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Research Article

An Implementation of the Generalized Differential Transform Scheme for Simulating Impulsive Fractional Differential Equations

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In this research study, the generalized differential transform scheme has been applied to simulate impulsive differential equations with the noninteger order. One specific tool of the implemented scheme is that it converts the problems into a recurrence equation that finally leads easily to the solution of the considered problem. The validity and reliability of this method have successfully been accomplished by applying it to simulate the solution of some equations. It is shown that the considered method is very suitable and efficient for solving classes of fractional-order initial value problems for impulsive differential equations and might find wide applications.

1. Introduction

Present, impulsive differential equations are treated as a basic system to explore the structures of various phenomena that are subjected to unexpected variations in their states. Many evolution processes which are simulated in applied sciences are defined by differential equations with the impulse effect. The theory and applications addressing such problems have been reported [1–6]. Recently, some interesting solutions' existence results for impulsive differential equations have been explored largely; we suggest the reader to [7–11] and the papers therein.

Over the last few years, the applications of fractional derivatives are sharply increasing and a huge quantity of mathematical systems has been explored by using these operators in different regions of science and engineering [12–18]. In the theory of fractional calculus, we talk about the noninteger orders of differential operators. The fractional calculus is just a generalization of classical calculus and uses similar methods and features, but is more useful in the application field. The memory effects and hereditary natures of different types of processes and materials can be studied by fractional-order operators much more accurately. These operators involve the complete history of that function in the given domain or span, which we say memory effects. That is why fractional-order operators are the best fit to describe dynamical systems or various real-life problems. Also, the nonlocal characteristic is one of the beauties of fractional operators. This justifies that the future state of a model depends not only upon the present stage but also upon all past states. All features make the importance of noninteger order systems and that is why an active area of research.

Nowadays, impulsive fractional differential equations, as generalizations of impulsive classical differential equations, are applied to model various important dynamical phenomena containing evolutionary structures specified by abrupt variations of the position at particular instants. Some recent developments in the stability, existence, and uniqueness of solutions for classes of impulsive fractional differential equations are investigated [19–27]. To date, a number of computational methods have been proposed to solve various types of noninteger order differential equations. Najafi and Allahviranloo [28] solved the linear and nonlinear fuzzy impulsive fractional differential equations by using the combination of reproducing kernel Hilbert space and fractional differential transform methods. A block-by-block numerical method is constructed for the impulsive fractional ordinary differential equations by carrying out a series of numerical examples in [29]. In [30], the Adomian decomposition method was applied to solve impulsive nonclassical type differential equations with the Caputo fractional operator. Very recently, a number of studies in the direction of efficiency and performance of various computational methods have been given by researchers [31–39]. In [40], authors have defined a novel fractional-order Lagrangian to study the motion of a beam on a nanowire. Some applications of noninteger order numerical methods in epidemiology and ecology can be seen [41, 42]. In the works presented in [20, 23, 26], we can observe a conflict between the obtained solutions for a particular impulsive fractional-order differential equation. In fact, they use different definitions for the Caputo fractional differential operators. However, to our knowledge, analytical or numerical techniques for solving impulsive fractional differential equations have not yet been sufficiently established. Therefore, the aim of this work is to implement the generalized differential transform scheme for analytically solving the initial value problem for the impulsive fractional differential equations:

$$\begin{cases} D_0^{\alpha}F(t) = f(t, F(t)), & \dfrac{t \in [0,T]}{\{t_1, t_2, \ldots, t_k\}}, \\ \Delta F(t_m) = F(t_m^+) - F(t_m^-) = I_m(F(t_m^-)), & m = 1, 2, \ldots, k, \\ F(0) = F_0, \end{cases} \tag{1}$$

where D_0^{α} is the Caputo fractional differential operator of order α with $0 < \alpha \leq 1$, described in Section 2, f: $[0,T] \times R \longrightarrow R$ is the appropriate continuous function, and I_m: $R \longrightarrow R$, t_m satisfying $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = T$ and $F_0 \in R$. The numbers t_m are called instants (or moments) of impulse, I_m shows the jump of the state t_m, and $F(t_m^+) = \lim_{h \longrightarrow 0+} F(t_m + h)$ and $F(t_m^-) = \lim_{h \longrightarrow 0-} F(t_m + h)$ specify the right and the left limits, respectively, of the state t_m.

2. Preliminaries and Notations

In this portion, first, we write few important definitions of fractional-order operators and generalized Taylor's formula. Next, we recall some relevant results which are applied in this research.

Definition 1. The Caputo fractional derivative operator of order α with $\alpha > 0$ is specified by

$$D_a^{\alpha} f(t) = J_a^{n-\alpha} D^n f(t), \tag{2}$$

where $n - 1 < \alpha \leq n$, $n \in N$, and D^n is the classical differential operator of order n and J^{μ} is the Riemann-Liouville fractional integral operator of order μ with $\mu > 0$.

Definition 2. The definition of Riemann-Liouville fractional integral operator of order μ with $\mu > 0$ is given by

$$J_a^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t - \tau)^{\mu - 1} f(\tau) d\tau, \quad t > a. \tag{3}$$

Brief discussion on the characteristics of the above given fractional derivative operators can be learned from [12–18].

It is worth mentioning that there are some differences between Caputo fractional differential operator D_a^{α}, given in Definition 1, and the usual integer differential operator D^n regarding the memory property. Caputo fractional operator of function f, $D_a^{\alpha} f(t)$, captures the complete history of the function f starting from $t = a$, while the classical derivative operator of the function f, $D^n f(t)$, only considers the nearby points. So, Caputo definition has long-term memory and long-span spatial interactions. In [43], the authors introduced the generalized Taylor's formula. This generalization is a derivation of a function as an infinite sum of terms that is simulated from the fractional derivative values of a function at a single point.

Theorem 1. *Assume that $(D_a^{\alpha})^m f(t) \in C(a,b]$ for $m = 0, 1, \ldots, k + 1$, where $0 < \alpha \leq 1$; then, we have [43]*

$$f(x) = \sum_{i=0}^{k} \frac{(t - a)^{\alpha i}}{\Gamma(\alpha i + 1)} \left((D_a^{\alpha})^i f \right)(a+) + R_k^{\alpha}(t, a), \tag{4}$$

where

$$R_k^{\alpha}(t, a) = \frac{\left((D_a^{\alpha})^{1+k} f \right)(\xi)}{\Gamma((1 + k)\alpha + 1)} (t - a)^{(1+k)\alpha}. \tag{5}$$

With $a \leq \xi \leq t$, for each $t \in (a, b]$, and D_a^{α} is the Caputo derivative operator of order α, with $(D_a^{\alpha})^k = D_a^{\alpha} \cdot D_a^{\alpha} \cdots D_a^{\alpha}$.

In some recent studies, number of results have been proposed to find the sufficient conditions regarding the solution existence for the Caputo-type IVPs for impulsive differential equations [20–22, 24, 25]. One of the most important results is given in the following theorem which establishes the connection between the IVP for the Caputo-type impulsive differential equation given in equation (1) and a class of integral equations.

Theorem 2. *Let $0 < \alpha \leq 1$ and f: $[0,T] \times R \longrightarrow R$ be Lebesgue measurable function with respect to t on $[0,T]$. A function $x(t)$ is a solution of IVP (1) if and only if $x(t)$ is a solution of the following integral equations [25]:*

$$x(t)=\begin{cases} x_0+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-v)f(v,x(v))\mathrm{d}v, & 0\le t\le t_1,\\ x_0+I_1\left(x\left(t_1^-\right)\right)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-v)^{1-\alpha}f(v,x(v))\mathrm{d}v, & t_1<t\le t_2,\\ \vdots \\ x_0+\displaystyle\sum_{i=1}^{m} I_i\left(x\left(t_i^-\right)\right)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-v)^{1-\alpha}f(v,x(v))\mathrm{d}v, & t_m<t\le T.\end{cases} \tag{6}$$

3. Generalized Differential Transform Scheme

The differential transform method (DTM), proposed by Zhou [44] in 1986, was given for simulating ordinary and partial differential equations. This scheme produces approximations based on an iterative method for calculating power series solutions in the form of initial value constraints of differential equation. The scheme, which is well addressed in [45, 46], can be taken as an alternative method for constructing the solution as formal Taylor series without linearization, discretization, perturbation, or large computational work. More recently, for solving the noninteger order differential equations, the DTM was generalized by using generalized Taylor's formula ([47]) to calculate the solutions of such equations in the terms of fractional power series. The new extension, which is known as the generalized differential transform method (GDTM), gives a useful feature for getting fractional power series expansions for the solutions of nonlinear systems having nonclassical derivatives. For understanding of the learners, we give a review on the GDTM. We define the generalized differential transformation of the m^{th} derivative of the function $f(x)$ as follows [48]:

$$F_\alpha(m)=\frac{1}{\Gamma(\alpha m+1)}\left[\left(D_{x_0}^{\alpha}\right)f(x)\right]|_{x=x_0'}, \tag{7}$$

where $0<\alpha\le 1$, $(D_{x_0}^{\alpha})^m=D_{x_0}^{\alpha}\cdot D_{x_0}^{\alpha}\cdots D_{x_0}^{\alpha}$, ($m$-times), and the generalized differential inverse transform of $F_\alpha(m)$ is defined as

$$f(x)=\sum_{m=0}^{\infty}F_\alpha(m)(x-x_0)^{\alpha m}. \tag{8}$$

When we put (7) into (8), applying the generalized Taylor's formula, we receive

$$\begin{aligned}\sum_{m=0}^{\infty}F_\alpha(m)(x-x_0)^{\alpha m}&=\sum_{m=0}^{\infty}\frac{(x-x_0)^{\alpha m}}{\Gamma(\alpha m+1)}\left[\left(D_{x_0}^{\alpha}\right)^m f(x)\right]|_{x=x_0}\\&=f(x).\end{aligned} \tag{9}$$

So, (8) is the inverse transformation of the generalized differential transform (7). The GDTM consisting of the generalized differential transformation (7) and its inverse transform (8) has increased the applications of the DTM to fractional differential equations. The basic simulations done by generalized differential transformation can be learned from [48], and the mostly applicable characteristics are specified by the following theorems.

Theorem 3. *If $G_\alpha(k)$, $V_\alpha(k)$, and $W_\alpha(k)$ are the generalized differential transformations of the functions $g(x)$, $v(x)$, and $w(x)$, respectively, then [48]*

(i) *If $g(x)=v(x)\mp w(x)$, then $G_\alpha(k)=V_\alpha(k)\mp W_\alpha(k)$;*

(ii) *If $g(x)=av(x)$, where $a\in R$, then $G_\alpha(k)=aV_\alpha(k)$;*

(iii) *If $g(x)=v(x)\cdot w(x)$, then $G_\alpha(k)=\sum_{l=0}^{k}V_\alpha(l)\cdot W_\alpha(k-l)$;*

(iv) *If $g(x)=D_{x_0}^{\alpha}v(x)$, then $G_\alpha(k)=(\Gamma(\alpha(k+1)+1)/\Gamma(\alpha k+1))V_\alpha(k+1)$;*

(v) *If $g(x)=(x-x_0)^{m\alpha}$, then $G_\alpha(k)=\delta(k-m)$, where $\delta(k)=\begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases}$.*

Theorem 4. *Suppose that $G_\alpha(n)$ and $V_\alpha(n)$ are the generalized differential transforms of the functions $g(x)$ and $v(x)$, simultaneously. Then, if $g(x)=D_{x_0}^{\beta}v(x)$, $m-1<\beta\le m$, where $D_{x_0}^{\alpha n}D_{x_0}^{\beta}v(x)=D_{x_0}^{\alpha n+\beta}v(x)$, for $n=0,1,2,\ldots$, then [48]*

$$G_\alpha(n)=\frac{\Gamma(\alpha n+1+\beta)}{\Gamma(\alpha n+1)}V_\alpha\left(\frac{n+\beta}{\alpha}\right). \tag{10}$$

4. GDTM for Impulsive Fractional Differential Equations

This section presents the applications of GDTM to solve IVP for the impulsive Caputo-type differential equations given in (1). The obtained piecewise continuous solutions of such IVPs demonstrate the performance and reliability of the method. Initially, one can verify that, using Definition 1 and Theorem 2, the solution of IVP (1) can be obtained as

$$y(t)=\begin{cases} y_1(t), & 0\le t\le t_1,\\ y_2(t), & t_1<t\le t_2,\\ \vdots \\ y_{m+1}(t), & t_m<t\le T,\end{cases} \tag{11}$$

where the solution component $y_k(t)$ satisfies IVP:

$$D_0^{\alpha} y_k(t) = f(t, y_k(t)), \quad t > 0, \tag{12}$$

For $k = 1, 2, \ldots, m+1$, respect to the initial constraints,

$$\begin{cases} y_1(0) = y_0, \\ y_2(0) = y_0 + I_1(y(t_i^-)), \\ y_3(0) = y_0 + I_1(y(t_i^-)) + I_2(y(t_2^-)), \\ \vdots \\ y_{m+1}(0) = y_0 + \sum_{i=1}^{m} I_i(y(t_i^-)). \end{cases} \tag{13}$$

The main steps of the GDTM for simulating the nonclassical differential equations are as follows: first, we employ the generalized differential transformation, specified in (7), to IVP (1); then, the output is a recurrence relation. Second, simulating this relation by applying the inverse generalized differential transformation, given in (8), we get the solution component $y_j(t)$ of IVP (1) as

$$y_j(t) = \sum_{k=0}^{\infty} Y_j(k) \cdot t^{\alpha k}, \tag{14}$$

where $Y_j(k)$ stratifies the recurrence relation:

$$\frac{\Gamma((1+k)\alpha + 1)}{\Gamma(k\alpha + 1)} Y_j(1+k) = F\big(k, Y_j(k)\big), \tag{15}$$

where $Y_1(0) = y_0$, $Y_{j+1}(0) = y_0 + \sum_{i=1}^{j} I_i(y(t_i^-))$, $j = 1, 2, \ldots, m$, and $F(k, Y_j(k))$ is the generalized differential transformation of the function $f(t, y_j(t))$. Now, by implementing the above analysis, piecewise continuous solutions of some illustrative IVPs for impulsive fractional-order differential equations are derived.

Furthermore, we will investigate the sufficient condition for the convergence of the series solution, given in (11). Based on these simulations, maximum absolute truncated error estimations for the solutions will also be addressed. Following the work presented in [49], we can establish the following results.

Theorem 5. *Let the solution of IVP (1) is obtained as given in (11), where the components $y_j(t)$ of the solution are evaluated as given in (14), and let $I_j = [t_{j-1}, t_j]$.*

(a) The series $\sum_{k=0}^{\infty} Y_j(k) \cdot t^{\alpha k}$, given in (14), converges if $\exists 0 < \gamma_j < 1$, such that $\|Y_j(k+1) \cdot t^{\alpha} / Y_j(k)\| < \gamma_j$, $\forall k \geq k_0$, for some $k_0 \in \mathbb{N}$ and $t \in I_j$, where $\|f(t)\| = \max_{t \in I_j} |f(t)|$, that is, the solution component $y_j(t)$ converges if $\lim_{k \to \infty} |Y_j(k+1) / Y_j(k)| \cdot \max_{t \in I_j} t^{\alpha} < 1$.

(b) Let the series $\sum_{k=0}^{\infty} Y_j(k) \cdot t^{\alpha k}$ converges to the solution component $y_j(t)$. If the truncated series $\sum_{k=0}^{n} Y_j(k) \cdot t^{\alpha k}$ is considered as an approximation to the solutions $y_j(t)$, then the maximum absolute truncated error is calculated as $\|y_j(t) - \sum_{k=0}^{n} Y_j(k) \cdot t^{\alpha k}\| < \gamma_j^{n-n_0+1} / 1 - \gamma_j \max_{t \in I_j} |Y_j(n_0) \cdot t^{\alpha n_0}|$, for any $n_0 \geq 0$, where $Y_j(n_0) \neq 0$.

Another way, we derive, for each $i \geq k_0$ and $t \in I_j$, the parameters $\gamma_j(i)$ as

$$\gamma_j(i+1) = \begin{cases} \left\| \dfrac{Y_j(i+1) \cdot t^{\alpha}}{Y_j(i)} \right\|, & \left\| Y_j(i) \right\| \neq 0, \\ 0, & \left\| Y_j(i) \right\| = 0. \end{cases} \tag{16}$$

$i \in \mathbb{N} \cup \{0\}$; then, the series $\sum_{k=0}^{\infty} Y_j(k) \cdot t^{\alpha k}$ converges to an exact solution, $y_j(t)$, when $0 \leq \gamma_j(i) < 1$, $\forall i \geq k_0$.

To show the reliability, applicability, and performance of this scheme as an efficient tool in obtaining series solutions, some initial value problems for impulsive Caputo-type differential equations will be examined in the following examples.

Example 1. First, we recall the following IVP for the impulsive differential equation in the sense of Caputo derivative [50, 51]:

$$\begin{cases} D_0^{(1/4)} g(t) = t, & \dfrac{t \in (0,2]}{\{1\}}, \\ g(1^+) = g(1^-) + 1, \\ g(0) = 0. \end{cases} \tag{17}$$

Using generalized differential transformation along with its properties, on both sides of (17), we get

$$G_j(k+1) = \frac{\Gamma(k/4 + 1)}{\Gamma(k/(4+5)/4)} [\delta(k-4)], \quad j = 1, 2, \tag{18}$$

where $G_1(0) = 0$ and $G_2(0) = 1$. Using the recurrence relation (18) and the transformed initial conditions, some initial components of the generalized differential transform solution for equation (17) can be written as follows:

$$\begin{aligned}
G_1(0) &= G_1(1) \\
&= G_1(2) \\
&= G_1(3) \\
&= G_1(4) \\
&= 0, \\
G_1(5) &= \frac{1}{\Gamma(9/4)}, \\
G_2(0) &= 1, \\
G_2(1) &= G_2(2) \\
&= G_2(3) \\
&= G_2(4) \\
&= 0, \\
G_2(5) &= \frac{1}{\Gamma(9/4)},
\end{aligned} \tag{19}$$

where $G_1(k) = G_2(k) = 0$, for $k > 5$. So, the solution to the IVP for the impulsive Caputo-type differential equation given in (17) can be obtained as

$$g(t) = \begin{cases} \dfrac{16}{5\Gamma(1/4)} t^{(5/4)}, & 0 \le t \le 1, \\ 1 + \dfrac{16}{5\Gamma(1/4)} t^{(5/4)}, & 1 < t \le 2, \end{cases} \tag{20}$$

which is the same solution of the initial value problem for the impulsive fractional differential (17) obtained in [50].

Example 2. We next adopt IVP for the impulsive Caputo-type differential equation:

$$\begin{cases} D_0^{\alpha} y(t) = 1 - y^2(t), & \dfrac{t \in (0,2]}{\{1\}}, \\ y(1^+) = y(1^-) + 2, \\ y(0) = 0, \end{cases} \tag{21}$$

where $0 < \alpha \le 1$. Using the generalized differential transformation to both sides of (21) and applying the features of the given transform, we obtain

$$Y_j(1+k) = \frac{\Gamma(k\alpha + 1)}{\Gamma(\alpha(1+k)+1)} \left\{ \delta(k) - \sum_{l=0}^{k} Y_j(l) Y_j(k-l) \right\}, \quad j = 1, 2, \tag{22}$$

where $Y_1(0) = 0$ and $Y_2(0) = 1$. Using the recurrence relation (22) and the transformed initial conditions, the approximate solution to IVP for the impulsive equation given in equation (21) can be derived as

$$y(t) = \begin{cases} y_1(t), & 0 \le t \le 1, \\ y_2(t), & 1 < t \le 2. \end{cases} \tag{23}$$

Here,

$$\begin{aligned}
y_1(t) &= \frac{1}{\Gamma(\alpha+1)} t^{\alpha} - \frac{1}{\Gamma(\alpha+1)^2} \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha} \\
&\quad + \frac{2}{\Gamma(\alpha+1)^3} \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)} t^{5\alpha} - \cdots, \\
y_2(t) &= 1.
\end{aligned} \tag{24}$$

Since $Y_2(0) = 1$ and $Y_2(k) = 0$, for $k \ge 2$. The exact solution of the Caputo-type differential equation $D_0^{\alpha} y(t) = 1 - y^2(t)$ with respect to the initial constraint $y(0) = 0$ when $\alpha = 1$ is $y(t) = ((e^{2t} - 1)/(e^{2t} + 1))$. In Table 1, the obtained numerical solutions $\sum_{k=0}^{n} Y_1(k) \cdot t^{\alpha k}$, when $n = 21$, for the solution component $y_1(t)$ of the initial value problems given in (21), over the interval $0 < t < 1$, are compared with those obtained in [50] using the second kind Chebyshev wavelet scheme, the numerical solutions obtained in [51] using the double perturbation collocation method, and the exact solutions when $\alpha = 1$. It is clear from Table 1 that our approximate solutions are very close in favor with the exact solutions and are much accurate as compared to the solutions given in [50, 51]. Definitely, the accuracy of our approach can be dramatically improved by simulating further terms of $y_1(t)$.

Example 3. Now, we consider the following initial value impulsive Caputo-type differential equation:

$$\begin{cases} D_0^{\alpha} y(t) = -y(t) + 0.5 y^2(t) + 1, & \dfrac{t \in (0,2]}{\{1\}}, \\ y(1^+) = y(1^-) + 1.25, \\ y(0) = 0, \end{cases} \tag{25}$$

where $0 < \alpha \le 1$. Using the generalized differential transformation on both the sides of (25) with the characteristics of the generalized differential transformation, we obtain

$$Y_j(k+1) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1)+1)} \left\{ -Y_j(k) + 0.5 \sum_{l=0}^{k} Y_j(l) Y_j(k-l) + \delta(k) \right\}, \quad j = 1, 2, \tag{26}$$

where $Y_1(0) = 0$ and $Y_2(0) = 1.25$. Applying the recurrence relation (27) and the transformed initial values, the approximate solution to the initial value impulsive fractional-order (25) can be derived by

$$y(t) = \begin{cases} y_1(t), & 0 \le t \le 1, \\ y_2(t), & 1 < t \le 2, \end{cases} \tag{27}$$

where

Table 1: Numerical outputs for equation (21) over $0 < t < 1$, when $\alpha = 1$.

t	Exact solution	Present method	Ref. [50]	Ref. [50]
0.1	0.099668	0.099668	0.099667	0.099694
0.2	0.197375	0.197375	0.197358	0.197437
0.3	0.291312	0.291313	0.291289	0.291345
0.4	0.379949	0.379949	0.379946	0.379928
0.5	0.462117	0.462117	0.462172	0.462074
0.6	0.537050	0.537050	0.537048	0.537033
0.7	0.604368	0.604368	0.604338	0.604397
0.8	0.664037	0.664037	0.664009	0.664082
0.9	0.716298	0.716300	0.716300	0.716314

$$
\begin{aligned}
y_1(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{1}{\Gamma(3\alpha+1)}\left(1 + \frac{\Gamma(2\alpha+1)}{2\Gamma(\alpha+1)^2}\right)t^{3\alpha} + \cdots, \\
y_2(t) &= 1.25 + \frac{17}{32}\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{17}{128}\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} + \cdots.
\end{aligned}
\tag{28}
$$

In Figures 1 and 2, we compute $\gamma_1(i)$'s and $\gamma_2(i)$'s (defined in (16)), for different values of α, that correspond to the solution components $y_1(t)$ and $y_2(t)$, respectively, of (25) where x-axis shows the index i and y-axis shows the parameter γ_i. From Figures 1 and 2, since $\gamma_1(i) < 1$ and $\gamma_2(i) < 1$ for $i \geq 1$, we observe that the components $y_1(t)$ and $y_2(t)$ of the series solution converge to the exact solution for (25), at $\alpha = 1$, 0.90 and 0.80. Also, we conclude that $\gamma_1(i)$'s and $\gamma_2(i)$'s are not less than 1, if $\alpha = 0.7$, and so, the series solution components may diverge when $\alpha = 0.7$. In the same way, we can perform the numerical simulations to show that the nonclassical type power series solution of (25) may diverge at the fractional-order.

For big value of n, if the truncated series $\sum_{k=0}^{n} Y_1(k)\cdot t^{\alpha k}$ and $\sum_{k=0}^{n} Y_2(k)\cdot t^{\alpha k}$ are used as approximations to the exact solution components $y_1(t)$ and $y_2(t)$ over $0 < t \leq 1$ and $1 < t \leq 2$, respectively, according to Figures 1 and 2, the max. absolute truncation error can be estimated as

$$
\left\| y_1(t) - \sum_{k=0}^{n} Y_1(k)\cdot t^{\alpha k} \right\| \leq
\begin{cases}
\dfrac{1}{1-0.6368}(0.6368)^n, & \alpha = 1, \\[2ex]
\dfrac{1}{\Gamma(1.9)}\dfrac{1}{1-0.7659}(0.7659)^n, & \alpha = 0.9, \\[2ex]
\dfrac{1}{\Gamma(1.8)}\dfrac{1}{1-0.9145}(0.9145)^n, & \alpha = 0.8,
\end{cases}
\tag{29}
$$

and

$$
\left\| y_2(t) - \sum_{k=0}^{n} Y_2(k)\cdot t^{\alpha k} \right\| \leq
\begin{cases}
\dfrac{1.25}{1-0.7545}(0.7545)^n, & \alpha = 1, \\[2ex]
\dfrac{1.25}{1-0.8389}(0.8389)^n, & \alpha = 0.9, \\[2ex]
\dfrac{1.25}{1-0.9257}(0.9257)^n, & \alpha = 0.8.
\end{cases}
\tag{30}
$$

As a result, we can observe that the domain of convergence becomes large as the order of the fractional derivative increases. The performed numerical simulations justify that the series solutions of given impulsive fractional-order differential equation may diverge when the fractional derivative order is $\alpha \ll 1$.

Example 4. Finally, we adopt the initial value problem for the impulsive Caputo-type differential equation.

$$
\begin{cases}
D_0^{\beta} y(t) = -t - y - t^2 y^2 + \dfrac{\left(te^{-t} + t^2 e^{-3t}\right)}{y}, & \dfrac{t \in (0,3]}{\{1,2\}}, \ 0 < \beta \leq 1, \\[2ex]
y(1^+) = 1 + y(1^-), \\[1ex]
y(2^+) = 1 + y(2^-), \\[1ex]
y(0) = 1.
\end{cases}
\tag{31}
$$

The exact solution of the system,

$$
\begin{cases}
D_0^{\beta} y(t) = -t - y - t^2 y^2 + \dfrac{\left(te^{-t} + t^2 e^{-3t}\right)}{y}, & t \in [0,1], \ 0 < \beta \leq 1, \\[2ex]
y(0) = 1,
\end{cases}
\tag{32}
$$

is given in [52, 53] by $y(t) = e^{-t}$ when $\beta = 1$. Using the characteristics of the generalized differential transformation of order α, (30) can be transformed to the following recurrence relation:

$$
\begin{aligned}
Y_j\left(\frac{k+\beta}{\alpha}\right) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k + \beta + 1)}\Bigg[-\delta\left(\frac{k-1}{\alpha}\right) - Y_j(k) \\
- \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} \delta\left(\frac{k_1 - 2}{\alpha}\right) Y_j(k_2 - k_1) Y_j(k - k_2) + F(k) \Bigg],
\end{aligned}
\tag{33}
$$

for $j = 1, 2, 3$, $F(k) = ([H(k) - \sum_{\lambda=0}^{k-1} F(\lambda)(k-\lambda)]/(0))$, $H(k) = P(k) + S(k)$, $P(k) = \sum_{p=0}^{k} E_1(p)\delta(k - p - (1/\alpha))$, and $S(k) = \sum_{p=0}^{k} E_2(p)\delta(k - p - (2/\alpha))$. Moreover, $E_1(k)$ and $E_2(k)$ represent the generalized transformations of e^{-t} and e^{-3t} that can be expressed, respectively, as follows:

$$
\begin{aligned}
E_1(k) &= \begin{cases}
\dfrac{(-1)^{\alpha k}}{(\alpha k)!}, & \alpha k \in \mathbb{Z}^+, \\[2ex]
0, & \alpha k \notin \mathbb{Z}^+,
\end{cases} \\
E_2(k) &= \begin{cases}
\dfrac{(-3)^{\alpha k}}{(\alpha k)!}, & \alpha k \in \mathbb{Z}^+, \\[2ex]
0, & \alpha k \notin \mathbb{Z}^+.
\end{cases}
\end{aligned}
\tag{34}
$$

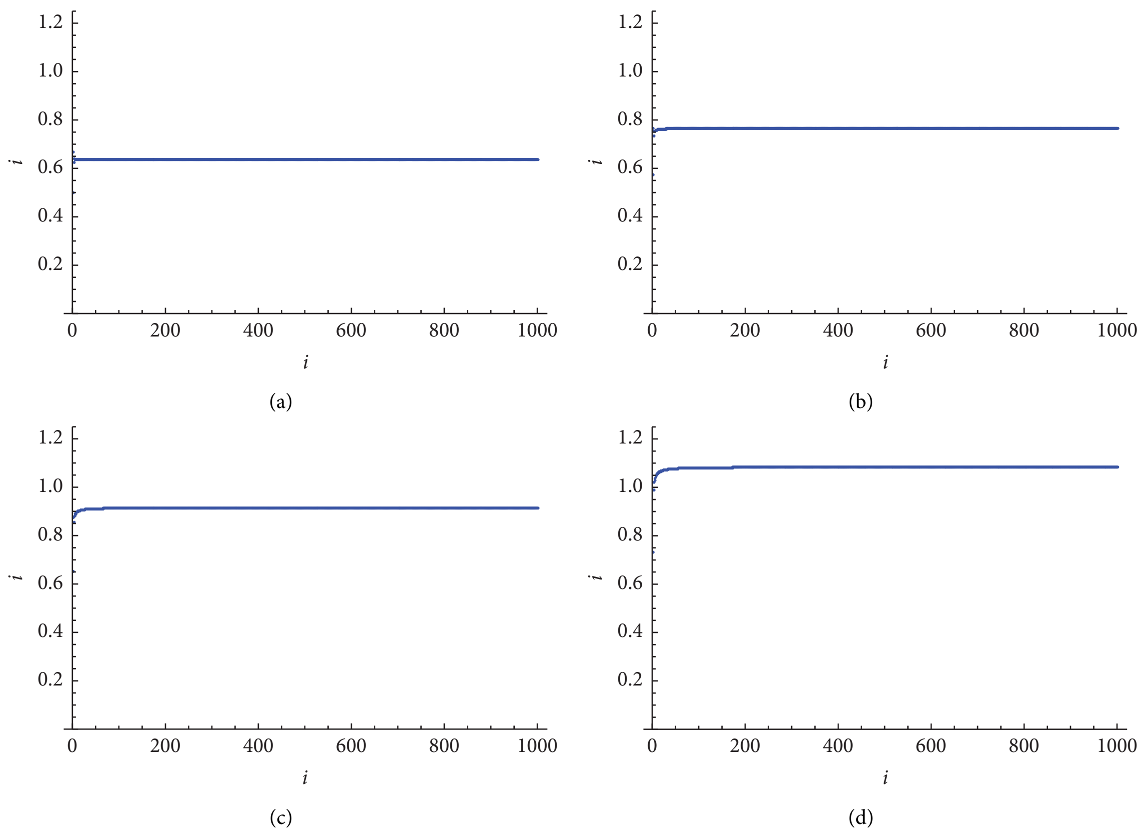

Figure 1: Numerical results of $\gamma_1(i)$'s for $y_1(t)$ of problem (26). (a) $\alpha = 1$. (b) $\alpha = 0.9$. (c) $\alpha = 0.8$. (d) $\alpha = 0.7$.

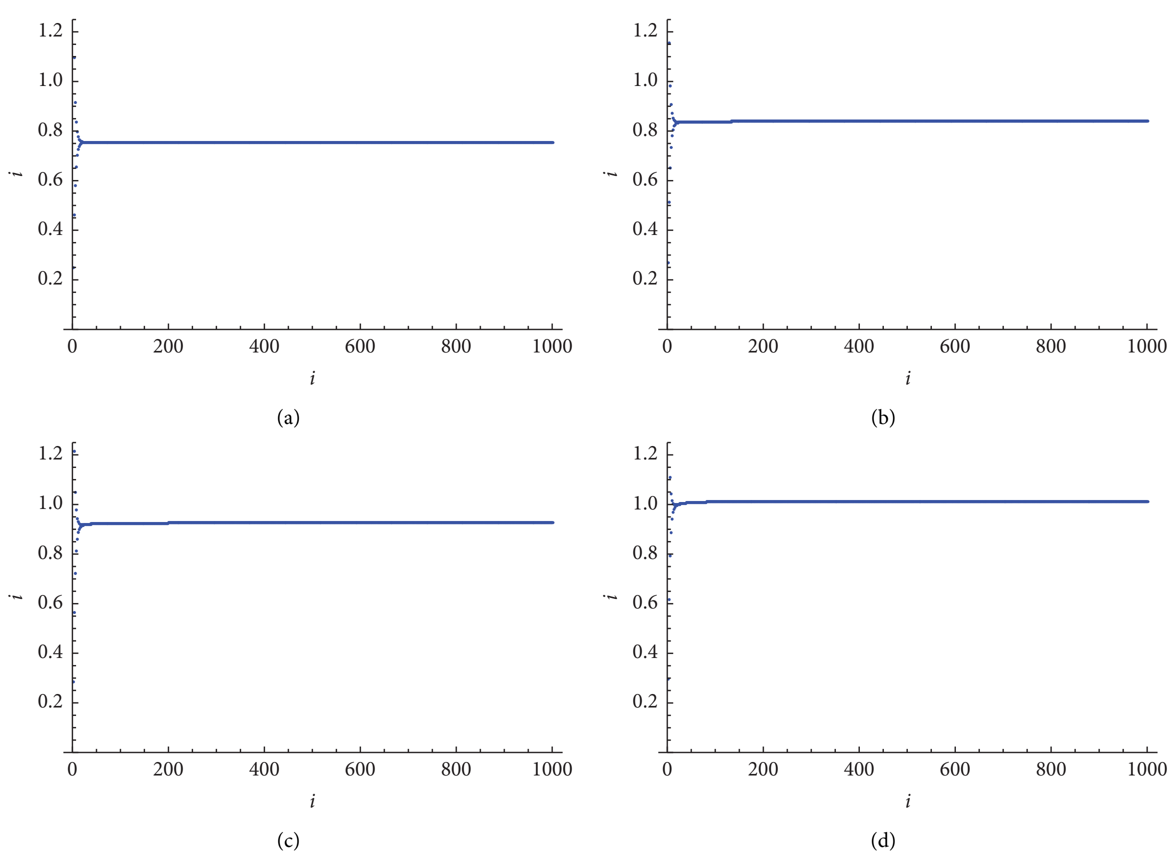

Figure 2: Numerical outputs of $\gamma_1(i)$'s for $y_2(t)$ of equation (26). (a) $\alpha = 1$. (b) $\alpha = 0.9$. (c) $\alpha = 0.8$. (d) $\alpha = 0.7$.

The initial condition and the jumps of the states of (30) are transformed as

$$Y_1(0)=1, Y_1(k)=0,\quad \text{for } k=1,2,\ldots,\left(\frac{\beta}{\alpha-1}\right),$$
$$Y_2(0)=2, Y_2(k)=0,\quad \text{for } k=1,2,\ldots,\left(\frac{\beta}{\alpha-1}\right), \tag{35}$$
$$Y_3(0)=3, Y_3(k)=0,\quad \text{for } k=1,2,\ldots,\left(\frac{\beta}{\alpha-1}\right).$$

For $\beta = 0.8$, solving the recurrence relation (33) using the transformed conditions given in equation (35) up to $k = 18$, the approximate solution to IVP for the impulsive Caputo-type differential given in equation (31) can be derived as

$$y(t)=\begin{cases} y_1(t), & 0\le t\le 1,\\ y_2(t), & 1<t\le 2,\\ y_3(t), & 2<t\le 3,\end{cases} \tag{36}$$

where

$$\begin{aligned} y_1(t) &= 1-1.0737t^{(4/5)}+0.69948t^{(8/5)}-0.33543t^{(12/5)}+0.48426t^{(13/5)}\\ &\quad -0.4261t^{(14/5)}+0.12892t^{(16/5)}-0.011358t^{(17/5)}+0.90275t^{(18/5)}+\cdots,\\ y_2(t) &= 2-2.14734t^{(4/5)}+1.39897t^{(8/5)}-0.298242t^{9/5}-0.670869t^{(12/5)}\\ &\quad +0.376645t^{(13/5)}-1.70424t^{(14/5)}+0.257842t^{(16/5)}-0.0550077t^{(17/5)}+\\ &\quad +3.63717t^{(18/5)}+\cdots,\\ y_3(t) &= 3-3.22101t^{(4/5)}+2.09845t^{(8/5)}-0.397656t^{9/5}-1.0063t^{(12/5)}+0.340774t^{(13/5)}\\ &\quad -3.83454t^{(14/5)}+0.386763t^{(16/5)}-0.0695575t^{(17/5)}+8.14028t^{(18/5)}+\cdots. \end{aligned} \tag{37}$$

For $\beta = 0.85$, the approximate solutions $y_1(t)$, $y_2(t)$, and $y_3(t)$ can be evaluated as

$$\begin{aligned} y_1(t) &= 1-1.0575t^{(17/20)}+0.64738t^{(17/10)}-0.28464t^{(51/20)}+0.44357t^{(27/10)}\\ &\quad -0.40114t^{(57/20)}+0.098657t^{(17/5)}+0.0091578t^{(71/20)}+0.81295t^{(37/10)}+\cdots,\\ y_2(t) &= 2-2.11503t^{(17/20)}+1.29476t^{(17/10)}-0.285815t^{(37/20)}-0.569273t^{(51/20)}+0.341673t^{(27/10)}\\ &\quad -1.60458t^{(57/20)}+0.197315t^{(17/5)}-0.0355107t^{(71/20)}+3.27489t^{(37/10)}+\cdots,\\ y_3(t) &= 3-3.17255t^{(17/20)}+1.94214t^{(17/10)}-0.381087t^{(37/20)}-0.85391t^{(51/20)}\\ &\quad +0.307706t^{(27/10)}-3.610t^{(57/20)}+0.295972t^{(17/5)}-0.0504003t^{(71/20)}+7.33025t^{(37/10)}+\cdots. \end{aligned} \tag{38}$$

While when $\beta = 1$, the approximate solutions $y_1(t)$, $y_2(t)$, and $y_3(t)$ can be evaluated as

$$\begin{aligned} y_1(t) &= 1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}+\frac{t^6}{720}-\frac{t^7}{5040}+\frac{tx^8}{40320}-\frac{t^9}{362880}+\frac{t^{10}}{3628800}-\cdots,\\ y_2(t) &= 2-2t+0.75t^2-1.41667t^3+2.11979t^4-1.54479t^5+1.59232t^6\\ &\quad -2.3723t^7+2.50693t^8-2.35509t^9+2.73001t^{10}-\cdots,\\ y_3(t) &= 3-3t+1.16667t^2\_3.27778t^3+5.16204t^4-4.02253t^5\\ &\quad +5.04799t^6+8.22138t^7+9.0091t^8-9.70313t^9+13.3388t^{10}-\cdots. \end{aligned} \tag{39}$$

Table 2: Absolute errors for $y_1(t)$ over $0 < t < 1$, when $\beta = 1$, for equation (31).

t	Exact solution	$\lvert y_{\text{exact}} - y_{\text{Chebyshev}} \rvert$	$\lvert y_{\text{exact}} - y_{\text{Padé}} \rvert$	$\lvert y_{\text{exact}} - y_{\text{GDTM}} \rvert$
0.0	1.0000000000	0	0	0
0.1	0.9048374180	2×10^{-10}	1×10^{-10}	0×10^{-10}
0.2	0.8187307531	8×10^{-9}	8×10^{-9}	0×10^{-10}
0.3	0.7408182207	8×10^{-8}	8×10^{-8}	0×10^{-10}
0.4	0.6703200460	4×10^{-7}	4×10^{-7}	0×10^{-10}
0.5	0.6065306597	1×10^{-6}	1×10^{-6},	0×10^{-10}
0.6	0.5488116361	4×10^{-6}	4×10^{-6}	1×10^{-9}
0.7	0.4965853038	9×10^{-6}	9×10^{-6}	5×10^{-10}
0.8	0.4493289641	2×10^{-5}	2×10^{-5}	2×10^{-9}
0.9	0.4065696597	4×10^{-5}	4×10^{-5}	7×10^{-9}
1.0	0.3678794412	6×10^{-5}	6×10^{-5}	2×10^{-8}

Table 2 provides the exact solution and the absolute errors for the approximate solutions of $y_1(t)$, over $0 < t < 1$, obtained using the GDTM, Chebyshev method [52], and Padé approximation method [53] when $\beta = 1$. From Table 2, we can conclude that the absolute errors are so small, and the approximate solutions simulated from GDTM are so closed to the exact solutions. Here, the comparison is made in the first subinterval because, to our knowledge, techniques for solving IVPs for impulsive Caputo-type differential equations have not yet been sufficiently introduced.

5. Conclusion

In this research, the application and performance of the generalized differential transform scheme to simulate initial value problems for Caputo-type differential equations are addressed. The reliability of the given technique has been demonstrated through some illustrative examples, and the obtained results show perfect agreements with other methods over the first subinterval. The sufficient condition for convergence of the scheme is presented. The main property of the method, as shown in this study, is that it deforms the impulsive differential equations of fractional order into a set of recurrence equations which gives several successive approximations, and hence, the procedure is direct and straightforward. This work illustrates the flexibility of the method as a tool to solve the classes of nonlinear problems containing fractional derivatives, effectively, easily, and accurately.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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