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Research Article

Three-Parameter Twin τ^2 Strength Theory for Brittle Materials under Multiaxial Stress State and Its Application

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The strength failure of brittle materials under complex stress is an important problem. Herein, we propose a novel three-parameter twin τ^2 strength theory considering the influence of hydrostatic pressure and normal stress on the principal shear-stress surface, derive a mathematical expression for the strength theory, and compare the theoretical predictions under several stress states with existing experimental data. The results show that different ultimate stress ratios, α and β, correspond to different strength theories for brittle materials. The principal stress σ_1 increases gradually with an increase in σ_2 (=σ_3) under the stress state $\sigma_1 > \sigma_2 = \sigma_3$; σ_1 (=σ_2) increases gradually with an increase in σ_3 under the stress state $\sigma_1 = \sigma_2 > \sigma3$. Furthermore, the biaxial compressive strength is considerably higher than the uniaxial compressive strength under the biaxial compressive stress for $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 = 0$. When σ_3 is fixed and σ_2 is relatively small under the stress states of $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 > 0$, the maximum principal stress σ_1 increases with the increasing σ_2. When σ_2 is relatively large and as σ_1 gradually decreases with the increasing σ_2, the effect law of intermediate principal stress σ_2 is obtained.

1. Introduction

With the rapid development of science and technology, brittle materials such as rock and concrete are being used more widely. To maximize the potential of engineering materials, it is necessary to gain further insight into the strength theories of rock-like brittle materials. The strength theories of materials are essential for materials science and form the basis for designing the various types of engineering structures. Hence, development and innovation in this area are of great academic significance, have a high application value in theoretical and practical engineering, and improve economic efficiency [1].

Currently, the commonly used material strength theories can be classified as one-, two-, three-, and four-parameter theories according to the number of formula parameters. There are even five-parameter theories for special cases. A multiparameter strength theory is required to describe the failure characteristics of some brittle materials with different tensile and compressive strengths. In the past century, based on the traditional maximum tensile stress, maximum strain, maximum shear stress, and shape strain energy theories [2], research on strength theories has garnered the attention of many scholars. Various strength theories that reflect the failure laws of brittle materials, such as rock and concrete, have been proposed. For example, Mohr [3] suggested the famous Mohr–Coulomb strength theory by considering the effects of the maximum principal stress σ_1 and minimum principal stress σ_3 on rock. The strength criterion of the Hoek–Brown strength theory [4–6] reflects the relationship between σ_1 and σ_3 based on the results of tests such as the triaxial confinement tests of rocks. This theory also considers factors such as the rock strength, stress state, and structural surface. The Hoek–Brown strength theory also describes the failures in fractured and anisotropic rock masses without considering the effects of the intermediate principal stresses on the material strength. Therefore, the theoretical results do not truly reflect the material stresses. Recently, new strength theories have been continuously proposed. For example, Ding et al. [7] summarized the classical strength theories and two types of modern strength theories for concrete and isotropic rocks. They divided each isotropic modern strength theory into shear stress, octahedral, and principal stress strength theories, and they discussed, compared, and reviewed these theories using relevant triaxial test data obtained from China and other countries. Jiang and Yang [8] proposed a three-dimensional Hoek–Brown failure criterion by introducing additional coefficients to explore failure in rock and concrete under multiaxial stress states. In addition, Wu et al. [9] studied the instantaneous equivalent Mohr–Coulomb parameters used to modify the Hoek–Brown criterion. They also discussed the rationality of the calculation method and the relationship between the instantaneous equivalent Mohr–Coulomb parameters and rock strength characteristics. Furthermore, Wu et al. [10] proposed an improved Hoek–Brown criterion by reviewing the development history of the Hoek–Brown criterion as well as its meridian and universal partial functions. Lu and Du [11] suggested a physical model for the unified strength theory and established a generalized nonlinear strength theory, in which two transformed stress spaces were proposed. Through testing, Lu concluded that this theory could be applied to the Mohr–Coulomb and Drucker–Prager criteria. Li et al. [12] introduced the critical confining pressure, and Singh intermediate principal stress methods proposed by Barton to improve the Hoek–Brown criterion, making it more suitable for predicting the triaxial strength of intact rock. Furthermore, You [13] developed linear and nonlinear unified strength theories directly based on principal stress to determine the material parameters for multiple strength criteria. Wang and Hiwada [14] established a multiaxial strength criterion for lightweight aggregate concrete based on the unified strength theory, and Yang et al. [15] proposed a generalized form of the twin shear strength criterion for rock and proposed a functional expression for it: $\tau_8 = g(\theta_\sigma) \cdot f(\sigma_m)$. Gao et al. [16] defined a Drucker–Prager series yield criterion that considers the influence coefficient of the intermediate principal stress and can maximize the strength potentials of materials compared with the conventional criterion. By introducing deformation parameters, Guo et al. [17] established a semitheoretical and semiempirical generalized unified strength theory that can reflect the high accumulation of strain energy in deep rock masses, integrity and elastic modulus of rock masses, strain energy released from seismic sources, and resistance energy of supporting structures. They proposed four threshold values (0.9, 1.0,1.0,1.2, and 1.9) for five levels of activity (i.e., expansion, yield or failure, weak rock burst, moderate rock burst, and high rock burst) in deep rock masses. The results showed that the generalized unified strength theory agrees well with actual scenarios. The Chinese scholar Yu et al. consecutively proposed the twin shear stress yield criterion, twin shear-stress strength theory, and unified strength theory between 1961 and 1990 [1, 18–22]. Yu published theses that proposed the twin shear stress yield criterion in 1961 and the generalized twin shear strength theory in 1985. Subsequently, a new system of unified strength theory was gradually developed, and improved strength theories for metals and other geotechnical materials were realized. Chen and Yu [23] offered a single-parameter double τ^2 yield criterion in 1994. This theory holds that the main factor that causes the yield failure of a material is the sum of the squares of the two principal shear stresses with the largest absolute values. Hence, until the sum of the squares of these two main shear stresses reaches a threshold determined by the properties of the material, the material yields and fails, regardless of its stress state. The two principal shear stresses are the maximum principal shear stress τ_{13} and the intermediate principal shear stress τ_{12} or τ_{23}, which are expressed mathematically as shown in the following equation:

$$\begin{cases} f_1 = \tau_{13}^2 + \tau_{12}^2 = C; & \tau_{12} \geq \tau_{23}, \\ f_2 = \tau_{13}^2 + \tau_{23}^2 = C; & \tau_{12} \leq \tau_{23}, \end{cases} \tag{1}$$

where C is the experimentally determined material constant. The single-parameter double τ^2 yield criterion considers the effect of the intermediate principal stress σ_2 on the material yield and failure, and the theory of equation (1) applies to plastic materials. Another two-parameter unified yield criterion was established by Chen et al. [24]; this criterion is mathematically expressed in the following equation:

$$\begin{cases} f_1 = \tau_{13}^2 + \tau_{12}^2 + A\tau_{13}\tau_{12} = B; & \tau_{12} \geq \tau_{23}, \\ f_2 = \tau_{13}^2 + \tau_{23}^2 + A\tau_{13}\tau_{12} = B; & \tau_{12} \leq \tau_{23}, \end{cases} \tag{2}$$

where A and B are material constants that are determined experimentally. The theory is a one-parameter yield criterion in the case of $A = 0$ (i.e., equation (1)). Subsequently, Chen et al. [25, 26] proposed a two-parameter τ^2 strength theory for brittle materials with different tensile and compressive ultimate strengths based on a two-parameter unified yield criterion, as shown in the following equation:

$$\begin{cases} f_1 = \tau_{13}^2 + \tau_{12}^2 + A\sigma_m = B; & \tau_{12} \geq \tau_{23}, \\ f_2 = \tau_{13}^2 + \tau_{23}^2 + A\sigma_m = B; & \tau_{12} \leq \tau_{23}, \end{cases} \tag{3}$$

where A and B are the experimentally determined material constants. $\sigma_m = 1/3(\sigma_1 + \sigma_2 + \sigma_3)$ is the average stress.

In 2005, Chen et al. [27] suggested a three-parameter τ^2 strength theory applicable to rock materials. This theory is mathematically expressed in the following equation:

$$\begin{cases} \tau_{13}^2 + \tau_{12}^2 + A\sigma_m^2 + B\sigma_m = C; & \tau_{12} \geq \tau_{23}, \\ \tau_{13}^2 + \tau_{23}^2 + A\sigma_m^2 + B\sigma_m = C; & \tau_{12} \leq \tau_{23}, \end{cases} \tag{4}$$

where A, B, and C are experimentally determined material constants. When $A = 0$ (i.e., equation (3)), it is a two-parameter τ^2 strength theory. Recently, Chen et al. gradually developed a new strength theory system based on the double τ^2 strength theory, and scholars such as Li [28], Sun [29], Kong et al. [30], and Chen et al. [31, 32] proposed a different three-parameter double τ^2 strength theory, which has been widely used for the strength analysis of brittle materials such as rock and concrete. Wang and Chen [33] successfully applied the double τ^2 strength theory for different purposes such as elastic-plastic constitutive modeling, tunnel rock burst prediction, and establishing and analyzing the constitutive integral algorithm.

Although a variety of strength theories have been proposed worldwide, they all have some limitations or defects.

For example, some theories are more applicable to specific types of material (e.g., metallic materials) and less applicable to others (e.g., brittle materials such as rock and concrete). In addition, some theories only consider the effects of the major and minor principal stresses without accounting for the effects of the intermediate principal stresses. As strength theories are extremely complex, it is impossible to successfully apply a single theory to all engineering materials. Therefore, further studies are essential. Thus, we derived a new expression based on the critical stress ratio for the three-parameter double τ^2 strength theory founded on the theoretical ideas of the single-parameter, two-parameter, and three-parameter double τ^2 strength theories. We conducted a comparative analysis with the experimental data of brittle materials such as rocks and concrete under various stress states. The results demonstrate that the new theory is feasible and supplementary to the development of strength theories and has great theoretical significance and application value.

2. Derivation of Strength Theory

According to the three-parameter twin τ^2 strength theory concept described in the Introduction section, the variation law of the limit surface of brittle materials is completely considered. Moreover, the change in the strength of the materials under the weighted combination of different hydrostatic pressure functions of the Tennessee and Compressive Meridian is considered to propose the new three-parameter twin τ^2 strength theory. This theory states that the failure of materials depends on the action of principal shear stresses ($\tau_{13,}$ $\tau_{12,}$ and τ_{23}). Meanwhile, the normal stress ($\sigma_{13,}$ $\sigma_{12,}$ and σ_{23}) and hydrostatic pressure (σ_m) on the surface of the principal shear stresses all affect the failure of materials. This theory is mathematically expressed as follows:

$$\begin{cases} \tau_{13}^2 + \tau_{12}^2 + A\left(\sigma_{13}^2 + \sigma_{12}^2 + \sigma_{23}^2\right) + B\sigma_m = C; & \tau_{12} \geq \tau_{23}, \\ \tau_{13}^2 + \tau_{23}^2 + A\left(\sigma_{13}^2 + \sigma_{12}^2 + \sigma_{23}^2\right) + B\sigma_m = C; & \tau_{12} \leq \tau_{23}, \end{cases} \tag{5}$$

where A, B, and C are the material coefficients, which are experimentally determined; $\tau_{13} = \sigma_1 - \sigma_3/2$, $\tau_{23} = \sigma_2 - \sigma_3/2$, and $\tau_{12} = \sigma_1 - \sigma_2/2$ are the maximum, intermediate, and minimum principal shear stresses, respectively; σ_m is the average stress (hydrostatic pressure), $\sigma_m = 1/3\,(\sigma_1 + \sigma_2 + \sigma_3)$, and σ_1, σ_2, and σ_3 are the maximum, intermediate, and minimum principal stresses, respectively; σ_{13}, σ_{23}, and σ_{12} are the corresponding normal stresses on the principal shear-stress surfaces $\tau_{13,}$ τ_{23}, and τ_{12}, respectively; and $\sigma_{13} = \sigma_1 + \sigma_3/2$, $\sigma_{23} = \sigma_2 + \sigma_3/2$, and $\sigma_{12} = \sigma_1 + \sigma_2/2$. When the principal shear stresses $\tau_{12} \geq \tau_{23}$, the theory considers the first expression given in equation (5), and when the principal shear stresses $\tau_{12} \leq \tau_{23}$, the theory considers the second expression given in equation (5). Thus, equation (5) can be expressed with the corresponding principal stress as follows:

$$\begin{cases} \frac{1}{4}(\sigma_1 - \sigma_3)^2 + \frac{1}{4}(\sigma_1 - \sigma_2)^2 + \frac{1}{4}A\left[(\sigma_1 + \sigma_3)^2 + (\sigma_1 + \sigma_2)^2 + (\sigma_2 + \sigma_3)^2\right] + \frac{1}{3}B(\sigma_1 + \sigma_2 + \sigma_3) = C; & \tau_{12} \geq \tau_{23}, \\ \frac{1}{4}(\sigma_1 - \sigma_3)^2 + \frac{1}{4}(\sigma_2 - \sigma_3)^2 + \frac{1}{4}A\left[(\sigma_1 + \sigma_3)^2 + (\sigma_1 + \sigma_2)^2 + (\sigma_2 + \sigma_3)^2\right] + \frac{1}{3}B(\sigma_1 + \sigma_2 + \sigma_3) = C; & \tau_{12} \leq \tau_{23}. \end{cases} \tag{6}$$

From equation (5), depending on the values of A and B, different strength theories can be obtained as follows.

For $A = B = 0$, equation (5) is a single-parameter twin τ^2 yield criterion. For $A = 0$, equation (5) is the generalized expression of the double-parameter twin τ^2 strength theory.

Generally, during the stress analysis of rock and concrete materials, the compressive stress is positive, and the tensile stress is negative. To determine the material coefficients, A, B, and C, Using the three special points in the experiment:

(1) The three principal stresses are $\sigma_1 = \sigma_c$ (σ_c is the ultimate stress under uniaxial compression), and because $\sigma_2 = \sigma_3 = 0$, equation (5) can be expressed as follows:

$$\frac{1}{2}\sigma_c^2 + \frac{1}{2}A\sigma_c^2 + \frac{1}{3}B\sigma_c = C. \tag{7}$$

(2) The three principal stresses are $\sigma_1 = \tau_k$, $\sigma_2 = 0$, and $\sigma_3 = -\tau_{\mathrm{k}}$ (τ_k is the ultimate stress of uniaxial compression) under pure shear ultimate stress, and because $\sigma_2 = (\sigma_1 + \sigma_1)/2$, equation (5) can be expressed as follows:

$$\frac{5}{4}\tau_k^2 + \frac{1}{2}A\tau_k^2 = C. \tag{8}$$

(3) The three principal stresses are $\sigma_1 = \sigma_2 = 0$ and $\sigma_3 = -\sigma_t$ (σ_t is the ultimate stress of uniaxial compression) under uniaxial tensile ultimate stress, and because $\sigma_2 > (\sigma_1 + \sigma_3)/2$, equation (5) can be expressed as follows:

$$\frac{1}{2}\sigma_t^2 + \frac{1}{2}A\sigma_t^2 - \frac{1}{3}B\sigma_t = C. \tag{9}$$

By substituting equations (7)–(9) in equation (5), the material coefficients are obtained as follows:

$$\begin{cases} A = \dfrac{5\tau_k^2 - 2\sigma_c\sigma_t}{2\left(\sigma_c\sigma_t - \tau_k^2\right)}, \\ B = \dfrac{9\tau_k^2\left(\sigma_t - \sigma_c\right)}{4\left(\sigma_c\sigma_t - \tau_k^2\right)}, \\ C = \dfrac{3\sigma_c\sigma_t\tau_k^2}{4\left(\sigma_c\sigma_t - \tau_k^2\right)}. \end{cases} \tag{10}$$

Assuming that the ultimate stress ratios of the material are α and β, where $\alpha = \sigma_c/\sigma_t$ and $\beta = \tau_k/\sigma_t$, equation (10) can be expressed as follows:

$$\begin{cases} A = \dfrac{5\beta^2 - 2\alpha}{2\left(\alpha - \beta^2\right)}, \\ B = \dfrac{9\beta^2\sigma_c\left(1 - \alpha\right)}{4\alpha\left(\alpha - \beta^2\right)}, \\ C = \dfrac{3\beta^2\sigma_c^2}{4\alpha\left(\alpha - \beta^2\right)}, \end{cases} \tag{11}$$

where the ultimate stresses σ_c, σ_t, and τ_k are experimentally determined. After obtaining the ultimate stress ratios α and β for different materials, a new three-parameter twin τ^2 strength theory that can be applied to different materials is obtained.

3. Limit Trace Equation and Limit Surface of Strength Theory

As the mechanical properties of various engineering materials are related to the magnitude of hydrostatic pressure, the stress space with the hydrostatic pressure axis as the principal axis is usually adopted in the study of strength theory and, particularly, in the study of the failure criterion and constitutive relation of brittle materials. The coordinates of the π-plane can be expressed using Cartesian (x, y) or cylindrical coordinates (r, θ) (Figure 1).

The relationship between the Cartesian coordinates of the π-plane and the principal stress is as follows:

$$\begin{cases} x = \dfrac{\sigma_3 - \sigma_2}{\sqrt{2}}, \\ y = \dfrac{2\sigma_1 - \sigma_2 - \sigma_3}{\sqrt{6}}, \\ z = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{\sqrt{3}}. \end{cases} \tag{12}$$

After transformation, we obtain

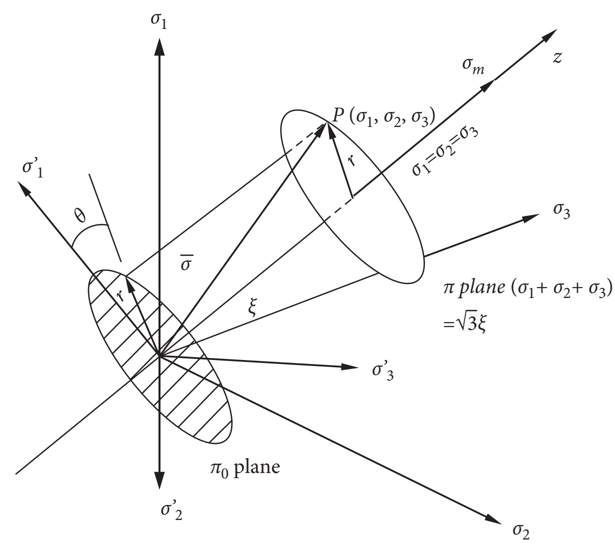

FIGURE 1: Principal stress space and the π plane.

$$\begin{cases} \sigma_1 = \dfrac{\sqrt{6}\,y + \sqrt{3}\,z}{3}, \\ \sigma_2 = \dfrac{2\sqrt{3}\,z - \sqrt{6}\,y - 3\sqrt{2}\,x}{6}, \\ \sigma_3 = \dfrac{2\sqrt{3}\,z - \sqrt{6}\,y + 3\sqrt{2}\,x}{6}. \end{cases} \tag{13}$$

As shown in Figure 2, the π-plane limit line of the four-parameter twin $\tau 2$ strength theory is similar to that of the three-parameter twin τ^2 strength theory. For $0 < A < 1$, the π-plane limit line approaches a curved triangle within the range of low hydrostatic pressures. With an increase in hydrostatic pressure, the limit line transitions to a nonequilateral dodecagon of the curved surface. When the hydrostatic pressure is sufficiently high, the graph approaches a regular dodecagon of the curved surface. When $A = 0$ and $B = 0$, the graph becomes a regular hexagon of the curved surface.

Assuming $\alpha = 2$ and $\beta = 1.8$, the space and π-plane limit lines can be obtained as shown in Figures 2 and 3, respectively.

The limit curve on the π-plane can also be expressed using cylindrical coordinates; the relationship between the cylindrical coordinates (ξ, r, and θ) and principal stresses ($\sigma 1$, $\sigma 2$, and $\sigma 3$) is as follows:

$$\begin{aligned} \xi &= |\mathrm{ON}| = \frac{1}{\sqrt{3}}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = \sqrt{3}\sigma_m, \\ r &= |\mathrm{NP}| = \frac{1}{\sqrt{3}}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_3 - \sigma_2\right)^2 + \left(\sigma_1 - \sigma_3\right)^2\right]^{1/2}, \\ \theta &= \arctan\frac{x}{y}. \end{aligned} \tag{14}$$

Therefore, the corresponding three principal stresses can be expressed as follows:

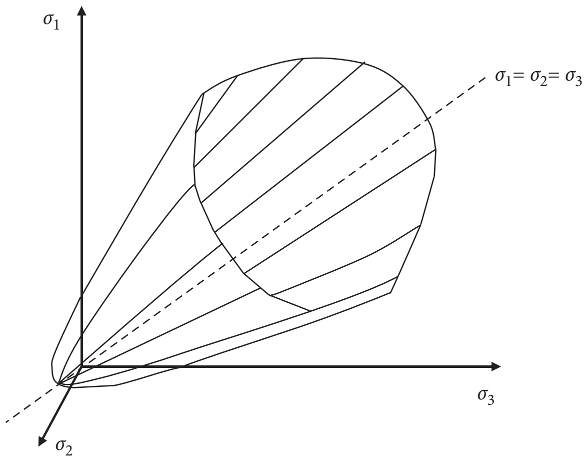

Figure 2: Space limit line.

$$\begin{cases} \sigma_1 = \dfrac{\xi}{\sqrt{3}} + \sqrt{\dfrac{2}{3}} r \cos\theta, \\ \sigma_2 = \dfrac{\xi}{\sqrt{3}} + \sqrt{\dfrac{2}{3}} r \cos\left(\theta - \dfrac{2\pi}{3}\right), \\ \sigma_3 = \dfrac{\xi}{\sqrt{3}} + \sqrt{\dfrac{2}{3}} r \cos\left(\theta + \dfrac{2\pi}{3}\right). \end{cases} \tag{15}$$

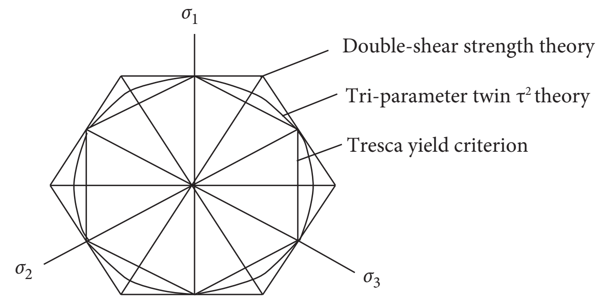

Figure 3: π-Plane limit line.

The corresponding three principal shear stresses can be expressed as follows:

$$\begin{cases} \tau_{13} = \sqrt{\dfrac{1}{6}} r \cos\theta - \sqrt{\dfrac{1}{6}} r \cos\left(\theta + \dfrac{2\pi}{3}\right) = \sqrt{\dfrac{1}{6}} r \cos\left(\theta - \dfrac{\pi}{6}\right), \\ \tau_{12} = \sqrt{\dfrac{1}{6}} r \cos\theta - \sqrt{\dfrac{1}{6}} r \cos\left(\theta - \dfrac{2\pi}{3}\right) = \sqrt{\dfrac{1}{6}} r \cos\left(\theta + \dfrac{\pi}{6}\right), \\ \tau_{23} = \sqrt{\dfrac{1}{6}} r \cos\left(\theta - \dfrac{2\pi}{3}\right) - \sqrt{\dfrac{1}{6}} r \cos\left(\theta + \dfrac{2\pi}{3}\right) = \sqrt{\dfrac{1}{2}} r \sin\theta. \end{cases} \tag{16}$$

By substituting equations (15) and (16) into equation (5), the cylindrical coordinates of the three-parameter twin τ^2 strength theory can be expressed as follows:

$$\begin{cases} \dfrac{1}{6} r^2 \cos^2\left(\theta - \dfrac{\pi}{6}\right) + \dfrac{1}{6} r^2 \cos^2\left(\theta + \dfrac{\pi}{6}\right) + \dfrac{1}{4} A\left(4\xi^2 + r^2\right) + B\dfrac{\xi}{\sqrt{3}} = C; \quad \sigma_2 \le \dfrac{1}{2}(\sigma_1 + \sigma_3), \\ \dfrac{1}{6} r^2 \cos^2\left(\theta - \dfrac{\pi}{6}\right) + \dfrac{1}{2} r^2 \sin^2\theta + \dfrac{1}{4} A\left(4\xi^2 + r^2\right) + B\dfrac{\xi}{\sqrt{3}} = C; \quad \sigma_2 \ge \dfrac{1}{2}(\sigma_1 + \sigma_3). \end{cases} \tag{17}$$

The angle θ shows the positions of the angular point of the limit trace on the π-plane. Substituting $\theta = 0^\circ$ and 60° into equation (17), the following equations can be obtained for the tensile and compressive meridians:

For $\theta = 0^\circ$,

$$\frac{1}{4} r^2 + \frac{1}{4} A\left(4\xi^2 + r^2\right) + B\frac{\xi}{\sqrt{3}} = C. \tag{18}$$

For $\theta = 60^\circ$,

$$\frac{1}{2} r^2 + \frac{1}{4} A\left(4\xi^2 + r^2\right) + B\frac{\xi}{\sqrt{3}} = C. \tag{19}$$

The tensile meridian ($\theta = 0^\circ$) and compressive meridian ($\theta = 60^\circ$) on the meridian plane are given by equations (18) and (19), respectively, as shown in Figure 4.

The figure shows that the meridian of the three-parameter twin τ^2 strength theory represents a partial parabola.

4. Theoretical Analysis and Experimental Verification of Strength

To analyze the variation law of the three-parameter twin τ^2 strength theory under a complex stress state, we discuss the theoretical prediction curves of this theory under several common stress states.

4.1. Stress State of $\sigma_1 > \sigma_2 = \sigma_3$. The stress state $\sigma_1 > \sigma_2 = \sigma_3$ is common in practical engineering, which can be obtained by simplifying the first part of equation (6).

$$\frac{1}{4}(\sigma_1 - \sigma_3)^2 + \frac{1}{4}(\sigma_1 - \sigma_2)^2 + \frac{1}{4} A\left[(\sigma_1 + \sigma_3)^2 + (\sigma_1 + \sigma_2)^2 + (\sigma_2 + \sigma_3)^2\right] + \frac{1}{3} B(\sigma_1 + \sigma_2 + \sigma_3) = C. \tag{20}$$

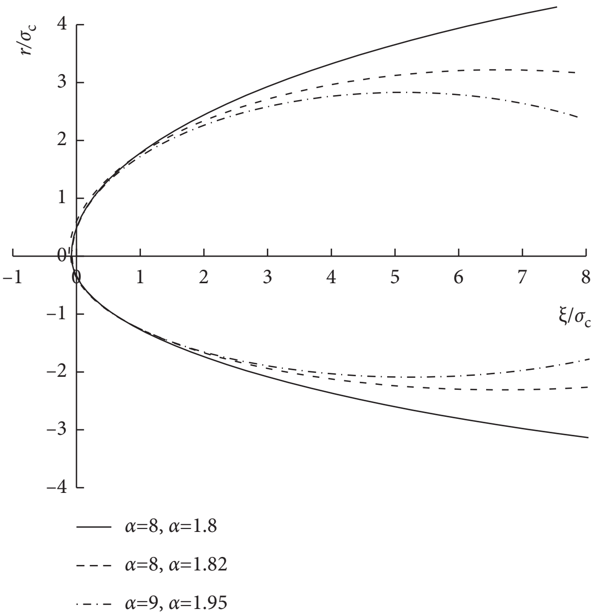

FIGURE 4: Meridian line.

K. Newman and J.-B. Newman [34] performed several experiments on plain concrete in 1971 under this stress state and obtained some experimental data. For plain concrete, the coefficients A, B, and C can be obtained by substituting $\alpha = 10$ and $\beta = 1.5$ into equation (11). Thereafter, this equation is substituted into equation (22). The theoretical prediction curve is compared with the experimental data, as shown in Figure 5.

4.2. Stress State of $\sigma_1 = \sigma_2 > \sigma_3$. For the stress state $\sigma_1 = \sigma_2 > \sigma_3$, the second part of equation (5) is simplified as follows:

$$\frac{1}{4}(\sigma_1 - \sigma_3)^2 + \frac{1}{4}A(\sigma_1 - \sigma_3)^2 + \frac{1}{3}B(2\sigma_1 + \sigma_3) + \frac{1}{9}C(2\sigma_1 + \sigma_3)^2 = D. \tag{21}$$

For the triaxial extrusion tensile state, three groups of experimental data of concrete were taken from Balmer [35], and the ultimate stress ratios α and β were 10 and 1.8, respectively. The comparison between the experimental data and theoretical prediction curve for the three-parameter twin τ^2 strength theory is shown in Figure 6.

The figure shows that the strength of concrete $\sigma_1/\sigma_c = \sigma_2/\sigma_c$ increases with an increase in σ_3/σ_c. This behavior indicates that the theoretical prediction curve of the three-parameter twin τ^2 strength theory is consistent with the experimental data.

4.3. Biaxial Stress States of $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 = 0$. By substituting $\sigma_3 = 0$, $\sigma_1 > 0$, and $\sigma_2 > 0$ into equation (6), we obtain the following equations:

$$\begin{cases} \frac{1}{4}\sigma_1^2 + \frac{1}{4}(\sigma_1 - \sigma_2)^2 + \frac{1}{4}A\left[\sigma_1^2 + (\sigma_1 + \sigma_2)^2 + \sigma_2^2\right] + \frac{1}{3}B(\sigma_1 + \sigma_2) = C & \tau_{12} \geq \tau_{23}, \\ \frac{1}{4}\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{4}A\left[\sigma_1^2 + (\sigma_1 + \sigma_2)^2 + \sigma_2^2\right] + \frac{1}{3}B(\sigma_1 + \sigma_2) = C & \tau_{12} \leq \tau_{23}. \end{cases} \tag{22}$$

Assuming $\alpha = 2$; $\beta = 2$ and $\alpha = 9$; and $\beta = 2$, the theoretical curve and Qu's experimental data [36] can be obtained, as shown in Figure 7.

4.4. Triaxial Stress States of $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 > 0$. Under the triaxial stress state $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 > 0$, rock materials were selected for comparative analysis. For example, we selected You's fitting analysis for the true triaxial tests of multiple rocks [13, 37] and Xu and Geng experimental data for weak sandstone [38]. The theoretical curve can be obtained from equation (5), as shown in Figure 8, where $\sigma_c = 30\,\text{MPa}$, $\alpha = 10$, and $\beta = 2.1$. In the figure, the white circle, black circle, and square represent the relationship between σ_1 and σ_2 at confining pressures of $\sigma_3 = \sigma_2$, $\sigma_3 = 5\,\text{MPa}$, and $\sigma_3 = 10\,\text{MPa}$, respectively.

4.5. σ–τ Compound Stress State. Combined torsion and bending deformation is a common stress phenomena in engineering components. When a member is subjected to a combined bending and torsional force, the relationship between the principal, normal σ, and shear stresses τ can be obtained based on material mechanics as follows:

$$\left.\begin{aligned} \sigma_1 &= \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \\ \sigma_2 &= 0 \\ \sigma_3 &= \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \end{aligned}\right\}. \tag{23}$$

On substituting equation (23) into equation (5), we obtain the following:

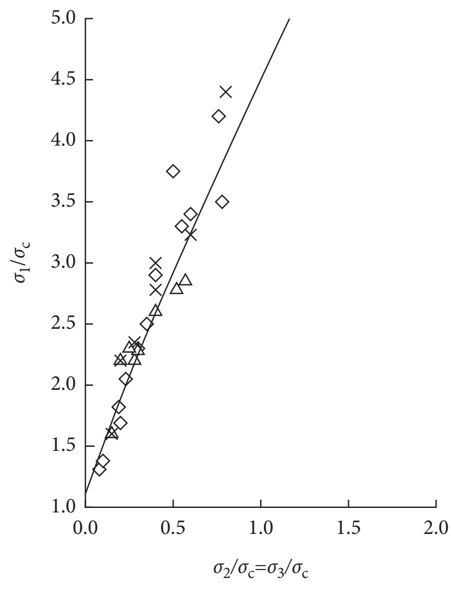

Figure 5: Experimental data and theoretical curves showing changes in concrete strength.

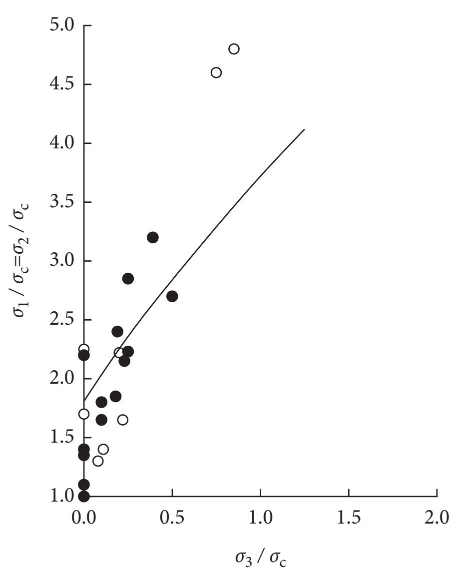

Figure 6: Theoretical curves and experimental data of three-axis extrusion.

$$\begin{cases} \frac{1}{4}\left(\sigma^2 + 4\tau^2\right) + \frac{1}{4}\left(\frac{1}{2}\sigma^2 + \tau^2 + \sigma\sqrt{\frac{1}{4}\sigma^2 + \tau^2}\right) + \frac{1}{2}A\left(\sigma^2 + \tau^2\right) + \frac{1}{3}B\sigma = C; & \sigma \leq 0, \\ \frac{1}{4}\left(\sigma^2 + 4\tau^2\right) + \frac{1}{4}\left(\frac{1}{2}\sigma^2 + \tau^2 - \sigma\sqrt{\frac{1}{4}\sigma^2 + \tau^2}\right) + \frac{1}{2}A\left(\sigma^2 + \tau^2\right) + \frac{1}{3}B\sigma = C; & \sigma \leq 0. \end{cases} \tag{24}$$

Under σ–τ compound stress, three groups of experimental data for concrete reported by Takao Okashima [39] were considered. Furthermore, $\alpha = 10$ and $\beta = 1.2$. For $\sigma_c = 20$, 30, or 40 MPa, the experimental data and theoretical prediction curve of the three-parameter twin τ^2 strength theory are shown in Figure 9.

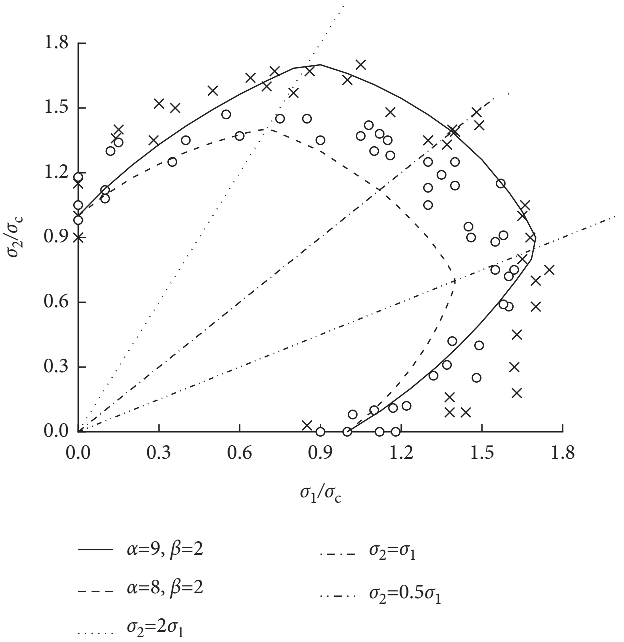

Figure 7: Experimental data and theoretical curves under biaxial compression.

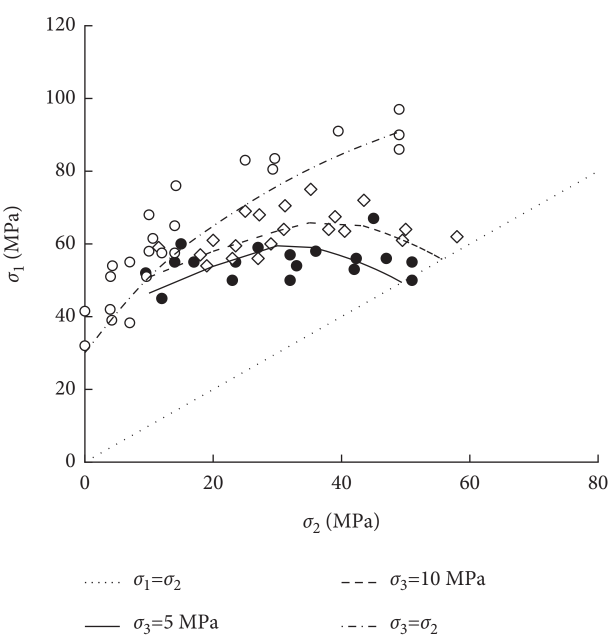

Figure 8: Experimental data and theoretical curves of the three-axis stress state.

The above analysis shows that the new three-parameter twin τ_2 strength theory prediction curve proposed herein agrees with the experimental data obtained for some brittle materials. This theory has the following characteristics: first, the proposed theory is deduced based on the influence of hydrostatic pressure and principal shear stresses, with a clear physical concept and a relatively simple mechanical model. Hence, this theory is convenient for use in engineering applications. Furthermore, note that the destruction of materials is related to the nature of materials. The ultimate stress ratios α and β of different materials are correspond to different theoretical expressions of strength. In addition, parameters A, B, and C are functions of the ultimate stress ratios α and β of brittle materials. It can be determined by stretching, compression, and shear experiments. Therefore, the proposed three-parameter τ^2 strength theory has a certain universality, improving and supplementing the strength theory of brittle materials such as rock and concrete.

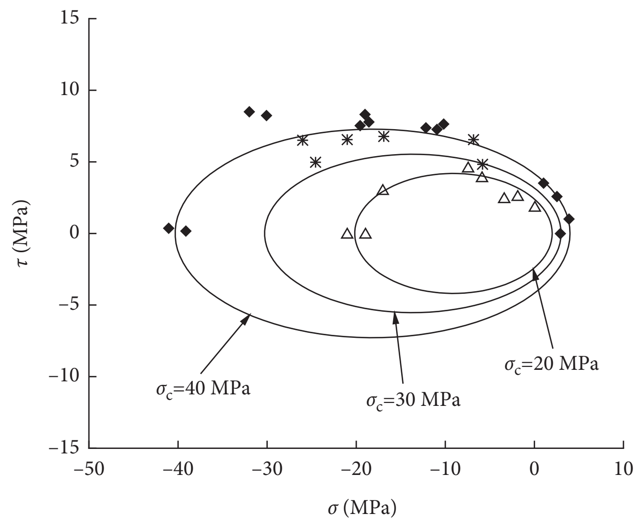

Figure 9: Theoretical curve of the σ–τ composite state.

5. Conclusions

Herein, we proposed a novel three-parameter twin τ^2 strength theory based on the twin τ^2 strength theory. The following conclusions can be obtained from our analyses:

(1) According to the proposed three-parameter twin τ^2 strength theory, the limit surface of the principal stress space is a curved surface that closes in the tensile zone, opens in the compression zone, and increases along the hydrostatic pressure axis. The meridian is an elliptic curve, and the shape and size of the limit surface are determined by the ultimate stress ratios, α and β, of various brittle materials.

(2) Under the stress state $\sigma_1 > \sigma_2 = \sigma_3$, σ_1 increases gradually with an increase in σ_2 $(=\sigma_3)$. Under the stress state $\sigma_1 = \sigma_2 > \sigma_3$, σ_1 $(=\sigma_2)$ increases gradually with an increase in σ_3. Furthermore, under the biaxial compressive stress $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 = 0$, the biaxial compressive strength is considerably larger than the uniaxial compressive strength. Under σ–τ compound stress, the theoretical value of three-parameter twin τ^2 strength is consistent with the experimental data.

(3) Under the stress state $\sigma_1 > 0$, $\sigma_2 > 0$, and $\sigma_3 > 0$, when σ_3 is set as a fixed value and σ_2 is relatively small, σ_1 increases with σ_2, whereas σ_1 decreases gradually with an increase in σ_2 if σ_2 is relatively large. The theoretical values are consistent with the experimental data.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

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References

[1] M. H. Yu, V. A. Kolupaev, Y. M. Li, and J. C. Li, “Advances in unified strength theory and its generalization,” *Procedia Engineering*, vol. 10, pp. 2508–2513, 2011.

[2] J. S. Xu, *Strength Theory and its Application*, Water Conservancy and electric power press, Beijing, 1998.

[3] O. Mohr, “Welche Umstande bedingen die elastizitatsgrenze und den Bruch eines materials,” *Zeitschrift des Vereins Deutscher Ingenieure*, vol. 44, pp. 1524–1530, 1900.

[4] E. Hoek and E. T. Brown, “Empirical strength criterion for rock masses,” *Journal of the Geotechnical Engineering Division*, vol. 106, no. 9, pp. 1013–1035, 1980.

[5] E. Hoek and E. T. Brown, “Strength of jointed rock masses,” *Géotechnique*, vol. 33, no. 3, pp. 187–223, 1983.

[6] E. Hoek and E. T. Brown, “Practical estimates of rock mass strength,” *International Journal of Rock Mechanics and Mining Sciences*, vol. 34, no. 8, pp. 1165–1186, 1997.

[7] F. X. Ding, X. Wu, P. Xiang, and Y. U. W. Zhi, “Reviews on strength theories of concrete and isotropic rock,” *Engineering Mechanics*, vol. 37, no. 2, pp. 1–15, 2020.

[8] H. Jiang and Y. Yang, “A three-dimensional Hoek–Brown failure criterion based on an elliptical Lode dependence,” *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 44, no. 18, pp. 2395–2411, 2020.

[9] S. C. Wu, M. Zhang, S. H. Zhang, and R. H. Jiang, “Study on determination method of equivalent Mohr-Coulomb strength parameters of a modified Hoek-Brown failure criterion,” *Rock and Soil Mechanics*, vol. 40, no. 11, pp. 4165–4177, 2019.

[10] S. Wu, S. Zhang, and G. Zhang, “Three-dimensional strength estimation of intact rocks using a modified Hoek-Brown criterion based on a new deviatoric function,” *International Journal of Rock Mechanics and Mining Sciences*, vol. 107, pp. 181–190, 2018.

[11] D. C. Lu and X. L. Du, “Research on nonlinear strength and failure criterion of rock material,” *Chinese Journal of Rock Mechanics and Engineering*, vol. 12, no. 12, pp. 2439–2408, 2013.

[12] B. Li, M. G. Xu, Y. Z. Liu, and W. Ping, “Modified Hoek-Brown strength criterion for intact rocks under the condition of triaxial stress test,” *Journal of Mining & Safety Engineering*, vol. 32, no. 6, pp. 1010–1016, 2015.

[13] M. Q. You, “Discussion on unified strength for rocks,” *Chinese Journal of Rock Mechanics and Engineering*, vol. 32, no. 2, pp. 258–265, 2013.

[14] L. C. Wang and K. Hiwada, “Multi-axial Strength criterion for lightweight aggregate (LWA) concrete based on the unified strength theory,” *Engineering Mechanics*, vol. 23, no. 5, pp. 125–131, 2006.

[15] X. Q. Yang, Z. D. Liu, and S. X. He, “Twin shear strength criterion for rock based on Traditional triaxial testing results,” *Engineering Mechanics*, vol. 20, no. 4, pp. 87–91, 2003.

[16] J. P. Gao, J. Q. Yang, and X. Sun, “Research on D-P series yield criteria considering the influence coefficient of double shear intermediate principal stress,” *Chinese Journal of Rock Mechanics and Engineering*, vol. 40, no. 6, pp. 1081–1091, 2021.

[17] J. Q. Guo, Q. D. Yang, and X. F. Lu, “Research on generalized unified strength theory of rock (mass) failure,” *Journal of Coal Science*, vol. 46, no. 12, pp. 3869–3882, 2021.

[18] M. H. Yu, “Twin shear stress yield criterion,” *International Journal of Mechanical Sciences*, vol. 25, no. 1, pp. 71–74, 1983.

[19] M. H. Yu, L. N. He, and L. Y. Song, “Double shear strength theory and its generalization,” *Science in China, Series A*, vol. 28, no. 12, pp. 1113–1120, 1985.

[20] M.-H. Yu and L.-N. He, “A new model and theory on yield and failure of materials under complex stress state,” in *Mechanical Behaviour of Materials-VI*, vol. 3, pp. 851–856, Pergamon Press, Oxford, UK, 1991.

[21] M. H. Yu, “Unified strength theory of rock-based materials and their applications,” *Chinese Journal of Geotechnical Engineering*, vol. 16, no. 2, pp. 1–10, 1994.

[22] M. H. Yu, “Rock, concrete strength a theory: past, present and development,” *Natural Science Development*, vol. 7, no. 6, pp. 653–660, 1997.

[23] S. L. Chen and B. Y. Yu, “Twin T^2 and τ^2 yield criteria and its generalization,” *Mechanics and Practice*, vol. 16, no. 5, pp. 60–62, 1994.

[24] S. L. Chen, B. Y. Yu, and L. J. Zhao, “Twin τ^2 yield criterion and its unified form,” *Journal of Shenyang University of Technology*, vol. 19, no. 4, pp. 41–45, 1997.

[25] S. L. Chen, “Twin τ^2 failure theory and its generation,” *Journal of Shenyang University of Technology*, vol. 17, no. 1, pp. 65–69, 1995.

[26] S. L. Chen, W. B. Bao, and S. J. Jin, “Twin τ^2 strength theory and its application to concrete material,” *Journal of China Three Gorges University*, vol. 25, no. 6, pp. 504–506, 2013.

[27] S. L. Chen, B. K. Ning, and W. B. Bao, “A tri-parameter twin τ^2 strength theory and its application for rock materials,” *Chinese Journal of Rock Mechanics and Engineering*, vol. 24, no. 7, pp. 1106–1109, 2005.

[28] Y. Y. Li, *Three Parameters of Twin τ^2 Strength Theory and its Application*, University of Technology, SY, China, 2016.

[29] Z. W. Sun, *Establishment and Application of Multiparameter Strength Theory of Brittle Materials*, Shenyang University of Technology, SY, China, 2017.

[30] Z -P. Kong, H. X. Sun, and S. L. Chen, “A nonlinear three-parameter strength criterion for rock material and its application,” *Rock and Soil Mechanics*, vol. 38, no. 12, pp. 1001–1007, 2017.

[31] S. L. Chen and Z. W. Sun, “Three-parameter strength theory and its application in brittle materials,” *Chinese Journal of Underground Space and Engineering*, vol. 14, no. 4, pp. 1042–1048, 2018.

[32] S. L. Chen, Y. Y. Li, and H. Zhou, "Three-parameter double τ^2 strength criterion based on ultimate stress ratio and its application," *Rock and Soil Mechanics*, vol. 39, no. 6, pp. 1948–1954, 2018.
[33] J.-X. Wang and S.-L. Chen, *Rock Strength, Constitutive Equations and Integration Algorithm: An Introduction*, Science Press, BJ, China, 2022.
[34] K. Newman and J.-B. Newman, "Failure theories and design criteria for plain concrete," *Solid Mechanics and Engineering Design*, pp. 963–995, Wiley Interscience, New York, NY, USA, 1971.
[35] G. G. Balmer, *Shearing Strength of concrete under High Triaxial Stress-Computation of Mohr's Envelope as a Curve*, Structural Research Laboratory, Denver, CO, USA, 1949.
[36] D.-J. Xu, "Mechanical properties of weak porous sandstone under general triaxial stress states," *Rock and Soil Mechanics*, vol. 3, no. 1, pp. 13–25, 1982.
[37] M. Q. You, "Fitting and evaluation of test data using unified strength theory," *Chinese Journal of Rock Mechanics and Engineering*, vol. 27, no. 11, pp. 2193–2204, 2008.
[38] D.-J. Xu and N.-G. Geng, "The variation law of rock strength with increase of intermediate stress," *Acta Mechanica Solida Sinica*, vol. 1, pp. 72–80, 1985.
[39] M.-H. Yu, *Concrete Strength Theory and Application*, Higher Education Press, BJ, China, 2022.