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Research Article

Alpha Power Generalized Inverse Rayleigh Distribution: Its Properties and Applications

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This manuscript is related with the development of Alpha Power Generalized Inverse Rayleigh (APGIR) Distribution. The suggested model provides fit of life time data more efficiently. Some of the important characteristics of the suggested model are obtained including moments, moment generating function, quantile, mode, order statistics, stress-strength parameter, and entropies. Parameter estimates are obtained by MLE technique. The performance of the suggested model is evaluated using real-world data sets. The findings of the simulation and real data sets suggest that the newly proposed model is superior to other current competitor models.

1. Introduction

Rayleigh distribution (RD) is a special model and a modified form of Weibull distribution when shape parameter equals 2. The RD has many applications in various disciplines including engineering and medical sciences, astronomy, and Physics. The RD has been well investigated in the literature. Some researchers have examined its significant properties [1–3]. Hoffman and Karst [4] studied characteristics of the RD and demonstrated how it can be used to analyze the responses of marine vehicles to wave excitation. Dyer and Whisenand [5] also demonstrated the use of RD in communication engineering. Polovko [6] showed how it can be applied to electro vacuum devices. There are various variants of RD recently introduced by researchers that may be used for fitting of data more adequately. Voda [7] proposed generalized Rayleigh (GR) distribution. Voda [8, 9] obtained the ML estimates of the RD. Bhattacharya and Tyagi [10] used RD for the analysis of medical data. Gomes et al. [11] suggested Kumaraswamy generalized Rayleigh (KGR) distribution. Merovci [12] presented transmuted Rayleigh (TR) distribution for investigating lifetime data. Cordeiro et al. [13] developed beta generalized Rayleigh (BGR) distribution. They also studied its main mathematical features. Leao et al. [14] proposed beta inverse Rayleigh (BIR) distribution. Ahmad et al. [15] offered transmuted inverse Rayleigh (TIR) distribution. Iriarte et al. [16] proposed slashed generalized Rayleigh (SGR) distribution. Lalitha and Mishra [17], Ariyawansa and Templeton [18], Howlader and Hossain [19], Sinha and Howlader [20], and Abd Elfattah et al. [21] are just few among others who contributed to RD.

Let X be a random variable having Rayleigh distribution. Symbolically, $X \sim R(\theta)$. Then, its CDF and PDF are

$$\begin{aligned} f(x) &= 2\theta^2 x \exp\left(-(\theta x)^2\right) x \geq 0, \theta > 0, \\ F(x) &= 1 - \exp\left(-(\theta x)^2\right) x \geq 0, \theta > 0, \end{aligned} \tag{1}$$

where θ represents scale parameter.

One important variant of RD is the Inverse Rayleigh Distribution (IRD), an important lifetime distribution. If X follows RD, then $(1/X)$ has the IRD. The PDF and CDF of IRD are provided by

$$g(y;\omega) = \frac{2\omega^2}{x^3}\exp\left(-\left(\frac{\omega}{x}\right)^2\right)x, \omega > 0,$$
$$G(x;\omega) = \exp\left(-\left(\frac{\omega}{x}\right)^2\right)x, \omega > 0. \quad (2)$$

It has several uses in different fields including reliability analysis, engineering, and medicine. Voda [22] used the IRD to estimate the lifetime distribution of many experimental units. Trayer [23] proposed the IRD to accommodate survival and reliability data. Voda [22] discussed several properties and derived expression of ML estimator for parameters of IRD. Mukarjee and Maitim [24] also studied some important statistical properties of IRD. Closed form expressions for some descriptive statistics of the IR distribution were developed by Gharraph [25]. Furthermore, Soliman et al. [26] and Gharraph [25] obtained parameter estimates of IRD using classical and Bayesian estimating approaches, respectively. Various extensions of the IRD are available in the literature. These generalized forms have been used in different disciplines comprising survival and reliability analysis and so on. Rehman and Dar [27], Ahmad et al. [15], and Leao et al. [14] developed EIR, TIR, and BIR distributions, respectively. ShuaibKhan [28] developed a modified form of IRD and discussed it in depth. Potdar and Shirke [29] added an additional shape parameter to scale family of distributions, resulting in generalized inverted scale family of distributions. These distributions fit the complex data better, and conclusions made from them appeared to be quite comprehensive. Mudholkar et al. [30], Gupta et al. [31], Nadarajah and Kotz [32], and Mudholkar and Srivastava [33] studied generalization of several distributions in various statistical publications, generally employed in reliability estimation.

Reshi et al. [34] analyzed scale parameter of Generalized Inverse Rayleigh (GIR) distribution. The GIR distribution is quite good at fitting lifetime data. Some of the applications of GIR distribution include reliability analysis, operations research, applied statistics, and communication engineering. Bakoban and Abu Baker [35] discussed many important characteristics of GIR distribution.

The PDF and CDF of GIR distribution are specified by

$$g(x) = \frac{2\gamma}{\theta^2 x^3}\exp\left(-(\theta x)^{-2}\right)\left[1-\exp\left(-(\theta x)^{-2}\right)\right]^{\gamma-1}, \quad x, \gamma, \theta > 0, \quad (3)$$

$$G(x) = 1-\left[1-\exp\left(-(\theta x)^{-2}\right)\right]^{\gamma}\gamma, \quad \theta > 0. \quad (4)$$

Here, γ and θ represent scale and shape parameter, respectively.

In statistical theory, new distributions have been developed in the last few decades by incorporating a spare parameter, employing generators, or mixing existing distributions [36]. The major goal of doing so is to improve the modelling flexibility of lifetime data when compared with existing distributions.

This article is about the development of new probability distribution, known as Alpha Power Generalized Inverse Rayleigh (APGIR) distribution. This new model is obtained using Alpha Power Transformation [37].

2. Alpha Power Transformation (APT)

The APT was proposed by Mahdavi and Kundu [37]. This technique can be used to develop new distributions by introducing a new parameter into available distributions.

The following is CDF and PDF of APT:

$$F_{\mathrm{APT}}(x) = \begin{cases} \dfrac{\alpha^{F(x)}-1}{\alpha-1}, & \text{if } \alpha > 0,\ \alpha \neq 1. \\ F(x), & \text{if } \alpha = 1, \end{cases} \quad (5)$$

and

$$f_{\mathrm{APT}}(x) = \begin{cases} \dfrac{\log\alpha}{\alpha-1} f(x)\,\alpha^{F(x)}, & \text{if } \alpha > 0,\ \alpha \neq 1, \\ f(x), & \text{if } \alpha = 1. \end{cases} \quad (6)$$

Initially, the proposed method of Mahdavi and Kundu [37] was used for the inclusion of additional parameter in exponential distribution. Later on, some other researchers used APT to some other distributions. Hassan et al. [38] used APT and proposed alpha power transformed extended exponential distribution. Nassar et al. [39] proposed Alpha Power Weibull distribution. Dina and Magdy [40] and Ihtisham et al. [41] introduced alpha power inverse Weibull (APIW) and alpha power Pareto (APP) distribution, respectively.

2.1. The Proposed Model. The main goal of this article is to develop a novel probability distribution termed as Alpha Power Generalized Inverse Rayleigh (APGIR) Distribution and to evaluate its flexibility in modelling life time data. The proposed model is a result of using the PDF and CDF of GIR distribution given in (3) and (4).

A random variable X is said to have Alpha Power Generalized Inverse Rayleigh distributed with three-parameters α, λ, and β if its PDF is given by

$$f_{\text{APGIR}}(x) = \begin{cases} \dfrac{\log \alpha}{\alpha - 1} \dfrac{2\beta}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right)\left[1 - \exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}, & x > 0,\ \alpha > 1, \beta.\lambda > 0, \\ f(x), & \alpha = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Definition 1. . A variable X follows Alpha Power Generalized Inverse Rayleigh distributed with CDF as follows:

$$F_{APGIR}(x) = \begin{cases} \dfrac{\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} - 1}{\alpha - 1}, & \alpha > 1, \\ 0, & \alpha = 1. \end{cases} \tag{8}$$

The following are APGIR Hazard Rate (HR) Function and Survival Function (SF):

$$h_{\text{APGIR}} = \frac{\text{pdf}}{\text{survivalfunction}},$$

$$h_{\text{APGIR}}(x) = \begin{cases} \dfrac{2\beta \log \alpha \ \exp\left(-(\lambda x)^{-2}\right)\left[1 - \exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}}{\lambda^2 x^3 \left(\alpha - \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right)}, & \alpha > 1, \end{cases} \tag{9}$$

$$S_{\text{APGIR}} = 1 - \text{CDF},$$

$$S_{\text{APGIR}}(x) = \begin{cases} \dfrac{\left(\alpha - \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right)}{(\alpha - 1)}, & \alpha > 1. \end{cases} \tag{10}$$

The functions PDF, CDF, HF, and SF are plotted in Figures 1(a), 1(b), 2(a), and 2(b), respectively.

Lemma 1. *If $f(x)$ is a decreasing function for $\alpha < 1$, then $f_{APGIR}(x)$ is also decreasing function.*

Proof. If $f(x)$ is differentiable function and $(\mathrm{d}/\mathrm{d}x)\log f(x) < 0$, then $f(x)$ is also decreasing function and vice versa.

Taking the first derivative of the following expression, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}x}\log f_{\text{APGIR}}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\left[\frac{\log \alpha}{\alpha - 1} \frac{2\beta}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right)\left[1 - \exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\log f_{\text{APTPR}}(x) = -\frac{3}{x} - \frac{2\exp\left(-(\lambda x)^{-2}\right)}{\lambda^2 x^3}\left[1 + (\beta - 1) + \beta \log \alpha\left(1 - \exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-1}\right]. \tag{11}$$

For non-negative and less than 1 values of α and for λ and $\beta > 0$, it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}x}\log f_{\text{APGIR}}(x) < 0. \tag{12}$$

Hence, for $\alpha < 1$, $f_{\text{APGIR}}(x)$ is decreasing function. □

Lemma 2. *If $f(x)$ is decreasing function for $\alpha < 1$, $f(x)$ is log-convex and then $h_{APGIR}(x)$ is decreasing function.*

Proof. If $\mathrm{d}^2/\mathrm{d}x^2$, $f(x)$ exist and $(\mathrm{d}^2/\mathrm{d}x^2)\log f(x) > 0$, then $f(x)$ is log-convex.

Differentiating (11), we get

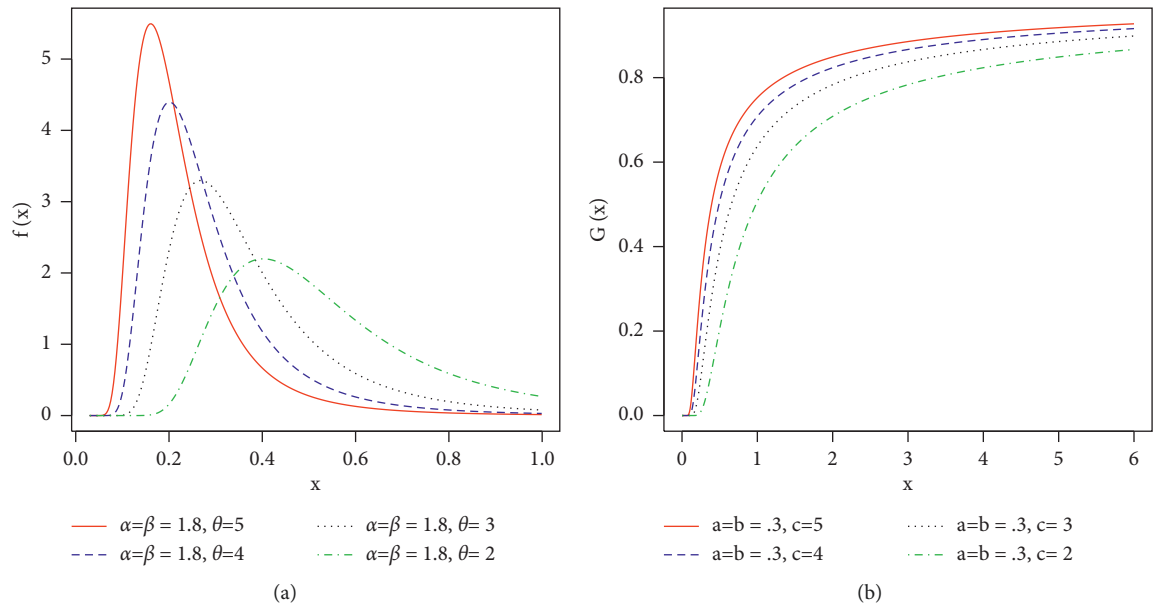

Figure 1: (a) Graph of PDF. (b) Graph of CDF.

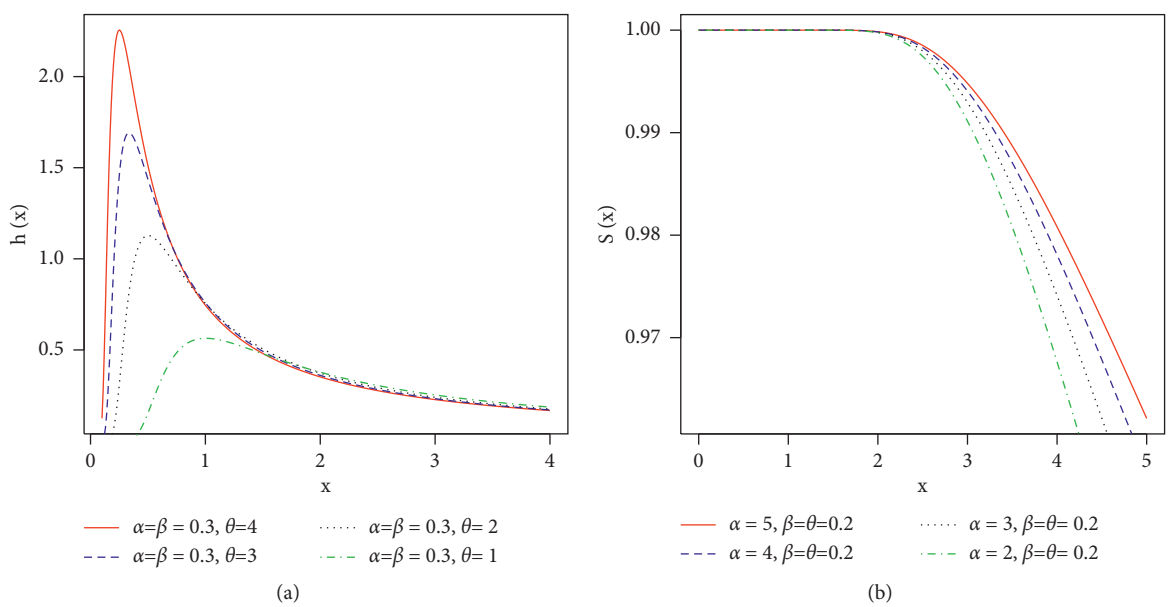

Figure 2: (a) Graphs of HF. (b) Graphs of SF.

$$\frac{d^2}{dx^2}\log f_{\text{APGIR}}(x) = \frac{3}{x^2} - \frac{2}{\lambda^2}\left[\begin{array}{c} \dfrac{\exp\left(-(\lambda x)^{-2}\right)\left(2-3\lambda^2x^2\right)}{\lambda^2x^6}\left\{1+(\beta-1)+\beta\log\alpha\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-1}\right\} \\ -\dfrac{\exp\left(-2(\lambda x)^{-2}\right)}{\lambda^2x^6}\left\{2\beta\log\alpha(\beta-1)\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-2}\right\} \end{array}\right]. \tag{13}$$

When α is non-negative and less than 1 and when λ and $\beta > 0$, then $(d^2/dx^2)\log f_{\text{APGIR}}(x) > 0$.

Thus, when $0 < \alpha < 1$, $f_{\text{APGIR}}(x)$ is log-convex [42]. □

2.2. Quantile Function (QF). Let $X \sim \text{APGIR}(\alpha, \lambda, \beta)$, then the QF is described by

$$x = F^{-1}(u), \tag{14}$$

where $u \sim U[0,1]$. The QF of APGIR distribution is

$$\frac{\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}-1}{\alpha-1}=u. \tag{15}$$

After simplification, we have

$$X_p=\frac{1}{\lambda}\left[-\log\left\{1-\left\{\frac{\log\alpha-\log\{u(\alpha-1)+1\}}{\log\alpha}\right\}^{1/\beta}\right\}\right]^{-1/2}. \tag{16}$$

2.3. Median. To obtain median, we have

$$\frac{\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}-1}{\alpha-1}=\frac{1}{2}. \tag{17}$$

After some calculations, we obtain the following result of median:

$$\text{Median}=m=\frac{1}{\lambda}\left[-\log\left\{1-\left\{\frac{2\log\alpha-(\alpha+1)}{2\log\alpha}\right\}^{1/\beta}\right\}\right]^{-1/2}. \tag{18}$$

2.4. Mode. To obtain mode, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}f_{\text{APGIR}}(x)=0\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\log\alpha}{\alpha-1}\ \frac{2\beta}{\lambda^2x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right)=0,$$

$$\begin{bmatrix}-3x^{-4}\exp\left(-(\lambda x)^{-2}\right)\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-1}\alpha^{1-\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta}}+\\ x^{-3}\left(\exp\left(-(\lambda x)^{-2}\right)\left(2\lambda(\lambda x)^{-3}\right)\right)\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-1}\alpha^{1-\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta}}\\ -x^{-3}\exp\left(-(\lambda x)^{-2}\right)\left((\beta-1)\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-2}\left(\exp\left(-(\lambda x)^{-2}\right)2\lambda(\lambda x)^{-3}\right)\right)\alpha^{1-\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta}}+\\ x^{-3}\exp\left(-(\lambda x)^{-2}\right)\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-1}\\ \alpha^{1-\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta}}\log\alpha\left(\beta\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta-1}\left(\exp\left(-(\lambda x)^{-2}\right)2\lambda(\lambda x)^{-3}\right)\right)\end{bmatrix}=0, \tag{19}$$

$$\frac{-3}{x}+\frac{2}{\lambda^2x^3}\left[1-(\beta-1)\exp\left(-(\lambda x)^{-2}\right)\left\{\frac{1}{\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)}\right\}+\beta\exp\left(-(\lambda x)^{-2}\right)\left\{1-\exp\left(-(\lambda x)^{-2}\right)\right\}\right]=0.$$

Equation (19) is satisfied by mode of APGIR distribution.

2.5. R^{th} Moment of APGIR Distribution. Let $X \sim$APGIR (α,λ,β), then the following is the rth moment:

$$\mu'_r=E(X^r)=\int_0^{\infty}x^r\frac{\log\alpha}{\alpha-1}\ \frac{2\beta}{\lambda^2x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\mathrm{d}x. \tag{20}$$

Put in (20) $1-\exp(-(\lambda x)^{-2})=y\Rightarrow 2/\lambda^2x^3\exp(-(\lambda x)^{-2})\mathrm{d}x=-\mathrm{d}y$, $x=1/\lambda[-\log(1-y)]^{-1/2}$, and $x\longrightarrow 0, y\longrightarrow 1$, $x\longrightarrow\infty, y\longrightarrow 0$.

$$\mu'_r=\frac{\beta\log\alpha}{(\alpha-1)}\left(\frac{1}{\lambda}\right)^r\int_0^1(-\log(1-y))^{\frac{-r}{2}}y^{\beta-1}\alpha^{1-y^{\beta}}\mathrm{d}y. \tag{21}$$

Let $y^{\beta}=z\Rightarrow\beta y^{\beta-1}\mathrm{d}y=\mathrm{d}z$, $\begin{matrix}y\longrightarrow 0, z\longrightarrow 0\\ y\longrightarrow 1, z\longrightarrow 1\end{matrix}$, and $y=z^{1/\beta}$.

$$(21)\Rightarrow\mu'_r=\frac{\alpha\log\alpha}{(\alpha-1)}\left(\frac{1}{\lambda}\right)^r\int_0^1\left(-\log\left(1-z^{1/\beta}\right)\right)^{-r/2}\alpha^{-z}\mathrm{d}z. \tag{22}$$

Using the following series representation in (22),

$$-\log\left(1-z^{1/\beta}\right)=\sum_{m=1}^{\infty}\frac{(-1)^m\left(-z^{1/\beta}\right)^m}{m}\text{ for }\left|z^{1/\beta}\right|<1,$$
$$\alpha^{-z}=\sum_{k=0}^{\infty}\frac{(-\log\alpha)^k}{k!}(z)^k, \tag{23}$$

to have

$$\mu'_r=\frac{\alpha\log\alpha}{(\alpha-1)}\left(\frac{1}{\lambda}\right)^r\sum_{k=0}^{\infty}(-\log\alpha)^k/k!\int_0^1\left(\sum_{m=1}^{\infty}(-1)^m\left(-z^{1/\beta}\right)^m/m\right)^{-r/2}(z)^k\mathrm{d}z. \tag{24}$$

The expression of μ'_r is incomplete integral; therefore, it can be solved approximately using numerical integration techniques.

2.6. Moment Generating Function (MGF). Let $X\sim$APGIR (α,λ,β), then MGF is defined as follows:

$$M_x(t)=E\left(e^{tx}\right)=\int_0^1 e^{tx}\frac{\log\alpha}{\alpha-1}\frac{2\beta}{\lambda^2x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\mathrm{d}x. \tag{25}$$

Using series notation $e^{tx}=\sum_{r=0}^{\infty}t^rx^r/r!$ in (25), we get

$$M_x(t)=\sum_{r=0}^{\infty}(t)^r/r!\int_0^{\infty}x^r\frac{2\beta}{\lambda^2x^3}\frac{\log\alpha}{(\alpha-1)}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\mathrm{d}x. \tag{26}$$

Utilize (24) in (26), we get

$$M_x(t)=\frac{\alpha\log\alpha}{(\alpha-1)}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^r}{r!}\frac{(-\log\alpha)^k}{\lambda^rk!}\int_0^1\left(\sum_{m=1}^{\infty}\frac{(-1)^m\left(-z^{1/\beta}\right)^m}{m}\right)^{-r/2}z^k\mathrm{d}z. \tag{27}$$

The result in equation (27) is incomplete integral, and it may be solved on the basis of numerical integration methods.

2.7. Mean Residual Life Function (MRLF). The MRLF is the average remaining life of a component that has survived till time t. Here, X is lifetime of an object with $f(x)$ and $S(x)$ provided in (7) and (10), respectively. The MRLF is given by

$$\mu(t)=\frac{1}{S(t)}\left(E(t)-\int_0^1 xf(x)\mathrm{d}x\right)-t,\quad t\geq 0, \tag{28}$$

where

$$\int_0^1 xf(x)\mathrm{d}x=\int_0^1 x\frac{\log\alpha}{\alpha-1}\frac{2\beta}{\lambda^2x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\mathrm{d}x. \tag{29}$$

Let $1-\exp(-(\lambda x)^{-2}) = y \Rightarrow (2/\lambda^2 x^3)\exp(-(\lambda x)^{-2})\mathrm{d}x = -\mathrm{d}y, x = (1/\lambda)(-\log(1-y))^{-1/2}$.

Then,

$$(30) \Rightarrow \int_0^t x f(x)\mathrm{d}x = -\frac{\log\alpha}{\lambda(\alpha-1)}\int_1^{1-\exp\left(-(\lambda t)^{-2}\right)} (-\log(1-y))^{-1/2}\beta y^{\beta-1}\alpha^{1-y^\beta}\mathrm{d}y. \tag{31}$$

Put $1-y^\beta = z \Rightarrow \beta y^{\beta-1}\mathrm{d}y = -\mathrm{d}z,\ y = (1-z)^{1/\beta}$ to have

$$\int_0^1 x f(x)\mathrm{d}x = \frac{\log\alpha}{\lambda(\alpha-1)}\int_0^{\left(1-\left(1-\exp\left(-(\lambda t)^{-2}\right)\right)^\beta\right)} \left(-\log\left(1-(1-z)^{1/\beta}\right)\right)^{-1/2}\alpha^z\mathrm{d}z. \tag{32}$$

Using the following series representation in (32), we have $-\log(1-(1-z)^{1/\beta}) = \sum_{k=1}^{\infty}(-1)^k(-(1-z)^{1/\beta})^k/k$ and $\alpha^z = \sum_{n=0}^{\infty}(\log\alpha)^n/n!(z)^n$:

$$\int_0^1 x f(x)\mathrm{d}x = \frac{\log\alpha}{\lambda(\alpha-1)}\sum_{n=0}^{\infty}\frac{(\log\alpha)^n}{n!}\int_0^{\left(1-\left(1-\exp\left(-(\lambda t)^{-2}\right)\right)^\beta\right)} \left(\sum_{k=1}^{\infty}(-1)^k\left(-(1-z)^{\frac{1}{\beta}}\right)^k/k\right)^{-1/2}(z)^n\mathrm{d}z. \tag{33}$$

The expression in (33) is an integral that is incomplete. This may be solved approximately using numerical integration techniques.

$$E(t) = \int_0^\infty t f(t) = \int_0^\infty t\frac{\log\alpha}{\alpha-1}\frac{2\beta}{\lambda^2 t^3}\exp\left(-(\lambda t)^{-2}\right)\left[1-\exp\left(-(\lambda t)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda t)^{-2}\right)\right]^\beta}\mathrm{d}t. \tag{34}$$

Insert $1-\exp(-(\lambda t)^{-2}) = y \Rightarrow (2/\lambda^2 t^3 \& Exponential E;)\ x\ p(-(\lambda\ t)^{-2})\mathrm{d}t = -\mathrm{d}y$ and $t = (1/\lambda)(-\log(1-y))^{-1/2}$ in (34) to have

$$E(t) = \frac{\beta\log\alpha}{\lambda(\alpha-1)}\int_0^1\left((-\log(1-y))^{-1/2}\right)y^{\beta-1}\alpha^{1-y^\beta}\mathrm{d}y. \tag{35}$$

Put $1-y^\beta = z \Rightarrow \beta y^{\beta-1}\mathrm{d}y = -\mathrm{d}z,\ y = (1-z)^{1/\beta}$ in (35) to have

$$E(t) = \frac{\log\alpha}{\lambda(\alpha-1)}\int_0^1\left(-\log\left(1-(1-z)^{1/\beta}\right)\right)^{-1/2}\alpha^z\mathrm{d}z. \tag{36}$$

Using the following series representation in (36), we have

$$\begin{aligned} -\log\left(1-(1-z)^{1/\beta}\right) &= \sum_{k=1}^{\infty}(-1)^k\left(-(1-z)^{1/\beta}\right)^k/k, \\ \alpha^z &= \sum_{n=0}^{\infty}(\log\alpha)^n/n!(z)^n, \\ E(t) &= \frac{\log\alpha}{\lambda(\alpha-1)}\sum_{n=0}^{\infty}\frac{(\log\alpha)^n}{n!}\int_0^1\left(\sum_{k=1}^{\infty}\frac{(-1)^k\left(-(1-z)^{1/\beta}\right)^k}{k}\right)^{-1/2}z^n\mathrm{d}z. \end{aligned} \tag{37}$$

Putting (10), (31), and (35) in (28), we get

$$\mu(t) = \frac{\log \alpha}{\lambda\left(\alpha - \alpha^{1-\left[1-e^{-(\lambda x)^{-2}}\right]^{\beta}}\right)} \sum_{n=0}^{\infty} \frac{(\log \alpha)^n}{n!} \left[\int_0^1 \left(\sum_{k=1}^{\infty} \frac{(-1)^k\left(-(1-z)^{1/\beta}\right)^k}{k}\right)^{-1/2} z^n \mathrm{d}z - \int_0^{\left(1-\left(1-e^{-(\lambda t)^{-2}}\right)^{\beta}\right)} \left(\sum_{k=1}^{\infty} \frac{(-1)^k\left(-(1-z)^{1/\beta}\right)^k}{k}\right)^{-1/2} z^n \mathrm{d}z\right] - t. \tag{38}$$

The result of $\mu(t)$ is an incomplete integral. Numerically, it can be approximated utilizing numerical integration techniques.

2.8. Order Statistics. Suppose $X_1, X_2, X_3, \ldots, X_n$ denote sample of size n. The corresponding order statistics are $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$. The PDF of ith order statistic is specified by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)\,[F(x)]^{i-1}\,[1-F(x)]^{(n-i)}. \tag{39}$$

Substituting $f(x)$ and $F(x)$ in (39), we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!\,(\alpha-1)^n} \frac{2\beta \log \alpha}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right)\left[\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} - 1\right]^{(i-1)} \left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} \cdot \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} \left[\alpha - \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right]^{n-i}. \tag{40}$$

We get PDF of the smallest order statistic by inserting $i = 1$ in (40), that is,

$$f_{1:n}(x) = \frac{n}{(\alpha-1)^n} \frac{2\beta \log \alpha}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} \left[\alpha - \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right]^{n-1}. \tag{41}$$

Put $i = n$ in (40), we acquire the PDF of the largest order statistic

$$f_{n:n}(x) = \frac{n}{(\alpha-1)^n} \frac{2\beta \log \alpha}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right)\left[\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} - 1\right]^{(n-1)} \left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}. \tag{42}$$

To get distribution of the median, substitute $i = n/2$ in (40) as

$$f_{\frac{n}{2}:n}(x) = \frac{n!}{(n/2-1)!\,(n-(n/2))!\,(\alpha-1)^n} \frac{2\beta \log \alpha}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right)\left[\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} - 1\right]^{(n/2-1)} \cdot \left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} \left[\alpha - \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right]^{n-n/2}. \tag{43}$$

2.9. Stress-Strength Parameter (SSP). Let X_1 and X_2 be two independent and identically distributed random variables. Suppose $X_1 \sim \mathrm{APGIR}(\alpha_1, \lambda, \beta_1)$ and $X_2 \sim \mathrm{APGIR}(\alpha_2, \lambda, \beta_2)$. The SSP is defined as follows:

$$R = \int_{-\infty}^{\infty} f_1(x) F_2(x) \mathrm{d}x. \tag{44}$$

The SSP is calculated, by incorporating (7) and (8) in the above equation:

$$R = \int_0^{\infty} \left(\frac{\log \alpha_1}{\alpha_1 - 1} \frac{2\beta_1}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right) \left[1 - \exp\left(-(\lambda x)^{-2}\right)\right]^{\beta_1 - 1} \alpha_1^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta_1}} \right) \left(\frac{\alpha_2^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta_2}} - 1}{\alpha_2 - 1} \right) \mathrm{d}x.$$

$$R = \int_0^{\infty} \left(\frac{\log \alpha_1}{\alpha_1 - 1} \frac{2\beta_1}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right) \left[1 - \exp\left(-(\lambda x)^{-2}\right)\right]^{\beta_1 - 1} \alpha_1^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta_1}} \right) \left(\frac{\alpha_2^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta_2}} - 1}{\alpha_2 - 1} \right) \mathrm{d}x - \frac{1}{(\alpha_2 - 1)}. \tag{45}$$

Substituting $1 - \exp(-(\lambda x)^{-2}) = y \Rightarrow (2/\lambda^2 x^3) \exp(-(\lambda x)^{-2})\, \mathrm{d}x = -\mathrm{d}y$ in (45), we have

$$R = \frac{\beta \alpha_1 \alpha_2 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)} \int_0^1 y^{\beta_1 - 1} \alpha_1^{-y^{\beta_1}} \alpha_2^{-y^{\beta_2}} \mathrm{d}y - \frac{1}{(\alpha_2 - 1)}. \tag{46}$$

Using series representation $\alpha_1^{-y^{\beta_1}} = \sum_{k=0}^{\infty} (-\log \alpha_1)^k / k! (y^{\beta_1})^k$ and $\alpha_2^{-y^{\beta_2}} = \sum_{m=0}^{\infty} (-\log \alpha_2)^m / m! (y^{\beta_2})^m$ in (46) and simplifying, we get the following final result for stress-strength parameter:

$$R = \frac{\beta \alpha_1 \alpha_2 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\log \alpha_1)^k (-\log \alpha_2)^m}{k! m! (\beta_1 (1 + k) + \beta_2 m)} - \frac{1}{(\alpha_2 - 1)}. \tag{47}$$

Lemma 3. *Let $X \sim APGIR(\alpha, \lambda, \beta)$, then final expression for Renyi entropy is given as follows:*

$$RE_X(\nu) = \frac{1}{(1 - \nu)} \left\{ \nu \log\left(\frac{2\beta \log \alpha}{\lambda^2 (\alpha - 1)} \right) - \log\left(\frac{\alpha^{\nu} \nu^k 2^{r-1}}{\lambda^{1-3\nu}} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\log \alpha)^k (-1)^{k+l}}{r! k!} \int_0^{\infty} t^{l+r} \left(1 - e^t\right)^m \mathrm{d}t \right) \right\}. \tag{48}$$

Proof. Renyi entropy is defined as

$$RE_X(\nu) = \frac{1}{1 - \nu} \log\left\{ \int_{-\infty}^{+\infty} f(x)^{\nu} \mathrm{d}x \right\},$$

$$RE_X(\nu) = \frac{1}{1 - \nu} \log\left[\int_0^{\infty} \left(\frac{\log \alpha}{\alpha - 1} \frac{2\beta}{\lambda^2 x^3} \exp\left(-(\lambda x)^{-2}\right) \left[1 - \exp\left(-(\lambda x)^{-2}\right)\right]^{\beta - 1} \alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}} \right)^{\nu} \right] \mathrm{d}x,$$

$$RE_X(\nu) = \frac{1}{(1 - \nu)} \left\{ \log\left(\frac{2\beta \log \alpha}{\lambda^2 (\alpha - 1)} \right)^{\nu} + \log\left(\int_0^{\infty} x^{-3\nu} \exp\left(-\nu (\lambda x)^{-2}\right) \left(1 - \exp\left(-(\lambda x)^{-2}\right)\right)^{\nu(\beta - 1)} \alpha^{\nu - \nu\left(1-\exp\left(-(\lambda x)^{-2}\right)\right)^{\beta}} \mathrm{d}x \right) \right\}. \tag{49}$$

Substitute $1 - \exp(-(\lambda x)^{-2}) = y$ to have

$$RE_X(\nu) = \frac{1}{(1 - \nu)} \left\{ \nu \log\left(\frac{2\beta \log \alpha}{\lambda^2 (\alpha - 1)} \right) + \log\left(\frac{\alpha^{\nu}}{2\lambda^{1-3\nu}} \int_0^1 [-\log(1 - y)]^{3(\nu - 1)/2} (1 - y)^{\nu - 1} y^{\nu(\beta - 1)} \alpha^{-\nu y^{\beta}} \mathrm{d}y \right) \right\}. \tag{50}$$

Using series representation $\alpha^{-\nu y^{\beta}} = \sum_{k=0}^{\infty} (-\log \alpha)^k / k! (\nu y^{\beta})^k$ in the above equation, we get

$$RE_X(\nu) = \frac{1}{(1-\nu)} \left\{ \nu \log\left(\frac{2\beta \log \alpha}{\lambda^2 (\alpha - 1)}\right) + \log\left(\frac{\alpha^{\nu} \nu^k}{2\lambda^{1-3\nu}} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^k}{k!} \int_0^1 [-\log(1-y)]^{3(\nu-1)/2} (1-y)^{\nu-1} y^{\nu(\beta-1)+\beta k} \mathrm{d}y \right) \right\}. \quad (51)$$

Using $\log(1-y) = t \Rightarrow \mathrm{d}y = (y-1)\mathrm{d}t \Rightarrow \mathrm{d}y = -e^t \mathrm{d}t$ in (51) and simplifying, we get

$$RE_X(\nu) = \frac{1}{(1-\nu)} \left\{ \nu \log\left(\frac{2\beta \log \alpha}{\lambda^2 (\alpha - 1)}\right) - \log\left(\frac{\alpha^{\nu} \nu^k 2^{r-1}}{\lambda^{1-3\nu}} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\log \alpha)^k (-1)^{k+l}}{r! k!} \int_0^1 t^{l+r} \left(1 - e^t\right)^m \mathrm{d}t \right) \right\}. \quad (52)$$

The expression of Renyi entropy is an incomplete integral. The solution of (52) is obtained on the basis of numerical integration techniques. □

Lemma 4. *The Mean Waiting Time (MWT) say $\overline{\mu}(t)$ is given by*

$$\overline{\mu}(t) = t - \frac{\log \alpha}{\lambda\left[\alpha^{1-\left[1-e^{-(\lambda t)^{-2}}\right]^{\beta}} - 1\right]} \sum_{n=0}^{\infty} \frac{(\log \alpha)^n}{n!} \int_0^{\left(1-\left(1-e^{-(\lambda t)^{-2}}\right)^{\beta}\right)} \left(\sum_{k=1}^{\infty} \frac{(-1)^k \left(-(1-z)^{1/\beta}\right)^k}{k} \right)^{-1/2} z^n dz\,. \quad (53)$$

Proof. The MWT of APGIR distribution is described as

$$\overline{\mu}(t) = t - \frac{1}{F(t)} \int_0^t x f(x) \mathrm{d}x. \quad (54)$$

Substituting the following results in (54),

$$\int_0^t x f(x) dx = \frac{\log \alpha}{\lambda(\alpha - 1)} \sum_{n=0}^{\infty} \frac{(\log \alpha)^n}{n!} \int_0^{\left(1-\left(1-e^{-(\lambda t)^{-2}}\right)^{\beta}\right)} \left(\sum_{k=1}^{\infty} \frac{(-1)^k \left(-(1-z)^{1/\beta}\right)^k}{k} \right)^{-1/2} (z)^n \mathrm{d}z, \quad (55)$$

and

$$F(t) = \frac{\alpha^{1-\left[1-\exp\left(-(\lambda t)^{-2}\right)\right]^{\beta}} - 1}{\alpha - 1}, \quad (56)$$

we obtain the required final expression as

$$\overline{\mu}(t) = t - \frac{(\alpha - 1)}{\left(\alpha^{1-\left[1-\exp\left(-(\lambda t)^{-2}\right)\right]^{\beta}} - 1\right)} \left[\frac{\log \alpha}{\lambda(\alpha - 1)} \sum_{n=0}^{\infty} \frac{(\log \alpha)^n}{n!} \int_0^{\left(1-\left(1-\exp\left(-(\lambda t)^{-2}\right)\right)^{\beta}\right)} \left(\sum_{k=1}^{\infty} \frac{(-1)^k \left(-(1-z)^{1/\beta}\right)^k}{k} \right)^{-1/2} (z)^n \mathrm{d}z \right]. \quad (57)$$

The expression for $\overline{\mu}(t)$ is an integral that is incomplete. The solution of (57) may be obtained by numerical integration techniques. □

Lemma 5. *The Shannon entropy (SE) expression is given as follows:*

$$S.E_x = \log\left[\left(\frac{\log\alpha}{\alpha-1}\right)^2 2\lambda\beta\sum_{n=0}^{\infty}\frac{(\log\alpha)^n 2^n}{n!}\int_0^1\left((1-z)\left((1-z)^{-1/\beta}-1\right)\right)z^n\left(\sum_{k=1}^{\infty}\frac{(-1)^k\left(-(1-z)^{1/\beta}\right)^k}{k}\right)^{3/2}\mathrm{d}z\right]. \tag{58}$$

Proof. The Shannon entropy is described by

$$S.E_x = E[-\log f(x)] = E\left[-\log\left\{\frac{\log\alpha}{\alpha-1}\ \frac{2\beta}{\lambda^2x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right\}\right]. \tag{59}$$

$$S.E_x = -\log\left[\left(\frac{2\beta\log\alpha}{\lambda^2(\alpha-1)}\right)E\left\{\frac{1}{x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right\}\right],$$

$$S.E_x = -\log\left[\left(\frac{2\beta\log\alpha}{\lambda^2(\alpha-1)}\right)\int_0^{\infty}\frac{\left\{\frac{1}{x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right\}}{\left\{\frac{\log\alpha}{\alpha-1}\ \frac{2\beta}{\lambda^2x^3}\exp\left(-(\lambda x)^{-2}\right)\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1}\alpha^{1-\left[1-\exp\left(-(\lambda x)^{-2}\right)\right]^{\beta}}\right\}}\mathrm{d}x\right]. \tag{60}$$

Putting $1-\exp(-(\lambda x)^{-2}) = y$ in (60), we get

$$S.E_x = \log\left[\left(\frac{2\beta}{\lambda^2}\left(\frac{\log\alpha}{\alpha-1}\right)^2\right)\int_0^1(1-y)\beta y^{\beta-1}\alpha^{1-y^{\beta}}\frac{1}{\left((1/\lambda)\{-\log(1-y)\}^{-1/2}\right)^3}y^{\beta-1}\alpha^{1-y^{\beta}}\mathrm{d}x\right]. \tag{61}$$

Insert in (61), $1-y^{\beta} = z \Rightarrow \beta y^{\beta-1}\mathrm{d}y = -\mathrm{d}z$, and $y = (1-z)^{1/\beta}$ to have

$$S.E_x = \log\left[\left(2\beta\lambda\left(\frac{\log\alpha}{\alpha-1}\right)^2\right)\int_0^1\left((1-z)\left((1-z)^{-1/\beta}-1\right)\right)\alpha^{2z}\left\{-\log\left(1-(1-z)^{1/\beta}\right)\right\}^{3/2}\mathrm{d}z\right]. \tag{62}$$

Using the following series in (62), $\alpha^{2z} = \sum_{m=0}^{\infty}(\log\alpha)^m(2z)^m/m!$ and $-\log(1-(1-z)^{1/\beta}) = \sum_{k=1}^{\infty}(-1)^k(-(1-z)^{1/\beta})^k/k$, for $|(1-z)^{1/\beta}| < 1$.

We get the Shannon entropy as

$$S.E_x = \log\left[\left(\frac{\log\alpha}{\alpha-1}\right)^2 2\lambda\beta\sum_{n=0}^{\infty}\frac{(\log\alpha)^n 2^n}{n!}\int_0^1\left((1-z)\left((1-z)^{-1/\beta}-1\right)\right)z^n\left(\sum_{k=1}^{\infty}\frac{(-1)^k\left(-(1-z)^{1/\beta}\right)^k}{k}\right)^{3/2}\mathrm{d}z\right]. \tag{63}$$

The integral in (63) may be solved approximately with the help of numerical integration techniques. □

3. Parameters Estimation

3.1. Maximum Likelihood Estimation. Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from APGIR (α, λ, β), then likelihood function is as follows:

$$l(\alpha,\lambda,\beta) = \left(\frac{2\beta \log \alpha}{\lambda^2(\alpha-1)}\right)^n \frac{1}{\prod_{i=1}^n x_i^3} e^{-\sum_{i=1}^n (\lambda x_i)^{-2}} \left[\prod_{i=1}^n \left(1-\exp\left(-(\lambda x)^{-2}\right)\right)\right]^{\beta-1} \alpha^{n-\sum_{i=1}^n \{1-\exp(-(\lambda x)^{-2})\}^{\beta}}. \tag{64}$$

Taking logarithm, (64) becomes

$$\begin{aligned}\log l(\alpha,\lambda,\beta) &= n\log(2\beta\log\alpha) - n\log\left\{\lambda^2(\alpha-1)\right\} - \sum_{i-1}^n \log x_i^3 - \sum_{i=1}^n (\lambda x_i)^{-2} + \\ &\cdot (\beta-1)\left[\sum_{i=1}^n \log\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)\right] + \log\alpha\left[n - \sum_{i=1}^n \left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta}\right].\end{aligned} \tag{65}$$

By differentiating (65) w.r.t $\alpha, \lambda,$ and β and equating to 0, we get the following equations:

$$\frac{\partial \log\ l(\alpha,\lambda,\beta)}{\partial\alpha} = \frac{n}{\alpha\log\alpha} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\left[n - \sum_{i=1}^n \left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta}\right] = 0, \tag{66}$$

$$\begin{aligned}\frac{\partial \log\ l(\alpha,\lambda,\beta)}{\partial\lambda} &= -\frac{2n}{\lambda} + 2\sum_{i=1}^n x_i(\lambda x_i)^{-3} - 2(\beta-1)\sum_{i=1}^n \left[\frac{x_i(\lambda x_i)^{-3}\exp\left(-(\lambda x_i)^{-2}\right)}{\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)}\right] \\ &+ 2\beta\ln\alpha\sum_{i=1}^n x_i(\lambda x_i)^{-3}\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta-1}\exp\left(-(\lambda x_i)^{-2}\right) = 0,\end{aligned} \tag{67}$$

$$\begin{aligned}\frac{\partial \log\ l(\alpha,\lambda,\beta)}{\partial\beta} &= \frac{n}{\beta} + \left[\sum_{i=1}^n \log\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)\right] \\ &- \log\alpha\left[\sum_{i=1}^n \left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta}\log\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)\right] = 0.\end{aligned} \tag{68}$$

We can get estimates of $\alpha, \lambda,$ and β by solving (64), (65), and (66) together. The Newton–Raphson technique was adopted for the solution of aforementioned equations. The ML estimators are asymptotically normally distributed, that is, $\sqrt{n}(\hat{\alpha}-\alpha, \hat{\lambda}-\lambda, \hat{\beta}-\beta) \sim N_3(0, \Sigma)$. The matrix $\sum$ is achieved by inverting the observed Fisher information matrix F as follows:

$$F = \begin{bmatrix} \frac{\partial^2 \log l}{\partial\alpha^2} & \frac{\partial^2 \log l}{\partial\alpha\,\partial\lambda} & \frac{\partial^2 \log l}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 \log l}{\partial\lambda\,\partial\alpha} & \frac{\partial^2 \log l}{\partial\lambda^2} & \frac{\partial^2 \log l}{\partial\lambda\,\partial\beta} \\ \frac{\partial^2 \log l}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \log l}{\partial\beta\,\partial\lambda} & \frac{\partial^2 \log l}{\partial\beta^2} \end{bmatrix}. \tag{69}$$

When we differentiate (64)–(66) w.r.t α, λ, and β, we get

$$\begin{aligned}
\frac{\partial^2 \log l}{\partial \alpha^2} &= \frac{-n(1+\log\alpha)}{(\alpha\log\alpha)^2} + \frac{n}{(\alpha-1)^2} - \frac{1}{\alpha^2}\left[n - \sum_{i=1}^{n}\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta}\right],\\
\frac{\partial^2 \log l}{\partial \lambda^2} &= \frac{2n}{\lambda^2} - 6\sum_{i=1}^{n}\frac{1}{x_i^2\lambda^4} - 2(\beta-1)\sum_{i=1}^{n}\frac{1}{x_i}\left[\frac{2\exp\left(-(\lambda x_i)^{-2}\right)}{x_i^2\lambda^6\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)}\right.\\
&\quad \left.-\frac{\exp\left(-(\lambda x_i)^{-2}\right)\left\{3\lambda^2\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right) - 2x_i^{-2}\exp\left(-(\lambda x_i)^{-2}\right)\right\}}{\lambda^6\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)}\right]+\\
&\quad \cdot 2\beta\log\alpha\sum_{i=1}^{n}x_i^2(\lambda x_i)^{-4}\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta-1}\exp\left(-(\lambda x_i)^{-2}\right)\left[-3+2(\lambda x_i)^{-2}\left\{1-\frac{(\beta-1)}{\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)}\right\}\right],\\
\cdot\frac{\partial^2 \log l}{\partial \beta^2} &= \frac{-n}{\beta^2} - \log\alpha\left[\sum_{i=1}^{n}\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta}\left\{\log\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)\right\}^2\right],\\
\cdot\frac{\partial^2 \log l}{\partial \alpha\,\partial \lambda} &= \frac{2\beta}{\alpha\lambda^3}\sum_{i=1}^{n}\frac{\exp\left(-(\lambda x_i)^{-2}\right)\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta-1}}{x_i^2},\\
\cdot\frac{\partial^2 \log l}{\partial \alpha\,\partial \beta} &= -\frac{1}{\alpha}\left[\sum_{i=1}^{n}\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta}\log\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)\right],\\
\cdot\frac{\partial^2 \log l}{\partial \lambda\,\partial \beta} &= -2\sum_{i=1}^{n}\left[\frac{x_i(\lambda x_i)^{-3}\exp\left(-(\lambda x_i)^{-2}\right)}{\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)}\right]\\
&\quad +2\log\alpha\sum_{i=1}^{n}x_i(\lambda x_i)^{-3}\left\{\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta-1} + \beta\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)^{\beta-1}\log\left(1-\exp\left(-(\lambda x_i)^{-2}\right)\right)\right\}.
\end{aligned} \tag{70}$$

The asymptotic $(1-\zeta)100\%$ confidence intervals for the parameters of suggested model are as follows:

$$\begin{aligned}
&\hat{\alpha} \pm Z_{\zeta/2}\sqrt{\Sigma_{11}},\\
&\hat{\lambda} \pm Z_{\zeta/2}\sqrt{\Sigma_{22}},\\
&\hat{\beta} \pm Z_{\zeta/2}\sqrt{\Sigma_{33}}.
\end{aligned} \tag{71}$$

Here, Z_{ζ} represents the upper ζ^{th} percentile of standard normal distribution.

3.2. Simulation Study. Simulation is used to obtain estimates, Mean Square Error (MSE), and Bias of parameters. The following expression of QF was used to develop $w = 100$ samples of size $n = 50, 90$, and 200, respectively:

$$X_p = \frac{1}{\lambda}\left[-\log\left\{1-\left\{\frac{\log\alpha - \log\{u(\alpha-1)+1\}}{\log\alpha}\right\}^{1/\beta}\right\}\right]^{-1/2}, \tag{72}$$

where $u \sim U[0,1]$. The following expression is used to calculate bias and MSE:

$$\begin{aligned}
\text{Bias} &= \frac{1}{W}\sum_{1=1}^{w}\left(\hat{b}_i - b\right).\\
\text{MSE} &= \frac{1}{W}\sum_{1=1}^{w}\left(\hat{b}_i - b\right)^2,
\end{aligned} \tag{73}$$

where $b = (\alpha, \lambda, \beta)$. For various choices of α, λ, and β, simulation results were obtained. Table 1 shows the simulated expected values of MSEs and bias. In Table 1, with increase in sample size, the consistency behavior may be easily observed as estimates approach their parametric values. Furthermore, as the sample size grows, the MSEs and bias of the estimates drop for all parameter combinations. As a result, we can infer that the MLE approach performs well when it comes to estimating the parameters of the APGIR distribution.

3.3. Applications. To see the performance and goodness of fit of the proposed mode, the suggested distribution has been fitted to two data sets. We found that the suggested model performed better than other Rayleigh distribution variants such as the Two-Parameter Rayleigh (TPR) distribution

Table 1: MSE and bias of MLE.

Parameter	N	MSE, $(\hat{\alpha})$	MSE $(\hat{\beta})$	MSE, $(\hat{\lambda})$	Bias $(\hat{\alpha})$	Bias $(\hat{\beta})$	Bias $(\hat{\lambda})$
$\alpha = 18.4510$ $\beta = 1.2510$ $\lambda = 0.6534$	50	45.6174	0.0388	0.3961	−1.0590	0.0641	−0.3221
	90	26.8081	0.0334	0.2198	−0.6299	0.0584	−0.1843
	200	23.1193	0.0137	0.1735	0.5317	0.0341	−0.1389
$\alpha = 17.4510$ $\beta = 1.4510$ $\lambda = 0.3534$	50	31.9016	0.0976	0.2236	−1.2408	0.0827	−0.3266
	90	18.4479	0.0454	0.2013	−0.7485	0.0507	−0.2920
	200	14.3389	0.0124	0.1530	−0.5075	−0.0053	−0.2155
$\alpha = 17.4510$ $\beta = 1.2510$ $\lambda = 0.6534$	50	44.2246	0.0463	0.3910	−1.8267	0.0683	−0.3208
	90	34.9931	0.0259	0.3383	−1.6335	0.0210	−0.2738
	200	17.5782	0.0133	0.1853	0.4444	0.0204	−0.1502
$\alpha = 17.4510$ $\beta = 1.3010$ $\lambda = 0.6534$	50	50.13503	0.0520	0.2882	−1.9696	0.0756	−0.2515
	90	33.5026	0.0334	0.2850	−1.7621	0.0333	−0.2255
	200	22.2469	0.0138	0.2051	−0.0078	0.0326	−0.1635
$\alpha = 17.4510$ $\beta = 1.4510$ $\lambda = 0.6034$	50	39.7837	0.0753	0.4067	−0.9935	0.0371	−0.3436
	90	37.8969	0.0406	0.3365	−0.2782	0.0349	−0.2836
	200	18.0537	0.0128	0.1725	0.1783	0.0062	−0.1470

proposed by Dey et al. [43], the MIR distribution suggested by Khan [28], the EIR distribution developed by Rehman and Dar [27], the GR distribution offered by Raqab and Madi [44], the TIR distribution by Ahmad et al. [15], and the GIR distribution by Potdar and Shirke [29].

Data set 1: the first set of data includes the survival times (in years) of 46 individuals who received just chemotherapy. These data are a subset of the data taken from the study by Bekker et al. [45]. The data points are 0.047,0.115, 0.121, 0.132, 0.164, 0.197,0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178,2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Data set 2: the failure times of 84 aircraft windshields are included in the second real data set. El-Bassiouny et al. [46] provided this information. The following are the data points:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

PDF of GIR distribution is as follows:

$$f(x) = \frac{2\beta}{\lambda^2 x^3}\exp\left(-(\lambda x)^{-2}\right) \cdot \left[1 - \exp\left(-(\lambda x)^{-2}\right)\right]^{\beta-1} x, \beta, \lambda > 0. \tag{74}$$

PDF of EIR distribution is as follows:

$$f(x) = \frac{2\beta\theta}{x^3}\ \exp\left(-\left(\frac{\beta\theta}{x^2}\right)\right),\ \theta,\ \beta, X > 0. \tag{75}$$

PDF of GR distribution is as follows:

$$f(x) = 2\alpha\gamma^2 x\ \exp\left(-(\gamma x)^2\right) \cdot \left(1 - \exp\left(-(\gamma x)^2\right)\right)^{\alpha-1}, \alpha, \gamma, X > 0. \tag{76}$$

PDF of TPR distribution is as follows:

$$f(x) = 2\alpha(x-\mu)\exp\left(-\alpha(x-\mu)^2\right), x > \mu,\ \alpha > 0. \tag{77}$$

PDF of MIR distribution is as follows:

$$f(x) = \left(\alpha + \frac{2\theta}{x}\right)\left(\frac{1}{x}\right)^2 \exp\left(-\frac{\alpha}{x} - \theta\left(\frac{1}{x}\right)^2\right), \alpha, \theta, X > 0. \tag{78}$$

PDF of TIR distribution is as follows:

$$f(x) = \frac{2\theta}{x^3}\exp\left(-\left(\frac{\theta}{x^2}\right)\right) \cdot \left(1 + \lambda - 2\lambda\ \exp\left(-\left(\frac{\theta}{x^2}\right)\right)\right), \theta, X > 0. \tag{79}$$

PDF of WR distribution is as follows:

$$f(x) = \alpha\beta\theta x\ \exp\left(\frac{\theta x^2}{2}\right)\left(\exp\left(\frac{\theta x^2}{2}\right) - 1\right)^{\beta-1} \cdot e^{-\alpha\left(\exp\left(\frac{\theta x^2}{2}\right)-1\right)^{\beta}}, \alpha, \theta, \beta, X > 0. \tag{80}$$

The APGIR model's results are compared with other Rayleigh distribution versions using well-known model selection criteria such as Akaike's Information Criteria (AIC), Consistent Akaike's Information Criteria (CAIC), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criteria (HQIC), and Kolmogorov–Smirnov (K-S) and their P values via the R programming language's Adequacy Model. The results are shown in Tables 2 and 3.

On the basis of several model selection criteria, Tables 2 and 3 show that our recommended distribution outperforms than other forms of Rayleigh distribution.

Table 2: Goodness of fit results for data set 1.

Distribution	MLE			AIC	CAIC	BIC	HQIC	K-S	P value
APGIR	10.4662	0.4091	8.4132	137.2571	137.8425	142.6771	139.2776	0.14051	0.3073
GIR	0.2720	7.4680		148.2952	148.5809	151.9086	149.6422	0.77484	$5.551e-16$
TIR	0.0414	−0.8352		211.8820	212.1677	215.4953	213.229	0.43156	$3.923e-08$
APEIR	10.8661	0.8934	0.0477	196.7339	197.3192	202.1538	198.7544	0.75677	$5.551e-16$
TPR	0.1898	−0.5973		138.6129	138.8986	142.2262	139.9599	0.21686	0.02449
EIR	0.1664	0.3224		234.1737	234.4594	237.787	235.5207	0.50772	$2.82e-11$

Table 3: Goodness of fit results for data set 2.

Distribution	MLE			AIC	CAIC	BIC	HQIC	K-S	P value
APGIR	17.4510	1.4510	0.6534	326.7218	326.9718	334.5373	329.8849	0.15139	0.02043
GIR	1.1357	−0.5285		353.6821	353.8058	358.8924	355.7908	0.17108	0.005739
TIR	1.9939	−0.8843		339.1298	339.2535	344.3401	341.2385	0.15364	0.01781
MIR	−2.0768	6.4548		338.6148	338.7385	343.8251	340.7235	0.40002	$2.531e-14$
EIR	7.9999	0.4095		354.4818	354.6055	359.6921	356.5905	0.18258	0.002544
APEIR	11.8642	0.2827	5.7878	339.3922	339.6422	347.2077	342.5553	0.78182	$2.2e-16$

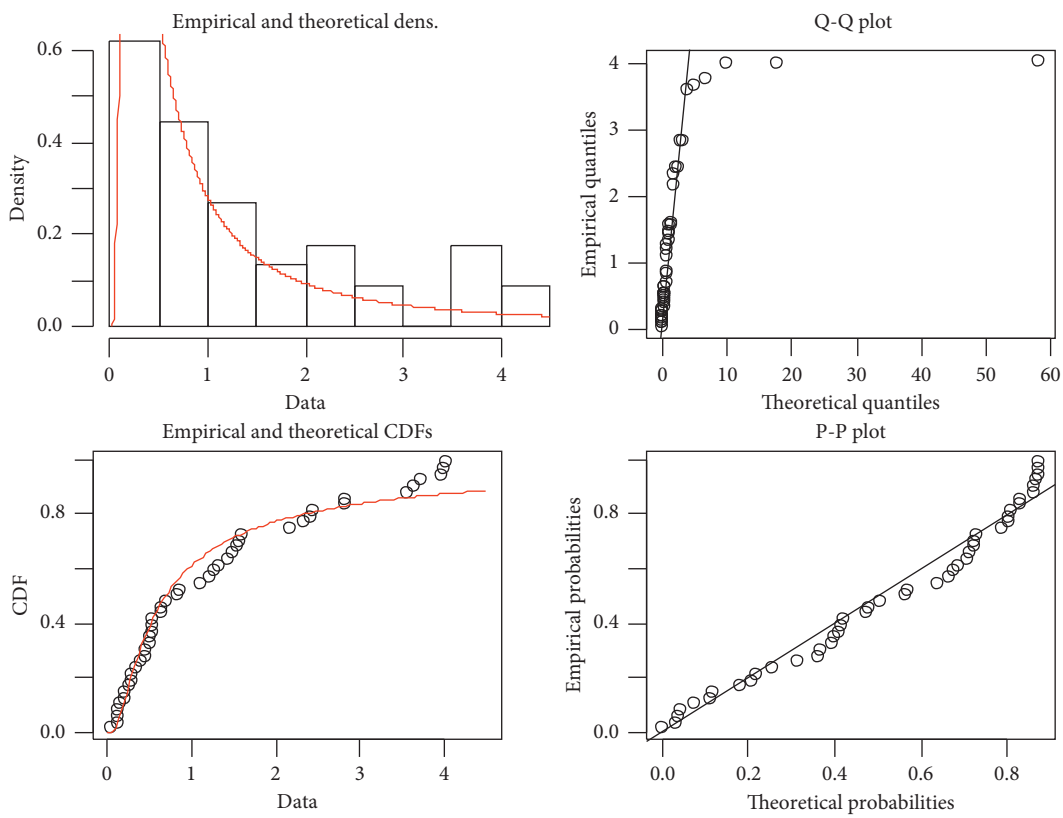

Figure 3: Graphs of QQ and PP plot for data set 1.

Figure 3 represents QQ and PP plot for data set 1. Figure 4 shows theoretical densities and CDFs for data set 1. The graphs clearly show better fit for data set 1. Figure 5 represents QQ and PP plot for data set 2. Figure 6 shows theoretical densities and CDFs for data set 2. It is clear from the figures that the data set 2 is better fitted by the proposed distribution.

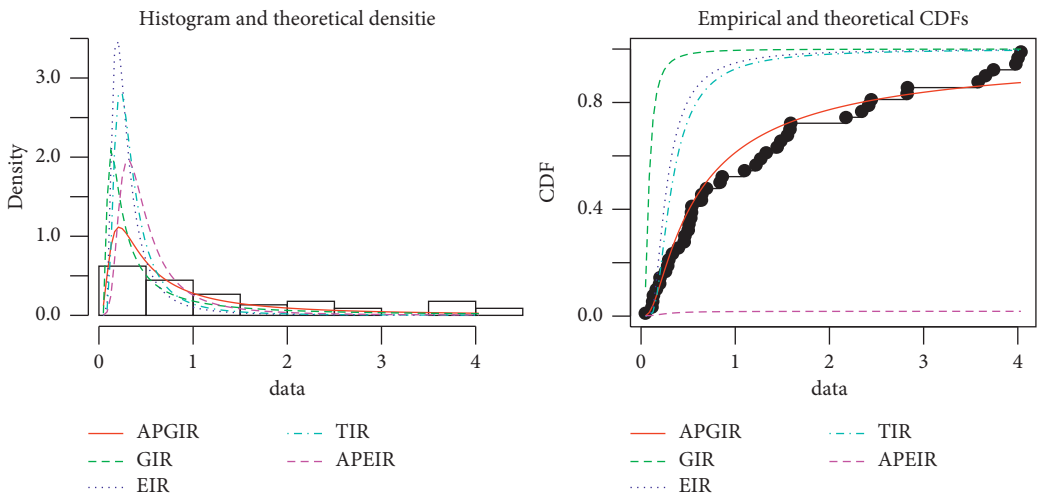

Figure 4: Graphs of theoretical densities and CDFs for data set 1.

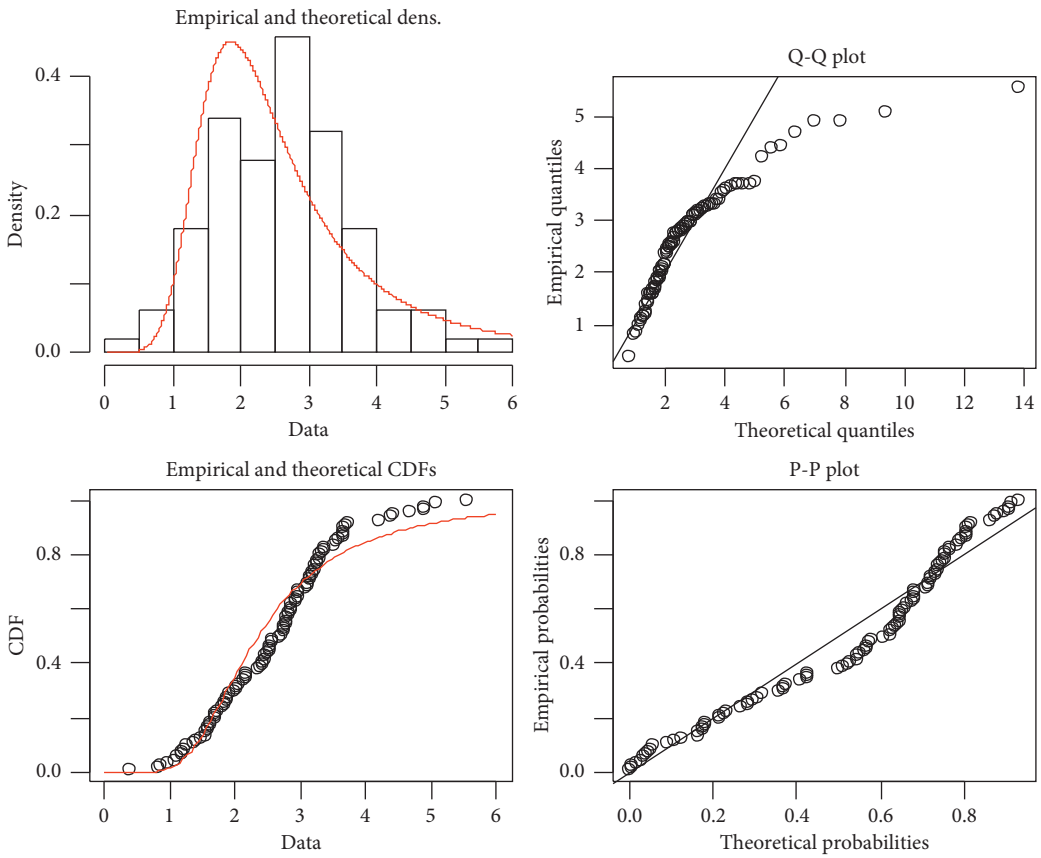

Figure 5: Graphs of QQ and PP plot for data set 2.

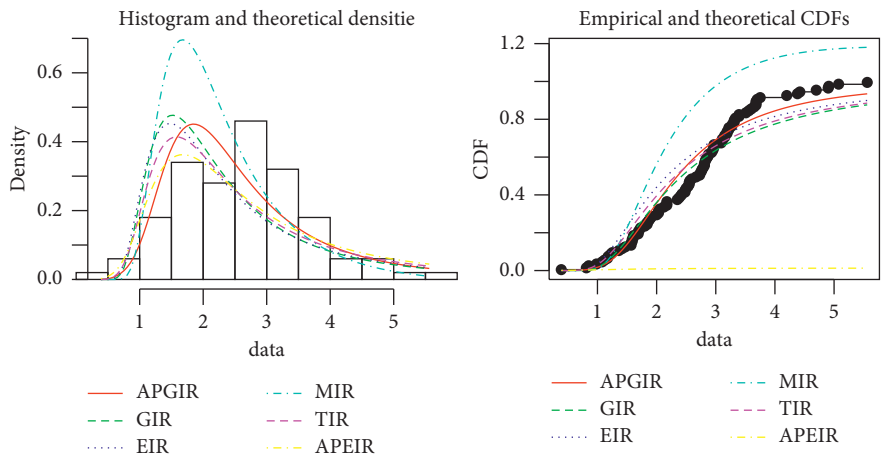

Figure 6: Graphs of theoretical densities and CDFs for data set 2.

4. Conclusion

In this paper, we have proposed a new distribution referred to as Alpha Power Generalized Inverse Rayleigh (APGIR) distribution. This distribution has been developed using APT with the input as Generalized Inverse Rayleigh. Several important mathematical properties including the moment generating function, order statistics, mean residual life function, mean waiting time, stress-strength parameter, expression for entropies, quantile function, and rth moment have been derived The parameter estimates were derived using the MLE technique. The consistency of MLE's was assessed using simulation studies. The performance of the proposed model was evaluated using two real data sets using some goodness of fit criteria. The results clearly reveal that our proposed model performs well as compared with other types of Rayleigh distribution available in the literature.

Data Availability

The data sets are included within the main body of the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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