

## Research Article

# On the Dynamics of a Discrete Fractional-Order Cournot–Bertrand Competition Duopoly Game

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A discrete fractional-order Cournot–Bertrand competition duopoly game is introduced based on the fractional-order difference calculus of the Caputo operator. The model is designed when players can make long memory decisions. The local stability of equilibrium points is discussed for the proposed model. Some numerical simulations explore the model's bifurcation and chaos by employing bifurcation diagrams, phase portraits, maximal Lyapunov exponents, and time series. According to our findings, the fractional-order parameter has an effect on the game's stability and dynamics.

## 1. Introduction

Game theory is one of the most interesting and complex topics that many researchers are interested in understanding. Game theory is concerned with predicting results for strategic games in which participants, for example, two or more firms competing on the market, have incomplete information on the intentions of others. It is known that the game theory is relevant to the study of corporate behavior in oligopolistic markets, for example, the decisions that companies must make in terms of production and pricing levels, as well as the amount of money invested in research and development. The decision-making mechanism has an important role to play in the process of adjusting the production and profits of firms. Firms typically use one of the following to change their market growth: naïve learning expectation, adaptive learning expectation, limited learning rationality, and local learning approximation. Discrete oligopoly dynamics based on company profit maximizations have been considered [1–7]. Furthermore, these models have been utilized to examine the dynamic characteristics of

competitive markets, which has been classified as steady state, periodic, and chaotic [8–14].

Fractional calculus, particularly discrete fractional calculus, has attracted substantial interest in recent decades due to its extensive significance in a wide range of scientific disciplines, including complex systems with memory and heredity. Researchers turned their attention to a discrete fractional calculus and tried to develop a complete theoretical framework for this subject. This is due to the importance of this field in many real issues, such as discrete adaptive systems, biological growth systems, and digital engineering systems, all of which contain memory [15–20]. The discrete difference analogues of classical Caputo and Riemann derivatives have been introduced in [21]. In addition, advances have been made in the study of fractional finite difference equations and the inclusion of fractional difference equations [22–26]. Recently, the stability of fractional time systems in a variety of real-world problems has been investigated. These studies have shown that discrete fractional systems are more realistic to process real systems and have a rich dynamic compared to discrete systems with

integer-order. Many studies have studied continuous fractional differentiation representing the effects of economic memory that have been presented [27–30] and the references that exist in them as well. The fractional-order difference equation, which is a natural extension of the integer-order difference equation, has long-term memory effects that have been explored in a few studies [25, 31–36]. Recently, the fractional difference calculus was used in the Cournot duopoly game [37] and the Bertrand duopoly game [38]. This reflects the long-term memory effects of Cournot–Bertrand dynamic games in the fractional-order form for the game. Xin et al. [37, 38] investigated the dynamics of the Cournot and Bertrand games, which indicated the market evolution of the two enterprises.

This work is especially interested in the novel discrete fractional-order Cournot–Bertrand duopoly game, which is a modification of the Cournot–Bertrand duopoly game with integer-order [39]. We aim to extend this game to a fractional case and to study the dynamics of the game. As we know, fractional-order calculus is a general form of integer-order calculus, so it has a higher representation for phenomena with a long memory. It can be shown that the fractional differentiation parameter is a vital indicator of the bifurcation path and the chaos that is created and disappeared. We will investigate the dynamics of the discrete fractional-order Cournot–Bertrand duopoly game such as the stability, bifurcation, and chaos of the proposed game. To analyze complexity of the game, explicit stability criteria [40, 41], asymptotic stability criteria [42] and the local stability regions of the boundary and Nash equilibrium points are provided through numerical simulation. The dynamic behavior of the proposed game is illustrated through an exploratory investigation of equilibrium point stability and numerical simulation. In this work, we are analyzing a Cournot–Bertrand duopoly game similar to Wang and Ma [39], but using a discrete fractional calculus. The equilibrium point structure dynamic reflects economic

explanations for the proposed game’s market of two enterprises.

The work is organized as seen follows. Section 2 describes the market dynamics of two enterprises using a discrete fractional-order Cournot–Bertrand duopoly game. The Nash equilibrium local stability conditions are established in Section 3. Using numerical simulations, we investigate local bifurcations, maximal Lyapunov exponents, and phase portraits of complex dynamics in Section 4. Section 5 contains a summary of the findings as well as a few observations.

## 2. Preliminaries

In this section, some preliminaries about fractional-order difference calculus are introduced. On an arbitrary time scale, dynamic behaviors and applications of fractional difference models were investigated in the last decade where delta difference equation was used [40–43].

Assume that a sequence  $u(n)$  is given, and the isolated time scale  $\aleph_a$  is represented in terms of real valued constant  $\tau$ , i.e.,  $\{\tau, \tau + 1, \tau + 2, \dots\}$ , such that  $u: \aleph_\tau \rightarrow \mathbb{R}$ . Also, the difference operator is denoted by  $\Delta$ , where  $\Delta u(n) = u(n + 1) - u(n)$ . Then, we summarize some of the basic definitions related to discrete fractional calculus as follows.

*Definition 1.* For  $\alpha > 0$ , the fractional sum of order  $\alpha$  is given by [21]

$$\Delta_\tau^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \sum_{m=\tau}^{t-\alpha} \frac{\Gamma(t-m)}{\Gamma(t-m-\alpha+1)} u(m), \quad t \in \aleph_{\tau+\alpha}. \tag{1}$$

*Definition 2.* The Caputo-like delta difference of order  $\alpha$  is defined by [21, 42]

$${}^C \Delta_\tau^\alpha u(t) = \Delta_\tau^{-(n-\alpha)} \Delta^n u(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{m=\tau}^{t-(n-\alpha)} \frac{\Gamma(t-m)}{\Gamma(t-m-n+\alpha+1)} \Delta^n u(m), \tag{2}$$

$$t \in \aleph_{\tau+\alpha}, n = [\alpha] + 1.$$

**Theorem 1** (See [21, 40–43]). *In order to solve the delta fractional difference equation,*

$$\begin{cases} {}^C \Delta_\tau^\alpha u(t) = f(t + \alpha - 1, u(t + \alpha - 1)), \\ \Delta^k u(t) = u_k, n = [\alpha] + 1, \quad k = 0, 1, \dots, n - 1. \end{cases} \tag{3}$$

As a result, the corresponding discrete integral equation is

$$u(t) = u_0(t) + \frac{1}{\Gamma(\alpha)} \sum_{m=\tau+n-\alpha}^{t-\alpha} (t - \sigma(m))^{\alpha-1} f(m + \alpha - 1, u(m + \alpha - 1)), \quad t \in \aleph_{\tau+n}, \tag{4}$$

where

$$u_0(t) = \sum_{k=0}^{n-1} \frac{\Gamma(t - \tau + 1)}{k! \Gamma(t - \tau - k + 1)} \Delta^k u(t). \quad (5)$$

*Remark 1.* If  $\tau = 0$ , we rewrite discrete integral equation in the following numerical form:

$$u(n) = u_0(t) + \frac{1}{\Gamma(\alpha)} \sum_{m=1}^n \frac{\Gamma(n - m + \alpha)}{\Gamma(n - m + 1)} f(u(m - 1)). \quad (6)$$

**Theorem 2** (See [40–42]). *The linear discrete-time fractional-order system is*

$${}^C \Delta_t^\alpha U(t) = GU(t + \alpha - 1), \quad (7)$$

where  $U(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ ,  $0 < \alpha \leq 1$ ,  $G \in \mathbb{R}^{n \times n}$ , and  $\forall t \in \mathbb{N}_{\tau+1-\alpha}$ . *The zero equilibrium of system (10) is asymptotically stable if*

$$\lambda \in \left\{ z \in \mathbb{C}: |z| < \left( 2 \cos \frac{|\arg z| - \pi}{2 - \alpha} \right)^\alpha, |\arg z| > \frac{\alpha\pi}{2} \right\}, \quad (8)$$

for all the eigenvalues  $\lambda$  of matrix  $G$ .

### 3. Discrete Fractional-Order Cournot–Bertrand Duopoly Game

According to traditional oligopoly models [1, 6], firms compete in the same strategic variable, such as output (Cournot) or price (Bertrand). A hybrid model, commonly known as the Cournot–Bertrand model [6, 39], permits certain enterprises to compete in output, while others compete in pricing. Real-world market observations that match Cournot–Bertrand behavior have bolstered the model’s validity and rapidly growing literature on advancements and applications. Long-term memory effects in dynamic oligopoly games are economically significant [34, 37, 38]. Therefore, we introduce the novel discrete fractional-order Cournot–Bertrand duopoly game, which is a modification of the Cournot–Bertrand duopoly game with integer order. As a consequence, it has a better representation of phenomena with a long memory in oligopoly game. The main goal is to establish out how the fractal parameter affects game dynamics including stability, bifurcation, and chaos.

We suggest a simple Cournot–Bertrand duopoly common oligopoly game [39]. Two enterprises, denoted by the letters  $i = 1, 2$ , produced differentiated goods with perfect replacements and set their product pricing based on the same market rule. Assume that  $p_i(t)$  and  $q_i(t)$  denote the goods price and quantity output of firm  $i$  for the period  $t \in \mathbb{Z}^+$ . The inverse demand functions for a variety of products 1 and 2 originate from the representative consumer maximization of the following utility function [39]:

$$U(q_1, q_2) = q_1 + q_2 - \frac{1}{2}(q_1^2 + 2dq_1q_2 + q_2^2), \quad (9)$$

subjected to restrictions on the budget  $p_1q_1 + p_2q_2 = M$ . Then, the inverse demand functions is defined as follows:

$$\begin{cases} p_1 = 1 - q_1 - dq_2, \\ p_2 = 1 - q_2 - dq_1, \end{cases} \quad (10)$$

where  $p_1$  and  $p_2$  represent the pricing of firm 1’s and firm 2’s items, respectively, and  $q_1$  and  $q_2$  are the quantities of products of company 1 and company 2. The parameter  $d \in [0, 1]$  is the product differentiation between two firms. Products are homogeneous goods when  $d = 1$ , and each firm has a monopoly case when  $d = 0$ . The demand system can be written in two strategic variables  $q_1$  and  $p_2$ .

$$\begin{cases} p_1 = 1 - d - (1 - d^2)q_1 + dp_2, \\ q_2 = 1 - p_2 - dq_1. \end{cases} \quad (11)$$

Consider that the two companies have the same unit cost  $c > 0$  and that the cost function has the same linear form:

$$C_i(q_i) = cq_i, i = 1, 2. \quad (12)$$

Thus, the profit functions for firms are given by

$$\begin{cases} \pi_1 = (1 - d - (1 - d^2)q_1 + dp_2 - c)q_1, \\ \pi_2 = (p_2 - c)(1 - p_2 - dq_1). \end{cases} \quad (13)$$

In the classical dynamical Cournot–Bertrand duopoly game, to decide the corresponding profit-maximizing, every player erroneously believes that its rival’s pricing in period  $(t + 1)$  is the same as in period  $(t)$ . Therefore, this type of game does not have a long memory effect. The traditional game will be introduced using discrete fractional-order calculus, and the two players will make decisions using a new dynamic adjustment mechanism with long memory and local marginal profit expectation. Thus, the marginal profit of two players is as follows [39]:

$$\begin{cases} \frac{\partial \pi_1}{\partial q_1} = ((1 - d - 2(1 - d^2)q_1 + dp_2) - c), \\ \frac{\partial \pi_2}{\partial p_2} = (1 + c - 2p_2 - dq_1). \end{cases} \quad (14)$$

Assume that the two businesses have limited information about the market demand function and also their price decision is based on a dynamic adjustment process with limited rationality and a long-term memory effect of marginal profit. In the next step, the firm decides to raise (reduce) the price of its product based on if the long-term marginal profit is greater (less) than zero. As a result, we utilize the dynamic adjustment process for the Cournot–Bertrand game as follows:

$$\begin{cases} \Delta^\alpha q_1 = \nu_1 q_1(t + \alpha - 1) \frac{\partial \pi_1}{\partial q_1}, \\ \Delta^\alpha p_2 = \nu_2 p_2(t + \alpha - 1) \frac{\partial \pi_2}{\partial p_2}, \end{cases} \quad (15)$$

where  $\nu_i$  is the speed of adjustment of firm  $i, i = 1, 2$  and  $\alpha \in (0, 1)$  denotes a fractional-order number, indicating the

long-term memory effect. Thus, the discrete fractional-order Cournot–Bertrand duopoly game is as follows:

$$\begin{cases} \Delta^\alpha q_1 = \nu_1 q_1 (t + \alpha - 1) (1 - c - d + d p_2 (t + \alpha - 1) - 2q_1 (t + \alpha - 1) + 2d^2 q_1 (t + \alpha - 1)), \\ \Delta^\alpha p_2 = \nu_2 p_2 (t + \alpha - 1) (1 + c - 2p_2 (t + \alpha - 1) - d q_1 (t + \alpha - 1)). \end{cases} \quad (16)$$

*Remark 2.* When  $\alpha = 1$ , the model (16) devolves to the Wang-Ma model [39]:

$$\begin{cases} q_1 (t + 1) = q_1 (t) + \nu_1 q_1 (t) (1 - c - d + d p_2 (t) - 2q_1 (t) + 2d^2 q_1 (t)), \\ p_2 (t + 1) = p_2 (t) + \nu_2 p_2 (t) (1 + c - 2p_2 (t) - d q_1 (t)). \end{cases} \quad (17)$$

We will show that the model game parameters, especially the long-term memory effect, have an effect on the long-term dynamic response of our novel game when compared to the Wang-Ma game [39].

In the next sections, several theoretical features corresponding to game (16) are investigated.

#### 4. The Equilibrium Points and Their Local Stability

We solve the following equation to find the equilibrium points of game system (16):

$$\begin{cases} (1 - c - d + d p_2 (t) - 2q_1 (t) + 2d^2 q_1 (t)) = 0, \\ (1 + c - 2p_2 (t) - d q_1 (t)) = 0. \end{cases} \quad (18)$$

Their four equilibria are  $E_0 = (0, 0)$ ,  $E_1 = (0, 1 + c/2)$ ,  $E_2 = (1 - c - d/2(1 - d^2), 0)$ , and  $E_* = (2 - 2c - d + c d/4 - 3d^2, 2 + 2c - d + c d - d^2 - 2cd^2/4 - 3d^2)$ . In economics, its equilibria mean

- (i) The equilibrium  $E_0$  is trivial equilibrium point
- (ii) The equilibrium point  $E_1$  implies that the best quantity of the first player is  $q_1^* = 0$  if the second

player sets its optimal product price  $p_2^* = 1 + c/2$ . Likewise, the second player's best price is  $p_2^* = 1 + c/2$  if the player uses a zero-price approach. Clearly,  $E_1$  is a border equilibrium point that corresponds to the pure monopoly.

- (iii) The  $E_2$  equilibria indicates that the best quantity of the first player is  $q_1^* = 1 - c - d/2(1 - d^2)$  if the second company determines the best good price  $p_2^* = 0$ . Likewise, the first company's best pricing is  $q_1^* = 1 - c - d/2(1 - d^2)$  if the company uses a zero-price approach. Clearly,  $E_2$  is a border equilibrium point.
- (iv) The equilibrium  $E_*$  indicates two enterprises will preserve their equilibrium quantity and pricing jointly since no enterprise can gain an advantage by deviating unilaterally from its own equilibrium. Clearly, the point  $E_*$  is a Nash equilibrium. The complexity of system (16) will be explored. First, the Jacobian matrix of system (16) is computed at a given equilibrium point  $\bar{E} = (q_1^*, p_2^*)$ , and it can be expressed as

$$J(q_1, p_2) = \begin{pmatrix} \nu_1(1 - c - d + d p_2^* + 4(d^2 - 1)q_1^*) & \nu_1 d q_1^* \\ -\nu_2 d p_2^* & \nu_2(1 + c - 4p_2^* - d q_1^*) \end{pmatrix}. \quad (19)$$

Then, the following theorems are presented to clarify linear stability for each equilibrium point in the model.

The trivial equilibrium point  $E_0$  is unstable.

*Proof.* The Jacobian matrix's eigenvalues at  $E_0$  can be demonstrated to be  $(\nu_2(1 + c), \nu_1(1 - c - d))$ , which indicate that one of them is always positive, and thus, the conditions of asymptotic stability in Theorem 2 are not satisfied.  $\square$

The equilibrium point  $E_1$  is asymptotically stable if

$$\begin{aligned} \nu_1(cd + 2 - 2c - d) < 0, \nu_2(1 + c) \\ < 2^\alpha, |\nu_1(cd + 2 - 2c - d)| < 2^{\alpha+1}. \end{aligned} \quad (20)$$

*Proof.* The eigenvalues of Jacobian matrix at  $E_1$  are  $(-\nu_2(1 + c), 1/2\nu_1(c d + 2 - 2c - d))$ , which means that the conditions of asymptotic stability in Theorem 2 are satisfied

if the second eigenvalue has a negative sign, and also, the modulus of the two eigenvalues is bounded by  $2^\alpha$  and  $2^{\alpha+1}$ , respectively. Figure 1 shows stability regions in some three and two-dimensional spaces of model's parameters, whereas

$$\begin{aligned} \nu_1(c+d-1) < 0, \nu_2(2cd^2 - cd - 2c + d^2 + d - 2) > 0, \quad 0 < d < 1, \\ |\nu_2(c+d-1)| < 2^\alpha, \\ \nu_2(2cd^2 - cd - 2c + d^2 + d - 2) < 2^{\alpha+1}(1-d^2). \end{aligned} \tag{21}$$

*Proof.* The eigenvalues of Jacobian matrix at  $E_2$  are  $(\nu_1(c+d-1), \nu_2(2cd^2 - cd - 2c + d^2 + d - 2)/2(d^2 - 1))$ , which means that the conditions of asymptotic stability in Theorem 2 are satisfied if the two eigenvalues have negative signs, and also, the modulus of the two eigenvalues is bounded by  $2^\alpha$  and  $2^{\alpha+1}(1-d^2)$ , respectively.  $\square$

However, detailed numerical examinations in space of promoters show that the aforementioned conditions cannot be simultaneously achieved at feasible values of parameters, and therefore, due to the impossibility of numerically satisfying the above conditions, the equilibrium point  $E_2$  is unstable.

Finally, the Nash equilibrium point  $E_*$  has long complicated expressions for its associated eigenvalues which

Figure 2 shows the resulting time series when the values of parameters are selected in stable regions of  $E_1$ .

The equilibrium  $E_2$  is asymptotically stable if and only if

$\square$  renders numerical investigations of regions of stability inevitable. Figure 3 shows stability regions in some three and two-dimensional spaces of model's parameters, whereas Figure 4 illustrates the resulting time series when the values of parameters are selected in stable regions of  $E_*$ .

### 5. Numerical Simulations

In this section, the complex dynamic features of the discrete fractional Cournot–Bertrand model (16) are investigated using various methods such as bifurcation diagrams, phase portraits, and MLE. The effects of major model parameters are investigated. For the present fractional discrete model (16) using theorem (4), the system (16) can be numerically rewritten as follows:

$$\begin{cases} q_1(n) = q_1(0) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^n \frac{\Gamma(n-i+\alpha)}{\Gamma(n-i+1)} \nu_1 q_1(i-1) ((1-c-d+dp_2(i-1) - 2q_1(i-1) + 2d^2 q_1(i-1))), \\ p_2(n) = p_2(0) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^n \frac{\Gamma(n-i+\alpha)}{\Gamma(n-i+1)} \nu_2 p_2(i-1) (1+c-2p_2(i-1) - dq_1(i-1)). \end{cases} \tag{22}$$

The initial point  $(q_1(0), p_2(0)) = (0.2, 0.1)$  is used in the following simulations. In particular, the complicated dynamics exhibited by the model are examined via using the bifurcation diagram, phase portraits, and maximal Lyapunov exponent (MLE). The Lyapunov exponent for a one-dimensional map can be calculated by calculating the average value for perturbations from the trajectory over a time interval. The Lyapunov exponents for an  $n$ -dimensional mapping can be obtained using the eigenvalues of the product of Jacobian matrices for integer-order systems. In order to approximate the values of Lyapunov exponents of the discrete fractional model (16), the Jacobian matrix method which has been introduced by Wu and Baleanu [44] can be employed [45, 46]. In the following part, the numerical analysis will look at the effects of the model's main parameters, as well as the effects of long-term memory and adjustment speeds.

First, the influences of parameter  $\nu_1$  in integer-order and fractional-order cases are explored. Figure 5 shows bifurcation diagrams and MLE plots versus parameter  $\nu_1$  along with examples of phase portraits at some selected values of parameters.

Second, the influences of parameter  $\nu_2$  in integer-order and fractional-order cases are explored. Figure 6 shows bifurcation diagrams and MLE plots versus parameter  $\nu_2$ , along with examples of phase portraits at some selected values of parameters.

Third, the influences of parameter  $d$  in integer-order and fractional-order cases are explored. Figure 7 shows bifurcation diagrams and MLE plots versus parameter  $d$  along with examples of phase portraits at some selected values of parameters.

Finally, the effects of fractional-order  $\alpha$  are explored. Figure 8 shows bifurcation diagrams and MLE plots versus parameter  $\alpha$  along with examples of phase portraits at some selected values of parameters.

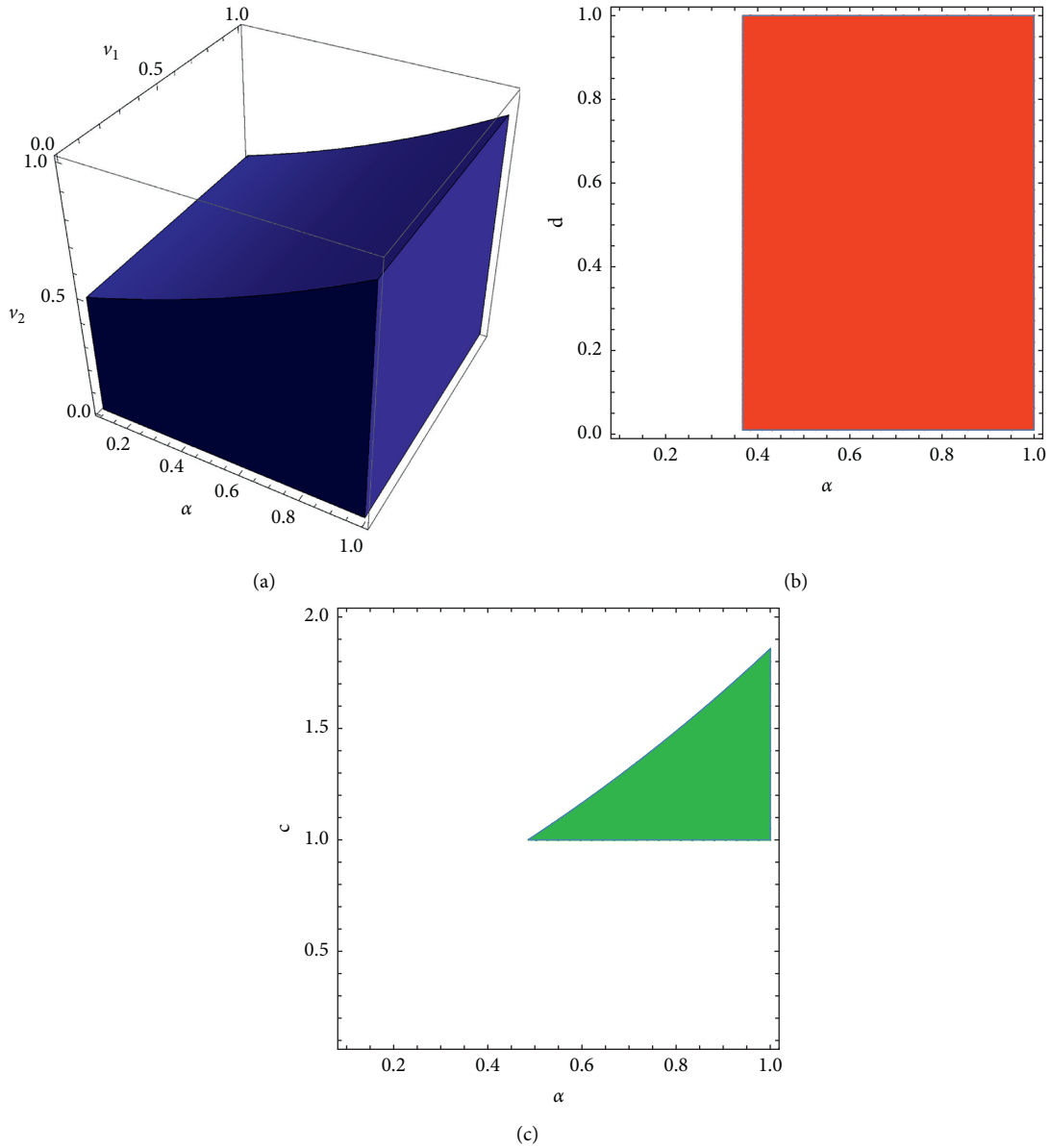


FIGURE 1: Stability regions of equilibrium point  $E_1$  in some three and two-dimensional spaces of model's parameters when (a)  $c = 1.15; d = 0.6$ , (b)  $c = 1.15; v_1 = 0.5; v_2 = 0.6$ , and (c)  $d = 0.5; v_1 = 0.7; v_2 = 0.7$ .

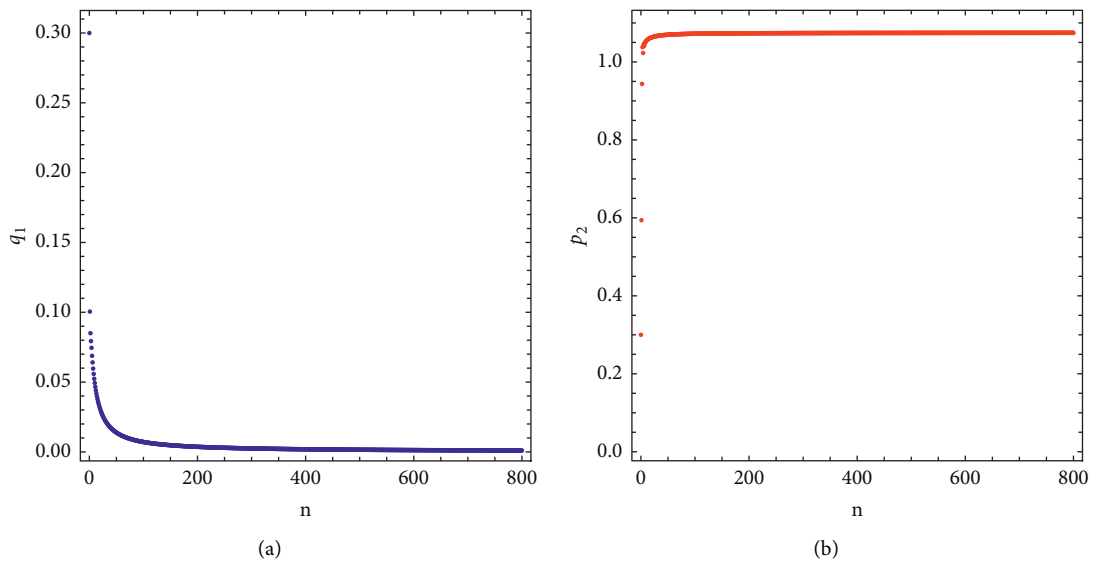


FIGURE 2: Time series of model (16) at  $c = 1.15, d = 0.5, v_1 = 0.7, v_2 = 0.7$ , and  $\alpha = 0.9$ .

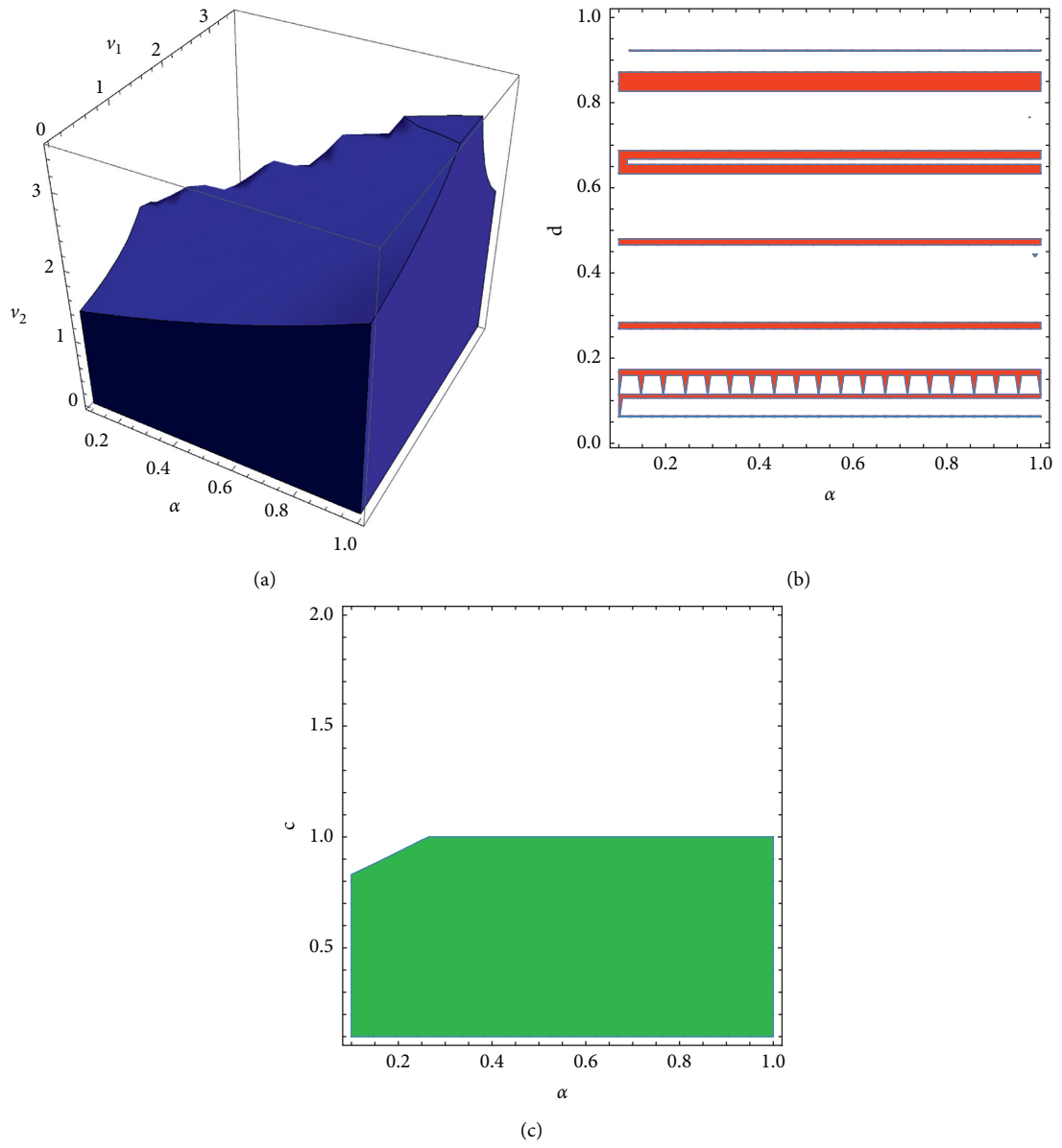


FIGURE 3: Stability regions of equilibrium point  $E_*$  in some three and two-dimensional spaces of model's parameters when (a)  $c = 0.15; d = 0.8$ , (b)  $c = 1; v_1 = 0.4; v_2 = 0.7$ , and (c)  $d = 0.5; v_1 = 0.3; v_2 = 0.6$ .

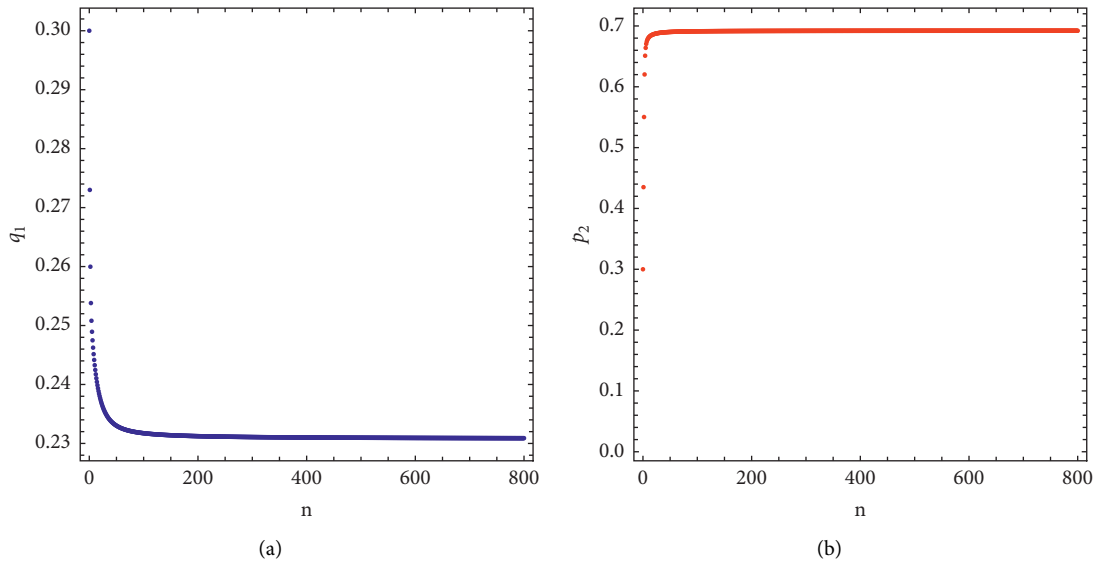


FIGURE 4: Time series of model (16) at  $c = 0.5, d = 0.5, \nu_1 = 0.3, \nu_2 = 0.6,$  and  $\alpha = 0.9.$

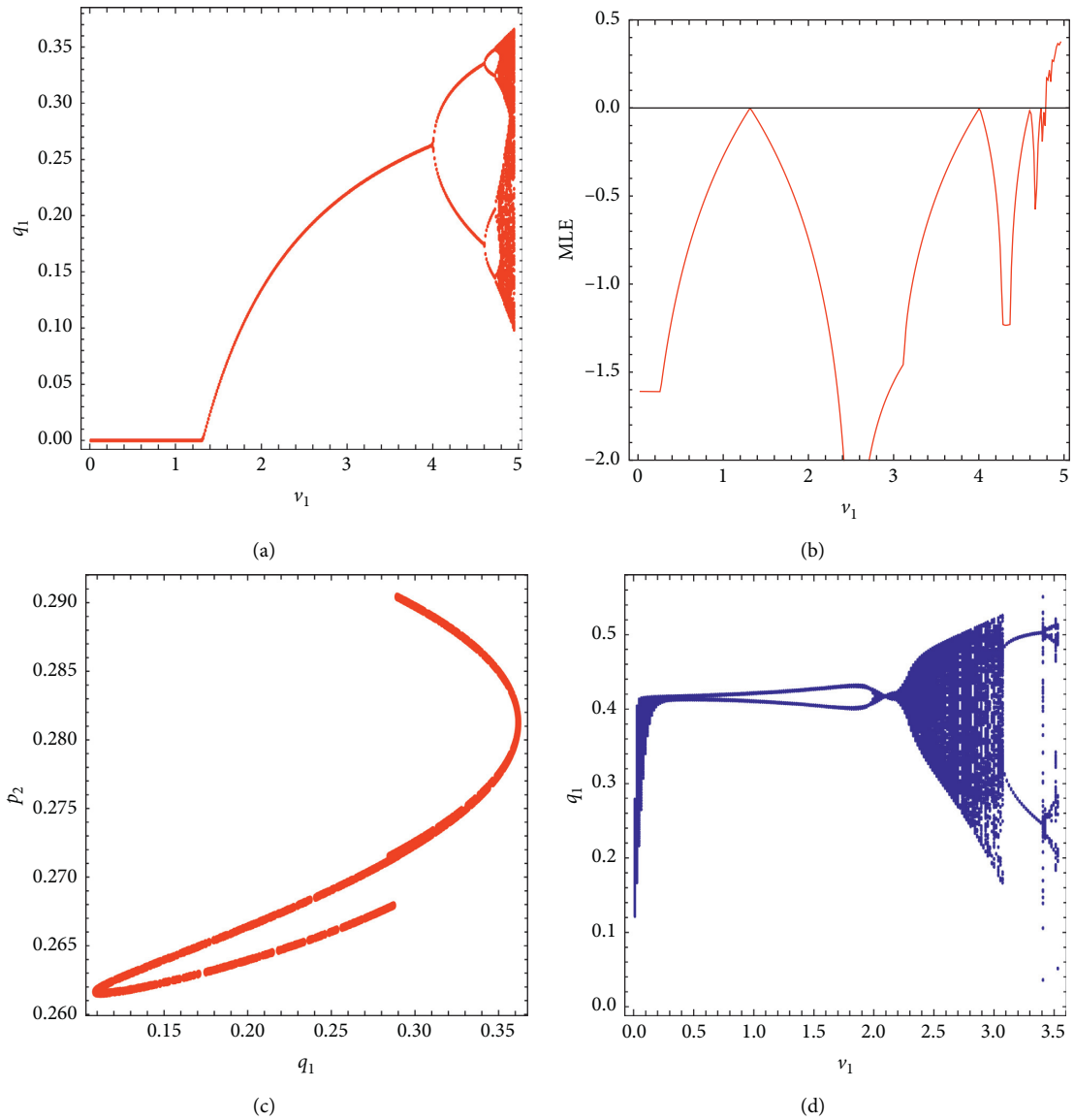


FIGURE 5: Continued.



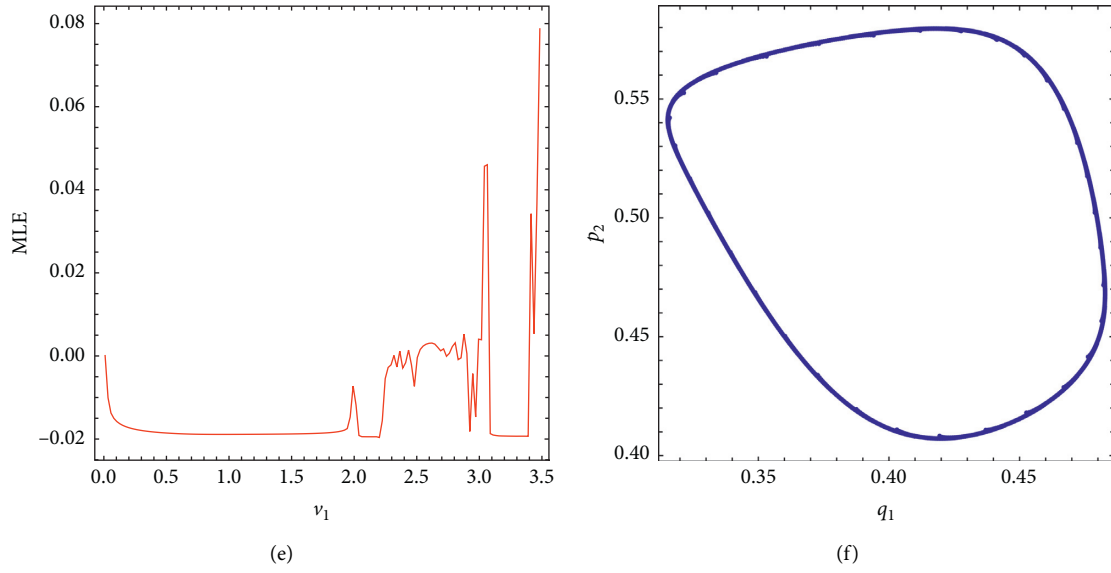


FIGURE 5: (a) Bifurcation diagram of system (16) vs.  $\nu_1$  at  $c = 0.1; d = 0.2; \nu_2 = 2; \alpha = 1$ . (b) MLE plot of system (16) vs.  $\nu_1$  at  $c = 0.1; d = 0.2; \nu_2 = 2; \alpha = 1$ . (c) Phase portrait of system (16) at  $c = 0.1; d = 0.2; \nu_1 = 4.9; \nu_2 = 2; \alpha = 1$ . (d)–(f) similar to (a)–(c) but for  $\alpha = 0.95$ .

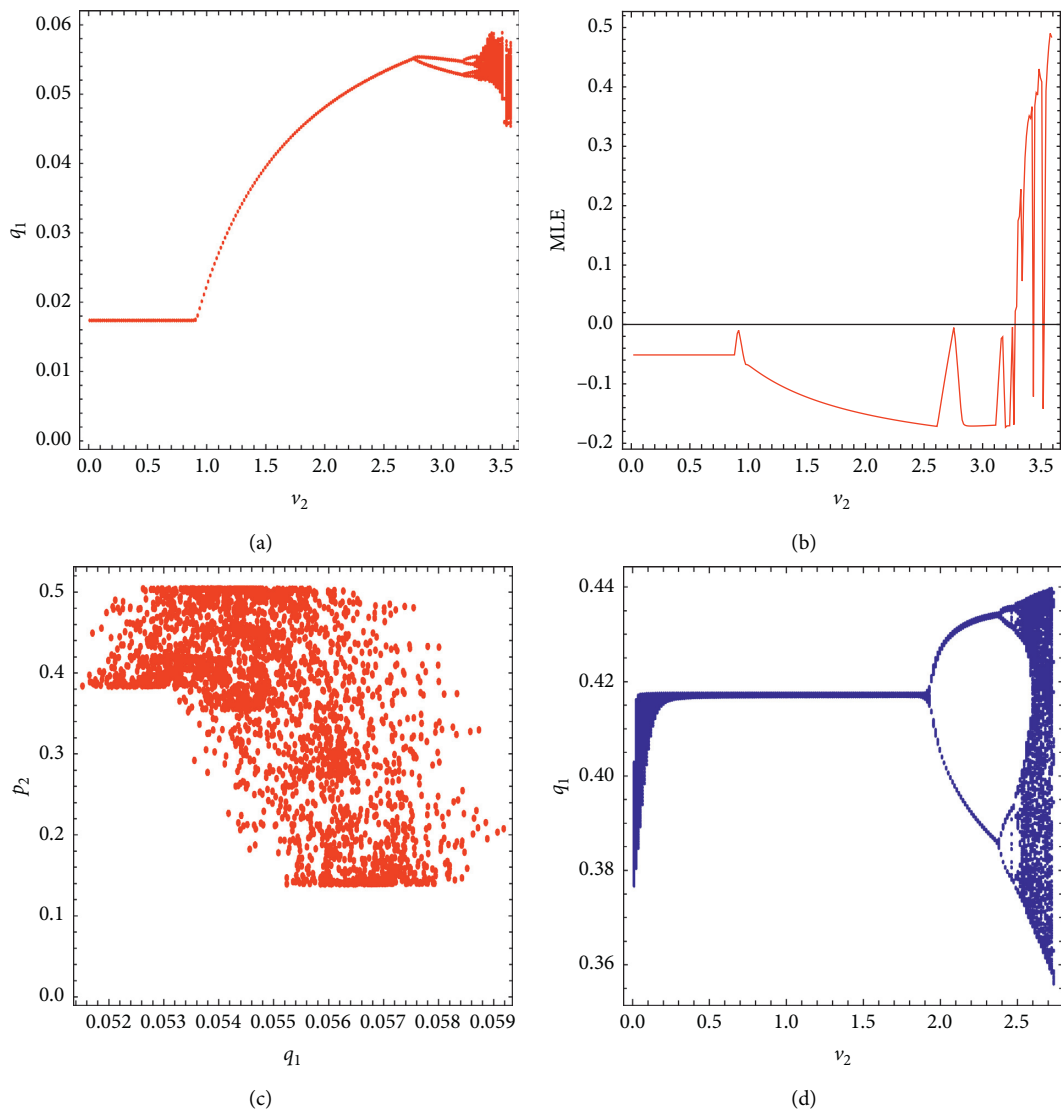


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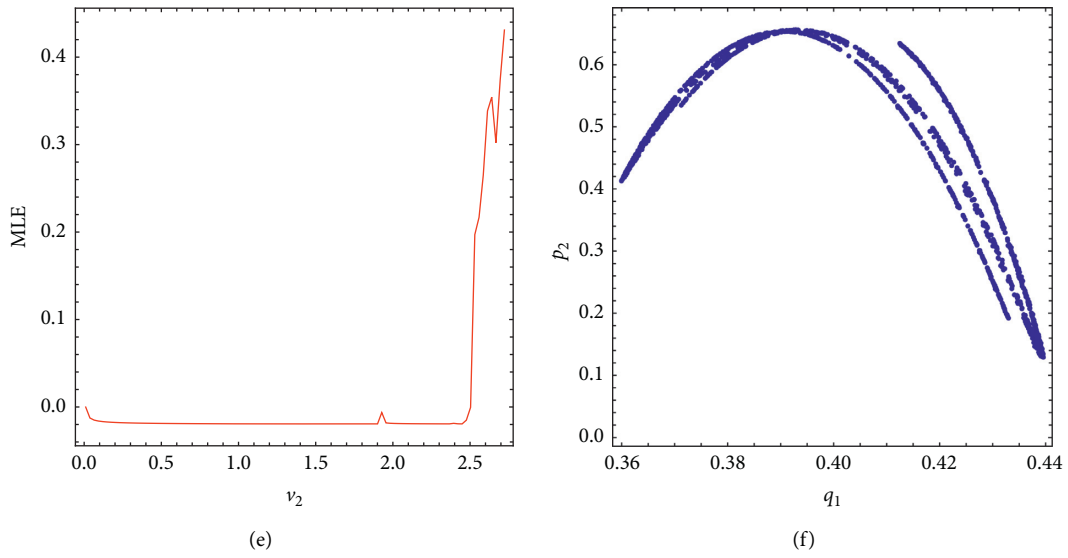


FIGURE 6: (a) Bifurcation diagram of system (16) vs.  $\nu_2$  at  $c = 0.1; d = 0.2; \nu_1 = 1.5; \alpha = 1$ . (b) MLE plot of system (16) vs.  $\nu_2$  at  $c = 0.1; d = 0.2; \nu_1 = 1.5; \alpha = 1$ . (c) Phase portrait of system (16) at  $c = 0.1; d = 0.2; \nu_1 = 1.5; \nu_2 = 3.4; \alpha = 1$ . (d)–(f) similar to (a)–(c) but for  $\alpha = 0.95$ .

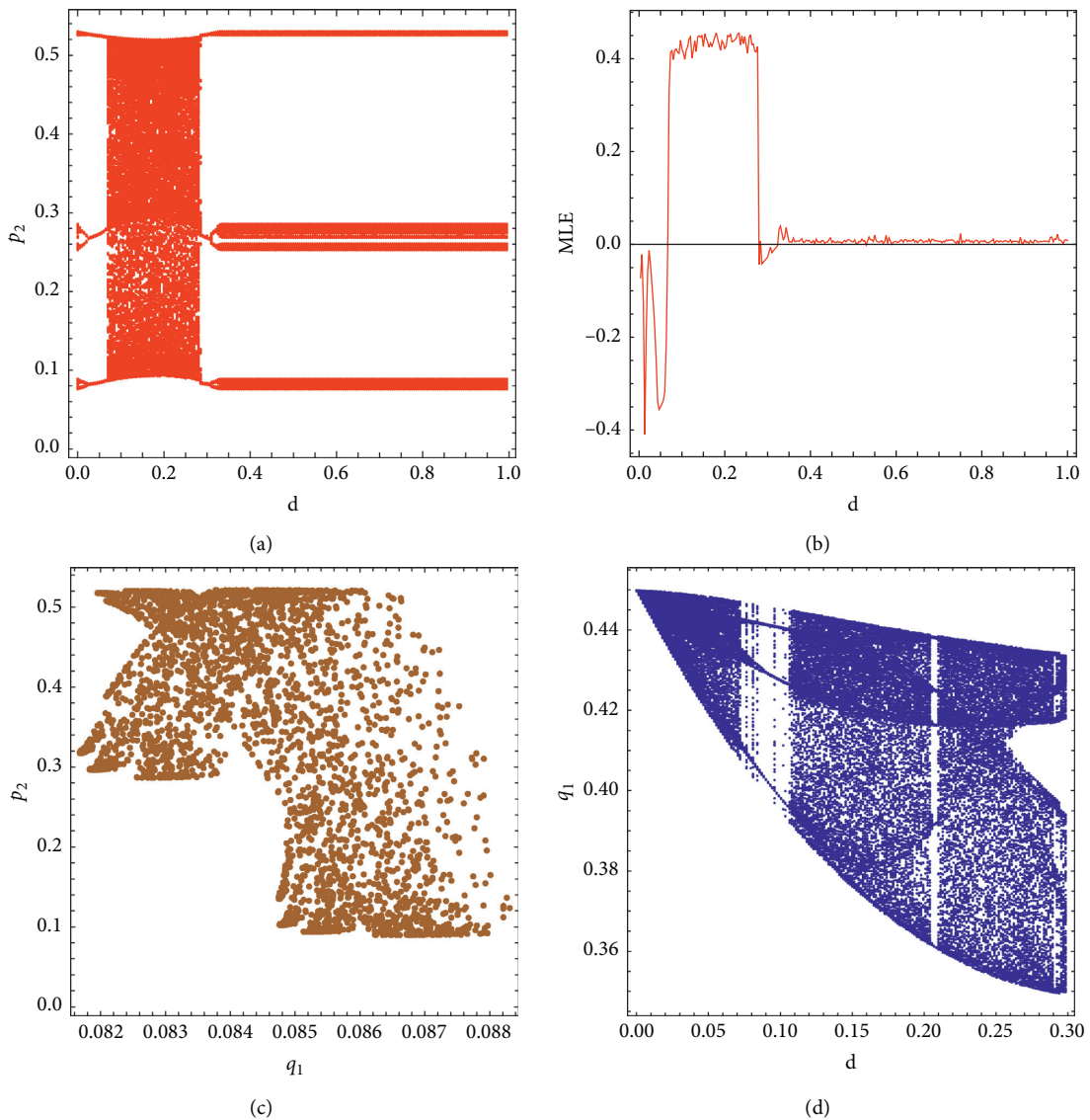


FIGURE 7: Continued.

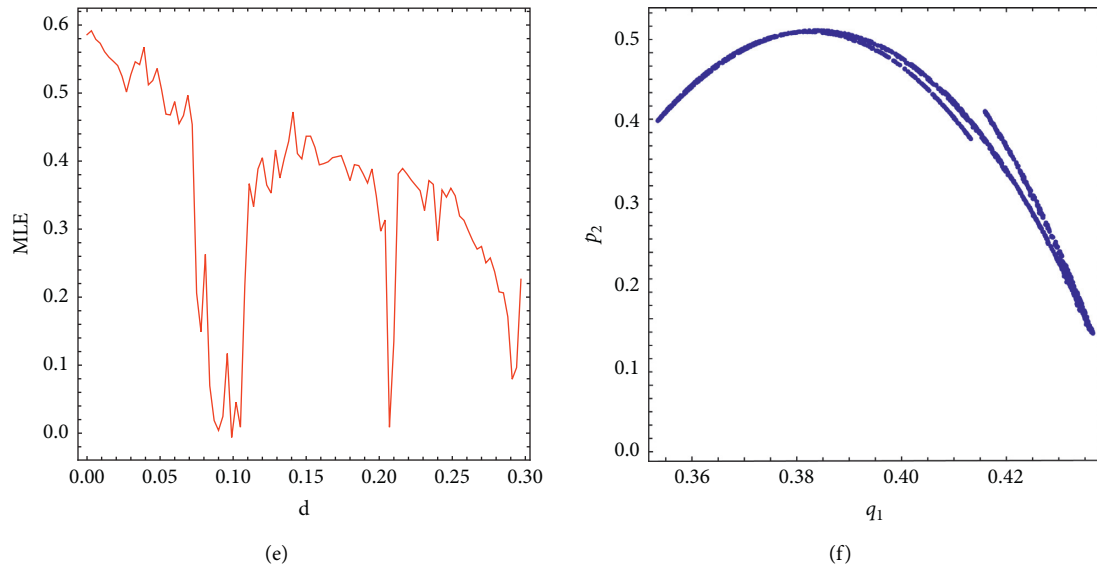


FIGURE 7: (a) Bifurcation diagram of system (16) vs.  $d$  at  $c = 0.1; \nu_1 = 1.5; \nu_2 = 3.5; \alpha = 1$ . (b) MLE plot of system (16) vs.  $d$  at  $c = 0.1; \nu_1 = 1.5; \nu_2 = 3.5; \alpha = 1$ . (c) Phase portrait of system (16) at  $c = 0.1; d = 0.1; \nu_1 = 1.5; \nu_2 = 3.5; \alpha = 1$ . (d)–(f) similar to (a)–(c) but for  $\alpha = 0.95$ .

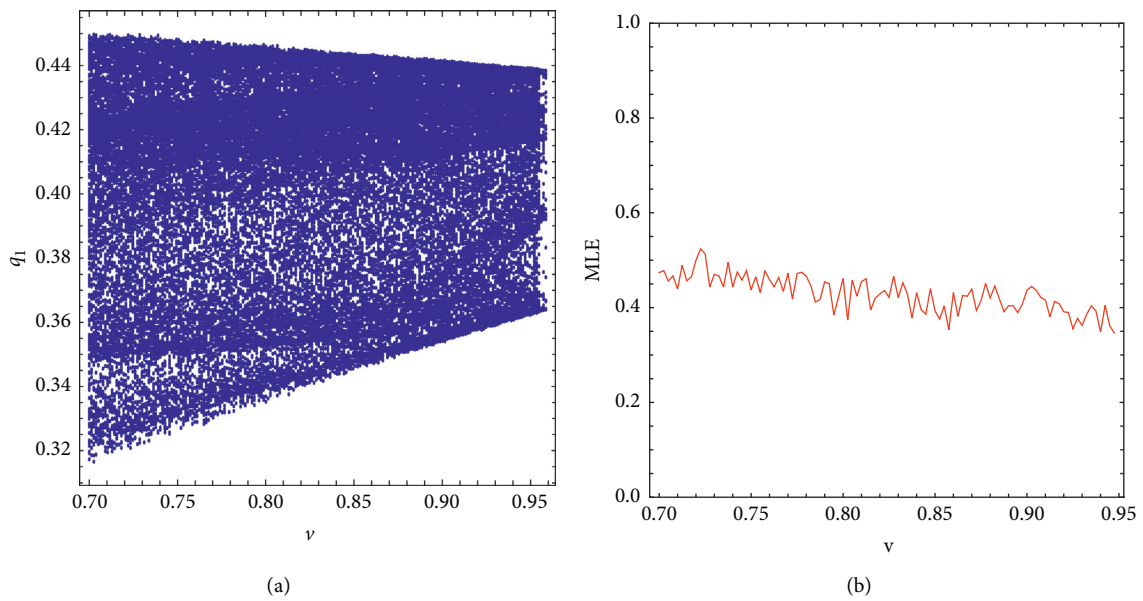


FIGURE 8: Continued.

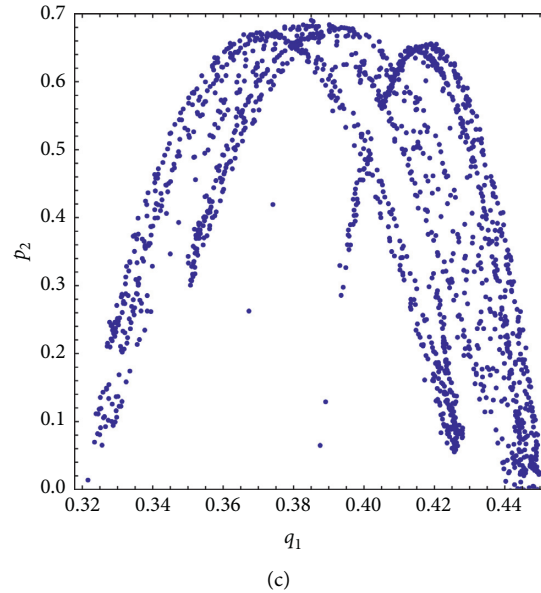


FIGURE 8: (a) Bifurcation diagram of system (16) vs.  $\alpha$  at  $c = 0.1; d = 0.2; \nu_1 = 1.5; \nu_2 = 2.67$ . (b) MLE plot of system (16) vs.  $\alpha$  at  $c = 0.1; d = 0.2; \nu_1 = 1.5; \nu_2 = 2.67$ . (c) Phase portrait of system (16) at  $c = 0.1; d = 0.2; \nu_1 = 1.5; \nu_2 = 2.67; \alpha = 0.7$ .

## 6. Conclusion

The new fractional-order Cournot–Bertrand game based on a long memory effect is proposed. The stability of the game’s equilibrium points, including the Nash equilibria, has been explored both qualitatively and numerically. Bifurcation, phase portraiture, time series, and maximal Lyapunov exponents’ diagrams have been used to analyze the complex dynamic characteristics of the proposed game. When we compared our new model to the Wang–Ma model [39], we found that the game parameters, especially the long-term memory influence, had an effect on the long-term dynamic response of our novel model. This is important for understanding the performance of the duopoly game with the long-term memory effect. According to our findings, the Cournot–Bertrand duopoly game with the long-term memory effect is more efficient than the duopoly game without long-term memory impact from economic viewpoint. As a consequence, we advise researchers to investigate the competitive games of long-term memory impact further.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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