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Research Article

Oscillation of Third-Order Nonlinear Generalized Difference Equation with Multiple Neutral Terms

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In this paper, the authors discuss the oscillatory behaviour of a third order generalized difference equation with multiple neutral terms. We have also applied Riccati transformation and Philo type technique to derive new oscillation criteria for the difference equation in question. Suitable examples are provided to validate our main results.

1. Introduction

The theory of difference equations has grown immensely over the past few decades. In the field of probability theory, statistical analysis, combinatorial analysis, electrical networks, and sociology, difference equations has emerged as mathematical models describing real life challenges.

In the recent past, the study of oscillation and nonoscillation for second order nonlinear difference equations has garnered a great deal of attention [1, 2]. The latest research also emphasizes the different kinds of difference equations, including ordinary, linear, nonlinear, superlinear, quasilinear, sublinear, delay, and neutral delay difference equations. Interestingly, one can refer the oscillatory behavior for sublinear neutral delay second and third order difference equations in [3, 4]. An investigation of the oscillation of the second-order quasilinear neutral delay difference equations can be seen in [5, 6]. The study of the Oscillation of second order half-linear difference equations has also been given in [7]. Importantly, the oscillation criteria for higher order neutral equations can be seen in [8].

The literature regarding the present study is referred in [9, 10]. In a revealing manner, tracking a maneuvering target, and fault diagnosis of wind turbine gearbox is performed in [11] by using the properties of Riccati difference equation. In [12], for the study of the Mittag-Leffer stability analysis of fractional discrete time neural networks, a class of semilinear fractional difference equations are used.

A few applications of specific kinds of nonlinear third order delay difference equations are prominent in the study of Mathematical Biology, Economics, and many other fields of Mathematics that involve discrete models [13–16]. Moreover, oscillatory solution of third order delay difference equations are used to remove speckle noise in the field of image processing which can be seen in [17]. The suitable smoothing filter for the edge mask computed using the third order difference equation is examined in [18].

Our main focus in this paper is on the oscillatory behavior of the third-order difference equations. In the earlier research, plethora of methods about the oscillatory property of third order difference equation were presented. For instance, in [19] the third order difference equation under consideration is as follows:

$$\Delta\left(a_\xi \Delta\left(b_\xi \Delta x_\xi\right)\right) + q_\xi f\left(x_{\xi-m+1}\right) = h_\xi, \xi \geq \xi_0, \qquad (1)$$

where $\{a_\xi\}, \{b_\xi\}, \{q_\xi\}$ are positive real sequences, m is positive integer, $f \in C(R, R)$ with $uf(u) > 0$ for $u \neq 0$ and $\sum_{\xi=\xi_0}^{\infty} 1/a_\xi = \sum_{\xi=\xi_0}^{\infty} 1/b_\xi = \infty$. Using the Riccati transformation technique, sufficient conditions for the existence of oscillatory solutions are provided. In [20], the authors considered the following equation:

$$\Delta\left(c_\xi \Delta\left(d_\xi \Delta\left(y_\xi + p_\xi y_{\xi-k}\right)\right)\right) + q_\xi f\left(x_{\xi-m}\right) = e_\xi, \xi \geq \xi_0 - m. \tag{2}$$

With $\{c_\xi\}, \{d_\xi\}$ as positive real sequences such that $\sum_{\xi=\xi_0}^{\infty} 1/c_\xi = \sum_{\xi=\xi_0}^{\infty} 1/d_\xi = \infty$. Some sufficient conditions are established for the oscillation of solutions of equation (2) by using Riccati transformation technique. In [21], the authors considered the nonlinear delay difference equation as follows:

$$\Delta\left(a_\xi \Delta\left(b_\xi \left(\Delta x_\xi\right)^{\alpha_1}\right)^{\alpha_2}\right) + q_\xi f\left(x_{\sigma_\xi}\right) = 0, \xi \geq \xi_0, \tag{3}$$

where $\{a_\xi\}, \{b_\xi\}, \{q_\xi\}$ are positive real sequences, $\{\sigma_\xi\}$ is sequence of integers such that $\sum_{\xi=\xi_0}^{\infty} 1/a_\xi^{1/\alpha_2} = 1/b_\xi^{1/\alpha_1} = \infty$. By reducing the order of the equation, the main results are obtained, as well as certain sufficient conditions are provided for the oscillation of solutions of equation (3) by adopting Riccati transformation.

Furthermore, in [4], the oscillation property of the generalized third order sublinear neutral delay difference equation of the form is as follows:

$$\Delta_\ell \left(\alpha_2(\xi) \Delta_\ell \left(\alpha_1(\xi) \Delta_\ell z(\xi)\right)\right) + q(\xi) x^\beta (k - \sigma\ell) = 0, \tag{4}$$

is discussed using Riccati type transformations.

For further study on third order difference equations, one can refer [22] also. Motivated by the above literature available on the oscillation criteria for different class of difference equations involving the conventional difference operator Δ, we wish to generalize the results for the more generalized difference equation involving the generalized difference operator Δ_ℓ. Hence, in this paper we consider the third-order non-linear generalized difference equations with multiple neutral terms of the form

$$\Delta_\ell \left(a_1(\xi) \left[\Delta_\ell \left(a_2(\xi) \left[\Delta_\ell x(\xi)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right) + \sum_{j=1}^{n} c_j(\xi) y^{\gamma_3}\left(g_j(\xi)\right) = 0, 0 < \xi_0 < \xi, \tag{5}$$

where $x(\xi) = y(\xi) + b(\xi) y(\overline{f}(\xi))$, $\gamma_i > 0$ for $i = 1, 2, 3$ is a ratio of odd positive integers. Here, Δ_ℓ is the generalized difference operator defined by $\Delta_\ell x(\xi) = x(\xi + \ell) - x(\xi) \equiv z(\xi)$, $\xi \in \mathbb{N}_\ell(\xi_0) = \{\xi_0, \xi_0 + \ell, \xi_0 + 2\ell, \cdots\}$, $\xi_0 \in [0, \infty$, $\ell \in (0, \infty)$ and its inverse is defined by the following equation:

$$x(\xi) = x(\xi_0 + j) + \sum_{r=0}^{[\xi - \xi_0 - j - \ell/\ell]} z(k_0 + j + r\ell), k \in \mathbb{N}_\ell(j). \tag{6}$$

For the validity of our discussion, we consider the following conditions on equation (5)

(i) $\{a_i(\xi)\}$ is a positive sequence and $\sum_{s=\xi_0}^{\infty} a_i^{-1/\gamma_{(3-i)}}(s) = \infty$, for $i = 1, 2$.

(ii) $b(\xi)$ is a real sequence with $1 < b(\xi)$.

(iii) $\{c_i(\xi)\} > 0$.

(iv) $\overline{f}(\xi)$ is strictly increasing with $\xi > \overline{f}(\xi)$, and $\lim_{\xi \longrightarrow \infty} \overline{f}(\xi) = \infty$.

(v) $\{g_j(\xi)\}$ with $\lim_{\xi \longrightarrow \infty} g_j(\xi) = \infty$ where $j = 1, 2, \ldots, n$.

(vi) $m_i(\xi) = [\xi - \overline{\xi_i} - \overline{j} - \ell/\ell]$, $\overline{\xi_i} = \xi_i + j$, and $j = \xi - \xi_0 - [\xi - \xi_0/\ell]\ell$.

We focus only on the solutions $y(\xi)$ of equation (5), defined for some $\xi_0 < \xi_1$ satisfying $\sup\{|y(\xi)|: \xi > K\} > 0$ for all $K > \xi$. We assume that a proper solution exists for equation (5). A proper solution $y(\xi)$ of equation (5) is said to be oscillatory if $y(\xi) y(\xi + \ell) < 0$ for large ξ and is non-oscillatory otherwise. Also, if all the solutions of equation (5) are oscillatory, then the equation itself is oscillatory.

Forthwith, in this paper we establish necessary conditions for the oscillation of solutions of third-order nonlinear generalized difference equations with multiple neutral terms. Many well-known oscillation criteria that have appeared in the prevailing research have been extended and improved by the results presented here.

2. Main Results

The following notations and lemmas are useful in order to prove our main results.

$$\phi_1(\xi) = \frac{1}{b\left(\overline{f}^{-1}(\xi)\right)} \left[1 - \frac{1}{b\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right)}\right], \tag{7}$$

$$\phi_2(\xi) = \frac{1}{b\left(\overline{f}^{-1}(\xi)\right)} \left[1 - \frac{R_3\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right), \xi_2\right)}{b\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right) R_3\left(\overline{f}^{-1}(\xi), \xi_2\right)}\right], \tag{8}$$

$$\Phi_1(\xi) = \sum_{j=1}^{n} c_j(\xi)\phi_1^{\gamma_3}(g_i(\xi)), \Phi_2(\xi) = \sum_{j=1}^{n} c_j(\xi)\phi_2^{\gamma_3}(g_i(\xi)),$$
$$R_1(\xi,\xi_1) = \sum_{r=0}^{m_1(\xi)} \frac{1}{a_1^{1/\gamma_2}\left(\overline{\xi_1}+r\ell\right)}, R_2(\xi,\xi_1) = \left(\frac{R_1(\xi,\xi_1)}{a_2(\xi)}\right)^{1/\gamma_1} \quad \text{for } \xi \geq \xi_1, \tag{9}$$
$$R_3(\xi,\xi_2) = \sum_{r_1=0}^{m_2(\xi)} R_2\left(\overline{\xi_2}+r_1\ell,\xi_1\right) \text{for } \xi \geq \xi_2 > \xi_1.$$

Lemma 1. *If $y(\xi)$ is an eventually positive solution of equation (5), then $x(\xi)$ satisfies either (C_1). $x(\xi)>0$, $\Delta_\ell x(\xi)>0$, $a_2(\xi)(\Delta_\ell x(\xi))^{\gamma_1}>0$ and $a_1(\xi)(a_2(\xi)(\Delta_\ell x(\xi))^{\gamma_1})^{\gamma_2} \leq 0$, or (C_2). $x(\xi)>0$, $\Delta_\ell x(\xi)<0$, $a_2(\xi)(\Delta_\ell x(\xi))^{\gamma_1}>0$, and $a_1(\xi)(a_2(\xi)(\Delta_\ell x(\xi))^{\gamma_1})^{\gamma_2} \leq 0$.*

Proof. Let $\{y(\xi)\}$ be a positive solution of equation (5) for every $\xi \geq \xi_0$. By defining $x(\xi)$, with $x(\xi) \geq y(\xi) > 0$ for $\xi \geq \xi_1 \in N_\ell(\xi_0)$, from equation (5), we have the following equation:

$$\Delta_\ell\left(a_1(\xi)[\Delta_\ell(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1})]^{\gamma_2}\right) = -\sum_{j=1}^{n} c_j(\xi) y^{\gamma_3}\left(g_j(\xi)\right) < 0. \tag{10}$$

This implies $a_1(\xi)[\Delta_\ell(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1})]^{\gamma_2}$ is decreasing on $[\xi_1,\infty$, which is either positive or negative. Furthermore, we must have to prove that $a_1(\xi)[\Delta_\ell(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1})]^{\gamma_2}>0$ for $\xi \geq \xi_1 \geq \xi_0$. Otherwise, we have a constant $L_1^{\gamma_2}>0$ in such a way that

$$\Delta_\ell\left(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1}\right) < -\frac{L_1}{a_1^{1/\gamma_2}(\xi)} < 0, \text{for} \xi \geq \xi_1. \tag{11}$$

Hence, by equation (6)

$$a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1} \leq a_2\left(\overline{\xi_1}\right)\left(\Delta_\ell x\left(\overline{\xi_1}\right)\right)^{\gamma_1} - L_1 \sum_{r=0}^{m_1(\xi)} \frac{1}{a_1^{1/\gamma_2}\left(\overline{\xi_1}+r\ell\right)}. \tag{12}$$

Letting $\xi \longrightarrow \infty$, then using condition (i), we have $\lim_{\xi \to \infty} a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1} = -\infty$. Subsequently, there exists a $\xi_2 \geq \xi_1$ also a constant $L_2^{\gamma_1}>0$, so that

$$a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1} < -L_2^{\gamma_1}, \text{for } \xi \geq \xi_2. \tag{13}$$

Dividing the above inequality by $a_2(\xi)$ and by applying summation from ξ_2 to $\xi-\ell$, we get the following equation:

$$x(\xi) < x\left(\overline{\xi_2}\right) - L_2 \sum_{t=0}^{m_2(\xi)} \frac{1}{a_2^{1/\gamma_2}\left(\overline{\xi_2}+t\ell\right)}. \tag{14}$$

Letting $\xi \longrightarrow \infty$ and using condition (i), we have $x(\xi) \longrightarrow -\infty$. Thus, $x(\xi)<0$ eventually which is contradictory to the fact $x(\xi)>0$, which gives $\Delta_\ell(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1})$ is positive, that is $[\Delta_\ell(a_2(\xi)(\Delta_\ell x(\xi)]^{\gamma_1})]^{\gamma_2}>0$ holds.

It can be known from $\Delta_\ell(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1})>0$, is a monotonically increasing in the interval $[\xi_2,\infty)$. Therefore $\Delta_\ell x(\xi)$ is ultimately either positive or negative. Hence, we obtain either (C_1) or (C_2) for $\{x(\xi)\}$, which completes the proof. □

Lemma 2. *Let equation (7) holds, and let $y(\xi)$ be an eventually positive solution of equation (5) with $x(\xi)$ satisfying condition (C_2) of Lemma 1. If*

$$\sum_{r=0}^{\infty} \frac{1}{a_2^{1/\gamma_1}\left(\overline{\xi_5}+r\ell\right)} \left(\sum_{s=r}^{m_4(\xi)} \frac{1}{a_1^{1/\gamma_2}\left(\overline{\xi_4}+s\ell\right)} \left(\sum_{t=s}^{m_3(\xi)} \Phi_1\left(\overline{\xi_3}+t\ell\right)\right)^{1/\gamma_2}\right)^{1/\gamma_1} = \infty, \tag{15}$$

Then, $y(\xi)$ of equation (5) converges to zero when $\xi \longrightarrow \infty$.

Proof. Let $y(\xi)$ be an eventually positive solution of equation (5). Then, there exits $\xi \in \mathbb{N}_\ell(\xi_0)$ such that, for

$\xi_1 \le \xi$, $0 < y(\xi)$, $0 < y(\overline{f}(\xi))$, $0 < y(g_1(\xi))$ for $i = 1, 2, \ldots, n$. From the definition of $x(\xi)$, we obtain the following equation:

$$\begin{aligned} y(\xi) &= \frac{1}{b\left(\overline{f}^{-1}(\xi)\right)}\left(x\left(\overline{f}^{-1}(\xi)\right) - y\left(\overline{f}^{-1}(\xi)\right)\right) \\ &= \frac{x\left(\overline{f}^{-1}(\xi)\right)}{b\left(\overline{f}^{-1}(\xi)\right)} - \frac{x\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right) - y\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right)}{b\left(\overline{f}^{-1}(\xi)\right)b\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right)} \\ &\ge \frac{x\left(\overline{f}^{-1}(\xi)\right)}{b\left(\overline{f}^{-1}(\xi)\right)} - \frac{x\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right)}{b\left(\overline{f}^{-1}(\xi)\right)x\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right)}. \end{aligned} \tag{16}$$

Through $\overline{f}(\xi) < \xi$, from (iv) and the fact that$x(\xi)$ is decreasing, we obtain the following equation:

$$x\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right) \le x\left(\overline{f}^{-1}(\xi)\right). \tag{17}$$

Using this in equation (16), we get the following equation:

$$y(\xi) \ge \phi_1(\xi)x\left(\overline{f}^{-1}(\xi)\right), \tag{18}$$

$$y(g_i(\xi)) \ge \phi_1(g_i(\xi))x\left(\overline{f}^{-1}(g_i(\xi))\right), i = 1, 2, \ldots, n, \tag{19}$$

for $\xi_2 \le \xi$. Using equation (19) in equation (5) we get the following equation:

$$\begin{aligned} &\Delta_\ell\left(a_1(\xi)\left[\Delta_\ell\left(a_2(\xi)\left[\Delta_\ell x(\xi)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right) \\ &\quad + \sum_{j=1}^{n} c_j(\xi)\phi_1^{\gamma_3}(g_i(\xi))x^{\gamma_3}\left(\overline{f}^{-1}(g_i(\xi))\right) \le 0, \end{aligned} \tag{20}$$

for $\xi_2 \le \xi$. From (iv)-(v), $x(\xi)$ is decreasing, and from equation (20) we obtain the following equation:

$$\begin{aligned} &\Delta_\ell\left(a_1(\xi)\left[\Delta_\ell\left(a_2(\xi)\left[\Delta_\ell x(\xi)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right) \\ &\quad + x^{\gamma_3}\left(\overline{f}^{-1}(\xi)\right)\sum_{j=1}^{n} c_j(\xi)\phi_1^{\gamma_3}(g_i(\xi)) \le 0, \text{for } \xi_2 \le \xi. \end{aligned} \tag{21}$$

Since $x(\xi) > 0$ and $\Delta_\ell x(\xi) < 0$, a constant M exists, such that

$$\lim_{\xi \longrightarrow \infty} x(\xi) = M < \infty, \tag{22}$$

where $M \ge 0$. If $M > 0$, then there exists $\xi_2 \le \xi_3$, such that $\xi_2 < \overline{f}^{-1}(\overline{f}^{-1}(\xi))$ and

$$x(\xi) = M, \text{for } \xi_3 \le \xi. \tag{23}$$

Summing twice equation (21) from $\xi_3 \le \xi_4 \le \xi$ to ∞, we get the following equation:

$$-\Delta_\ell z(\xi) \ge \frac{M^{\gamma_3/\gamma_1\gamma_2}}{a_2^{1/\gamma_1}(\xi_4)}\left(\sum_{s=0}^{\infty}\frac{1}{a_1^{1/\gamma_2}(\xi_4 + s\ell)}\left(\sum_{t=s}^{m_3(\xi)}\Phi_1\left(\overline{\xi_3} + t\ell\right)\right)^{1/\gamma_2}\right)^{1/\gamma_1}. \tag{24}$$

Summing the above inequality from ξ_5 to ∞, we obtain the following equation:

$$z(\xi) \ge M^{\gamma_3/\gamma_1\gamma_2}\sum_{r=0}^{\infty}\frac{1}{a_2^{1/\gamma_1}\left(\overline{\xi_5} + r\ell\right)}\left(\sum_{s=r}^{m_4(\xi)}\frac{1}{a_1^{1/\gamma_2}\left(\overline{\xi_4} + s\ell\right)}\left(\sum_{t=s}^{m_3(\xi)}\Phi_1\left(\overline{\xi_3} + t\ell\right)\right)^{1/\gamma_2}\right)^{1/\gamma_1}, \tag{25}$$

which is contradictory to equation (15), and hence it is true that $M = 0$. Hence, $\lim_{\xi \longrightarrow \infty} x(\xi) = 0$. Since $0 < y(\xi) \le x(\xi)$ for $\xi_1 \le \xi < \infty$, it is clear that $\lim_{\xi \longrightarrow \infty} y(\xi) = 0$. □

Theorem 1. *Let $g_j(\xi) \le \overline{f}(\xi)$ for $j = 1$ to n, $\Phi_1(\xi) > 0$, $\Phi_2(\xi) > 0$ and equation (15) hold. If there exists functions $h(\xi)$ and $H(\xi) \in C(N_\ell(\xi_0), \mathbb{R})$ such that $a_2(\xi)H(\xi) \in C(N_\ell(\xi_0), \mathbb{R})$ and*

$$\sum_{s=0}^{\infty}\left[A\left(\overline{\xi_3} + s\ell\right) - M\frac{B^{1+\gamma_1\gamma_2}\left(\overline{\xi_3} + s\ell\right)}{C^{\gamma_1\gamma_2}\left(\overline{\xi_3} + s\ell\right)}\right] = \infty, \text{for } \xi_2 < \xi_3 \le \xi, \tag{26}$$

where

$$M = \frac{(\gamma_1\gamma_2)^{\gamma_1\gamma_2}}{(1 + \gamma_1\gamma_2)^{1+\gamma_1\gamma_2}}, \tag{27}$$

$$\psi(\xi) = \begin{cases} N_1, & \text{if } \gamma_1\gamma_2 \le \gamma_3, \\ N_2(R_3(\xi, \xi_2))^{\gamma_3/\gamma_1\gamma_2 - 1}, & \text{if } \gamma_3 < \gamma_1\gamma_2, \end{cases} \tag{28}$$

$$A(\xi) = -h(\xi)\Delta_\ell\left(a_1(\xi)H(\xi)\right) + h(\xi)\sum_{j=1}^{n} c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3}\left(\frac{R_3\left(\overline{f}^{-1}\left(g_j(\xi)\right),\xi_2\right)}{R_3(\xi,\xi_2)}\right)^{\gamma_3}$$
$$+\frac{\gamma_3}{\gamma_1\gamma_2}h(\xi)R_3(\xi,\xi_2)\psi(\xi)\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}, \tag{29}$$

$$B(\xi) = \frac{\Delta_\ell h(\xi)}{h(\xi+\ell)} + \left(\gamma_3 + \frac{\gamma_3}{\gamma_1\gamma_2}\right)h(\xi)R_3(\xi,\xi_2)\psi(\xi)\frac{\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2}}{h(\xi+\ell)}, \tag{30}$$

$$C(\xi) = \gamma_3\frac{h(\xi)R_3(\xi,\xi_2)\psi(\xi)}{\left(h(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}}, \tag{31}$$

for $\xi_2 < \xi$, *then every solution of equation (5) is either oscillatory or approaches zero as* ξ *tends to* ∞.

Proof. Let equation (5) have a nonoscillatory solution $\{y(\xi)\}$ on $\mathbb{N}_\ell(\xi_0)$. Let us assume that there exists $\xi_1 \in \mathbb{N}_\ell(\xi_0)$ such that, for $\xi \geq \xi_1$, $y(f(\xi)) > 0$, and $y(g_j(\xi)) > 0$, equations (7) and (8) hold and $x(\xi)$ satisfies either (C_1) or (C_2) for $j = 1$ to n. Equation (16) follows from the proof of Lemma 2 with the assumption that (C_1) holds. Since $a_1(\xi)[\Delta_\ell(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1})]^{\gamma_2}$ is decreasing, we observe that

$$a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1} = a_2\left(\overline{\xi_1}\right)\left[\Delta_\ell z\left(\overline{\xi_1}\right)\right]^{\gamma_1} + \sum_{r=0}^{m_1(\xi)}\Delta_\ell\left(a_2\left(\overline{\xi_1}+r\ell\right)\left[\Delta_\ell x\left(\overline{\xi_1}+r\ell\right)\right]^{\gamma_1}\right)$$
$$\geq \sum_{r=0}^{m_1(\xi)}\frac{a_1^{1/\gamma_2}\left(\overline{\xi_1}+r\ell\right)\Delta_\ell\left(a_2\left(\overline{\xi_1}+r\ell\right)\left[\Delta_\ell x\left(\overline{\xi_1}+r\ell\right)\right]^{\gamma_1}\right)}{a_1^{1/\gamma_2}\left(\overline{\xi_1}+r\ell\right)}$$
$$\geq a_1^{1/\gamma_2}(\xi+\ell)\Delta_\ell\left(a_2(\xi+\ell)[\Delta_\ell x(\xi+\ell)]^{\gamma_1}\right)R_1(\xi,\xi_1). \tag{32}$$

From the above equation, with $\xi \geq \xi_2 > \xi_1$, we have the following equation:

$$\Delta_\ell\left(\frac{a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1}}{R_1(\xi,\xi_1)}\right) = \frac{a_1^{-1/\gamma_2}(\xi)\left[a_1^{1/\gamma_2}(\xi)R_1(\xi,\xi_1)\Delta_\ell\left(a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1}\right) - a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1}\right]}{R_1(\xi,\xi_1)R_1(\xi+\ell,\xi_1)} \leq 0. \tag{33}$$

Thus, $a_2(\xi)[\Delta_\ell x(\xi)]^{\gamma_1}[R_1(\xi,\xi_1)]^{-1}$ is decreasing for $\xi \geq \xi_2$ and further, this fact yields as follows:

$$x(\xi) = x\left(\overline{\xi_2}\right) + \sum_{r_1=0}^{m_2(\xi)}\frac{\Delta_\ell x\left(\overline{\xi_2}+r_1\ell\right)}{R_2\left(\overline{\xi_2}+r_1\ell,\xi_1\right)}R_2\left(\overline{\xi_2}+r_1\ell,\xi_1\right) \geq \frac{R_3(\xi,\xi_2)}{R_2(\xi,\xi_1)}\Delta_\ell x(\xi). \tag{34}$$

From the above equation, with $\xi \geq \xi_3 > \xi_2$, we obtain the following equation:

$$\Delta_\ell\left(\frac{x(\xi)}{R_3(\xi,\xi_2)}\right) = \frac{R_3(\xi,\xi_2)\Delta_\ell x(\xi) - x(\xi)R_2(\xi,\xi_1)}{R_3(\xi,\xi_2)R_3(\xi+\ell,\xi_2)} \leq 0. \tag{35}$$

which implies that $x(\xi)(R_3(\xi,\xi_2))^{-1}$ is decreasing for $\xi \geq \xi_3$.

Next, in view of the fact that $x(\xi)(R_3(\xi,\xi_2))^{-1}$ is decreasing for $\xi \geq \xi_3$ and $\xi > \overline{f}(\xi)$ or $\overline{f}^{-1}(\xi) \leq \overline{f}^{-1}(\overline{f}^{-1}(\xi))$, yields

$$x\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right) \leq \frac{R_3\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right), \xi_2\right) x\left(\overline{f}^{-1}(\xi)\right)}{R_3\left(\overline{f}^{-1}(\xi), \xi_2\right)}. \tag{36}$$

Using equation (34) in (15), we obtain the following equation:

$$y(\xi) \geq \frac{1}{b\left(\overline{f}^{-1}(\xi)\right)}\left[1 - \frac{R_3\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right), \xi_2\right)}{b\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right)\right) R_3\left(\overline{f}^{-1}(\xi), \xi_2\right)}\right] x\left(\overline{f}^{-1}(\xi)\right) = \phi_2(\xi) x\left(\overline{f}^{-1}(\xi)\right). \tag{37}$$

This implies that

$$y\left(g_j(\xi)\right) \geq \phi_2\left(g_j(\xi)\right) x\left(\overline{f}^{-1}\left(g_j(\xi)\right)\right), \tag{38}$$

For $j = 1$ to n and for all $\xi \geq \xi_3$. Using equation (38) in equation (5) we get the following equation:

$$\Delta_\ell\left(a_1(\xi)\left[\Delta_\ell\left(a_2(\xi)\left[\Delta_\ell x(\xi)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right) + \sum_{j=1}^{n} c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3} x^{\gamma_3}\left(\overline{f}^{-1}\left(g_j(\xi)\right)\right) \leq 0, \tag{39}$$

Define the Riccati transformation

$$w(\xi) = h(\xi)\left[\frac{a_1(\xi)\left[\Delta_\ell\left(a_2(\xi)\left[\Delta_\ell x(\xi)\right]^{\gamma_1}\right)\right]^{\gamma_2}}{x^{\gamma_3}(\xi)} + a_1(\xi) H(\xi)\right] \text{for } \xi_1 \leq \xi. \tag{40}$$

Now,

$$\begin{aligned} \Delta_\ell w(\xi) \leq{}& \frac{\Delta_\ell h(\xi)}{h(\xi + \ell)} w(\xi + \ell) + h(\xi) \Delta_\ell\left(a_1(\xi) H(\xi)\right) - h(\xi) \sum_{j=1}^{n} c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3} \frac{x^{\gamma_3}\left(\overline{f}^{-1}\left(g_j(\xi)\right)\right)}{x^{\gamma_3}(\xi)} \\ & - \gamma_3 h(\xi) a_1(\xi + \ell)\left[\Delta_\ell\left(a_2(\xi + \ell)\left[\Delta_\ell x(\xi + \ell)\right]^{\gamma_1}\right)\right]^{\gamma_2} \frac{\Delta_\ell x(\xi)}{(x(\xi + \ell))^{\gamma_3 + 1}}. \end{aligned} \tag{41}$$

Using the fact $x(\xi)(R_3(\xi, \xi_2))^{-1}$ is nonincreasing for $\xi_3 \leq$, and $\overline{f}(\xi) \geq g_j(\xi)$ implies $\overline{f}^{-1}(g_j(\xi)) \leq \xi$, which gives the following equation:

$$\frac{x\left(\overline{f}^{-1}\left(g_j(\xi)\right)\right)}{x(\xi)} \geq \frac{R_3\left(\overline{f}^{-1}\left(g_j(\xi)\right), \xi_2\right)}{R_3(\xi, \xi_2)}, j = 1, 2, \ldots, n, \tag{42}$$

for $\xi_4 \leq \xi$. Substituting equations (32), (34), (40) and (42) into equation (41), we get the following equation:

$$\Delta_\ell w(\xi) \leq \frac{\Delta_\ell h(\xi)}{h(\xi+\ell)} w(\xi+\ell) + h(\xi)\Delta_\ell (a_1(\xi)H(\xi)) - h(\xi)\sum_{j=1}^{n} c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3}\left(\frac{R_3\left(\overline{f}^{-1}\left(g_j(\xi)\right),\xi_3\right)}{R_3(\xi,\xi_3)}\right)^{\gamma_3} - \gamma_3 h(\xi) R_2(\xi,\xi_1)\left(x(\xi+\ell)^{\gamma_3/\gamma_1\gamma_2-1}\left(\frac{w(\xi+\ell)}{h(\xi+\ell)} - a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}. \tag{43}$$

Next, to compute $(x(\xi)^{\gamma_3/\gamma_1\gamma_2-1}$, consider the following cases. □

Case 1. Assume $\gamma_1\gamma_2 \leq \gamma_3$. Without loss of generality we assume that there exists a constant n_1 and $\Delta_\ell x(\xi) > 0$, where

$$n_1 := x(\xi_1) \leq x(\xi) \text{for } \xi_2 \leq \xi, \tag{44}$$

which implies that

$$N_1 := n_1^{\gamma_3/\gamma_1\gamma_2-1} \leq x^{\gamma_3/\gamma_1\gamma_2-1}(\xi) \text{for } \xi_2 \leq \xi. \tag{45}$$

Case 2. Assume $\gamma_1\gamma_2 > \gamma_3$. From equation (35), there exists a constant n_2, such that

$$\frac{x(\xi)}{R_3(\xi,\xi_2)} \leq \frac{x(\xi)}{R_3(\xi_3,\xi_2)} := n_2 \quad \text{for } \xi_3 \leq \xi. \tag{46}$$

Hence,

$$\left(x(\xi)^{\gamma_3/\gamma_1\gamma_2-1} \geq n_2^{\gamma_3/\gamma_1\gamma_2-1}\left(R_3(\xi,\xi_2)\right)^{\gamma_3/\gamma_1\gamma_2-1} := N_2\left(R_3(\xi,\xi_2)\right)^{\gamma_3/\gamma_1\gamma_2-1} \quad \text{for } \xi_3 \leq \xi. \tag{47}$$

Combining equation (43) with equations (45) and (47), we get the following equation:

$$\Delta_\ell w(\xi) \leq \frac{\Delta_\ell h(\xi)}{h(\xi+\ell)} w(\xi+\ell) + h(\xi)\Delta_\ell (a_1(\xi)H(\xi)), - h(\xi)\sum_{j=1}^{n} c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3}\left(\frac{R_3\left(\overline{f}^{-1}\left(g_j(\xi)\right),\xi_2\right)}{R_3(\xi,\xi_2)}\right)^{\gamma_3} - \gamma_3 h(\xi) R_3(\xi,\xi_2)\psi(\xi)\left(\frac{w(\xi+\ell)}{h(\xi+\ell)} - a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}. \tag{48}$$

By the inequality

$$A^{1+1/\beta} - (A-B)^{1+1/\beta} \leq B^{1/\beta}\left(\left(1+\frac{1}{\beta}\right)A - \frac{1}{\beta}B\right) \text{for } 1 \leq \beta. \tag{49}$$

We obtain the following equation:

$$\left(\frac{w(\xi+\ell)}{h(\xi+\ell)} - a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1} \geq \left(\frac{w(\xi+\ell)}{h(\xi+\ell)}\right)^{1/\gamma_1\gamma_2+1} + \frac{1}{\gamma_1\gamma_2}\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1} - \left(1+\frac{1}{\gamma_1\gamma_2}\right)\frac{\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}}{h(\xi+\ell)} w(\xi+\ell). \tag{50}$$

Substituting equation (50) into equation (48), we obtain the following equation:

$$
\begin{aligned}
\Delta_\ell w(\xi) \le{}& \frac{\Delta_\ell h(\xi)}{h(\xi+\ell)} w(\xi+\ell) + h(\xi)\Delta_\ell\left(a_1(\xi)H(\xi)\right)\\
&- h(\xi)\sum_{j=1}^{n} c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3}\left(\frac{R_3\left(\overline{f}^{-1}\left(g_j(\xi)\right),\xi_2\right)}{R_3(\xi,\xi_2)}\right)^{\gamma_3}\\
&- \gamma_3 h(\xi)R_3(\xi,\xi_2)\psi(\xi)\left(\frac{w(\xi+\ell)}{h(\xi+\ell)}\right)^{1/\gamma_1\gamma_2+1}\\
&- \frac{\gamma_3}{\gamma_1\gamma_2} h(\xi)R_3(\xi,\xi_2)\psi(\xi)\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}\\
&+ \gamma_3 h(\xi)R_3(\xi,\xi_2)\psi(\xi)\left(1+\frac{1}{\gamma_1\gamma_2}\right)\frac{\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2}}{h(\xi+\ell)} w(\xi+\ell),
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\Delta_\ell w(\xi) \le{}& h(\xi)\Delta_\ell\left(a_1(\xi)H(\xi)\right) - h(\xi)\sum_{j=1}^{n} c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3}\left(\frac{R_3\left(\overline{f}^{-1}\left(g_j(\xi)\right),\xi_2\right)}{R_3(\xi,\xi_2)}\right)^{\gamma_3}\\
&- \frac{\gamma_3}{\gamma_1\gamma_2} h(\xi)R_3(\xi,\xi_2)\psi(\xi)\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}\\
&+ \left[\frac{\Delta_\ell h(\xi)}{h(\xi+\ell)} + \left(\gamma_3+\frac{\gamma_3}{\gamma_1\gamma_2}\right) h(\xi)R_3(\xi,\xi_2)\psi(\xi)\frac{\left(a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2}}{h(\xi+\ell)}\right] w(\xi+\ell)\\
&- \gamma_3\frac{h(\xi)R_3(\xi,\xi_2)\psi(\xi)}{\left(h(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}}\left(w(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1},
\end{aligned}
\tag{52}
$$

Using equation (29), (30), and (31), the above equation can be expressed as follows:

$$
\Delta_\ell w(\xi) = -A(\xi) + B(\xi)w(\xi+\ell) - C(\xi)\left(w(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}. \tag{53}
$$

By applying the inequality

$$
Xw - Yw^{1+1/\beta} \le \frac{\beta^\beta}{(1+\beta)^{1+\beta}}\frac{X^{1+\beta}}{Y^\beta}, \tag{54}
$$

We obtain the following equation:

$$
\Delta_\ell w(\xi) \le -A(\xi) + \frac{(\gamma_1\gamma_2)^{\gamma_1\gamma_2}}{(1+\gamma_1\gamma_2)^{1+\gamma_1\gamma_2}}\frac{B^{1+\gamma_1\gamma_2}(\xi)}{C^{\gamma_1\gamma_2}(\xi)}. \tag{55}
$$

Summing the above inequality from ξ_3 to ξ yields

$$
\sum_{s=0}^{m_3(\xi)}\left[A\left(\overline{\xi_3}+s\ell\right) - M\frac{B^{1+\gamma_1\gamma_2}\left(\overline{\xi_3}+s\ell\right)}{C^{\gamma_1\gamma_2}\left(\overline{\xi_3}+s\ell\right)}\right] \le w\left(\overline{\xi_3}\right) - w(\xi) < w\left(\overline{\xi_3}\right), \tag{56}
$$

which is contradictory to equation (66). With the assumption (C_2), and by Lemma 2, $y(\xi)$ converges to zero as $\xi \longrightarrow \infty$. The proof is now completed.

Now, we shall establish oscillation criteria and the convergence of solutions to equation (5) based on Philos type technique. Let us define the functions d, D: $\mathbb{N}_\ell \times \mathbb{N}_\ell \longrightarrow \mathbb{R}$, such that

(i) $D(\xi,\xi) = 0$ with $\xi \ge \xi_0 \ge 0$;
(ii) $D(\xi,\xi_1) > 0$ with $\xi > \xi_1 \ge \xi_0$;
(iii) $\Delta_{\ell(\xi_1)} D(\xi,\xi_1) = D(\xi,\xi_1+\ell) - D(\xi,\xi_1) \le 0$ for $\xi > \xi_1 \ge \xi_0$, and

$$
\Delta_{\ell(\xi_1)} D(\xi,\xi_1) + B(\xi_1)D(\xi,\xi_1) = -d(\xi,\xi_1). \tag{57}
$$

Theorem 2. *Let $g_j(\xi) \le f(\xi)$ for $j = 1$ to n, $\Phi_1(\xi) > 0$, $\Phi_2(\xi) > 0$, and equation (15) hold. If there exists functions $h(\xi)$ and $H(\xi) \in C(N_\ell(\xi_0),\mathbb{R})$ such that $a_2(\xi)H(\xi) \in C(N_\ell(\xi_0),\mathbb{R})$ and*

$$\lim_{\xi\longrightarrow\infty}\sup\frac{1}{D(\xi,\xi_1)}\sum_{s=0}^{m_1(\xi)}\left[D\left(\xi,\overline{\xi_1}+s\ell\right)A\left(\overline{\xi_1}+s\ell\right)-M\frac{\left(d\left(\xi,\overline{\xi_1}+s\ell\right)\right)^{1+\gamma_1\gamma_2}}{C^{\gamma_1\gamma_2}(\overline{\xi}+s\ell)D^{\gamma_1\gamma_2}(\xi,\overline{\xi}+s\ell)}\right]=\infty, \tag{58}$$

for all $\xi_0\le\xi_1\le\xi$, where $h(\xi)$, $A(\xi)$, $B(\xi)$ and $C(\xi)$ are defined as in Theorem 1, then as ξ tends to ∞, every solution of equation (5) is either oscillatory or tends to zero.

Proof. Let (5) has a nonoscillatory solution $\{y(\xi)\}$ on $\mathbb{N}_\ell(\xi_0)$. Without loss of generality we assume that there exists $\xi_1\in\mathbb{N}_\ell(\xi_0)$ such that, for $\xi\ge\xi_1$, $y(\xi)$, $y(\overline{f}(\xi))>0$ and $y(g_j(\xi))>0$, equations (7) and (8) holds and $x(\xi)$ satisfies either (C_1) or (C_2) for $j=1$ to n. Suppose that (C_1) is true. Equation (52) is obtained by using the same arguments as in the proof of Theorem 1. Given equation (40), the inequality (53) assumes the following form:

$$A(\xi)\le-\Delta_\ell w(\xi)+B(\xi)w(\xi+\ell)-C(\xi)(w(\xi+\ell))^{1/\gamma_1\gamma_2+1}. \tag{59}$$

Multiplying equation (59) by $D(\xi,s)$ and summing the resulting inequality from ξ_1 to $\xi-\ell$ for every $\xi_1\le\xi$, we obtain the following equation:

$$\begin{aligned}\sum_{s=0}^{m_1(\xi)}D\left(\xi,\overline{\xi_1}+s\ell\right)A\left(\overline{\xi_1}+s\ell\right)\le&-\sum_{s=0}^{m_1(\xi)}D\left(\xi,\overline{\xi_1}+s\ell\right)\Delta_\ell w\left(\overline{\xi_1}+s\ell\right)\\&+\sum_{s=0}^{m_1(\xi)}B\left(\overline{\xi_1}+s\ell\right)D\left(\xi,\overline{\xi_1}+s\ell\right)w\left(\overline{\xi_1}+s\ell+\ell\right)\\&-\sum_{s=0}^{m_1(\xi)}C\left(\overline{\xi_1}+s\ell\right)D\left(\xi,\overline{\xi_1}+s\ell\right)\left(w\left(\overline{\xi_1}+s\ell+\ell\right)\right)^{1/\gamma_1\gamma_2+1}.\end{aligned} \tag{60}$$

Applying summation by parts, we get the following equation:

$$\begin{aligned}\sum_{s=0}^{m_1(\xi)}D\left(\xi,\overline{\xi_1}+s\ell\right)A\left(\overline{\xi_1}+s\ell\right)\le&D(\xi,\xi_1)w(\xi_1)-\sum_{s=0}^{m_1(\xi)}C(\overline{\xi}+s\ell)D(\xi,\overline{\xi}+s\ell)(w(\overline{\xi}+s\ell+\ell))^{1/\gamma_1\gamma_2+1}\\&+\sum_{s=0}^{m_1(\xi)}\left[\Delta_{\ell(\xi_1)}D\left(\xi,\overline{\xi_1}+s\ell\right)+B\left(\overline{\xi_1}+s\ell\right)D\left(\xi,\overline{\xi_1}+s\ell\right)\right]w\left(\overline{\xi_1}+s\ell+\ell\right),\\\le&D(\xi,\xi_1)w(\xi_1)-\sum_{s=0}^{m_1(\xi)}C(\overline{\xi}+s\ell)D(\xi,\overline{\xi}+s\ell)(w(\overline{\xi}+s\ell+\ell))^{1/\gamma_1\gamma_2+1}+\sum_{s=0}^{m_1(\xi)}d\left(\xi,\overline{\xi_1}+s\ell\right)w\left(\overline{\xi_1}+s\ell+\ell\right),\\\le&D(\xi,\xi_1)w(\xi_1)+\sum_{s=0}^{m_1(\xi)}\left[d\left(\xi,\overline{\xi_1}+s\ell\right)w\left(\overline{\xi_1}+s\ell+\ell\right)-C(\overline{\xi}+s\ell)D(\xi,\overline{\xi}+s\ell)(w(\overline{\xi}+s\ell+\ell))^{1/\gamma_1\gamma_2+1}\right].\end{aligned} \tag{61}$$

Using equation (54), the above inequality becomes as follows:

$$\sum_{s=0}^{m_1(\xi)}D\left(\xi,\overline{\xi_1}+s\ell\right)A\left(\overline{\xi_1}+s\ell\right)\le D(\xi,\xi_1)w(\xi_1)+\sum_{s=0}^{m_1(\xi)}\left[M\frac{\left(d\left(\xi,\overline{\xi_1}+s\ell\right)\right)^{1+\gamma_1\gamma_2}}{C^{\gamma_1\gamma_2}(\overline{\xi}+s\ell)D^{\gamma_1\gamma_2}(\xi,\overline{\xi}+s\ell)}\right], \tag{62}$$

and this implies that

$$\frac{1}{D(\xi,\xi_1)}\sum_{s=0}^{m_1(\xi)}\left[D\left(\xi,\overline{\xi_1}+s\ell\right)A\left(\overline{\xi_1}+s\ell\right)-M\frac{\left(d(\xi,\overline{\xi_1}+s\ell\right)^{1+\gamma_1\gamma_2}}{C^{\gamma_1\gamma_2}(\overline{\xi}+s\ell)D^{\gamma_1\gamma_2}(\xi,\overline{\xi}+s\ell)}\right]\leq w(\xi_1), \quad (63)$$

which stands contrary to equation (58). If we assume (C_2), then by Lemma 2, we get the desired result. □

Corollary 1. *Assume that all the conditions of Theorem 2 are met and if equation (58) is replaced by the following conditions*

$$\lim_{\xi\longrightarrow\infty}\sup\frac{1}{D(\xi,\xi_1)}\sum_{s=0}^{m_1(\xi)}D\left(\xi,\overline{\xi_1}+s\ell\right)A\left(\overline{\xi_1}+s\ell\right)=\infty,$$

$$\lim_{\xi\longrightarrow\infty}\sup\frac{1}{D(\xi,\xi_1)}\sum_{s=0}^{m_1(\xi)}M\frac{\left(d(\xi,\overline{\xi_1}+s\ell\right)^{1+\gamma_1\gamma_2}}{C^{\gamma_1\gamma_2}(\overline{\xi}+s\ell)D^{\gamma_1\gamma_2}(\xi,\overline{\xi}+s\ell)}<\infty, \quad (64)$$

then also, every solution of equation (5) either tends to zero as $\xi\longrightarrow\infty$ or oscillatory.

Remark 1. One may choose $\{D(\xi,\xi^*)\}$ in a proper manner to develop various oscillation criteria for equation (5). Some alternatives are as follows:

$$\begin{aligned}&D(\xi,\xi*)=(\xi-\xi^*)_\ell^\lambda,\lambda\geq 1,\xi_0\leq\xi^*\leq\xi,\\&D(\xi,\xi*)=\left(\log\frac{\xi+1}{\xi^*+1}\right)_\ell^\lambda,\lambda\geq 1,\xi_0\leq\xi^*\leq\xi,\end{aligned} \quad (65)$$

where $n_\ell^\lambda=n(n-\ell)(n-2\ell)\cdots(n-(\lambda-1)\ell)$.

Subsequently, the oscillation criteria for equation (5) are established in the case $g_j(\xi)\geq f(\xi)$.

Theorem 3. *Let $g_j(\xi)\geq f(\xi)$ for $j=1$ to n and $\Phi_1(\xi)>0$, $\Phi_2(\xi)>0$, and equation (15) hold. If there exists functions $h(\xi)$ and $H(\xi)\in C(N_\ell(\xi_0),\mathbb{R})$ such that $a_2(\xi)H(\xi)\in C(N_\ell(\xi_0),\mathbb{R})$ and*

$$\sum_{s=0}^{\infty}\left[A^*\left(\overline{\xi_3}+s\ell\right)-M\frac{B^{1+\gamma_1\gamma_2}\left(\overline{\xi_3}+s\ell\right)}{C^{\gamma_1\gamma_2}\left(\overline{\xi_3}+s\ell\right)}\right]=\infty,\text{for }\xi_2<\xi_3\leq\xi. \quad (66)$$

where

$$\begin{aligned}A^*(\xi)&=h(\xi)\Delta_\ell(a_1(\xi)H(\xi))-h(\xi)\Phi_2(\xi)\\&+\frac{\gamma_3}{\gamma_1\gamma_2}h(\xi)R_3(\xi,\xi_2)\psi(\xi)(a_1(\xi+\ell)H(\xi+\ell))^{1/\gamma_1\gamma_2+1},\end{aligned} \quad (67)$$

for all $\xi_0\leq\xi_1\leq\xi$, where $h(\xi),A(\xi)$, $B(\xi)$, and $C(\xi)$ are defined in Theorem 1, then every solution to equation (5) is either oscillatory or tends to zero, as ξ approaches ∞.

Proof. Let equation (5) have a nonoscillatory solution $\{y(\xi)\}$ on $\mathbb{N}_\ell(\xi_0)$. Without loss of generality we assume that there exists $\xi_1\in\mathbb{N}_\ell(\xi_0)$ such that, for $\xi\geq\xi_1$, $y(\xi)$, $y(\overline{f}(\xi))>0$ and $y(g_j(\xi))>0$, equation (7) and (8) holds and $x(\xi)$ satisfies either (C_1) or (C_2) for $j=1$ to n. Suppose that (C_1) is valid. Using the procedure as followed in the proof of Theorem 1, the process is smooth in arriving equation (52). By considering the fact that $\overline{f}(\xi)\leq g_j(\xi)$, we have the following equation:

$$\xi\leq\overline{f}^{-1}\left(g_j(\xi)\right),j=1\text{ to }n. \quad (68)$$

Consequently, given that $x(\xi)$ is increasing, we get the following inequality:

$$\frac{x\left(\overline{f}^{-1}\left(g_j(\xi)\right)\right)}{x(\xi)}\geq 1,j=1\text{ to }n. \quad (69)$$

Using equation (69) in equation (48), we obtain the following equation:

$$\begin{aligned}\Delta_\ell w(\xi)&\leq\frac{\Delta_\ell h(\xi)}{h(\xi+\ell)}w(\xi+\ell)+h(\xi)\Delta_\ell(a_1(\xi)H(\xi))-h(\xi)\sum_{j=1}^{n}c_j(\xi)\left(\phi_2\left(g_j(\xi)\right)\right)^{\gamma_3}\\&-\gamma_3h(\xi)R_3(\xi,\xi_2)\psi(\xi)\left(\frac{w(\xi+\ell)}{h(\xi+\ell)}-a_1(\xi+\ell)H(\xi+\ell)\right)^{1/\gamma_1\gamma_2+1}\\&\leq-A^*(\xi)+B(\xi)w(\xi+\ell)-C(\xi)(w(\xi+\ell))^{1/\gamma_1\gamma_2+1},\end{aligned} \quad (70)$$

where $B(\xi)$ and $C(\xi)$ are presented as in Theorem 1. By adopting the same procedure as in Theorem 1, we can complete the proof. □

Corollary 2. *Let $g_j(\xi)\geq\overline{f}(\xi)$ for $j=1$ to n and $\Phi_1(\xi)>0$, $\Phi_2(\xi)>0$, and equation (15) holds. If there exist functions $h(\xi)$ and $H(\xi)\in C(N_\ell(\xi_0),\mathbb{R})$ such that $a_2(\xi)H(\xi)\in C(N_\ell(\xi_0),\mathbb{R})$ and*

$$\lim_{\xi\longrightarrow\infty}\sup \frac{1}{D(\xi,\xi_1)}\sum_{s=0}^{m_1(\xi)}\left[D\left(\xi,\overline{\xi_1}+s\ell\right)A^*\left(\overline{\xi_1}+s\ell\right)-M\frac{\left(d\left(\xi,\overline{\xi_1}+s\ell\right)\right)^{1+\gamma_1\gamma_2}}{C^{\gamma_1\gamma_2}(\overline{\xi}+s\ell)D^{\gamma_1\gamma_2}(\xi,\overline{\xi}+s\ell)}\right]=\infty, \tag{71}$$

for all $\xi_0\le\xi_1\le\xi$, where $h(\xi)$, $B(\xi)$,and $C(\xi)$ are defined in Theorem 1, $A^*(\xi)$ is defined in Theorem 3, then every solution of equation (5) is either oscillatory or approaches zero as ξ tends to ∞.

3. Examples

Example 1. Assume the third-order nonlinear generalized difference equation with multiple neutral terms

$$\Delta_\ell\left(\Delta_\ell\left(\Delta_\ell\left(y(\xi)+20y(\xi-2\ell)\right)^{1/3}\right)^3\right)+\sum_{j=1}^{j=4}c_j(\xi)y\left(g_j(\xi)\right)=0,\xi\ge4\ell. \tag{72}$$

Here $a_1(\xi)=a_2(\xi)=1$, $b(\xi)=20$, $\gamma_1=1/3$, $\gamma_2=3$, $\gamma_3=1$, $c_j(\xi)=42j(j+3)>0$, $\overline{f}(\xi)=\xi-2\ell<\xi$,and $g_j(\xi)=\xi-j\ell$, which implies $\lim_{\xi\longrightarrow\infty}\overline{f}(\xi)=\lim_{\xi\longrightarrow\infty}\overline{f}(\xi-2\ell)=\infty$ and $\lim_{\xi\longrightarrow\infty}g_1(\xi)=\lim_{\xi\longrightarrow\infty}(\xi)=\infty$. Then, we obtain the following equation:

$$\begin{aligned}
R_1(\xi,\xi_1)&=\frac{\xi-\xi_1}{\ell}\Rightarrow R_1(\xi,2\ell)=\frac{\xi-2\ell}{\ell},\\
R_2(\xi,\xi_1)&=\left(\frac{\xi-\xi_1}{\ell}\right)^3\Rightarrow R_2(\xi,2\ell)=\left(\frac{\xi-2\ell}{\ell}\right)^3,\\
R_3(\xi,\xi_2)&=\frac{(\xi-\xi_2)^2(\xi-\xi_2-\ell)^2}{4\ell^4}\Rightarrow R_3(\xi,2\ell)=\frac{\left(\xi^2-5\xi\ell+6\ell^2\right)^2}{4\ell^4},\\
\phi_1(\xi)&=\frac{19}{400}\ge0,\\
\phi_2(\xi)&=\frac{1}{20}\left(1-\frac{(\xi+\ell)^2(\xi+2\ell)^2}{20\xi^2(\xi-\ell)^2}\right)\ge0,\\
\Phi_1(\xi)&=119.7\ge0,\\
\Phi_2(\xi)&=\sum_{j=1}^{4}\frac{21}{10}\left(j^2+3j\right)\left(1-\frac{(\xi-j\ell+\ell)^2(\xi-j\ell+2\ell)^2}{20(\xi-j\ell)^2(\xi-j\ell-\ell)^2}\right)\ge0,
\end{aligned} \tag{73}$$

where $M=1/4$ and $\phi(\xi)=N_1$. If we choose $h(\xi)=\xi$ and $H(\xi)=\xi$, it is easy to verify all the conditions of Theorem 1. Hence, all solutions of equation (72) are oscillatory. Infact $\{y(\xi)\}=\left\{(-1)^{[3\xi/\ell]}\right\}$ is one such oscillatory solution.

Example 2. Assume the third-order non-linear generalized difference equation with multiple neutral terms

$$\Delta_\ell\left[\Delta_\ell\left((\xi-2\ell)\left[\Delta_\ell\left(y(\xi)+2y(\xi-2\ell)\right)\right]^3\right)\right]^{1/5}+\sum_{j=1}^{2}\frac{6^{3/5}}{\ell}(2\xi-(5-2j)\ell^{1/5}y^{3/5}(\xi-(4-2j)\ell)=0. \tag{74}$$

Here $a_1(\xi)=1$, $a_2(\xi)=\xi-2\ell$, $b(\xi)=2$, $\gamma_1=3$, $\gamma_2=1/5$, $\gamma_3=3/5$, $c_1(\xi)=6^{3/5}(2\xi-3\ell)^{1/5}/\ell$, $c_2(\xi)=6^{3/5}(2\xi-\ell)^{1/5}/\ell$, $\overline{f}(\xi)=\xi-2\ell<\xi$, $g_1(\xi)=\xi-2\ell$, and $g_2(\xi)=\xi$, which implies $\lim_{\xi\longrightarrow\infty}\overline{f}(\xi)=\lim_{\xi\longrightarrow\infty}\overline{f}(\xi-2\ell)=\infty$, $\lim_{\xi\longrightarrow\infty}g_1(\xi)=\lim_{\xi\longrightarrow\infty}(\xi-2\ell)=\infty$ and $\lim_{\xi\longrightarrow\infty}g_2(\xi)=\lim_{\xi\longrightarrow\infty}\xi=\infty$. Then, we obtain $R_1(\xi,\xi_1)=R_1(\xi,2\ell)=\xi-2\ell/\ell$, $R_2(\xi,\xi_1)=R_2(\xi,2\ell)=1$, $R_3(\xi,\xi_2)=R_3(\xi,2\ell)=\xi-2\ell/\ell$,

$$\begin{aligned}
&R_3\left(\overline{f}^{-1}(\xi), 2\ell\right) = R_3(\xi + 2\ell, 2\ell) = \frac{\xi}{\ell},\\
&R_3\left(\overline{f}^{-1}\left(\overline{f}^{-1}(\xi)\right), 2\ell\right) = R_3(\xi + 4\ell, 2\ell) = \frac{\xi + 2\ell}{\ell},\\
&\phi_1(\xi) = \frac{1}{4} \geq 0,\\
&\phi_2(\xi) = \frac{\xi - 2\ell}{4\xi} \geq 0,\\
&\Phi_1(\xi) = \left(\frac{6}{4}\right)^{3/5} \times \frac{1}{\ell}\left[(2\xi - 3\ell)^{1/5} + (2\xi - \ell)^{1/5}\right] \geq 0,\\
&\Phi_2(\xi) = \left(\frac{6(\xi - 4\ell)}{4(\xi - 2\ell)}\right)^{3/5}\frac{(2\xi - 3\ell)^{1/5}}{\ell} + \left(\frac{6(\xi - 2\ell)}{4\xi}\right)^{3/5}\frac{(2\xi - \ell)^{1/5}}{\ell} \geq 0.
\end{aligned} \tag{75}$$

where $M = 1/4$ and $\psi(\xi) = N_1$. If we choose $h(\xi) = \xi$ and $(\xi) = \xi$, it is observed that all the conditions of Theorem 1 are satisfied. Hence, all solutions of equation (74) are oscillatory. We observed that $\{y(\xi)\} = \left\{(-1)^{[5\xi/\ell]}\right\}$ is one such oscillatory solution of equation (74).

Example 3. Consider the third-order nonlinear generalized difference equation with multiple neutral terms

$$\Delta_\ell\left[\Delta_\ell\left[(\xi - 2\ell)\left[\Delta_\ell\left(y(\xi) + 3y(\xi - 2\ell)\right)\right]\right]^3\right] + \sum_{i=1}^{5} c_i(\xi) y^3(g_i(\xi)) = 0. \tag{76}$$

Here $a_1(\xi) = 1$, $a_2(\xi) = \xi - 2\ell$, $b(\xi) = 3$, $\gamma_1 = 1$, $\gamma_2 = 3$, $\gamma_3 = 3$, $c_1(\xi) = 27\ell^3 + 441/2\ell(\xi - \ell)^2 - 189/2\ell^2(\xi - \ell)$, $c_2(\xi) = -152\ell^3 + 132\ell^2\xi - 768\ell\xi^2$, $c_3(\xi) = 250\ell^3 + \ell(\xi + \ell)^2$, $c_4(\xi) = -152\ell^3 - 132\ell^2(\xi + 2\ell) - 768\ell(\xi + 2\ell)^2$, $c_5(\xi) = 27\ell^3 + 441/2\ell(\xi + 3\ell)^2 + 189/2\ell^2(\xi + 3\ell)$, $\overline{f}(\xi) = \xi - 2\ell < \xi$, $g_1(\xi) = \xi - \ell$, $g_2(\xi) = \xi$, $g_3(\xi) = \xi + \ell$, $g_4(\xi) = \xi + 2\ell$, and $g_5(\xi) = \xi + 3\ell$ with $h(\xi) = H(\xi) = \xi$, then it is easy to verify that all the conditions of Theorem 1 are satisfied. Infact $\{y(\xi)\} = \{\ell/\xi\}$ is one such solution of equation (74) which converges to zero as $\xi \longrightarrow \infty$.

4. Conclusion

As part of this study, we established newly enhanced oscillation criteria for the third order nonlinear generalized difference equation with multiple neutral terms, which is new in the literature. Implementation of Riccati transformation and Philos type technique enhances the quality of our findings. Our findings are well supported by the presentation of few examples.

5. Suggestion for Future Research

By defining $x(\xi) = y(\xi) + \sum_{i=1} b_i(\xi) y(\overline{f}_i(\xi))$, the research can be extended for various class of generalized difference equation.

Data Availability

There are no synthetic data used for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and all authors read the manuscript and approved the final submission.

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