absolute error and \( l_1 \) gives the solutions in various fractional-order graphs for \( \mu (\xi, \mathfrak{F}) \). The behavior of exact solution and analytical solution for \( \nu (\xi, \mathfrak{F}) \) are seen in Figures 2(a) and 2(b). From the figures, it is clear that the presented solution methodology is in good agreement with exact solution.

**Problem 2.** Consider the generalized coupled Hirota-Satsuma KdV system.

\[
\frac{\partial^3 \mu}{\partial \xi^3} + \frac{3 \mu}{\partial \xi} + \frac{3 \partial \nu}{\partial \xi} + (\nu \varphi), \\
\frac{\partial^3 \nu}{\partial \xi^3} = \frac{3 \partial \nu}{\partial \xi} - \frac{3 \partial \mu}{\partial \xi} (\nu \varphi), \\
\frac{\partial^3 \ell}{\partial \xi^3} = \frac{3 \partial \ell}{\partial \xi} - \frac{3 \partial \nu}{\partial \xi} (\nu \varphi), 0 < \varphi \leq 1,
\]

with an initial source
\[
\begin{align*}
\mu (\xi, 0) &= \frac{1}{3} + 2 \text{tanh}^3 \xi, \\
\nu (\xi, 0) &= \frac{3}{2} \text{tanh} \xi, \\
\ell (\xi, 0) &= \frac{8}{3} \text{tanh} \xi.
\end{align*}
\]
Figure 1: Exact solution, analytical solution, absolute error, and various fractional-order solutions for $\mu(\xi, 3)$ of Problem 1.

Figure 2: Exact solution and analytical solution for $v(\xi, 3)$ Problem 1.
On taking the Yang transform of (49), we get

\[
\frac{1}{1 + \varphi (\omega - 1)} \mathcal{Y} [\mu (\xi, \mathfrak{A})] - \omega \mu (\xi, 0) = \mathcal{Y} \left[ \frac{1}{2} \frac{\partial^3 \mu}{\partial \xi^3} - 3 \frac{\partial \mu}{\partial \xi} + 3 \frac{\partial}{\partial \xi} (v \mu) \right],
\]

\[
\frac{1}{1 + \varphi (\omega - 1)} \mathcal{Y} [\nu (\xi, \mathfrak{A})] - \omega \nu (\xi, 0) = \mathcal{Y} \left[ 3 \mu \frac{\partial \nu}{\partial \xi} - \frac{\partial^3 \nu}{\partial \xi^3} \right],
\]

\[
\frac{1}{1 + \varphi (\omega - 1)} \mathcal{Y} [\ell (\xi, \mathfrak{A})] - \omega \ell (\xi, 0) = \mathcal{Y} \left[ \frac{3}{2} \mu \frac{\partial \ell}{\partial \xi} - \frac{\partial^3 \ell}{\partial \xi^3} \right].
\]

When we use the Yang transform, we obtain

\[
\mu (\xi, \mathfrak{A}) = \frac{1}{3} + 2 \tanh \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left[ \frac{1}{2} \frac{\partial^3 \mu}{\partial \xi^3} - 3 \frac{\partial \mu}{\partial \xi} + 3 \frac{\partial}{\partial \xi} (v \mu) \right] \right],
\]

\[
\nu (\xi, \mathfrak{A}) = \tanh \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left[ 3 \mu \frac{\partial \nu}{\partial \xi} - \frac{\partial^3 \nu}{\partial \xi^3} \right] \right],
\]

\[
\ell (\xi, \mathfrak{A}) = \frac{8}{3} \tanh \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left[ \frac{3}{2} \mu \frac{\partial \ell}{\partial \xi} - \frac{\partial^3 \ell}{\partial \xi^3} \right] \right],
\]

Assume that the solutions \( \mu (\xi, \mathfrak{A}) \), \( \nu (\xi, \mathfrak{A}) \), and \( \ell (\xi, \mathfrak{A}) \) in the form of infinite series are given by

\[
\mu (\xi, \mathfrak{A}) = \sum_{m=0}^{\infty} \mu_m (\xi, \mathfrak{A}), \quad \nu (\xi, \mathfrak{A}) = \sum_{m=0}^{\infty} \nu_m (\xi, \mathfrak{A}), \quad \ell (\xi, \mathfrak{A}) = \sum_{m=0}^{\infty} \ell_m (\xi, \mathfrak{A}),
\]

where \( \mu \mu_\xi = \sum_{m=0}^{\infty} A_m \), \( (v) \xi = \sum_{m=0}^{\infty} B_m \), \( \mu \nu_\xi = \sum_{m=0}^{\infty} C_m \), \( \mu \ell_\xi = \sum_{m=0}^{\infty} D_m \) are the so-called Adomian polynomials that represent the nonlinear terms, and so equation (52) is rewritten as follows:

\[
\sum_{m=0}^{\infty} \mu_m (\xi, \mathfrak{A}) = \frac{1}{3} + 2 \tanh \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left[ \frac{1}{2} \frac{\partial^3 \mu}{\partial \xi^3} - 3 \frac{\partial \mu}{\partial \xi} + 3 \frac{\partial}{\partial \xi} (v \mu) \right] \right],
\]

\[
\sum_{m=0}^{\infty} \nu_m (\xi, \mathfrak{A}) = \tanh \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left[ 3 \sum_{m=0}^{\infty} C_m - \frac{\partial^3 \nu}{\partial \xi^3} \right] \right],
\]

\[
\sum_{m=0}^{\infty} \ell_m (\xi, \mathfrak{A}) = \frac{8}{3} \tanh \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left[ \sum_{m=0}^{\infty} D_m - \frac{\partial^3 \ell}{\partial \xi^3} \right] \right].
\]

The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (22),
\[ A_0 = \mu_0 \mu_0, \]
\[ A_1 = \mu_1 \mu_0 + \mu_0 \mu_1, \]
\[ A_2 = \mu_2 \mu_0 + \mu_1 \mu_1 + \mu_0 \mu_2, \]
\[ B_0 = \gamma_0 \ell_0 + E_0 \gamma_0, \]
\[ B_1 = \left( \gamma_0 \ell_0 + \gamma_1 \ell_1 \right) + \left( \ell_0 \gamma_0 + \ell_0 \gamma_1 \right), \]
\[ B_2 = \left( \gamma_0 \ell_0 + \gamma_1 \ell_1 + \gamma_2 \ell_2 \right) + \left( \ell_0 \gamma_0 + \ell_1 \gamma_1 + \ell_2 \gamma_2 \right). \] (55)
\[ C_0 = \mu_0 \gamma_0, \]
\[ C_1 = \mu_1 \gamma_0 + \mu_0 \gamma_1, \]
\[ C_2 = \mu_2 \gamma_0 + \mu_1 \gamma_1 + \mu_0 \gamma_2, \]
\[ D_0 = \mu_0 \ell_0, \]
\[ D_1 = \mu_1 \ell_0 + \mu_0 \ell_1, \]
\[ D_2 = \mu_2 \ell_0 + \mu_1 \ell_1 + \mu_0 \ell_2. \]

As a result, when comparing the two sides of equation (54), we get

\[ \mu_0 (\xi, \mathfrak{F}) = -\frac{1}{3} + 2 \tanh^2 \xi, \]
\[ \gamma_0 (\xi, \mathfrak{F}) = \tanh \xi, \] (56)
\[ \ell_0 (\xi, \mathfrak{F}) = \frac{8}{3} \tanh \xi. \]

For \( m = 0, \)
\[ \mu_1 (\xi, \mathfrak{F}) = 4 \text{sech}^2 \xi \tanh \xi \{1 + \varphi \mathfrak{F} - \varphi\}, \]
\[ \gamma_1 (\xi, \mathfrak{F}) = \text{sech}^2 \xi \{1 + \varphi \mathfrak{F} - \varphi\}, \] (57)
\[ \ell_1 (\xi, \mathfrak{F}) = \frac{8}{3} \text{sech}^2 \xi \{1 + \varphi \mathfrak{F} - \varphi\}. \]

For \( m = 1, \)

\[ \mu_2 (\xi, \mathfrak{F}) = 4 \text{sech}^3 \xi \{1 - 3 \tanh^2 \xi\} \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \right\}, \]
\[ \gamma_2 (\xi, \mathfrak{F}) = -\text{sech}^3 \xi \tanh \xi \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \right\}, \] (58)
\[ \ell_2 (\xi, \mathfrak{F}) = \frac{8}{3} \text{sech}^2 \xi \tanh \xi \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \right\}. \]

The approximate solution to the series is written as follows:

\[ \mu (\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \mu_m (\xi, \mathfrak{F}) = \mu_0 (\xi, \mathfrak{F}) + \mu_1 (\xi, \mathfrak{F}) + \mu_2 (\xi, \mathfrak{F}) + \cdots, \]
\[ \gamma (\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \gamma_m (\xi, \mathfrak{F}) = \gamma_0 (\xi, \mathfrak{F}) + \gamma_1 (\xi, \mathfrak{F}) + \gamma_2 (\xi, \mathfrak{F}) + \cdots, \]
\[ \ell (\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \ell_m (\xi, \mathfrak{F}) = \ell_0 (\xi, \mathfrak{F}) + \ell_1 (\xi, \mathfrak{F}) + \ell_2 (\xi, \mathfrak{F}) + \cdots, \]
\[ \mu (\xi, \mathfrak{F}) = -\frac{1}{3} + 2 \tanh^2 \xi + 4 \text{sech}^2 \xi \tanh \xi \{1 + \varphi \mathfrak{F} - \varphi\} + 4 \text{sech}^2 \xi \{1 - 3 \tanh^2 \xi\} \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 \right\} + \frac{\varphi^2 \mathfrak{F}^2}{2} + \cdots, \] (59)
\[ \gamma (\xi, \mathfrak{F}) = \tanh \xi + \text{sech}^2 \xi \{1 + \varphi \mathfrak{F} - \varphi\} \right. - \text{sech}^2 \xi \tanh \xi \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \right\} + \cdots, \]
\[ \ell (\xi, \mathfrak{F}) = \frac{8}{3} \tanh \xi + \frac{8}{3} \text{sech}^2 \xi \{1 + \varphi \mathfrak{F} - \varphi\} \right. - \frac{8}{3} \text{sech}^2 \xi \tanh \xi \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \right\} + \cdots. \]
We achieve the exact solution by putting \( \varphi = 1 \).

\[
\mu(\xi, \mathcal{A}) = \frac{1}{3} + 2\tanh^2(\mathcal{A} + \xi),
\]

\[
\nu(\xi, \mathcal{A}) = \tanh(\mathcal{A} + \xi), \tag{60}
\]

\[
\ell(\xi, \mathcal{A}) = \frac{8}{3}\tanh(\mathcal{A} + \xi).
\]

VITM analytical results are as follows.

For equation (49), we have the iteration formula as

\[
\mu_{m+1}(\xi, \mathcal{A}) = \mu_m(\xi, \mathcal{A}) - \gamma^{-1}\left[ (1 + \varphi(\omega - 1))\gamma\left\{ 1 \frac{\partial^2 \mu_m}{\partial \xi^2} - 3\mu_m \frac{\partial \nu_m}{\partial \xi} + 3 \frac{\partial}{\partial \xi}(\nu_m \ell_m) \right\} \right],
\]

\[
\nu_{m+1}(\xi, \mathcal{A}) = \nu_m(\xi, \mathcal{A}) - \gamma^{-1}\left[ (1 + \varphi(\omega - 1))\gamma\left\{ 3\mu_m \frac{\partial \nu_m}{\partial \xi} - \frac{\partial^3 \nu_m}{\partial \xi^3} \right\} \right], \tag{61}
\]

\[
\ell_{m+1}(\xi, \mathcal{A}) = \ell_m(\xi, \mathcal{A}) - \gamma^{-1}\left[ (1 + \varphi(\omega - 1))\gamma\left\{ 3\mu_m \frac{\partial \ell_m}{\partial \xi} - \frac{\partial^3 \ell_m}{\partial \xi^3} \right\} \right],
\]

where

\[
\mu_0(\xi, \mathcal{A}) = \frac{1}{3} + 2\tanh^2\xi,
\]

\[
\nu_0(\xi, \mathcal{A}) = \tanh\xi, \tag{62}
\]

\[
\ell_0(\xi, \mathcal{A}) = \frac{8}{3}\tanh\xi.
\]

\[
\mu_1(\xi, \mathcal{A}) = \mu_0(\xi, \mathcal{A}) - \gamma^{-1}\left[ (1 + \varphi(\omega - 1))\gamma\left\{ 1 \frac{\partial^2 \mu_0}{\partial \xi^2} - 3\mu_0 \frac{\partial \nu_0}{\partial \xi} + 3 \frac{\partial}{\partial \xi}(\nu_0 \ell_0) \right\} \right],
\]

\[
\mu_1(\xi, \mathcal{A}) = 4\sech^2\xi \tanh\{1 + \varphi(\mathcal{A} - \varphi)\},
\]

\[
\nu_1(\xi, \mathcal{A}) = \nu_0(\xi, \mathcal{A}) - \gamma^{-1}\left[ (1 + \varphi(\omega - 1))\gamma\left\{ 3\mu_0 \frac{\partial \nu_0}{\partial \xi} - \frac{\partial^3 \nu_0}{\partial \xi^3} \right\} \right],
\]

\[
\nu_1(\xi, \mathcal{A}) = \sech^2\xi \{1 + \varphi(\mathcal{A} - \varphi)\},
\]

\[
\ell_1(\xi, \mathcal{A}) = \ell_0(\xi, \mathcal{A}) - \gamma^{-1}\left[ (1 + \varphi(\omega - 1))\gamma\left\{ 3\mu_0 \frac{\partial \ell_0}{\partial \xi} - \frac{\partial^3 \ell_0}{\partial \xi^3} \right\} \right],
\]

\[
\ell_1(\xi, \mathcal{A}) = \frac{8}{3}\sech^2\xi \{1 + \varphi(\mathcal{A} - \varphi)\},
\]

\[
\mu_2(\xi, \mathcal{A}) = \mu_1(\xi, \mathcal{A}) - \gamma^{-1}\left[ (1 + \varphi(\omega - 1))\gamma\left\{ 1 \frac{\partial^2 \mu_1}{\partial \xi^2} - 3\mu_1 \frac{\partial \nu_1}{\partial \xi} + 3 \frac{\partial}{\partial \xi}(\nu_1 \ell_1) \right\} \right],
\]

\[
\mu_2(\xi, \mathcal{A}) = 4\sech^2\xi \left\{1 - 3\tanh^2\xi\right\} \left\{(1 - \varphi)2\varphi(\mathcal{A} + (1 - \varphi)^3 + \frac{\varphi^2\mathcal{A}^2}{2}\right\},
\]
We achieve the exact solution by putting \( \varphi = 1 \).

\[
\mu(\xi, 3) = \frac{1}{3} + 2\tanh^2(3 + \xi),
\]

\[
\nu(\xi, 3) = \tanh(3 + \xi),
\]

\[
\ell(\xi, 3) = \frac{8}{3}\tanh(3 + \xi).
\]

The analytical solution and exact solution for \( \mu(\xi, 3) \) of example 2 at \( \varphi = 1 \) and \(-5 \leq \xi \leq 5\) are shown in Figures 3(a) and 3(b), whereas Figures 3(c) and 3(d) show the absolute error and the solution at various fractional orders. The graphical behavior of exact solution and analytical solution for \( \nu(\xi, 3) \) is shown in Figures 4(a) and 4(b), while Figures 4(c) and 4(d) show the absolute error and the solution at different fractional orders. Figures 5(a) and 5(b) give the solution graph for \( \ell(\xi, 3) \). From the results of the figures, it is confirmed that our method solution converges quickly towards exact solution.

Problem 3. We consider the coupled Burgers equation in one dimension.

\[
\begin{align*}
\frac{CF \varphi \mu}{\partial \xi} &= \frac{\partial^2 \mu}{\partial \xi^2} + 2\mu \frac{\partial \mu}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu \nu), \\
\frac{CF \varphi \nu}{\partial \xi^2} &= \frac{\partial^2 \nu}{\partial \xi^2} + 2\nu \frac{\partial \nu}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu \nu), 0 < \varphi \leq 1,
\end{align*}
\]

with an initial source

\[
\mu(\xi, 0) = \cos \xi, \nu(\xi, 0) = \cos \xi.
\]

On taking the Yang transform of (65), we get

\[
\begin{align*}
\frac{1}{1 + \varphi(\omega - 1)} \mathcal{Y}[\mu(\xi, 3)] - \omega \mu(\xi, 0) &= \mathcal{Y} \left[ \frac{\partial^2 \mu}{\partial \xi^2} + 2\mu \frac{\partial \mu}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu \nu) \right], \\
\frac{1}{1 + \varphi(\omega - 1)} \mathcal{Y}[\nu(\xi, 3)] - \omega \nu(\xi, 0) &= \mathcal{Y} \left[ \frac{\partial^2 \nu}{\partial \xi^2} + 2\nu \frac{\partial \nu}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu \nu) \right].
\end{align*}
\]
Assume that the solutions \( \mu(\xi, \mathfrak{H}) \) and \( \nu(\xi, \mathfrak{H}) \) in the form of infinite series are given by

\[
\mu(\xi, \mathfrak{H}) = \sum_{m=0}^{\infty} \mu_m(\xi, \mathfrak{H}), \quad \nu(\xi, \mathfrak{H}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{H}),
\]

(69)

where \( \mu_m = \sum_{m=0}^{\infty} A_m, \quad (\mu \nu)_m = \sum_{m=0}^{\infty} B_m, \) and \( \nu \nu_m = \sum_{m=0}^{\infty} c_m \) are the so-called Adomian polynomials that represent nonlinear terms, and so equation (68) is rewritten as
The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (22).

\[
\sum_{m=0}^{\infty} \mu_m(\xi, \mathfrak{I}) = \cos \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left( \frac{\partial^2 \mu}{\partial \xi^2} + 2 \sum_{m=0}^{\infty} \mathcal{A}_m - \sum_{m=0}^{\infty} \mathcal{B}_m \right) \right].
\]

\[
\sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{I}) = \cos \xi + \mathcal{Y}^{-1} \left[ (1 + \varphi (\omega - 1)) \mathcal{Y} \left( \frac{\partial^2 \nu}{\partial \xi^2} + \sum_{m=0}^{\infty} \mathcal{C}_m - \sum_{m=0}^{\infty} \mathcal{R}_m \right) \right].
\]
We achieve the exact solution by putting $\varphi = 1$.

\[ (\xi, \mathfrak{F}) = \cos \xi; \]
\[ \nu_0 (\xi, \mathfrak{F}) = \cos \xi. \]  

For $m = 0$,
\[ \mu_1 (\xi, \mathfrak{F}) = -\cos \xi[1 + \varphi \mathfrak{F} - \varphi], \]
\[ \nu_1 (\xi, \mathfrak{F}) = -\cos \xi[1 + \varphi \mathfrak{F} - \varphi]. \]  

For $m = 1$,
\[ \mu_2 (\xi, \mathfrak{F}) = \cos \xi \left(1 - \varphi \right) 2 \varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2}, \]
\[ \nu_2 (\xi, \mathfrak{F}) = \cos \xi \left(1 - \varphi \right) 2 \varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2}. \]  

The approximate solution to the series is written as

\[
\begin{align*}
\mu (\xi, \mathfrak{F}) &= \sum_{m=0}^{\infty} \mu_m (\xi, \mathfrak{F}) = \mu_0 (\xi, \mathfrak{F}) + \mu_1 (\xi, \mathfrak{F}) + \mu_2 (\xi, \mathfrak{F}) + \cdots, \\
\nu (\xi, \mathfrak{F}) &= \sum_{m=0}^{\infty} \nu_m (\xi, \mathfrak{F}) = \nu_0 (\xi, \mathfrak{F}) + \nu_1 (\xi, \mathfrak{F}) + \nu_2 (\xi, \mathfrak{F}) + \cdots, \\
\mu (\xi, \mathfrak{F}) &= \cos \xi - \cos \xi[1 + \varphi \mathfrak{F} - \varphi] + \cos \xi \left(1 - \varphi \right) 2 \varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} + \cdots, \\
\nu (\xi, \mathfrak{F}) &= \cos \xi - \cos \xi[1 + \varphi \mathfrak{F} - \varphi] + \cos \xi \left(1 - \varphi \right) 2 \varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} + \cdots.
\end{align*}
\]
\[ \mu(\xi, \mathfrak{S}) = \cos\xi\left(1 - \mathfrak{S}^2 + \frac{\mathfrak{S}^4}{2} - \cdots\right), \]

\[ \nu(\xi, \mathfrak{S}) = \cos\xi\left(1 - \mathfrak{S}^2 + \frac{\mathfrak{S}^4}{2} - \cdots\right). \]

(76)

In a closed form, \( \mu(\xi, \mathfrak{S}) = \cos(\xi)\exp^{-\mathfrak{S}} \) and \( \nu(\xi, \mathfrak{S}) = \cos(\xi)\exp^{-\mathfrak{S}} \).

VITM analytical results as follows.

For equation (65), we have the iteration formula as

\[ \mu_{m+1}(\xi, \mathfrak{S}) = \mu_m(\xi, \mathfrak{S}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \mu_m}{\partial \xi^2} + 2\mu_0 \frac{\partial \mu_m}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_m \nu_m) \right\} \right], \]

(77)

\[ \nu_{m+1}(\xi, \mathfrak{S}) = \nu_m(\xi, \mathfrak{S}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \nu_m}{\partial \xi^2} + 2\nu_0 \frac{\partial \nu_m}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_m \nu_m) \right\} \right], \]

where

\[ \mu_0(\xi, \mathfrak{S}) = \cos\xi, \]

\[ \nu_0(\xi, \mathfrak{S}) = \cos\xi. \]

(78)

\[ \mu_1(\xi, \mathfrak{S}) = \mu_0(\xi, \mathfrak{S}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \mu_0}{\partial \xi^2} + 2\mu_0 \frac{\partial \mu_0}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_0 \nu_0) \right\} \right], \]

\[ \mu_1(\xi, \mathfrak{S}) = -\cos\xi[1 + \varphi \mathfrak{S} - \varphi], \]

\[ \nu_1(\xi, \mathfrak{S}) = \nu_0(\xi, \mathfrak{S}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \nu_0}{\partial \xi^2} + 2\nu_0 \frac{\partial \nu_0}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_0 \nu_0) \right\} \right], \]

\[ \nu_1(\xi, \mathfrak{S}) = -\cos\xi[1 + \varphi \mathfrak{S} - \varphi], \]

\[ \mu_2(\xi, \mathfrak{S}) = \mu_1(\xi, \mathfrak{S}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \mu_1}{\partial \xi^2} + 2\mu_1 \frac{\partial \mu_1}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_1 \nu_1) \right\} \right], \]

\[ \mu_2(\xi, \mathfrak{S}) = \cos\xi \left\{ (1 - \varphi)2\varphi \mathfrak{S} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{S}^2}{2} \right\}, \]

(79)

\[ \nu_2(\xi, \mathfrak{S}) = \nu_1(\xi, \mathfrak{S}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \nu_1}{\partial \xi^2} + 2\nu_1 \frac{\partial \nu_1}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_1 \nu_1) \right\} \right], \]

\[ \nu_1(\xi, \mathfrak{S}) = \cos\xi \left\{ (1 - \varphi)2\varphi \mathfrak{S} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{S}^2}{2} \right\}, \]

\[ \mu(\xi, \mathfrak{S}) = \cos\xi - \cos\xi[1 + \varphi \mathfrak{S} - \varphi] + \cos\xi \left\{ (1 - \varphi)2\varphi \mathfrak{S} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{S}^2}{2} \right\} + \cdots, \]

\[ \nu(\xi, \mathfrak{S}) = \cos\xi - \cos\xi[1 + \varphi \mathfrak{S} - \varphi] + \cos\xi \left\{ (1 - \varphi)2\varphi \mathfrak{S} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{S}^2}{2} \right\} + \cdots. \]

We achieve the exact solution by putting \( \varphi = 1 \).
\[
\mu(\xi, 3) = \cos \left(1 - 3 + \frac{3^2}{2} - \cdots\right),
\]
\[
\nu(\xi, 3) = \cos \left(1 - 3 + \frac{3^2}{2} - \cdots\right).
\]

In a closed form, \(\mu(\xi, 3) = \cos(\xi)\exp^{-3}\) and \(\nu(\xi, 3) = \cos(\xi)\exp^{-3}\).

In Figure 6, we display solution graphs 6(a) and 6(b) at \(\psi = 1\) for \(\mu(\xi, 3), \nu(\xi, 3)\) in the domain \(-5 \leq \xi \leq 5\) and Figures 6(c) and 6(d) show the solution graph of the absolute error and various fractional-order solutions. It is verified from figures and tables that our solution is closely related with the exact solution.

Problem 4. We consider the coupled Burgers equation in two dimensions.

\[
\frac{\partial \mu}{\partial \xi} = \frac{\partial^2 \mu}{\partial \xi^2} + \frac{\partial \mu}{\partial \psi} + 2\frac{\partial \mu}{\partial \xi} \frac{\partial \nu}{\partial \xi} (\mu\nu),
\]
\[
\frac{\partial \nu}{\partial \xi} = \frac{\partial^2 \nu}{\partial \xi^2} + \frac{\partial \nu}{\partial \psi} + 2\frac{\partial \nu}{\partial \xi} \frac{\partial \mu}{\partial \xi} (\mu\nu), 0 < \psi \leq 1,
\]
with an initial source

\[
\mu(\xi, \omega, 0) = \cos(\xi + \omega), \nu(\xi, \omega, 0) = \cos(\xi + \omega).
\]

On taking the Yang transform of (81), we get

\[
\frac{1}{(1 + \psi(\omega - 1))} \mathcal{Y}[\mu(\xi, 3)] - \omega \mu(\xi, 0) = \mathcal{Y} \left[ \frac{\partial^2 \mu}{\partial \xi^2} + \frac{\partial \mu}{\partial \psi} + 2\frac{\partial \mu}{\partial \xi} \frac{\partial \nu}{\partial \xi} (\mu\nu) \right],
\]
\[
\frac{1}{(1 + \psi(\omega - 1))} \mathcal{Y}[\nu(\xi, 3)] - \omega \nu(\xi, 0) = \mathcal{Y} \left[ \frac{\partial^2 \nu}{\partial \xi^2} + \frac{\partial \nu}{\partial \psi} + 2\frac{\partial \nu}{\partial \xi} \frac{\partial \mu}{\partial \xi} (\mu\nu) \right].
\]

We obtain (when we use the Yang transform).

Assume that the solutions \(\mu(\xi, \omega, 3)\) and \(\nu(\xi, \omega, 3)\) in the form of infinite series are given by

\[
\mu(\xi, \omega, 3) = \sum_{m=0}^{\infty} \mu_m(\xi, \omega, 3), \quad \nu(\xi, \omega, 3) = \sum_{m=0}^{\infty} \nu_m(\xi, \omega, 3),
\]

where \(\mu_m = \sum_{m=0}^{\infty} A_m, (\mu\nu)_m = \sum_{m=0}^{\infty} \mathcal{B}_m, \) and \(\nu\nu = \sum_{m=0}^{\infty} C_m\) are the so-called Adomian polynomials that represent the nonlinear terms, and so equation (84) is rewritten as

\[
\sum_{m=0}^{\infty} \mu_m(\xi, \omega, 3) = \cos(\xi + \omega) + \mathcal{Y}^{-1} \left[ (1 + \psi(\omega - 1)) \mathcal{Y} \left[ \frac{\partial^2 \mu}{\partial \xi^2} + \frac{\partial \mu}{\partial \psi} + 2\frac{\partial \mu}{\partial \xi} \frac{\partial \nu}{\partial \xi} (\mu\nu) \right] \right],
\]
\[
\sum_{m=0}^{\infty} \nu_m(\xi, \omega, 3) = \cos(\xi + \omega) + \mathcal{Y}^{-1} \left[ (1 + \psi(\omega - 1)) \mathcal{Y} \left[ \frac{\partial^2 \nu}{\partial \xi^2} + \frac{\partial \nu}{\partial \psi} + 2\frac{\partial \nu}{\partial \xi} \frac{\partial \mu}{\partial \xi} (\mu\nu) \right] \right].
\]

The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (22).
\[ A_0 = \mu_0 \mu_0, \]
\[ A_1 = \mu_1 \mu_0 + \mu_0 \mu_1, \]
\[ A_2 = \mu_2 \mu_0 + \mu_1 \mu_1 + \mu_0 \mu_2, \]
\[ B_0 = \mu_0 \gamma_0 + \gamma_0 \mu_0, \]
\[ B_1 = (\mu_0 \gamma_0 + \mu_1 \gamma_0) + (\gamma_1 \mu_0 + \gamma_0 \mu_1), \]
\[ B_2 = (\mu_2 \gamma_0 + \mu_1 \gamma_1 + \mu_0 \gamma_2) + (\gamma_1 \mu_0 + \gamma_0 \mu_1 + \mu_0 \gamma_0), \]
\[ C_0 = \gamma_0 \gamma_0, \]
\[ C_1 = \gamma_1 \gamma_0 + \gamma_0 \gamma_1, \]
\[ C_2 = \gamma_2 \gamma_0 + \gamma_1 \gamma_1 + \gamma_0 \gamma_2. \]

As a result, when comparing the two sides of equation (54), we get

\[ \mu_0 (\xi, \omega, \mathfrak{F}) = \cos (\xi + \omega), \]
\[ \nu_0 (\xi, \omega, \mathfrak{F}) = \cos (\xi + \omega). \] (88)

For \( m = 0, \)
\[ \mu_1 (\xi, \omega, \mathfrak{F}) = -2 \cos (\xi + \omega) \left[ 1 + \varphi \mathfrak{F} - \varphi \right], \]
\[ \nu_1 (\xi, \omega, \mathfrak{F}) = -2 \cos (\xi + \omega) \left[ 1 + \varphi \mathfrak{F} - \varphi \right]. \] (89)

For \( m = 1, \)
\[ \mu_2 (\xi, \omega, \mathfrak{F}) = 2 \cos (\xi + \omega) \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \right\}, \]
\[ \nu_2 (\xi, \omega, \mathfrak{F}) = 2 \cos (\xi + \omega) \left\{ (1 - \varphi)2\varphi \mathfrak{F} + (1 - \varphi)^2 + \frac{\varphi^2 \mathfrak{F}^2}{2} \right\}. \] (90)

Figure 6: Exact solution, analytical solution, absolute error, and various fractional-order solutions for \( \mu (\xi, \mathfrak{F}) \) and \( \nu (\xi, \mathfrak{F}) \) of Problem 3.
The approximate solution to the series is written as follows:

\[
\begin{align*}
\mu(\xi, \omega, \mathcal{F}) &= \sum_{m=0}^{\infty} \mu_m(\xi, \mathcal{F}) \\
&= \mu_0(\xi, \mathcal{F}) + \mu_1(\xi, \mathcal{F}) + \mu_2(\xi, \mathcal{F}) + \cdots, \\
\nu(\xi, \omega, \mathcal{F}) &= \sum_{m=0}^{\infty} \nu_m(\xi, \mathcal{F}) \\
&= \nu_0(\xi, \mathcal{F}) + \nu_1(\xi, \mathcal{F}) + \nu_2(\xi, \mathcal{F}) + \cdots, \\
\end{align*}
\]

(91)

We achieve the exact solution by putting \( \varphi = 1 \).

\[
\begin{align*}
\mu(\xi, \omega, \mathcal{F}) &= \cos(\xi + \omega) \left( 1 - 2\mathcal{F} + \frac{4\mathcal{F}^2}{2!} - \cdots \right), \\
\nu(\xi, \omega, \mathcal{F}) &= \cos(\xi + \omega) \left( 1 - 2\mathcal{F} + \frac{4\mathcal{F}^2}{2!} - \cdots \right). \\
\end{align*}
\]

(92)

In a closed form, \( \mu(\xi, \omega, \mathcal{F}) = \cos(\xi + \omega) \exp^{-2\mathcal{F}} \) and \( \nu(\xi, \omega, \mathcal{F}) = \cos(\xi + \omega) \exp^{-2\mathcal{F}} \). VITM analytical results as follows.

For equation (81), we have the iteration formula as

\[
\begin{align*}
\mu_{m+1}(\xi, \omega, \mathcal{F}) &= \mu_m(\xi, \mathcal{F}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \mu_m}{\partial \xi^2} + \frac{\partial^2 \mu_m}{\partial \omega^2} + 2\mu_m \frac{\partial \mu_m}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_m \nu_m) \right\} \right], \\
\nu_{m+1}(\xi, \omega, \mathcal{F}) &= \nu_m(\xi, \mathcal{F}) - \mathcal{Y}^{-1} \left[ (1 + \varphi(\omega - 1)) \mathcal{Y} \left\{ \frac{\partial^2 \nu_m}{\partial \xi^2} + \frac{\partial^2 \nu_m}{\partial \omega^2} + 2\nu_m \frac{\partial \nu_m}{\partial \xi} - \frac{\partial}{\partial \xi} (\mu_m \nu_m) \right\} \right], \\
\end{align*}
\]

(93)

where

\[
\begin{align*}
\mu_0(\xi, \omega, \mathcal{F}) &= \cos(\xi + \omega), \\
\nu_0(\xi, \omega, \mathcal{F}) &= \cos(\xi + \omega). \\
\end{align*}
\]

(94)

For \( m = 0, 1, 2, \cdots \).
Figure 7: Exact solution, analytical solution, absolute error, and various fractional-order solutions for $\mu(\xi, \mathcal{I})$ and $\nu(\xi, \mathcal{I})$ of Problem 4.
\[
\mu_1(\xi, \omega, \mathfrak{A}) = \mu_0(\xi, \mathfrak{A}) - \mathcal{Y}^{-1}\left[(1 + \wp(\omega - 1))\mathcal{Y}\left\{\frac{\partial^2 \mu_0}{\partial \xi^2} + \frac{\partial^2 \mu_0}{\partial \omega^2} + 2\mu_0 \frac{\partial \mu_0}{\partial \xi} (\mu_0 \nu_0)\right\}\right],
\]

\[
\mu_1(\xi, \omega, \mathfrak{A}) = -2\cos(\xi + \omega)[1 + \wp\mathfrak{A} - \wp],
\]

\[
\nu_1(\xi, \omega, \mathfrak{A}) = \nu_0(\xi, \mathfrak{A}) - \mathcal{Y}^{-1}\left[(1 + \wp(\omega - 1))\mathcal{Y}\left\{\frac{\partial^2 \nu_0}{\partial \xi^2} + \frac{\partial^2 \nu_0}{\partial \omega^2} + 2\nu_0 \frac{\partial \nu_0}{\partial \xi} (\mu_0 \nu_0)\right\}\right],
\]

\[
\nu_1(\xi, \omega, \mathfrak{A}) = -2\cos(\xi + \omega)[1 + \wp\mathfrak{A} - \wp],
\]

\[
\mu_2(\xi, \omega, \mathfrak{A}) = \mu_1(\xi, \mathfrak{A}) - \mathcal{Y}^{-1}\left[(1 + \wp(\omega - 1))\mathcal{Y}\left\{\frac{\partial^2 \mu_1}{\partial \xi^2} + \frac{\partial^2 \mu_1}{\partial \omega^2} + 2\mu_1 \frac{\partial \mu_1}{\partial \xi} (\mu_1 \nu_1)\right\}\right],
\]

\[
\mu_2(\xi, \omega, \mathfrak{A}) = 2\cos(\xi + \omega) \left\{(1 - \wp)2\wp\mathfrak{A} + (1 - \wp)^2 + \frac{\wp^2 \mathfrak{A}^2}{2}\right\},
\]

\[
\nu_2(\xi, \omega, \mathfrak{A}) = \nu_1(\xi, \mathfrak{A}) - \mathcal{Y}^{-1}\left[(1 + \wp(\omega - 1))\mathcal{Y}\left\{\frac{\partial^2 \nu_1}{\partial \xi^2} + \frac{\partial^2 \nu_1}{\partial \omega^2} + 2\nu_1 \frac{\partial \nu_1}{\partial \xi} (\mu_1 \nu_1)\right\}\right],
\]

\[
\nu_2(\xi, \omega, \mathfrak{A}) = 2\cos(\xi + \omega) \left\{(1 - \wp)2\wp\mathfrak{A} + (1 - \wp)^2 + \frac{\wp^2 \mathfrak{A}^2}{2}\right\},
\]

\[
\mu(\xi, \omega, \mathfrak{A}) = \cos(\xi + \omega) - 2\cos(\xi + \omega)[1 + \wp\mathfrak{A} - \wp] + 2\cos(\xi + \omega) \left\{(1 - \wp)2\wp\mathfrak{A} + (1 - \wp)^2 + \frac{\wp^2 \mathfrak{A}^2}{2}\right\} - \cdots,
\]

\[
\nu(\xi, \omega, \mathfrak{A}) = \cos(\xi + \omega) - 2\cos(\xi + \omega)[1 + \wp\mathfrak{A} - \wp] + 2\cos(\xi + \omega) \left\{(1 - \wp)2\wp\mathfrak{A} + (1 - \wp)^2 + \frac{\wp^2 \mathfrak{A}^2}{2}\right\} - \cdots.
\]

We achieve the exact solution by putting \(\wp = 1\).

\[
\mu(\xi, \omega, \mathfrak{A}) = \cos(\xi + \omega) \left\{1 - 2\mathfrak{A} + \frac{4\mathfrak{A}^2}{2!} - \cdots\right\},
\]

\[
\nu(\xi, \omega, \mathfrak{A}) = \cos(\xi + \omega) \left\{1 - 2\mathfrak{A} + \frac{4\mathfrak{A}^2}{2!} - \cdots\right\}.
\]

In a closed form, \(\mu(\xi, \omega, \mathfrak{A}) = \cos(\xi + \omega)\exp^{-2\mathfrak{A}}\) and \(\nu(\xi, \omega, \mathfrak{A}) = \cos(\xi + \omega)\exp^{-2\mathfrak{A}}\). In Figure 7, Figures 7(a) and 7(b) demonstrate the layout of exact solution and analytical solution, whereas the graphical view for various fractional orders is demonstrated with the help of Figures 7(c) and 7(d). Finally, it is clear from the figures and tables that the proposed methods have sufficient degree of accuracy and quick convergence towards the exact solution.

6. Conclusion

YDM and VITM were used for solving the coupled nonlinear partial differential equations. Comparing the results of these methods with the exact solution shows that the proposed methods are extremely simple and easy to handle nonlinear terms. The obtained results converge quickly in the form of series towards the exact solution. Four nonlinear systems are solved, which shows that solutions are in strong agreement with the exact solution with suggested techniques. It is confirmed that the proposed methods need less computational work, which shows fast convergence. Furthermore, YDM and VITM are very effective and efficient for finding approximate analytic solutions for a wide range of real-world problems arising in engineering and science.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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