Research Article

Stability Analysis of Stochastic Differential Equation Driven by G-Brownian Motion under Non-Lipschitz Condition

Li Ma 1,2 and Yujing Li 1,2

1Key Laboratory of the Ministry of Education, Hainan Normal University, Hainan 570203, China
2Department of Mathematics and Statistics, Hainan Normal University, Hainan 570203, China

Correspondence should be addressed to Yujing Li; lyj1622141477@163.com

Received 4 April 2022; Accepted 11 July 2022; Published 1 September 2022

Academic Editor: Zhiguo Yan

Copyright © 2022 Li Ma and Yujing Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is devoted to studying the p-th moment exponential stability for a class of stochastic differential equation (SDE) driven by G-Brownian motion under non-Lipschitz condition. The delays considered in this paper are time-varying delays \( \tau_i(t) (1 \leq i \leq 3) \). Since the coefficients are non-Lipschitz, the normal enlargement on the coefficients is not available and the Gronwall inequality is not suitable in this case. By Biham inequality and Itô integral formula, it is pointed out that there exists a constant \( \tau^* \) such that the p-th moment exponential stability holds if the time-varying delays are smaller than \( \tau^* \).

1. Introduction

In recent years, much effort has been made to develop the theory of sublinear expectations connected with the volatility uncertainty and the so-called G-Brownian motion, which is an efficient way to incorporate the unknown volatility into financial model. G-Brownian motion has a very rich and interesting new structure which non-trivially generalizes the classical one. Stochastic dynamic equations based on G-Brownian motion have been studied by several authors [1–5]. Among them, the existence-uniqueness, stability, moment estimations, continuity, and differentiability of solution with respect to the initial value were studied in detail. In [6], Zhu et al. got the p-th moment exponential stability for a class of stochastic delay systems driven by G-Brownian motion under Lipschitz condition and linear growth condition. In [7], Gao et al. got the almost sure exponential stability of stochastic differential delay equations driven by Brownian motion under Lipschitz condition and linear growth condition.

Let \( p > 0 \). In this paper, we will prove the p-th moment exponential stability problem of solutions for a class of stochastic delay non-linear systems driven by G-Brownian motion under non-Lipschitz condition. Throughout this paper, the following notations will be used. Let \( \mathbb{R}^d \) be the \( d \)-dimensional Euclidean space and \( |x| \) be the Euclidean norm of a vector \( x \). Let \( R = (−∞, +∞) \) and \( R_+ = [0, +∞) \). Let \( \{F_t\}_{t \geq 0} \) be a filtration generated by G-Brownian motion \( \{B(t), t \geq 0\} \). Let \( \tau > 0 \) and \( C([-\tau, 0]; \mathbb{R}^d) \) denote the family of continuous functions \( \phi \) from \( [-\tau, 0] \) to \( \mathbb{R}^d \) with the uniform norm \( \|\phi\|_\infty = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)| \). Let \( E \) be the G-expectation. For \( t \geq 0 \) and \( p > 0 \), denote by \( L^p_F(\Omega; C([-\tau, 0]; \mathbb{R}^d)) \) the family of all \( F_t \)-measurable \( \mathbb{R}^d \)-valued stochastic variables \( \xi \) such that \( E|\xi|^p < \infty \) and denote by \( L^p_F(\Omega; C([-\tau, 0]; \mathbb{R}^d)) \) the family of all \( F_t \)-measurable, \( C([-\tau, 0]; \mathbb{R}^d) \)-valued stochastic variables \( \xi = \{\xi(\theta) = -\tau \leq \theta \leq 0\} \) such that \( \sup_{-\tau \leq \theta \leq 0} E|\xi(\theta)|^p < \infty \). Define \( M^p_G(\mathbb{R}^d) = \{\xi(\omega) = \sum_{j=1}^{N} \xi_j(\omega) \eta^j(t_{j,i}) = \xi_j \in L^p_F(\Omega; \mathbb{R}^d), -\tau = t_0 < t_1 < \cdots < t_N = T, \forall T \geq 0\} \), let \( M^p_G([-\tau, 0]; \mathbb{R}^d) \) be the composition of \( M^p_G(\mathbb{R}^d) \) under the norm \( \|\xi\|_{M^p_G([-\tau, T]; \mathbb{R}^d)} = (1/T \int_{-\tau}^{T} E|\xi(\theta)|^p d\theta)^{1/p} \). Let \( C_{b, lip}(\mathbb{R}^d) \) be the real-valued space of bounded and Lipschitz continuous functions. Let \( \mathbb{R}_+^d = \mathbb{R}^d - [0] \) and \( V \) be a set of all Borel measures on \( (\mathbb{R}_+^d, B_0(\mathbb{R}_+^d)) \). Let \( G_T(\mathbb{R}_+^d \times \mathbb{R}_+^d; \mathbb{R}_+\mathbb{R}_+^d) \) denote the family of all non-negative functions \( V(y, t) \) on \( \mathbb{R}_+^d \times \mathbb{R}_+ \), which are continuously twice differentiable in \( y \) and once differentiable in \( t \).

In this paper, we will discuss the following stochastic delay non-linear system driven by G-Brownian motion:
\[ dx(t) = b(t, x(t - \tau_1(t)))dt + h(t, x(t - \tau_2(t)))d\langle B \rangle(t) + \sigma(t, x(t - \tau_3(t)))dB(t), \] 

(1)

for \( t \geq t_0 \geq 0 \) with the initial data \( x(t_0 + \theta) = \xi(\theta) \) for \(-\tau \leq \theta \leq 0\), where \( \xi \in L^2_{\mathbb{F}^t} (\Omega; C([-\tau, 0]; \mathbb{R}^d)) \), \( b, h, \sigma \in M^2_{\mathbb{R}^d}(\Omega, [-\tau, T], \mathbb{R}^d) \).

\( \langle B \rangle(t) \) is the quadratic variation process of the G-Brownian motion \( B(\cdot) \) with \( G(a) = 1/2E[aB(1)^2] = 1/2(\sigma^* a^* + \sigma a^*) \), for \( a \in \mathbb{R} \), where \( a^* = \max\{a, 0\}, a^- = \max\{-a, 0\}, \sigma^2 = E[B(1)^2], \overline{a}^2 = -E[B(1)^2], \tau_1(t), \tau_2(t), \tau_3(t) \) are time delays satisfying \( 0 \leq \tau_1(t) \leq \bar{\tau}_1, 0 \leq \tau_2(t) \leq \tau_2, 0 \leq \tau_3(t) \leq \tau_3 \), \( \tau = \max\{\tau_1, \tau_2, \tau_3\} \).

We will study the stability of system 1 under the following conditions.

**Assumption 1.** Let \( \rho > 0 \). There exists a concave function \( \rho(x) : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that the following conditions hold.

1. \((H_1)\) \( \rho(x) \) is a continuous non-decreasing function satisfying \( \rho(0^+) = 0 \) and \( \int_0^1 1/r + \rho(r)dr = +\infty \).

2. \((H_2)\) There is a constant \( C > 0 \) such that for all \( t \geq 0, x, y \in \mathbb{R}^d \),

\[
|b(t, x) - b(t, y)|^p + |h(t, x) - h(t, y)|^p + |\sigma(t, x) - \sigma(t, y)|^p \leq Cp(|x - y|^p).
\]

3. \((H_3)\) \( \rho(t_0) = 0, h(t_0) = 0, \sigma(t_0) = 0 \).

**Remark 1.** In fact, under conditions \((H_1)-(H_2)\), we have

\[
\lim_{r \to 0^+} [r^p + \rho(r)] = 0, +\infty = \int_0^1 1/r + \rho(r)dr \leq \int_0^1 1/p(r)dr;
\]

therefore, \( \int_0^1 1/p(r)dr = +\infty \).

**Remark 2.** \((H_1)-(H_3)\) are valid if \( p = 1, \rho(x) = \sqrt{x} \), and \( \sigma(t, x) = h(t, x) = b(t, x) \equiv \sqrt{x} \) on \([-\infty, \infty]\). By 8, Example 1.12, for all \( x, y \geq 0, \) we have \( |\sqrt{x} - \sqrt{y}|/\sqrt{|x - y|} \leq C \). Hence, \( \sigma, b, h \) are not Lipschitz continuous but 1/2-Hölder continuous.

\[
\mathbb{E}|x(t)|^p \leq G_1^{-1}\left(G_1^2 \mathbb{E}\|x\|^p \right) + A(p, t - t_0),
\]

(3)

\[
\mathbb{E}\left( \sup_{0 \leq s \leq \tau} |x(t + s) - x(t)|^p \right) \leq M_1(p, \tau) \rho \left[ G_1^{-1}\left(G_1^2 \mathbb{E}\|x\|^p \right) + A(p, t + \tau - t_0) \right],
\]

(4)

\[
\mathbb{E}\left( \sup_{t_0 \leq s \leq t} |x(s)|^p \right) \leq G_2^{-1}\left(G_2^2 \mathbb{E}\|x\|^p \right) + A_2(p, t - t_0).
\]

(5)

(2) For \( 0 < p < 2, \)

\[
\mathbb{E}|x(t)|^p \leq \left[ G_1^{-1}\left(G_1^2 \mathbb{E}\|x\|^p \right) + A(2, t - t_0) \right]^{p/2},
\]

(6)

\[
\mathbb{E}\left( \sup_{0 \leq s \leq \tau} |x(t + s) - x(t)|^p \right) \leq \left[ M_1(2, \tau) \cdot \rho \left[ G_1^{-1}\left(G_1^2 \mathbb{E}\|x\|^p \right) + A(2, t + \tau - t_0) \right] \right]^{p/2},
\]

(7)

**2. Main Results and Proofs**

It is known from [9] that (1) has a unique solution under the conditions of \((H_1)-(H_2)\). In what follows, we present an important lemma to give some moment estimations on the solution \( x(t) \).

**Lemma 1.** Let Assumption 1 hold. For any initial data \( \xi \in L^2_{\mathbb{F}^t} (\Omega; C([-\tau, 0]; \mathbb{R}^d)) \), let \( x(t; t_0, \xi) = x(t) \) be the solution of (1). Then, for all \( t \geq t_0 \), the following holds.

(1) For \( p \geq 2, \)

(2) For \( 0 < p < 2, \)

(3) For \( p < 0, \)

(4) For \( \rho > 0, \)

(5) For \( \rho < 0, \)
where $G_1(r) = \int_1^r 1/s + p(s)\,ds$, $G_2(r) = \int_1^r 1/\rho(s)\,ds$, for $r > 0$, and $G_1^{-1}$ is the inverse function of $G_1$. $C_2(p) = (p - 1)/2\rho(p - 1)\,ds$, $C_1(p) = C_2(1 + p \, \bar{\sigma}^2)$, $A(p, t - t_0) = \max\{C_1(p), C_2(p)\} \cdot (t - t_0)$, $M_1(p, \tau) = 3\rho^{-1}\,C_1\rho \cdot (1 + \bar{\sigma}^2 \tau + \bar{\sigma}^2 \tau (p^2/2)(p - 1)^{p/2})$.

**Proof.** (1) By G-Itô formula, for any $u > t_0$, we have

\[
|x(u)|^p - |x(t_0)|^p = \mathbf{E}\left(\int_{t_0}^u |x(s)|^{p-2} x^T(s)b(s, x(s - \tau_1(s)))\,ds + \int_{t_0}^u p|x(s)|^{p-2}x(s)\sigma(s, x(s - \tau_3(s)))\,dB(s) \right) \\
+ \int_{t_0}^u p|x(s)|^{p-2}x^T(s)h(s, x(s - \tau_2(s)))\,d\langle B \rangle(s) + \frac{p}{2} \int_{t_0}^u |x(s)|^{p-2}\sigma(s, x(s - \tau_3(s)))^2 \,d\langle B \rangle(s) \\
+ \frac{1}{2} p(p - 2) \int_{t_0}^u |x(s)|^{p-4} |x(s)\sigma(s, x(s - \tau_3(s)))|^2 \,d\langle B \rangle(s) = I_1 + I_2 + I_3 + I_4 + I_5.
\]

By Young inequality, $(H_1)$–$(H_3)$, Jensen inequality, and 6, Lemma 2.4(iii),

\[
\mathbf{E}
\left| I_1 \right| \leq p \mathbf{E}
\left( \int_{t_0}^u |x(s)|^{p-1} |x(s)| \,ds \right) \leq p \mathbf{E}
\left( \int_{t_0}^u \frac{P - 1}{P} |x(s)|^{p-1} \,ds \right) \leq (p - 1) \int_{t_0}^u \mathbf{E}
\left| x(s) \right|^p \,ds + C \int_{t_0}^u \mathbf{E}
\left| \sigma(s, x(s - \tau_1(s))) \right|^p \,ds.
\]

Similarly, by Young inequality, $(H_1)$–$(H_3)$, and 6, Lemma 2.4(iii),

\[
\mathbf{E}
\left| I_3 \right| \leq p \mathbf{E}
\left( \int_{t_0}^u |x(s)|^{p-1} |h(s, x(s - \tau_2(s)))| \,ds \right) \leq p \mathbf{E}
\left( \int_{t_0}^u \frac{P - 1}{P} |x(s)|^{p-1} \,ds \right) \leq (p - 1) \mathbf{E}
\left( \int_{t_0}^u |x(s)|^p \,ds \right) + C \int_{t_0}^u \mathbf{E}
\left( |x(s - \tau_2(s))|^p \right) \,ds,
\]

\[
\mathbf{E}
\left| I_4 \right| \leq \frac{1}{2} p(p - 2) \mathbf{E}
\left( \int_{t_0}^u |x(s)|^{p-2} \sigma(s, x(s - \tau_3(s)))^2 \,d\langle B \rangle(s) \right) \leq \frac{1}{2} p(p - 2) \mathbf{E}
\left( \int_{t_0}^u \left( \frac{P - 2}{P} |x(s)|^p + \frac{2}{P} |\sigma(s, x(s - \tau_3(s)))|^p \right) \,ds \right) \leq \frac{1}{2} (p - 2) \mathbf{E}
\left( \int_{t_0}^u \mathbf{E}
\left| x(s - \tau_3(s)) \right|^p \,ds \right).
\]
\[
\hat{E} |x(u)|^p - \hat{E} |x(t_0)|^p \leq \left[ (p - 1) + \frac{1}{2} \sigma(p - 1) \sigma \right] \int_{t_0}^u \hat{E} |x(s)|^p ds + C \int_{t_0}^1 \rho \left( \hat{E} |x(t - \tau_1(s))|^p \right) ds \\
+ C \sigma^2 \int_{t_0}^u \rho \left( \hat{E} |x(t - \tau_2(s))|^p \right) ds + C(p - 1) \sigma^2 \int_{t_0}^u \rho \left( \hat{E} |x(t - \tau_3(s))|^p \right) ds,
\]

and

\[
\sup_{t_0 \leq t \leq u \leq t_0 + \tau G(t)} \hat{E} |x(u)|^p \leq \sup_{t_0 \leq t \leq u \leq t_0 + \tau G(t)} \hat{E} |x(t_0)|^p + \max \{ C_1(p), C_2(p) \} \int_{t_0}^u \left[ K \left( \sup_{t_0 - \tau \leq s \leq t} \hat{E} |x(u)|^p \right) \right] ds,
\]

where \( C_1(p) = (p - 1) + 1/2p(p - 1)\sigma^2, \quad C_2(p) = C[1 + p\sigma^2]. \) Let \( K(r) = r + \rho(r); \) then,

\[
\sup_{t_0 \leq t \leq u \leq t_0 + \tau G(t)} \hat{E} |x(u)|^p \leq \sup_{t_0 \leq t \leq u \leq t_0 + \tau G(t)} \hat{E} |x(u)|^p + \hat{E} |\xi|^p \leq 2\hat{E} |\xi|^p + \max \{ C_1(p), C_2(p) \} \int_{t_0}^u \left[ K \left( \sup_{t_0 - \tau \leq s \leq t} \hat{E} |x(u)|^p \right) \right] ds.
\]

By \((H_1), K: R_+ \rightarrow R,\) is a continuous non-decreasing function such that \( K(t) > 0 \) for all \( t > 0. \) Define \( G_1(r) = \int_r^\infty 1/K(s) ds \) on \( r > 0) \) and denote its inverse function by \( G_1^{-1}. \) By \((H_1),\) one sees that \( \lim_{t \to 0} G_1(e) = -\infty \) and \( G_1(\infty) = +\infty, \) that is, \( \text{Dom}(G_1^{-1}) = (-\infty, +\infty). \) So, by 11, Theorem 2.3 (Bihari’s inequality) or 12, Theorem D,

\[
|x(t + u) - x(t)|^p \leq 3^{p-1} \left[ \int_t^{t+u} b(s, x(s - \tau_1(s))) ds \right]^p + 3^{p-1} \left[ \int_t^{t+u} h(s, x(s - \tau_2(s))) d\langle B \rangle(s) \right]^p + 3^{p-1} \left[ \int_t^{t+u} \sigma(s, x(s - \tau_3(s))) ds \right]^p = I_1 + I_2 + I_3.
\]

By Hölder inequality,

\[
\hat{E} \left( \sup_{\tau \geq t \geq t} I_1 \right) \leq 3^{p-1} \tau^{p-1} \hat{E} \left( \int_t^{t+\tau} |b(s, x(s - \tau_1(s)))|^p ds \right) \leq C3^{p-1} \tau^{p-1} \int_t^{t+\tau} \rho \left( \hat{E} |x(s - \tau_1(s))|^p \right) ds \leq C3^{p-1} \tau^{p-1} \sup_{t_0 - \tau \leq s \leq t} \rho \left( \hat{E} |x(s)|^p \right).
\]
By 6, Lemma 2.5,

\[
\hat{E}\left(\sup_{0 \leq s \leq T} I_2\right) = 3^{P-1} \hat{E}\left(\sup_{0 \leq s \leq T} \left| \int_t^{t+\tau} \sigma(s, x(t_1(s))) dB(s) \right|^P\right)
\leq 3^{P-1} \sigma^2 \tau^P \hat{E}\left(\int_t^{t+\tau} |h| s, x(t_2(s))) ds\right)^P
\leq 3^{P-1} \sigma^2 \tau^P \int_t^{t+\tau} \rho\left(\hat{E}|x(s-t_1)|^P\right) ds
\leq 3^{P-1} \sigma^2 \tau^P \int_t^{t+\tau} \rho\left(\hat{E}|x(s-t_2)|^P\right) ds
\leq 3^{P-1} \sigma^2 \tau^P \int_t^{t+\tau} \rho\left(\hat{E}|x(s-t_3)|^P\right) ds
\leq 3^{P-1} \sigma^2 \tau^P \int_t^{t+\tau} \rho\left(\hat{E}|x(u)|^P\right).
\]

(20)

By 6, Lemma 2.6,

\[
\hat{E}\left(\sup_{0 \leq s \leq T} I_3\right) = 3^{P-1} \hat{E}\left(\sup_{0 \leq s \leq T} \int_t^{t+\tau} \sigma(s, x(t_1(s))) dB(s) \right|^P\right)
\leq 3^{P-1} \sigma^2 \tau^P \left(\frac{p^3}{2(p-1)}\right) \hat{E}\left(\int_t^{t+\tau} |\sigma(s, x(t_1(s)))|^P ds\right)
\leq C3^{P-1} \sigma^2 \tau^P \left(\frac{p^3}{2(p-1)}\right) \int_t^{t+\tau} \rho\left(\hat{E}|x(s-t_1)|^P\right) ds
\leq C3^{P-1} \sigma^2 \tau^P \int_t^{t+\tau} \rho\left(\hat{E}|x(s-t_2)|^P\right) ds
\leq C3^{P-1} \sigma^2 \tau^P \int_t^{t+\tau} \rho\left(\hat{E}|x(s-t_3)|^P\right) ds
\leq C3^{P-1} \sigma^2 \tau^P \int_t^{t+\tau} \rho\left(\hat{E}|x(u)|^P\right).
\]

(21)

Hence, by (3) and (17),

\[
\hat{E}\left(\sup_{0 \leq s \leq T} |x(t+u) - x(t)|^P\right) \leq \hat{E}\left(\sup_{0 \leq s \leq T} I_1\right) + \hat{E}\left(\sup_{0 \leq s \leq T} I_2\right) + \hat{E}\left(\sup_{0 \leq s \leq T} I_3\right) \leq M_1(p, \tau) \sup_{t_2 - \tau \leq s \leq t_2 + \tau} \rho\left(\hat{E}|x(u)|^P\right)
\leq M_1(p, \tau) \rho\left(\sup_{t_2 - \tau \leq s \leq t_2 + \tau} \hat{E}|x(u)|^P\right) \leq M_1(p, \tau) \rho\left[G_1^{-1}\left(G_1\left(2\hat{E}|x|^P\right) + A(p, t + \tau - t_2)\right)\right],
\]

(22)

where \( M_1(p, \tau) = 3^{P-1} C \tau^{P-1}\left[1 + \tau^2 \tau + \tau^2 \tau (p^3/2) (p-1)^{P/2}\right] \), so inequality (4) is proved.

Finally, we will prove (5). Similar to the proof of (3), it is easy to get that

\[
\hat{E}\left(\sup_{t_2 \geq s \geq t_0} |x(u)|^P\right) \leq 4^{P-1} \hat{E}|x(t_0)|^P + 4^{P-1} C (t-t_0) \int_{t_0}^{t} \rho\left(\hat{E}|x(s-t_1)|^P\right) ds
\]

\[
+ 4^{P-1} C \tau^P (t-t_0)^P \int_{t_0}^{t} \rho\left(\hat{E}|x(s-t_2)|^P\right) ds
\]

\[
+ 4^{P-1} C \tau^P (t-t_0)^P \left(\frac{p^3}{2(p-1)}\right) \int_{t_0}^{t} \rho\left(\hat{E}|x(s-t_3)|^P\right) ds
\]

\[
\leq 4^{P-1} \hat{E}|x(t_0)|^P + M_2(p, t-t_0) \int_{t_0}^{t} \rho\left(\sup_{t_2 - \tau \leq s \leq t_2 + \tau} |x(r)|^P\right) ds,
\]

(23)
where \( M_2(p, t) = 4p^{-1}Ct^{-1}[1 + \sigma^2p + \sigma^2p^2t/(p - 1)^2p^2] \). Define \( G_2(r) = \int_1^r 1/p(s)ds \), on \( r > 0 \), and let \( G_2^{-1} \) be the inverse function of \( G_2 \). By \((H_1)\), one sees that
\[
\sup_{t, y \in \mathbb{R}^d} \hat{E}[x(t)]^p \leq G_2^{-1}\left( G_2\left( 4p^{-1}\hat{E}[\xi]^p \right) + M_2(p, t - t_0) \cdot (t - t_0) \right) = G_2^{-1}\left( G_2\left( 4p^{-1}\hat{E}[\xi]^p \right) + A_2(p, t - t_0) \right).
\]
where \( A_2(p, t - t_0) = M_2(p, t - t_0) \cdot (t - t_0) \).

(2) Firstly assume that \( \xi \in \mathcal{C}([-\tau, 0]; \mathbb{R}^d) \). Notice \( 0 < p < 2 \) and by H"older inequality,
\[
\hat{E}[x(t)]^p \leq \left[ \hat{E}[x(t)]^{p/2} \right]^{p/2} \leq \left[ G_1^{-1}\left( G_1\left( 2\hat{E}[\xi]^p \right) + A(2, t - t_0) \right) \right]^{p/2}.
\]
Next, for any \( \xi \in L^p_p(\Omega; \mathcal{C}([-\tau, 0] \times \mathbb{R}^d)) \), it follows from the property of conditional expectation that for \( t \geq t_0 \),
\[
\hat{E}[x(t)]^p \leq \hat{E}\left( \hat{E}[x(t)]^p | F_{t_0} \right).
\]

Consider the following equation:
\[
dy(t) = b(t, y(t))dt + h(t, y(t))dB(t) + \sigma(t, y(t))dB(t),
\]
where the initial data \( y(t_0) = y_0 \in L^p_p(\Omega; \mathbb{R}^d) \) and \( t \geq t_0 \). Denote the solution of (27) by \( y(t; t_0, y_0) \). In particular, \( y(t; t_0 + \tau, x(t_0 + \tau)) \) means the solution of (27) with initial data \( x(t_0 + \tau) \) and initial time \( t_0 + \tau \).

\[\text{Definition 1 (see [13])}. \quad \text{For each} \ V \in C^1\left( \mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+ \right), \text{define an operator} \ L \text{ from} \ \mathbb{R}^d \times \mathbb{R}_+ \text{to} \ \mathbb{R} \text{ by}
\]
\[
LV(t, y) = V_t(t, y) + V_y(t, y)b(t, y) + G\left( 2V_y(t, y)h(t, y) + \sigma^T(t, y)V_y(t, y)\sigma(t, y) \right).
\]

\[\text{Lemma 2}. \quad \text{(see 9, Theorem 4.1)}. \quad \text{Assume that there exist} \ V \in C^1\left( \mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+ \right) \text{ and constants} \ C_4 > C_3 > 0 \text{ and} \lambda_1 > 0 \text{ such that}
\]
\[
(1) \ |V(y)|^p \leq C_3 \left( C_3 |y|^p \right), \quad \text{for all} \ t \geq t_0, \ y \in \mathbb{R}^d.
\]

(2) \( LV(t, y(t)) \leq -\lambda_1 V(t, y(t)) \).

Then, the solution of system (28) is \( p \)-th moment exponentially stable in the following sense:
\[
\sup_{t \to \infty} \log(\hat{E}[y(t)]^p) / t \leq -\lambda_1,
\]
\[
\hat{E}[y(t; t_0 + \tau, x(t_0 + \tau)) - x(t)]^p \leq G_2^{-1}\left( G_1\left( S(p, t - t_0, \tau) \right) + M_4(p) \cdot (t - t_0 - \tau) \right).
\]

(2) \( 0 < p < 2 \),
\[
\hat{E}[y(t; t_0 + \tau, x(t_0 + \tau)) - x(t)]^p \leq \left[ G_1^{-1}\left[ G_1\left( S(2, t - t_0, \tau) \right) + M_4(2) \cdot (t - t_0 - \tau) \right] \right]^{p/2}.
\]
where \( S(p, t - t_0, \tau) = C_6(p) \cdot (t - t_0 - \tau) \cdot \rho[M_3(p, \tau) \cdot \rho[G_1^{-1}(G_1(2E[\|p\|^2] + A(p, t - t_0)))]M_3(p, \tau) = C3^{p-1} (\tau p^{-1} + \tau^2 p^{-1} + \tau^3 p^{-1} (p^2/2 (p - 1))^{p-1})C_2(p) = \rho - 1 + p/22^2 \rho, C_6(p) = C2^{p-1} (p^2 + 1), M_4(p) = \max[C_3(p), C_6(p)].

Proof. (1) For simplicity, let \( y(t; t_0 + \tau, x(t_0 + \tau)) = y(t). \) Then, by G-Itô formula, for \( t = t_0 + \tau, \)

\[
|x(t) - y(t)|^p = p \int_{t_0 + \tau}^t |x(s) - y(s)|^{p-2} [x(s) - y(s)]^T [b(s, x(s - \tau_1(s))) - b(s, y(s))] ds \\
+ p \int_{t_0 + \tau}^t |x(s) - y(s)|^{p-2} [x(s) - y(s)]^T [\sigma(s, x(s - \tau_3(s))) - \sigma(s, y(s))] dB(s) \\
+ p \int_{t_0 + \tau}^t |x(s) - y(s)|^{p-2} [x(s) - y(s)]^T [h(s, x(s - \tau_2(s))) - h(s, y(s))] d\langle B \rangle (s) \\
+ \frac{p}{2} (p - 2) p \int_{t_0 + \tau}^t |x(s) - y(s)|^{p-4} [x(s) - y(s)]^T [\sigma(s, x(s - \tau_3(s))) - \sigma(s, y(s))]^2 ds \\
= I_1 + I_2 + I_3 + I_4 + I_5.
\]

By Young inequality, Assumption 1, and 6, Lemma 2.4(iii)].

\[
\hat{E} I_1 = p \hat{E} \int_{t_0 + \tau}^t |x(s) - y(s)|^{p-2} (x(s) - y(s))^T (b(s, x(s - \tau_1(s))) - b(s, y(s))) ds \\
\leq p \left( \hat{E} \int_{t_0 + \tau}^t [x(s) - y(s)]^{p-1} ds + 1/p^p (b(s, x(s - \tau_1(s))) - b(s, y(s)))^p ds \right),
\]

\[
\leq (p - 1) \int_{t_0 + \tau}^t \hat{E} |x(s) - y(s)|^p ds + 2^{p-1} \int_{t_0 + \tau}^t \hat{E} |b(s, x(s - \tau_1(s))) - b(s, x(s))^p ds \\
+ 2^{p-1} \int_{t_0 + \tau}^t \hat{E} |b(s, x(s)) - b(s, y(s))^p ds \\
\leq (p - 1) \int_{t_0 + \tau}^t \hat{E} |x(s) - y(s)|^p ds + C2^{p-1} \int_{t_0 + \tau}^t \hat{E} [\rho(|x(s) - x(s - \tau_1(s))|^p)] ds + C2^{p-1} \int_{t_0 + \tau}^t \hat{E} [\rho(|x(s) - y(s)|^p)] ds \\
\leq (p - 1) \int_{t_0 + \tau}^t \hat{E} |x(s) - y(s)|^p ds + C2^{p-1} \int_{t_0 + \tau}^t \rho \hat{E} [\rho(|x(s) - y(s)|^p)] ds + C2^{p-1} \int_{t_0 + \tau}^t \rho \hat{E} [\rho(|x(s) - y(s)|^p)] ds.
\]

On the other hand, by (3), for \( t_0 + \tau \leq s \leq t, \)
\[
\hat{E}|x(s) - x(s - \tau_1(s))|^p \leq 3^{p-1}\hat{E}\int_{s-\tau_1(s)}^s b(u, x(u - \tau_1(u))) du \right|^p + 3^{p-1}\hat{E}\int_{s-\tau_1(s)}^s h(u, x(u - \tau_2(u))) d\langle B\rangle(s)
\]
\[
+ 3^{p-1}\hat{E}\int_{s-\tau_1(s)}^s \sigma(u, x(u - \tau_3(u))) dB(u) \right|^p
\]
\[
\leq 3^{p-1}\tau_1^pC \int_{s-\tau_1(s)}^s \rho\left(\hat{E}|x(u - \tau_1(u))|^p\right) du + 3^{p-1}C\tau_2^p \int_{s-\tau_1(s)}^s \rho\left(\hat{E}|x(u - \tau_2(u))|^p\right) du
\]
\[
+ C3^{p-1}\tau_3^p \int_{s-\tau_1(s)}^s \rho\left(\hat{E}|x(u - \tau_3(u))|^p\right) du
\]
\[
\leq M_3(p, \tau) \int_{s-\tau_1(s)}^s \rho\left(\sup_{t_0+\rho(t\geq s)} \hat{E}|x(v)|^p\right) du \leq M_3(p, \tau) \cdot \tau \cdot \rho\left[G_1^{-1}\left(G_1\left(2\hat{E}\|\xi\|_p^p\right) + A(p, t - t_0)\right)\right],
\]
(34)

where \( M_3(p, \tau) = C3^{p-1}(\tau_1^p + \tau_2^p + \tau_3^p) (p^3/2) (p - 1)^p \). Similarly, for \( i = 2, 3 \) and \( t_0 + \tau \leq s \leq t \).

\[
\hat{E}|x(s) - x(s - \tau_1(s))|^p \leq M_3(p, \tau) \cdot \tau \cdot \rho\left[G_1^{-1}\left(G_1\left(2\hat{E}\|\xi\|_p^p\right) + A(p, s - t_0)\right)\right].
\]
(35)

Notice that \( A(p, s - t_0) \) is increasing with respect to \( s \), so

\[
\hat{E}I_1 \leq (p - 1) \int_{t_0 + \tau}^t \hat{E}|x(s) - y(s)|^p ds + C2^{p-1}\int_{t_0 + \tau}^t \rho\left(\hat{E}|x(s) - y(s)|^p\right) ds
\]
\[
+ C2^{p-1}(t - t_0 - \tau)\rho\left[M_3(p, \tau) \cdot \tau \cdot \rho\left[G_1^{-1}\left(G_1\left(2\hat{E}\|\xi\|_p^p\right) + A(p, t - t_0)\right)\right]\right].
\]
(36)

Notice that \( \hat{E}I_2 = 0 \). By 6, Lemma 2.4(i),

\[
\hat{E}I_3 = p\hat{E}\int_{t_0 + \tau}^t |x(s) - y(s)|^{p-1} (x(s) - y(s))^T h(s, x(s - \tau_2(s))) - h(s, y(s))| d\langle B\rangle(s)
\]
\[
\leq p\sigma\left(\int_{t_0 + \tau}^t |x(s) - y(s)|^{p-1} |h(s, x(s - \tau_2(s))) - h(s, y(s))| ds\right)
\]
\[
\leq p\sigma\left(\int_{t_0 + \tau}^t \frac{p-1}{p} |x(s) - y(s)|^p + \frac{1}{p} |h(s, x(s - \tau_2(s))) - h(s, y(s))|^p ds\right)
\]
\[
\leq (p - 1)\sigma^p \int_{t_0 + \tau}^t \hat{E}|x(s) - y(s)|^p ds + \sigma^p 2^{p-1}\int_{t_0 + \tau}^t \hat{E}|h(s, x(s - \tau_2(s))) - h(s, y(s))|^p ds
\]
\[
+ 2^{p-1}\sigma^p \int_{t_0 + \tau}^t \hat{E}|h(s, x(s)) - h(s, y(s))|^p ds
\]
\[
\leq (p - 1)\sigma^p \int_{t_0 + \tau}^t \hat{E}|x(s) - y(s)|^p ds + C2^{p-1}\sigma^p \int_{t_0 + \tau}^t \hat{E}|\rho(x(s) - x(s - \tau_2(s)))|^p ds
\]
\[
+ C2^{p-1}\sigma^p \int_{t_0 + \tau}^t \hat{E}|\rho(x(s) - y(s))|^p ds,
\]
(37)
\[
\leq (p - 1)\sigma^2 \int_{t_0 + \tau}^t \hat{E} |x(s) - y(s)|^p ds + C2^p - 1 \sigma^2 \int_{t_0 + \tau}^t \rho \left( \hat{E} |x(s) - x(s - \tau_3 (s))|^p \right) ds
+ C2^p - 1 \sigma^2 \int_{t_0 + \tau}^t \rho \left( \hat{E} |x(s) - y(s)|^p \right) ds
\]

(38)

Similarly,

\[
\hat{I}_2 = \frac{1}{2} P(p - 2) \hat{E} \int_{t_0 + \tau}^t |x(s) - y(s)|^{p - 2} \left[ (x(s) - y(s))^T \left( \sigma(s, x(s - \tau_3 (s))) - \sigma(s, y(s)) \right) \right]^2 d\langle B \rangle (s)
\]

\[
\leq \frac{1}{2} P(p - 2)\sigma^2 \left( \hat{E} \int_{t_0 + \tau}^t \frac{P - 2}{P} |x(s) - y(s)|^p + \frac{2}{P} |\sigma(s, x(s - \tau_3 (s))) - \sigma(s, y(s))|^p \right) ds
\]

\[
\leq \frac{1}{2} P(p - 2)\sigma^2 \int_{t_0 + \tau}^t \hat{E} |x(s) - y(s)|^p ds + C(p - 2)\sigma^2 2^p - 1 \int_{t_0 + \tau}^t \hat{E} |x(s) - x(s - \tau_3 (s))|^p ds
+ C2^p - 1 \sigma^2 \int_{t_0 + \tau}^t \rho \left( \hat{E} |x(s) - y(s)|^p \right) ds
\]

(39)
Then,

\[ \hat{E}|x(t) - y(t)|^p \leq C_5(p) \int_{t_0}^{t} \hat{E}|x(s) - y(s)|^p ds + C_6(p) \int_{t_0}^{t} \rho\left(\hat{E}|x(s) - y(s)|^p\right) ds \]

\[ + C_6(p) \cdot (t - t_0 - \tau) \rho\left(\mathcal{L}_1(p, \tau) \cdot \mathcal{I}_1(p, \tau) \cdot \mathcal{G}_1\left(\frac{\hat{g}^2(2\mathcal{E}\|\xi\|_0^p) + A(p, t - t_0)}{2}\right)\right) \]

\[ \leq M_4(p) \int_{t_0}^{t} \hat{E}|x(s) - y(s)|^p + \rho\left(\hat{E}|x(s) - y(s)|^p\right) ds \]

\[ + C_6(p) \cdot (t - t_0 - \tau) \rho\left(\mathcal{L}_1(p, \tau) \cdot \mathcal{I}_1(p, \tau) \cdot \mathcal{G}_1\left(\frac{\hat{g}^2(2\mathcal{E}\|\xi\|_0^p) + A(p, t - t_0)}{2}\right)\right), \]

where \( M_4(p) = \max\{C_5(p), C_6(p)\} \), \( C_6(p) = C2^{p - 1}(p\sigma^2 + 1), C_5(p) = (p - 1) + p^2 - p/2\sigma^2 \). By Lemma 2, we omit the details.

Next, the proof of case 0 < \( p < 2 \) is similar to that of Lemma 1 (2); hence, we omit the details.

### Theorem 1

Let Assumption 1 and the conditions of Lemma 2 hold. Then, there is a positive number \( \tau^* \) such that for any initial data \( x(\xi) \in L^p_{\xi 0}(\Omega; C(\mathbb{R}^n; \mathbb{R}^n)) \) with \( \mathbb{E}\|\xi\|_0^p \geq 1 \), the solution of system 1 is \( p \)-th moment exponentially stable provided \( \tau < \tau^* \). In practice, we can find \( \tau^* \) according to the following three cases.

1. \( p \geq 2 \), \( 0 < \delta < 1 \) and take \( T \) satisfying the following equation:

\[ \frac{4^{p - 1}C_4e^{-\lambda_1T\mathbb{E}\|\xi\|_0^p}}{C_3} = \delta. \]

Then, \( \tau^* \) is the unique root to the equation (in \( \tau \)).

2. \( 1 < p < 2 \), \( 0 < \delta < 1 \) and take \( T \) satisfying the following equation:

\[ \frac{2^{2p - 2}C_4e^{-\lambda_1T\mathbb{E}\|\xi\|_0^p}}{C_3} = \delta. \]

Then, \( \tau^* \) is the unique root to the equation (in \( \tau \)).

3. \( 0 < p < 1 \), \( 0 < \delta < 1 \) and take \( T \) satisfying the following equation:

\[ \frac{2^{0.5p}C_4e^{-\lambda_1T\mathbb{E}\|\xi\|_0^p}}{C_3} = \delta. \]

Then, \( \tau^* \) is the unique root to the equation (in \( \tau \)).
\[
1 = \frac{C_4 e^{-\lambda T}}{C_3} \left[ G_{t}(G_{1}(G_{1}(2E\|\xi_0^p) + A(2, t)) + A(2)) \right]^{p/2} + \frac{C_4 e^{-\lambda T}}{C_3} \left[ G_{1}(S(2, \tau + T, t)) + M_4(2) \cdot T \right]^{p/2}
\]

(47)

\[
+ M_{\tau}^{p/2}(2, t) \cdot \left\{ p \left[ G_{1}(G_{1}(2E\|\xi_0^p) + A(2, T + 2t)) \right] \right\}^{p/2}.
\]

**Proof.** (1) For any initial data \( \xi \in L^p_{\tau} (\Omega; [-\tau, 0]; \mathbb{R}^n) \), write \( x(t; t_0, \xi) = x(t) \) for \( t \geq t_0 \). Write \( y(t_0 + \tau + T; t_0 + \tau, x(t_0 + \tau + T) = y(t_0 + \tau + T) \). Then, it follows from (3) and Lemma 2 that

\[
E|y(t + \tau + T)|^p \leq E|x(t + \tau + T)|^p + 2^{p-1}E|y(t + \tau + T) - y(t + \tau + T)|^p
\]

(49)

By (28) and (48) and the elementary inequality \(|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)\),

\[
\hat{E}\|x(t_0 + 2\tau + T)\|_T^p = \sup_{t_0 + \tau + T \in \Omega; t_0 + \tau + T} \hat{E}|x(t)|^p.
\]

Then, combining (4) and (48),

\[
\hat{E}\|x(t_0 + 2\tau + T)\|_T^p \leq 2^{p-1}\hat{E}|x(t_0 + \tau + T)|^p + 2^{p-1}\hat{E}\left\{ \sup_{t_0 + \tau + T} |x(t_0 + \tau + T) - x(t_0 + \tau + T + \delta)|^p \right\}
\]

(50)

\[
\leq 4^{p-1}\frac{C_4 e^{-\lambda T}}{C_3} G_{1}(G_{1}(2E\|\xi_0^p) + A(\tau, p)) \cdot p \left[ G_{1}(G_{1}(2E\|\xi_0^p) + A(\tau, T + 2\tau)) \right].
\]

Let

\[
F(\tau) = 4^{p-1}\frac{C_4 e^{-\lambda T}}{C_3} G_{1}(G_{1}(2E\|\xi_0^p) + A(\tau, p)) + 4^{p-1}G_{1}(S(\tau, \tau + T, p)) + M_4(p)T
\]

(51)

It is clear that \( F(\tau) \) is a continuously increasing function of \( \tau \geq 0 \), \( F(\tau) = \infty \), and \( F(0) = 4^{p-1}C_4/ C_3 e^{-\lambda T}E\|\xi_0^p = \delta \leq 1 \), so the equation \( F(\tau) = 1 \) (in \( \tau \)) has a unique root \( \tau^* > 0 \).

Fix \( \tau \in (0, \tau^* \) and \( \xi \in L^p_{\tau} (\Omega; [-\tau, 0]; \mathbb{R}^n) \) arbitrarily. Then, \( F(\tau) < 1 \), and so \( \log F(\tau) < 0 \). Take \( \gamma = -\log F(\tau)/(\tau^2 + \tau + \log E\|\xi_0^p / \tau + T) \), and then it is clear that \( \gamma > 0 \) and \( F(\tau) = e^{-\gamma(\tau^2 + \tau + \log E\|\xi_0^p / \tau + T) \|\xi_0^p}. From (49), we have
which means that the $p$-th moment of the solution at time $t_0 + 2 \tau + T$ can be dominate by the product of $e^{-\gamma(2\tau + T)}\hat{E}\|\xi\|^p$ and the upper bound of the $p$-th moment of the initial value.

Now, we consider the solution $x(t)$ of (1) when $t \geq t_0 + 2 \tau + T$. By the uniqueness of the solution, we could look at $x(t_0 + 2 \tau + T)$ as the solution of (1) at time $t_0 + 2 \tau + T$ with initial value $x(t_0 + 2 \tau + T)$ and initial time $t_0 + 2 \tau + T$, which has continuous paths. So, by (52),

$$
\mathop{\sup}_{t_0 + i(2\tau + T) \in (t_0 + 2\tau + T)} |x(t)|^p \leq G_2^{-1} \left( G_2 \left( 4^{-1} e^\gamma (2\tau + T) \hat{E}\|\xi\|^p \right) + A_2 (p, 2\tau + T) \right).
$$

(55)

When $i \to \infty$, $G_2 \left( 4^{-1} e^{-i\gamma (2\tau + T) \hat{E}\|\xi\|^p} \right)$ goes to $G_2(0^+) = -\infty$, and note that $G_2$ is continuous and

$$
\lim_{i \to \infty} \frac{G_2^{-1} \left( G_2 \left( 4^{-1} e^{-i\gamma (2\tau + T) \hat{E}\|\xi\|^p} \right) + A_2 (p, 2\tau + T) \right)}{G_2^{-1} \left( G_2 \left( 4^{-1} e^{-i\gamma (2\tau + T) \hat{E}\|\xi\|^p} \right) \right)} = 1.
$$

(56)

Therefore, by (5) and (54), for all $i = 0, 1, \ldots$. Moreover, it is clear that (54) holds for $i = 0$. Therefore, by (5) and (54), for all $i \geq 0,

$$
\mathop{\sup}_{t_0 + i(2\tau + T) \in (t_0 + 2\tau + T)} |x(t)|^p \leq G_2^{-1} \left( G_2 \left( 4^{-1} e^{-i\gamma (2\tau + T) \hat{E}\|\xi\|^p} \right) + A_2 (p, 2\tau + T) \right).
$$

(57)

So, we can easily prove that

$$
\lim_{i \to \infty} \frac{\log G_2^{-1} \left( G_2 \left( 4^{-1} e^{-i\gamma (2\tau + T) \hat{E}\|\xi\|^p} \right) + A_2 (p, 2\tau + T) \right)}{t_0 + i(2\tau + T)} = 0.
$$

(58)
(2) By (6) and Lemma 2,

\[ E\left| y(t_0 + \tau + T) \right|^p \leq \frac{C_4}{C_3} e^{-\lambda T} E\left| x(t_0 + \tau) \right|^p \leq \frac{C_4}{C_3} e^{-\lambda T} \left[ G_1^{-1} \left( \widehat{G}_1 \left( 2 \mathbb{E} \| \xi \|^2_\tau \right) \right) + A(2, \tau) \right]^{\frac{p}{2}}. \]  

By (29), for all \( t \geq t_0 + \tau, \)

\[ E\left| y(t; t_0 + \tau, x(t_0 + \tau)) - x(t) \right|^p \leq \left[ G_1^{-1} \left( G_1 \left( S(2, t - t_0, \tau) \right) + M_4(2) (t - t_0 - \tau) \right) \right]^{\frac{p}{2}}. \]  

By (59) and (60) and the elementary inequality \( |a + b|^p \leq 2^{p-1} (|a|^p + |b|^p), \)

\[ \hat{E}\|x(t_0 + \tau + T)\|^p \leq 2^{p-1} \hat{E}\|x(t_0 + \tau + T)\|^p + 2^{p-1} \hat{E}\|x(t_0 + \tau + T) - y(t_0 + \tau + T)\|^p \]

\[ \leq \frac{2^{p-1} C_4}{C_3} e^{-\lambda T} \left[ G_1^{-1} \left( G_1 \left( 2 \mathbb{E} \| \xi \|^2_\tau \right) \right) + A(2, \tau) \right]^{\frac{p}{2}} + 2^{p-1} \left[ G_1^{-1} \left( G_1 \left( S(2, \tau + T, \tau) \right) + M_4(2) \cdot T \right) \right]^{\frac{p}{2}}. \]  

By (61) and (7),

Let

\[ F(\tau) = \frac{2^{p-2} C_4}{C_3} e^{-\lambda T} \left[ G_1^{-1} \left( G_1 \left( 2 \mathbb{E} \| \xi \|^2_\tau \right) \right) + A(2, \tau) \right]^{\frac{p}{2}} + 4^{p-1} \left[ G_1^{-1} \left( G_1 \left( S(2, \tau + T, \tau) \right) + M_4(2) \cdot T \right) \right]^{\frac{p}{2}} \]

\[ + 2^{p-1} \cdot M^{p/2}_1(2, \tau) \cdot \left\{ \rho \left[ G_1^{-1} \left( G_1 \left( 2 \mathbb{E} \| \xi \|^2_\tau \right) \right) + A(2, T + 2\tau) \right] \right\}^{\frac{p}{2}}. \]  

Then, \( F(\tau) \) is a continuously increasing function of \( \tau > 0, \)

\( F(+\infty) = +\infty, \) and \( F(0) = 2^{2p-2} C_4/C_3 e^{-\lambda T} \mathbb{E}\| \xi \|^2_\delta = \delta < 1. \) 

So, the equation \( F(\tau) = 1 \) (in \( \tau \)) has a unique root \( \tau^* > 0. \) The rest of the proof is the same as that in part (1) and is omitted.
(3) Clearly, (59) and (61) also hold for $0 < p \leq 1$. Then, by (59) and (61) and $|a + b|^p \leq |a|^p + |b|^p$,

\[
\hat{E}|x(t_0 + \tau + T)|^p \leq \hat{E}|y(t_0 + \tau + T)|^p + \hat{E}|x(t_0 + \tau + T) - y(t_0 + \tau + T)|^p \leq \frac{C_2}{C_1} e^{-\lambda T} \left[ G_1^{-1}(G_1(2\hat{E}\|\xi\|_T^2 + A(2, \tau))) + |G_1^{-1}(G_1(S(2, \tau + T, \tau)) + M_4(2T))|^{p/2} \right]
\]

\[
\hat{E}\|x(t_0 + 2\tau + T)\|^p \leq \hat{E}|x(t_0 + \tau + T)|^p + \hat{E}\left( \sup_{0 \leq s \leq T} |x(t_0 + \tau + T) - x(t_0 + \tau + T + s)|^p \right) \leq \frac{C_2}{C_1} e^{-\lambda T} \left[ G_1^{-1}(G_1(2\hat{E}\|\xi\|_T^2 + A(2, \tau))) + |G_1^{-1}(G_1(S(2, \tau + T, \tau)) + M_4(2T))|^{p/2} \right]
\]

\[
+ M_1^{p/2}(2, \tau) \cdot \rho \left( |G_1^{-1}(G_1(2\hat{E}\|\xi\|_T^2 + A(2, T + 2\tau))|\right) \right)^{p/2} .
\]

Let

\[
F(\tau) = \frac{C_2}{C_1} e^{-\lambda T} \left[ G_1^{-1}(G_1(2\hat{E}\|\xi\|_T^2 + A(2, \tau))) + |G_1^{-1}(G_1(S(2, \tau + T, \tau)) + M_4(2T))|^{p/2} \right]
\]

\[
+ M_1^{p/2}(2, \tau) \cdot \rho \left( |G_1^{-1}(G_1(2\hat{E}\|\xi\|_T^2 + A(2, T + 2\tau))|\right) \right)^{p/2} .
\]

It is clear that $F(\tau)$ is a continuously increasing function of $\tau > 0$ and $F(0) = 2^{0.5p} C_2/C_1 e^{-\lambda T} \hat{E}\|\xi\|_T^2 = \delta < 1$. So, the equation $F(\tau) = 1$ (in $\tau$) has a unique root $\tau^* > 0$. Similar to the proof of part (2), from the definition of $F(\tau)$, we know that (58) holds for $0 < p \leq 1$.

From the above discussion, system 1 is $p$-th moment exponentially stable and the proof is finished. \(\square\)

Remark 4. In [9], the existence and uniqueness of solution to stochastic delay non-linear systems driven by G-Lévy processes are proved under non-Lipschitz condition, which is similar to Assumption 1. So, we wonder whether the corresponding results similar to Theorem 1 hold if there is another Lévy term in equation (1). By careful calculation, we can get Lemmas 1 and 3 with a little revision of some parameters. However, we cannot prove the $p$-th moment exponential stability of the solution by the method presented here. The main reason is that in equation (53), we take $x(t_0 + 2\tau + T)$ as an initial point of the new solution $x(t)$, which is continuous, so we could use (52). However, the solution of stochastic delay non-linear systems driven by G-Lévy processes is not continuous, so the formula cannot hold. In the future, we will look for other methods to study the $p$-th order moment exponential stability of stochastic delay non-linear system driven by G-Lévy process under the conditions of $(H_1)$–$(H_3)$.

3. Example

We consider the following one-dimensional stochastic differential equation driven by G-Brownian motion:

\[
\begin{cases}
   dy(t) = -2y(t)dt - \frac{1}{2} \left( \frac{|\sin t|}{\sqrt{1 + t^2}} + \frac{\sin^2 t}{2(1 + t^2)} \right) y(t) dB(t) + \frac{1}{\sqrt{2}} \left( 1 + \frac{|\sin t|}{\sqrt{1 + t^2}} \right) y(t) d\langle B \rangle(t), \\
y(0) = 1,
\end{cases}
\]

where $B$ is a G-Brownian motion with $\pi = 0.1, \sigma = 0.01$. Let $b(t, y(t)) = -2y(t)$, $h(t, y(t)) = -1/2(|\sin t|/\sqrt{1 + t^2} + \sin^2 t/2(1 + t^2))$, $\sigma(t, y(t)) = (1 + |\sin t|/\sqrt{1 + t^2}) y(t)$, $V(t, y) = y^2$. Then,
\[
V_y(t, y(t))h(t, y(t)) = -4y^2(t),
\]
and by [9], we have
\[
V_y(t, y(t))h(t, y(t)) = \left( \frac{|\sin t|}{\sqrt{1 + t^2}} - \frac{\sin^2 t}{2(1 + t^2)} \right) y^2(t),
\]
\[
V_{yy}(t, y(t))\sigma^2(t, y(t)) = \left( 1 + \frac{|\sin t|}{\sqrt{1 + t^2}} \right)^2 y^2(t),
\]
(68)
\[
LV(t, y) = -4y^2 + G\left[ \frac{2|\sin t|}{\sqrt{1 + t^2}} - \frac{\sin^2 t}{(1 + t^2)} \right] y^2 + \left( 1 + \frac{|\sin t|}{\sqrt{1 + t^2}} \right)^2 y^2 \leq -3y^2.
\]
(69)

So, by taking \(C_2 = 1/2, C_3 = 2, p = 2, \lambda_1 = 3\), the solution of system (67) is \(p\)-th moment exponentially stable by Lemma 2.

Next, consider the following stochastic delay non-linear system driven by G-Brownian motion:

\[
\begin{aligned}
&dx(t) = -2x(t - \tau_1(t))dt - \frac{1}{2} \left( \frac{|\sin t|}{\sqrt{1 + t^2}} + \frac{\sin^2 t}{2(1 + t^2)} \right) x(t - \tau_2(t))d\langle B \rangle(t) + \frac{1}{\sqrt{2}} \left( 1 + \frac{|\sin t|}{\sqrt{1 + t^2}} \right)x(t - \tau_3(t))dB(t), \\
x(0) &= 1.
\end{aligned}
\]

By Remark 2, we have \(\rho(x) = x\) and \(C = 1\), so \(G_1(x) = \ln(x)/2, G_2^{-1}(x) = e^{2x}\). Let \(\delta = 100^{-1}\) then, \(T = \ln(1/400) \cdot (-1/3) = 1.997155, G_1(2) = \left\lfloor \frac{2}{1/2sds} = 0.3 \right\rfloor,\) \(A(2, \tau) = \tau \cdot \max\{1 + \sqrt{\bar{\sigma}}, C(1 + 2\bar{\sigma})\} = (1 + 2\sqrt{\bar{\sigma}})\) \(\tau = 1.02\tau, C_2(2) = C(4\bar{\sigma}^2 + 2) = 2.04, M_2(2, \tau) = (\tau + T) \cdot \max\{1 + \sqrt{\bar{\sigma}}, C(1 + 2\bar{\sigma})\} = 1.02(\tau + T), M_4(2) = \max\{1 + \sqrt{\bar{\sigma}}, 2.04\} = 2.04, S(2, \tau + T, \tau) = 2.04 \cdot T \cdot \left\lfloor (3\tau + 0.0015\tau^2) \cdot \tau \cdot [G_1^{-1}(G_2(2) + 2.04(\tau + T))]\right\rfloor, M_1(2, \tau) = 3\tau^2(1 + 0.0005\tau), A(2, 2\tau + T) = (2\tau + T) \cdot \max\{1 + \sqrt{\bar{\sigma}}, C(1 + 2\bar{\sigma})\} = 1.02(2\tau + T).

By Theorem 1, the solution of system (67) is \(p\)-th moment exponentially stable provided \(r = \max\{\tau_1(t), \tau_2(t), \tau_3(t)\} < \tau^*\) and \(\tau^*\) is the unique root to the equation (in \(r\)):

\[
1 = 100^{-1} \left\lfloor \exp\left(2.04 \cdot \left(3\tau + 0.0015\tau^2\right) \cdot \tau \cdot \left\lfloor \exp\left(2.04 \cdot \left(3\tau + 0.0015\tau^2\right)\right)\right\rfloor\right\rfloor \cdot \exp(4.08 \cdot T) + 6 \cdot \tau^2 \cdot (1 + 0.0005\tau) \cdot \left\lfloor \exp\left(2.04 \cdot \left(2\tau + T\right)\right)\right\rfloor,
\]
(71)

so \(\tau^* \approx 1.7 \times 10^{-3}\).

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


[12] H. Zhang, Generalization and Application of Gronwall Bellman Bihari Type Delay Integral Inequality, Qufu Normal University, China Jining, 2008.