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Research Article

On the FAULT-TOLERANT Resolvability in Line Graphs of Dragon and Kayak Paddles Graphs

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Because of its wide range of applications, metric resolvability has been used in chemical structures, computer networks, and electrical circuits. It has been applied as a node (sensor) in an electric circuit. The electric circuit will not be able to flow current if one node (sensor) in that chain becomes faulty. The fault-tolerant selfstable circuit is a circuit that permits the current flow even if one of the nodes (sensors) becomes faulty. If the removal of any node from a resolving set (RS) of the circuit is still a RS, then the RS of the circuit is considered a fault-tolerant resolving set (FTRS) and the fault-tolerant metric dimension (FTMD) is its minimum cardinality. Even though the problem of finding the exact values of MD in line graphs seems to be even harder, the FTMD for the line graphs was first discussed by Guo et al. [13]. Ahmad et al. [5] determined the precise value of the MD for the line graph of the kayak paddle graph. We calculate the precise value of the FTMD for the line graph in this family of graphs. The FTMD is a more generalized invariant than the MD. We also consider the problem of obtaining a precise value for this parameter in the line graph of the dragon graph. It is concluded that these families have a constant FTMD.

1. Introduction and Preliminaries

Network topology is the graphical depiction of electric circuits. Because convoluted electric circuits (networks) are difficult to work on and study in their natural state, network topology is developed to make them simple and intelligible. Using this technique, any electric circuit (network) can be changed (moulded) into its corresponding graph; open circuits replace current sources, while short circuits replace passive parts and voltage sources. Short circuits are termed branches in network topology and edges in graph theory conceptualization, while open circuits are called nodes in network topology and vertices in pure mathematical graph theory. The following is the formal definition of a graphical depiction of an electric circuit (network).

Definition 1. Let $\lambda(V(\lambda), E(\lambda))$ be an electric circuit, where $V(\lambda)$ and $E(\lambda)$ are the sets of nodes (vertices) and branches (edges), respectively. The order of the electric circuit is $|V(\lambda)|$ and the size of the circuit is $|E(\lambda))|$.

Figure 1 demonstrates the example of the electric circuit and its equivalent graph.

Slater and Harary described the concept of resolving sets (RSs) independently in graphs [1–3], respectively. Metric basis have been used in robot navigation [4], chemical structures [5], and computed networks [6]. An electric circuit will stop working if one node of the circuit becomes faulty. Hernando et al. [7] established the new invariant FTRS to resolve such complications. If the removal of one node from the RS results in another RS, then the RS is called the FTRS. In this situation, a FTRS solves the problem by efficiently flowing the current in the circuit when one of the nodes stops working. The minimum cardinality of FTRS is known as the FTMD. Due to the generalized invariant, the FTMD produces more efficient results than MD. Due to this fact, researchers started to give attention to computing the exact values of the FTMD for different families of graphs.

Hernando et al. [7] discussed the invariant FTMD for the tree graphs and computed the upper bound $\beta'(G) \leq \beta(G)(1 + 2.5^{\beta(G)-1})$ for any graph G. Raza et al. [8],

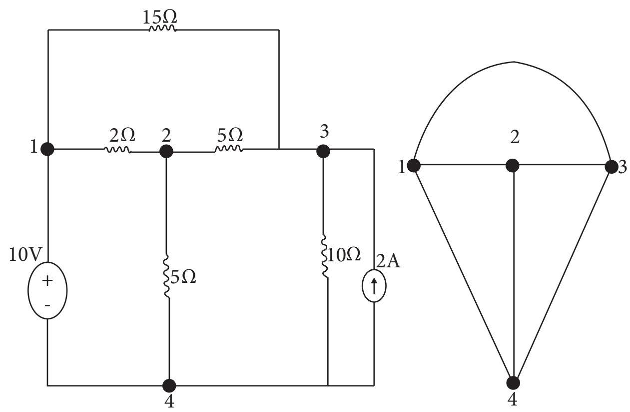

Figure 1: Electric circuit and its equivalent graph.

Zheng et al. [9], and Afzal et al. [10] applied the concept of the FTMD in some families of convex polytopes and computed their exact values. Basak et al. [11] calculated this parameter for the graph $C_n(1,2,3)$, and Saha et al. [12] generalized the results for the graph $C_n(1,2,3,4)$. Hayat et al. [13] and Prabhu et al. [14] applied this invariant to different computer networks and found their upper bounds. Somasundari et al. [15], Azeem et al. [16], Ahmad et al. [17], and Nadeem et al. [18] used the FTMD on different chemical structures and computed the exact values of this parameter. Laxman in [19] computed the lower bound of the FTMD for the cube of the path graph. Koam et al., in [20], calculated the MD and FTMD of the hollow coronoid chemical structure. Wang et al., in [21], considered the problem of finding the FTMD of three types of ladder graphs. Sharma and Bhat, in [22], calculated the FTMD of three families of the double antiprism graphs, which equals to 4. For more applications of the FTMD, see [23–25].

Voronov [26] and Raza et al. [27] determined some important upper bounds for the king's and extended Petersen graphs, respectively. Guo et al. [28] computed the FTMD for the line graphs of the families of necklace and prism graphs. Faheem et al. [29] calculated this invariant for the subdivision graphs of the same families of graphs. Simic et al. [30] and Saha et al. [31] determined the precise value of the FTMD for the grid and square of grid graphs, respectively. Ahmad et al. [32] calculated this parameter for $P(n,2)\odot K_1$ graph, which equals to 4. Hussain et al. [33] applied the idea of the FTMD to some families of gear graphs. For more applications and results about the FTMD in engineering, we refer [34–36].

The following are some essential terminologies and definitions that assist in calculating our primary results.

Definition 2. The degree "$d_\lambda(\xi)$" of "ξ" is the cardinality of branches that is incident to a node in $\xi\in V(\lambda)$.

Definition 3. The minimum cardinality of the branches, between $\xi_1-\xi_2$ path, is known as the distance $d_\lambda(\xi_1,\xi_2)$ between $\xi_1,\xi_2\in V(\lambda)$.

Definition 4. Let $\kappa=\{\xi_1,\xi_2,\ldots,\xi_t\}\subset V(\lambda)$; then, the absolute difference code have t -vector $(|d_\lambda(\mu_1,\xi_1)-d_\lambda(\mu_2,\xi_1)|,\ldots,|d_\lambda(\mu_1,\xi_t)-d_\lambda(\mu_2,\xi_t)|)$ for any $\mu_1,\mu_2\in V(\lambda)$ with respect to κ, denoted by $A\ D((\mu_1,\mu_2)|\kappa)$.

Definition 5. Let $\kappa=\{\xi_1,\xi_2,\ldots,\xi_t\}\subset V(\lambda)$; then, the t-order distance code $r(\mu|\kappa)$ for a node $\mu\in V(\lambda)$ is $(d_\lambda(\mu,\xi_1),d_\lambda(\mu,\xi_2),\ldots,d_\lambda(\mu,\xi_t))$ with respect to κ. If the distance codes for every nodes of the circuit are unique, then the set κ is said to be a RS of the circuit λ. Moreover, if the absolute difference codes for any two nodes of the circuit have at least one nonzero with respect to κ, then κ is called the RS. The minimum cardinality of κ is called the MD, denoted by $\beta(\lambda)$.

Definition 6. Any RS $\kappa\prime$ of the circuit λ is known as the FTRS of the circuit if $\kappa'\setminus\{\xi\}$ is again a RS of the circuit, where $\xi\in\kappa'$. Moreover, if the absolute difference codes for any two nodes of the circuit have at least two nonzeros with respect to $\kappa\prime$, then $\kappa\prime$ is called the FTRS. The minimum cardinality of κ' is called the FTMD, denoted by $\beta'(\lambda)$.

Definition 7. The line graph of the circuit λ is a new circuit $L(\lambda)$, whose nodes are the branches of λ and two branches v_1 and v_2 have a common end node in λ if and only if they are connected in $L(\lambda)$.

Following are some important bounds for $\beta'(\lambda)$ which are presented.

Lemma 1 (see [37]). *Let λ be any graph; then, $\beta(\lambda)<\beta'(\lambda)$.*

Lemma 2 (see [37]). *Let $\lambda\neq P_n$ be any graph; then, $\beta'(\lambda)\geq 3$.*

Lemma 3. *If the FTMD of any graph λ is 3 and $\{\xi_1,\xi_2,\xi_3\}\subset V(\lambda)$ is a FTRS in λ, then the degrees of the vertices ξ_1,ξ_2,ξ_3 are no more than 3.*

2. The Fault-Tolerant Resolvability of the Line Graph of Dragon Graph

Let C_n be a cycle with edge set $E(C_n)=\{e_1,e_2,\ldots,e_n\}$; also, let P_{m+1} be a path with edge set $E(P_{m+1})=\{f_1,f_2,\ldots,f_m\}$. Dragon graph $T_{n,m}$ is shown in Figure 2.

To compute our required results, we convert the graph $T_{n,m}$ into their line graph $L(T_{n,m})$. The line graph $L(T_{n,m})$ of the dragon graph consists of a cycle of nodes $\{e_1,e_2,\ldots,e_n\}$ and the path of nodes $\{f_1,f_2,\ldots,f_m\}$, as shown in Figure 3.

The result of the MD of $L(T_{n,m})$ is presented below.

Theorem 1. *For any integers $m\geq 1$ and $n\geq 4$, we have $\beta(L(T_{n,m}))=2$.*

Proof. For even integers $n\geq 4$, it can be easily verify that $\kappa=\{e_1,e_{n+4/2}\}\subset V(L(T_{n,m}))$ is the metric generator of $L(T_{n,m})$. For odd integers $n\geq 5$, it can also be verified that $\kappa=\{e_1,e_{n+1/2}\}\subset V(L(T_{n,m}))$ is the metric generator of $L(T_{n,m})$. So, $\beta(L(T_{n,m}))=2$.

Now, we will compute the FTMD for $L(T_{n,m})$. □

Theorem 2. *For any integers $m\geq 1$ and $n\geq 4$, we have $\beta'(L(T_{n,m}))=3$.*

Proof. To calculate our required results, the following are the cases. □

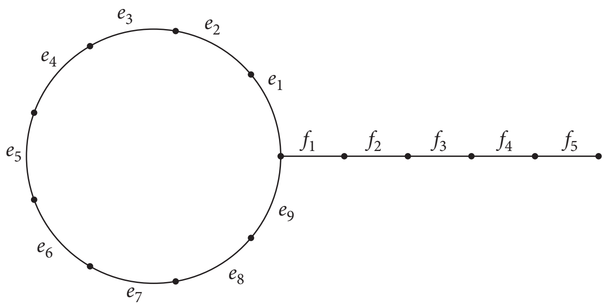

FIGURE 2: Dragon graph $T_{9,5}$.

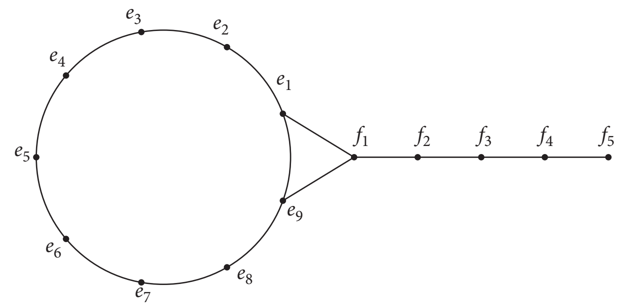

FIGURE 3: Line graph of dragon graph $T_{9,5}$.

Case 1. If n is odd.

Take $\kappa' = \{e_2, e_{n+1/2}, e_{n+3/2}\} \subset V(L(T_{n,m}))$ for odd integers $n \geq 5$. The distance codes of the nodes e_k, where $1 \leq k \leq n$, are

$$r\left(e_k|\kappa\prime\right) = \begin{cases} \left(1, \dfrac{-1+n}{2}, \dfrac{-1+n}{2}\right), & \text{if } k = 1, \\ \left(-2+k, \left|\dfrac{n+2k-1}{2}\right|, \dfrac{-2k+n+3}{2}\right), & \text{if } 2 \leq k \leq \dfrac{3+n}{2}, \\ \left(-k+n+2, \left|\dfrac{-n+2k-1}{2}\right|, \dfrac{-n+2k-3}{2}\right), & \text{if } \dfrac{5+n}{2} \leq k \leq n. \end{cases} \tag{1}$$

The distance codes for the nodes f_k are $r(f_k|\kappa') = (k+1, -1+n+2k/2, -3+n+2k/2)$, for $1 \leq k \leq m$.

From the above codes, we can conclude that the absolute difference codes for every pair of nodes have at least two nonzero in their 3-vector. So, $\beta'(L(T_{n,m})) \leq 3$. From Lemma 1 and Theorem 1, we have $\beta'(L(T_{n,m})) \geq 3$. Hence, $\beta'(L(T_{n,m})) = 3$.

Case 2. If n is even.

Take $\kappa' = \{e_1, e_{n+4/2}, f_1\} \subset V(L(T_{n,m}))$ for every even integers $n \geq 4$. The distance codes of the nodes e_k, where $1 \leq k \leq n$, are

$$r(e_k|\kappa') = \begin{cases} \left(0, \dfrac{-2+n}{2}, 1\right), & \text{if } k = 1, \\ \left(-1+k, \left|\dfrac{-n+2k-4}{2}\right|, k\right), & \text{if } 2 \leq k \leq \dfrac{n}{2}, \\ \left(-k+n+1, \left|\dfrac{-n+2k-4}{2}\right|, -k+n+1\right), & \text{if } \dfrac{n+2}{2} \leq k \leq n. \end{cases} \tag{2}$$

The distance codes for the nodes f_k with respect to W' are $r(f_k \mid \kappa') = (k, -4+n+2k/2, -1+k)$, for $1 \leq k \leq m$.

From the above codes, we can conclude that the absolute difference codes for every pair of nodes have at least two nonzero in their 3-vector. This shows that $\beta'(L(T_{n,m})) \leq 3$, and from Lemma 2, $\beta'(L(T_{n,m})) \geq 3$. Hence, $\beta'(L(T_{n,m})) = 3$.

3. The Fault-Tolerant Resolvability of the Line Graph of Kayak Paddles Graph

Kayak paddles graph $KP(l, m, n)$ is a graph made up of two cycles C_l and C_m having size l and m joined by a path of length n. We label the branches of cycle C_l by $\{e_1, e_2, \ldots, e_l\}$, the branches of cycle C_m by $\{f_1, f_2, \ldots, f_m\}$, and the

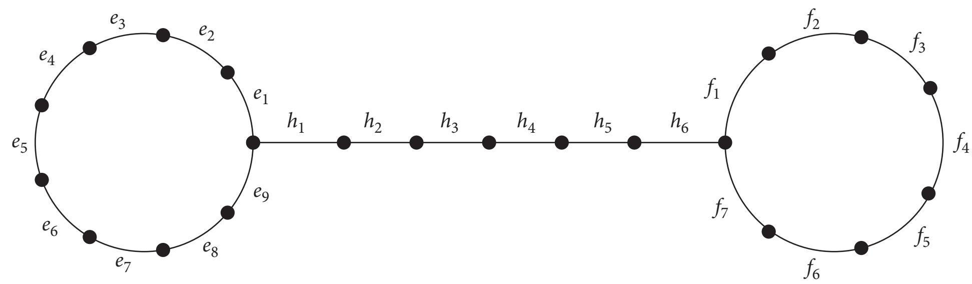

Figure 4: Kayak paddle graph $KP(9,7,6)$.

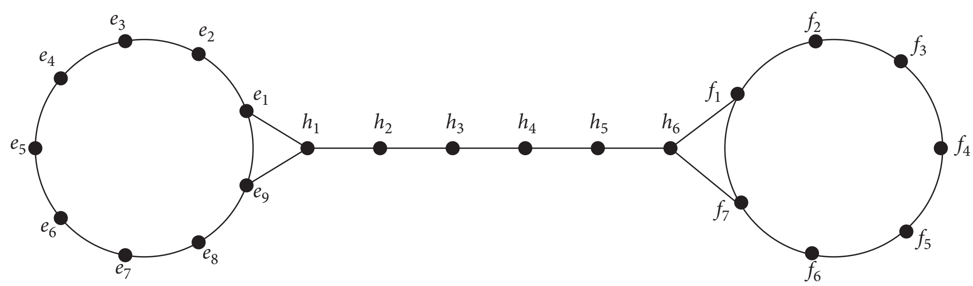

Figure 5: Line graph $L(KP(9,7,6))$.

branches of path joining these cycles by $\{h_1, h_2, \ldots, h_n\}$, as shown in Figure 4.

To compute our required results, we need to convert the graph $KP(l,m,n)$ into the graph $L(KP(l,m,n))$. The line graph of kayak paddles graph $L(KP(l,m,n))$ consists of cycle C_l with nodes $\{e_1, e_2, \ldots, e_l\}$, the cycle C_m with nodes $\{f_1, f_2, \ldots, f_m\}$, and the nodes of path joining these cycles $\{h_1, h_2, \ldots, h_n\}$, as shown in Figure 5.

The known result about $\beta(L(KP(l,m,n)))$ is presented below.

Theorem 3 (see [38]). *For any integers $n \geq 2$ and $l, m \geq 3$, we have $\beta(L(KP(l,m,n))) = 2$.*

Now, we will compute $\beta'(L(KP(l,m,n)))$.

Theorem 4. *For any integers $n \geq 2$ and $l, m \geq 3$, we have $\beta'(L(KP(l,m,n))) = 4$.*

Proof. To calculate our required results, the following are the cases. □

Case 3. If l and m both are even.

Take $\kappa' = \{e_1, e_2, f_1, f_2\} \subset V(L(KP(l,m,n)))$ for both even integers $l, m \geq 4$. The distance codes of the nodes e_k, where $1 \leq k \leq l$, are

$$r(e_k \mid \kappa') = \begin{cases} (-1+k, |-k+2|, k+n, 1+n+k), & \text{if } 1 \leq k \leq \dfrac{l}{2}, \\ \left(\dfrac{l}{2}, \dfrac{-2+l}{2}, \dfrac{l+2n}{2}, \dfrac{2+l+2n}{2}\right), & \text{if } k = \dfrac{2+l}{2}, \\ (-k+l+1, -k+l+2, -k+l+n+1, -k+2+l+n), & \text{if } \dfrac{4+l}{2} \leq k \leq l. \end{cases} \quad (3)$$

The distance codes for the nodes f_k, where $1 \leq k \leq m$, are

$$r(f_k \mid \kappa') = \begin{cases} (k+n, 1+n+k, -1+k, |-2+k|), & \text{if } 1 \le k \le \dfrac{m}{2}, \\ \left(\dfrac{m+2n}{2}, \dfrac{2+m+2n}{2}, \dfrac{m}{2}, \dfrac{-2+m}{2}\right), & \text{if } k = \dfrac{2+m}{2}, \\ (-k+1+m+n, -k+2+m+n, -k+m+1, -k+2+m), & \text{if } \dfrac{m+4}{2} \le k \le m. \end{cases} \quad (4)$$

The distance codes for the nodes h_k are $r(h_k \mid \kappa') = (k, 1+k, -k+1+n, -k+2+n)$, for $1 \le k \le n$.

From the above codes, we can conclude that the absolute difference codes for every pair of nodes have at least two nonzeros in their 4-vector. This shows that $\beta'(L(KP(l,m,n))) \le 4$, but, in Lemma 2, $\beta'(L(KP(l,m,n))) \ge 3$.

Now, to prove that $\beta'(L(KP(l,m,n))) \ge 4$, suppose contrary that $\beta'(L(KP(l,m,n))) = 3$, and according to Lemma 3, we have the following conditions:

(i) Let $\kappa\prime = \{e_i, e_j, e_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i < j < k \le l$; then, $A\ D((f_1, f_m) | \kappa\prime) = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

(ii) Let $\kappa\prime = \{e_i, f_j, f_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i \le l$ and $1 \le j < k \le m$; then, $A\ D((e_1, e_l) | \kappa\prime) = (1,0,0)$. So, $\kappa\prime$ is not FTRS.

(iii) Let $\kappa\prime = \{e_i, e_j, h_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i < j \le l$ and $1 \le k \le n$; then, $A\ D((f_1, f_m) | \kappa') = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

(iv) Let $\kappa' = \{e_i, h_j, h_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i \le l$ and $1 \le j < k \le n$; then, $A\ D((f_1, f_m) | \kappa') = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

(v) Let $\kappa'\{h_i, h_j, h_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i < j < k \le n$; then, $A\ D((e_1, e_l) | \kappa\prime) = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

From the above discussion, we conclude that there is no FTRS with cardinality 3. This shows that $\beta'(L(KP(l,m,n))) \ge 4$. Hence, $\beta'(L(KP(l,m,n))) = 4$.

Case 4. If l and m both are odd.

(1) Let $l = 3$, and for any odd integers $m \ge 3$, take $\kappa\prime = \{e_1, e_3, f_1, f_{m+3/2}\} \subset V(L(KP(3,m,n)))$. The distance codes for the nodes e_k, where $1 \le k \le 3$, are

$$r(e_k \mid \kappa') = \begin{cases} \left(-1+k, 1, k+n, \dfrac{-3+m+2k+2n}{2}\right), & \text{if } 1 \le k \le 2, \\ \left(1, 0, 1+n, \dfrac{-1+m+2n}{2}\right), & \text{if } k = 3. \end{cases} \quad (5)$$

The distance codes for the nodes f_k, where $1 \le k \le m$, are

$$r(f_k \mid \kappa') = \begin{cases} \left(1+n, 1+n, 0, \dfrac{-1+m}{2}\right), & \text{if } k = 1, \\ \left(k+n, n+k, -1+k, \dfrac{-2k+m+3}{2}\right), & \text{if } 2 \le k \le \dfrac{1+m}{2}, \\ \left(-k+1+m+n, -k+1+m+n, -k+m+1, \dfrac{-3-m+2k}{2}\right), & \text{if } \dfrac{3+m}{2} \le k \le m. \end{cases} \quad (6)$$

The distance codes for the nodes h_k are $r(h_k \mid \kappa') = (k, k, -k+n+1, -2k-1m+2n/2)$, for $1 \le k \le n$.

(2) Take $\kappa' = \{e_1, e_2, f_1, f_2\} \subset V(L(KP(l,m,n)))$, for both odd integers $l, m \ge 5$. The distance codes for the nodes e_k, where $1 \le k \le l$, are

$$r(e_k \mid \kappa') = \begin{cases} (-1+k, |-2+k|, n+k, 1+n+k), & \text{if } 1 \le k \le \dfrac{l+1}{2}, \\ \left(\dfrac{-1+l}{2}, \dfrac{-1+l}{2}, \dfrac{-1+l+2n}{2}, \dfrac{1+l+2n}{2}\right), & \text{if } k = \dfrac{3+l}{2}, \\ (-k+1+l, -k+l+2, -k+l+n+1, -k+2+l+n), & \text{if } \dfrac{5+l}{2} \le k \le l. \end{cases} \tag{7}$$

The distance codes for the nodes f_k, where $1 \le k \le m$, are

$$r(f_k \mid \kappa') = \begin{cases} (k+n, 1+n+k, -1+k, |-k+2|), & \text{if } 1 \le k \le \dfrac{1+m}{2}, \\ \left(\dfrac{-1+m+2n}{2}, \dfrac{1+m+2n}{2}, \dfrac{-1+m}{2}, \dfrac{-1+m}{2}\right), & \text{if } k = \dfrac{3+m}{2}, \\ (-k+1+m+n, -k+2+m+n, -k+1+m, -k+2+m), & \text{if } \dfrac{5+m}{2} \le k \le m. \end{cases} \tag{8}$$

The distance codes for the nodes h_k are $r(h_k|\kappa') = (k, 1+k, -k+1+n, -k+2+n)$, for $1 \le k \le n$.

From the above codes, we can conclude that the absolute difference codes for every pair of nodes have at least two nonzeros in their 4-vector. This shows that $\beta'(L(KP(l, m, n))) \le 4$, but in Lemma 2, $\beta'(L(KP(l, m, n))) \ge 3$.

Now, to prove that $\beta'(L(KP(l, m, n))) \ge 4$, suppose contrary that $\beta'(L(KP(l, m, n))) = 3$, and according to Lemma 3, we have the following conditions:

(i) Let $\kappa' = \{e_i, e_j, e_k\} \subset V(L(KP(l, m, n)))$, for $1 \le i < j < k \le l$; then, $A\ D((f_1, f_m)|\kappa') = (0, 0, 0)$. So, $\kappa\prime$ is not FTRS.

(ii) Let $\kappa' = \{e_i, f_j, f_k\} \subset V(L(KP(l, m, n)))$; for $1 \le i \le l$ and $1 \le j < k \le m$, then

$$A\ D\Big((e_1, e_l)|\kappa\prime\Big) = \begin{cases} (0, 0, 0), & \text{if } i = \dfrac{l+1}{2}, \\ (1, 0, 0), & \text{else.} \end{cases} \tag{9}$$

So, $\kappa\prime$ is not FTRS.

(iii) Let $\kappa' = \{e_i, e_j, h_k\} \subset V(L(KP(l, m, n)))$; for $1 \le i < j \le l$ and $1 \le k \le n$, then $A\ D((f_1, f_m)|\kappa\prime) = (0, 0, 0)$. So, $\kappa\prime$ is not FTRS.

(iv) Let $\kappa' = \{e_i, h_j, h_k\} \subset V(L(KP(l, m, n)))$; for $1 \le i \le l$ and $1 \le j < k \le n$, then $A\ D((f_1, f_m)|\kappa\prime) = (0, 0, 0)$. So, $\kappa\prime$ is not FTRS.

(v) Let $\kappa' = \{h_i, h_j, h_k\} \subset V(L(KP(l, m, n)))$; for $1 \le i < j < k \le n$, then $A\ D((e_1, e_l)|\kappa\prime) = (0, 0, 0)$. So, $\kappa\prime$ is not FTRS.

From the above discussion, we conclude that there is no FTRS with cardinality 3. This shows that $\beta'(L(KP(l, m, n))) \ge 4$. Hence, $\beta'(L(KP(l, m, n))) = 4$.

Case 5. If m is even and l is odd.

(i) Let $l = 3$, and for any even integers $m \ge 4$, take $\kappa\prime = \{e_1, e_3, f_1, f_2\} \subset V(L(KP(3, m, n)))$. The distance codes for the nodes e_k, where $1 \le k \le 3$, are

$$r\Big(e_k|\kappa\prime\Big) = \begin{cases} (-1+k, 1, k+n, 1+n+k), & \text{if } 1 \le k \le 2, \\ (1, 0, 1+n, 2+n), & \text{if } k = 3. \end{cases} \tag{10}$$

The distance codes for the nodes f_k, where $1 \le k \le m$, are

$$r\Big(f_k|\kappa\prime\Big) = \begin{cases} (n+k, k+n, -1+k, |-k+2|), & \text{if } 1 \le k \le \dfrac{m}{2}, \\ \left(\dfrac{m+2n}{2}, \dfrac{m+2n}{2}, \dfrac{m}{2}, \dfrac{-2+m}{2}\right), & \text{if } k = \dfrac{2+m}{2}, \\ (-k+1+m+n, -k+1+m+n, -k+m+1, -k+m+2), & \text{if } \dfrac{4+m}{2} \le k \le m. \end{cases} \tag{11}$$

The distance codes for the nodes h_k are $r(h_k|\kappa') = (k, k, -k+n+1, -k+n+2)$, for $1 \le k \le n$.

(2) Take $\kappa' = \{e_1, e_2, f_1, f_2\} \subset V(L(KP(l,m,n)))$ for any odd $l \ge 5$ and even $m \ge 4$ integers. The distance codes for the nodes e_k, where $1 \le k \le l$, are

$$r\left(e_k|\kappa\prime\right) = \begin{cases} (-1+k, |-k+2|, k+n, 1+n+k), & \text{if } 1 \le k \le \dfrac{1+l}{2}, \\ \left(\dfrac{-1+l}{2}, \dfrac{-1+l}{2}, \dfrac{-1+l+2n}{2}, \dfrac{1+l+2n}{2}\right), & \text{if } k = \dfrac{l+3}{2}, \\ (-k+l+1, -k+l+2, -k+1+l+n, -k+2+l+n), & \text{if } \dfrac{l+5}{2} \le k \le l. \end{cases} \tag{12}$$

The distance codes for the nodes f_k, where $1 \le k \le m$, are

$$r(f_k|\kappa') = \begin{cases} (k+n, 1+n+k, -1+k, |-k+2|), & \text{if } 1 \le k \le \dfrac{m}{2}, \\ \left(\dfrac{m+2n}{2}, \dfrac{2+m+2n}{2}, \dfrac{m}{2}, \dfrac{-2+m}{2}\right), & \text{if } k = \dfrac{2+m}{2}, \\ (-k+1+m+n, -k+2+m+n, -k+1+m, -k+2+m), & \text{if } \dfrac{4+m}{2} \le k \le m. \end{cases} \tag{13}$$

The distance codes for the nodes h_W are $r(h_k|W\prime) = (k, 1+k, -k+1+n, -k+2+n)$, for $1 \le k \le n$.

From the above codes, we can conclude that the absolute difference codes for every pair of nodes have at least two nonzeros in their 4-vector. This shows that $\beta'(L(KP(l,m,n))) \le 4$, but, in Lemma 2, $\beta'(L(KP(l,m,n))) \ge 3$.

Now, to prove $\beta'(L(KP(l,m,n))) \ge 4$, suppose contrary that $\beta'(L(KP(l,m,n))) = 3$, and according to Lemma 3, we have the following conditions:

(i) Let $\kappa\prime = \{e_i, e_j, e_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i < j < k \le l$; then, $A\,D((f_1, f_m)|\kappa\prime) = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

(ii) Let $\kappa\prime = \{e_i, f_j, f_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i \le l$ and $1 \le j < k \le m$; then,

$$A\,D\left((e_1, e_l)|\kappa\prime\right) = \begin{cases} (0,0,0), & \text{if } i = \dfrac{l+1}{2}, \\ (1,0,0), & \text{else.} \end{cases} \tag{14}$$

So, $\kappa\prime$ is not FTRS.

(iii) Let $\kappa' = \{e_i, e_j, h_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i < j \le l$ and $1 \le k \le n$; then, $A\,D((f_1, f_m)|\kappa\prime) = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

(iv) Let $\kappa\prime = \{e_i, h_j, h_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i \le l$ and $1 \le j < k \le n$; then, $A\,D((f_1, f_m)|\kappa\prime) = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

(v) Let $\kappa\prime = \{h_i, h_j, h_k\} \subset V(L(KP(l,m,n)))$, for $1 \le i < j < k \le n$; then, $A\,D((e_1, e_l)|\kappa\prime) = (0,0,0)$. So, $\kappa\prime$ is not FTRS.

From the above discussion, we conclude that there is no FTRS with cardinality 3. This shows that $\beta'(L(KP(l,m,n))) \ge 4$. Hence, $\beta'(L(KP(l,m,n))) = 4$.

4. Conclusion

We conclude that, in the case of the line graph for the dragon graph, the FTMD is exactly one more than its MD, and it exactly doubles the MD in the case of the line graph of kayak paddles graph [39–44].

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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