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Research Article

On System of Nonlinear Sequential Hybrid Fractional Differential Equations

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In this study, the existence and uniqueness of the solution for a system consisting of sequential fractional differential equations that contain Caputo–Hadamard (CH) derivative are verified. To study the existence and uniqueness of these solutions, some of the most important results from the fixed point theorems in Banach space were used. A practical example is also given to support the theoretical side that was obtained.

1. Introduction

Many problems in various fields can be successfully formulated by fractional differential equations, such as theoretical physics, Biology, viscosity, electrochemistry, and other physical processes (see [1–7].) In the last decade, the fractional differential equation has attracted the attention of mathematicians, physicists, and engineers as well [8, 9].

A fractional differential equation is an equation that contains fractional derivatives and differentials of some mathematical functions and appears in the form of variables. The goal of solving these equations is to find these mathematical functions whose derivatives achieve these equations. Before starting to search for solutions to these equations, studying the conditions of existence and uniqueness is a major matter. To study these conditions, most researchers use the most important fixed point theorems in Banach space, such as Banach contraction principle and Leray Schauder's theorem (see [10–24]).

In 2014, Zhang et al. [19] published a study investigating the existence results for

$$\left\{ {}^{H}D_{1}^{p}\left(\frac{x(t)}{g(t,x(t))}\right) \in G(t,x(t)), \quad t \in [1,e], x(1) = x\prime(e) = 0, \right. \tag{1}$$

where ${}^{H}D_{1}^{p}$, $p \in (1,2]$ is the Hadamard fractional derivative, $g \in C([1,e]\times\mathbb{R}, \mathbb{R}\backslash\{0\})$.

In 2016, Algoudi et al. [25] published a study investigating the existence results for the following boundary value problem (sequential Hadamard type):

$$\begin{cases} \left({}^{H}D_{1}^{p} + \lambda^{H}D_{1}^{p-1}\right)x(t) = f_{1}\left(t, x(t), y(t), {}^{H}D_{1}^{r}y(t)\right), \\ \left({}^{H}D_{1}^{q} + \lambda^{H}D_{1}^{q-1}\right)y(t) = f_{2}\left(t, x(t), {}^{H}D_{1}^{v}(t), y(t)\right), \quad y(\eta), y(1) = 0, \quad y(e) = {}^{H}I^{\theta_{2}}x(\xi), \\ \qquad x(1) = 0, \quad x(e) = {}^{H}I^{\theta_{1}} \end{cases} \tag{2}$$

where ${}^{H}D_{1}^{(\cdot)}$, $p, q \in (1,2]$, $r, \nu \in (0,1)$ is the Hadamard fractional derivative and ${}^{H}I^{\theta_1}$ is the Hadamard fractional integral with order $\theta_1, \theta_2 > 0$, $f_1, f_2 \in C([1,e] \times \mathbb{R}^3, \mathbb{R})$, $\eta, \xi \in [1,e]$.

Some researchers went deeper into their research and verified the stability of the solutions to these equations (see [26, 27]). Furthermore, many specialists in the field have paid attention to hybrid fractional differential equations; the importance of fractional hybrid differential equations is that they have a different dynamic than ordinary differential equations and that the hybrid type describes the nonlinear relationship in the derivative of the hybrid function (see [28–31]).

Some focus on having solutions to a system of equations (see [32–36]).

Based on what has been studied in the articles mentioned above, existence and uniqueness of the following nonlinear coupled differential equations are investigated. Unlike the previous studies, the main results of this article are different; that is, we generalize the problem mentioned in [36] by converting one single fractional differential equation into a system, using different fractional derivatives; here, we consider the problems in the context of sequential type.

$$\begin{cases} \left({}^{H}D_{a^+}^{q} + \lambda {}^{H}D_{a^+}^{q-1}\right)\left(\dfrac{x(t)}{f(t,x(t),y(t))}\right) = \psi(t,x(t),y(t)), \\ \left({}^{H}D_{a^+}^{q} + \mu {}^{H}D_{a^+}^{q-1}\right)\left(\dfrac{y(t)}{g(t,x(t),y(t))}\right) = \varphi(t,x(t),y(t)), \\ x(a^+) = x\prime(a^+) = 0, \\ y(a^+) = y\prime(a^+) = 0, \end{cases} \tag{3}$$

where ${}^{H}D^{\gamma}$, $\gamma = \{p,q\}$ is the Hadamard fractional derivative of order $1 < p, q < 2$, $a \le t \le T$, $f, g \in C([a,T] \times \mathbb{R}^2, \mathbb{R}\backslash\{0\})$, and $\lambda, \mu \in \mathbb{R}$.

In this work, we will follow the steps of researchers and specialists in the field. By organizing our results on existence of a solution to the problem as follows, Section 2 contains some fundamental results of fractional calculus and important result for establishing our main results. In Section 3, we introduce our main results. In Section 4, a practical example shows the applicability of our results is given. In Section 5, conclusion and future work are presented.

2. Preliminaries

In this section, we introduce some useful definitions, lemmas, and notations of fractional calculus.

Definition 1 (see [36]). The Hadamard fractional integral of order θ for a continuous function ψ: $[a,\infty) \longrightarrow \mathbb{R}$ is defined as

$$ {}^{H}I^{\theta}\psi(t) = \frac{1}{\Gamma(\theta)} \int_a^t \left(\ln \frac{t}{x}\right)^{\theta-1} \frac{\psi(x)}{x} \mathrm{d}x, \quad \theta > 0. \tag{4}$$

Definition 2 (see [36]). The Hadamard fractional derivative of order $\theta > 0$ for a continuous function ψ: $[a,\infty) \longrightarrow \mathbb{R}$ is defined as

$$\begin{aligned} {}^{H}D^{\theta}\psi(t) &= \delta^n\left({}^{H}I^{\theta}\psi\right)(t) \\ &= \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \frac{1}{\Gamma(n-\theta)} \int_a^t \left(\ln\frac{t}{r}\right)^{n-\theta-1} \frac{\psi(r)}{r}\mathrm{d}r, \quad n-1 < \theta < n,\ n = [\theta]+1, \end{aligned} \tag{5}$$

where $\delta = t(\mathrm{d}/\mathrm{d}t)$, $[\theta]$ denotes the integer part of the real number θ.

Let $C([a,T],\mathbb{R})$ denote the Banach space of all real valued continuous functions defined on $[a,T]$ and $C_{\delta}^{n}([a,T],\mathbb{R})$ denote the Banach space of all real valued functions φ such that $\delta^n\varphi \in C([a,T],\mathbb{R})$ (see [38]).

Lemma 1 (see [37]). *Let $z \in C_{\delta}^{n}([a,T],\mathbb{R})$, where $C_{\delta}^{n}[a,T] = \{z\colon [a,T] \longrightarrow \mathbb{R}\colon\ \delta^n z \in C[a,T]\}$.*

Then,

$$ {}^{H}I^{\theta}\left({}^{H}D^{\theta}z\right)(t) = z(t) - \sum_{j=1}^{n} l_j\left(\ln\frac{t}{a}\right)^{\theta-j}. \tag{6}$$

Lemma 2. *Given* $x \in C_\delta^2([a,T],\mathbb{R})$, *and* $h_1 \in C([a,T],\mathbb{R})$, *and*

$$\begin{cases} \left({}^H D_{a^+}^p + \lambda {}^H D_{a^+}^{p-1}\right)\left(\dfrac{x(t)}{f(t,x(t),y(t))}\right) = h_1(t), \quad 1 < p < 2, 0 < a \le t \le T, \\ x(a^+) = x'(a^+) = 0, \quad \lambda \in \mathbb{R}. \end{cases} \tag{7}$$

Then, the solution of problem (7) is given by

$$x(t) = f(t,x(t),y(t)) \times \left[t^{-\mu}\int_a^t s^{\mu-1H} I_{a^+}^{p-1}\varphi(s,x(s),y(s))\mathrm{d}s\right]. \tag{8}$$

Proof. Applying ${}^H I_{a^+}^p$ to (2), we get

$$\begin{aligned} {}^H I_{a^+}^p h_1(t) &= \left(\frac{x(t)}{f(t,x(t),y(t))}\right) + b_1\left(\ln\frac{t}{a}\right)^{p-1} \\ &+ b_2\left(\ln\frac{t}{a}\right)^{p-2} + \lambda {}^H I_{a^+}^1\left(\left(\frac{x(t)}{f(t,x(t),y(t))}\right) + c_1\left(\ln\frac{t}{a}\right)^{p-2}\right), \end{aligned} \tag{9}$$

where $b_1, b_2, c_1 \in \mathbb{R}$. The condition $x(a^+) = 0$ implies that $b_2 = 0$. The first derivative of (4) is calculated as follows:

$$\begin{aligned} \frac{1}{t}{}^H I_{a^+}^p h_1(t) &= \left(\frac{x(t)}{f(t,x(t),y(t))}\right)' + b_1(p-1)\left(\ln\frac{t}{a}\right)^{p-2} \\ &+ \frac{\lambda}{t}\left(\left(\frac{x(t)}{f(t,x(t),y(t))}\right) + c_1\left(\ln\frac{t}{a}\right)^{p-2}\right). \end{aligned} \tag{10}$$

Note that $x\prime(a^+) = 0$ implies $(x(t)/f(t,x(t),y(t)))'|_{t=a^+} = 0$, and $b_1 = 0 = c_1$.

The integrating factor $\eta(t) = e^{\int(\lambda/t)\mathrm{d}t}$; then, multiplying $\eta(t)$ by (10), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^\lambda\left(\frac{x(t)}{f(t,x(t),y(t))}\right)\right) = t^{\lambda-1H} I_{a^+}^{p-1} h_1(t), \tag{11}$$

then integrating (11) and again using $x(a^+) = 0$, we conclude that

$$\left(\frac{x(t)}{f(t,x(t),y(t))}\right) = \left[t^{-\lambda}\int_a^t s^{\lambda-1H} I_{a^+}^{p-1} h_1(s)\mathrm{d}s\right], \tag{12}$$

consequently,

$$x(t) = f(t,x(t),y(t)) \times \left[t^{-\lambda}\int_a^t s^{\lambda-1H} I_{a^+}^{p-1} h_1(s)\mathrm{d}s\right]. \tag{13}$$

In a like manner of Lemma 2, one can easily find the solution $y(t)$ as

$$y(t) = g(t,x(t),y(t)) \times \left[t^{-\lambda}\int_a^t s^{\lambda-1H} I_{a^+}^{p-1} h_1(s)\mathrm{d}s\right]. \tag{14}$$

□

Remark 1. For $\lambda = 0$, the solution is still valid, as ${}^H I^\theta h_1(t) = {}^H I^{1H} I^{\theta-1} h_1(t) = \int_a^t (I^{\theta-1} h_1(x)/x)\mathrm{d}x$ similar logic is applied for the case when $\mu = 0$. Here, these cases will not be taken for consideration in this study.

3. Main Results

In this section, we will present the main results to be obtained from this study.

The space $H = \{(x(t), y(t))\colon (x,y) \in C_\delta^2 \times C_\delta^2\}$ is a Banach space with the norm defined as $\|(x,y)\|_H = \|x\| + \|y\|\ \forall (x,y) \in H$. Based on Lemma 1, we define an operator $\aleph\colon H \longrightarrow H$ as

$$\aleph(x,y)(t) = \begin{pmatrix} \aleph_1(x,y)(t) \\ \aleph_2(x,y)(t) \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned}\aleph_1(x,y)(t) &= f(t,x(t),y(t))\times\left[t^{-\mu}\int_a^t s^{\mu-1H}I_{a^+}^{p-1}\varphi(s,x(s),y(s))\mathrm{d}s\right],\\ \aleph_2(x,y)(t) &= g(t,x(t),y(t))\times\left[t^{-\mu}\int_a^t s^{\mu-1H}I_{a^+}^{q-1}\varphi(s,x(s),y(s))\mathrm{d}s\right].\end{aligned}\tag{16}$$

To obtain our results for problem (3), we assume that the following conditions hold:

$(C1)$ Assume that both f, g are continuous and $\exists\lambda_f,\lambda_g>0$ such that

$$|f(t,x,y)|\le\lambda_f,\quad |g(t,x,y)|\le\lambda_g\forall(t,x,y)\in[a,T]\times\mathbb{R}^2.\tag{17}$$

$(C2)$ Suppose that ψ,φ are continuous and $\exists v_i,\ \tau_i>0,\ (i=1,2)$ such that

$$\begin{aligned}&\left|\psi(t,x_1,y_1)-\psi(t,x_2,y_2)\right|\le v_1\left|x_1-x_2\right|+v_2\left|y_1-y_2\right|,\\ &\left|\varphi(t,x_1,y_1)-\varphi(t,x_2,y_2)\right|\le\tau_1\left|x_1-x_2\right|+\tau_2\left|y_1-y_2\right|,\forall t\in[a,T],\quad x_i,\ y_i\in\mathbb{R},\ (i=1,2).\end{aligned}\tag{18}$$

$(C3)$ $\exists\omega_0,\ \theta_0>0$, and $\omega_i,\theta_i\ge 0\,(i=1,2)$ such that

$$\begin{aligned}&|\psi(t,x,y)|\le\omega_0+\omega_1|x|+\omega_2|y|,\\ &|\varphi(t,x,y)|\le\theta_0+\theta_1|x|+\theta_2|y|,\quad\forall t\in[a,T],\ x,\ y\in\mathbb{R},\ (i=1,2).\end{aligned}\tag{19}$$

$(C4)$ Define a bounded subset $S\subset H$, that is, $\exists\sigma_i>0,\ (i=1,2)$ such that $|\psi(t,x(t),y(t))|\le\sigma_1$, and $|\varphi(t,x(t),y(t))|\le\sigma_2$, $\forall(x,y)\in S$.

Using $(C4)$, note that

$$\begin{aligned}\left|t^{-\lambda}\int_a^t s^{\lambda-1H}I_{a^+}^{p-1}|\psi(s,x(s),y(s))|\mathrm{d}s\right| &= t^{-\lambda}\int_a^t s^{\lambda-1H}I_{a^+}^{p-1}|\psi(s,x(s),y(s))|\mathrm{d}s,\\ &= t^{-\lambda}\int_a^t s^{\lambda-1H}I_{a^+}^{p-1}|\psi(s,x(s),y(s))|\mathrm{d}s\le\sigma_1\frac{(\ln t/a)^{p-1}\left|1-(a/t)^{\lambda}\right|}{|\lambda|\Gamma(p)}.\end{aligned}\tag{20}$$

To ease our computations, we set

$$\begin{aligned}\Lambda_1 &= \sup_{a\le t\le T}\left\{^{-\lambda}\int_a^t s^{\mu-1}\int_a^s\left(\ln\frac{s}{r}\right)^{q-2}\frac{\mathrm{d}r}{r}\mathrm{d}s\right\}\le\frac{(\ln T/a)^{p-1}\left|1-(a/T)^{\lambda}\right|}{|\lambda|\Gamma(p)},\\ \Lambda_2 &= \sup_{a\le t\le T}\left\{t^{-\mu}\int_a^t s^{\mu-1}\int_a^s\left(\ln\frac{s}{r}\right)^{q-2}\frac{\mathrm{d}r}{r}\mathrm{d}s\right\}\le\frac{(\ln T/a)^{q-1}\left|1-(a/T)^{\mu}\right|}{|\mu|\Gamma(q)}.\end{aligned}\tag{21}$$

Theorem 1. *Assume that* $(C1)$ *and* $(C2)$ *hold if* $[\lambda_f\Lambda_1(v_1+v_2)+\lambda_g\Lambda_2(\tau_1+\tau_2)]<1$. *Then, problem (1) has a unique solution.*

Proof. Consider $\aleph$ defined by (9) and let $\overline{\mathfrak{B}_\gamma}=\{(x,y)\in H\colon\|(x,y)\|\le\gamma\}$ be a closed ball in H with $\gamma\ge\lambda_f\Lambda_1N_\psi+\lambda_g\Lambda_2N_\varphi/1-(\lambda_f\Lambda_1(v_1+v_2)+\lambda_g\Lambda_2(\tau_1+\tau_2))$, where $N_\psi=\sup_{a\le t\le T}|\psi(t,0,0)|$, $N_\varphi=\sup_{a\le t\le T}|\varphi(t,0,0)|$.

Observe that $|\psi(t,x,y)|=|\psi(t,x,y)-\psi(t,0,0)+\psi(t,0,0)|\le v_1\|x\|+v_2\|y\|+N_\psi\le(v_1+v_2)\gamma+N_\psi$.

First, we show that $\aleph\overline{\mathfrak{B}_\gamma}\subset\overline{\mathfrak{B}_\gamma}$. For any $(x,y)\in\overline{\mathfrak{B}_\gamma}$, $t\in[a,T]$, we have

$$\begin{aligned}\left|\aleph_1(x,y)(t)\right| &= |f(t,x(t),y(t))| \times \left|t^{-\lambda}\int_a^t s^{\lambda-1H}I_{a^+}^{p-1}\psi(s,x(s),y(s))\mathrm{d}s\right|,\\ &= \lambda_f \times t^{-\lambda}\int_a^t s^{\lambda-1H}I_{a^+}^{p-1}|\psi(s,x(s),y(s))|\mathrm{d}s,\\ &\le \lambda_f\left((v_1+v_2)\gamma + N_\psi\right)\times \sup_{a\le t\le T}\left\{^{-\lambda}\int_a^t s^{\lambda-1}\int_a^s\left(\ln\frac{s}{r}\right)^{p-2}\frac{dr}{r}\mathrm{d}s\right\},\\ &\le \lambda_f\Lambda_1\left[(v_1+v_2)\gamma + N_\psi\right].\end{aligned} \tag{22}$$

In a same manner, we find that

$$\left\|\aleph_2(x,y)\right\| \le \lambda_g\Lambda_2\left[(\tau_1+\tau_2)\gamma + N_\varphi\right]. \tag{23}$$

From (14) and (15), we deduce that $\|\aleph(x,y)\|\le\gamma$ Next for $(x_1,y_1),(x_2,y_2)\in H, \forall t\in[a,T]$, we have

$$\begin{aligned}&\left|\aleph_1(x_1,y_1)(t)-\aleph_1(x_2,y_2)(t)\right| \le \lambda_f\\ &\quad\times\sup_{a\le t\le T}\left\{t^{-\lambda}\int_a^t s^{\lambda-1H}I_{a^+}^{p-1}\left|\psi(s,x_1(s),y_1(s))-\psi(s,x_2(s),y_2(s))\right|\mathrm{d}s\right\}\\ &\quad\le \lambda_f\left(v_1\|x_1-x_2\| + v_2\|y_1-y_2\|\right)\sup_{a\le t\le T}\left\{^{-\lambda}\int_a^t s^{\lambda-1}\int_a^s\left(\ln\frac{s}{r}\right)^{p-2}\frac{\mathrm{d}r}{r}\mathrm{d}s\right\},\\ &\quad\le \lambda_f\Lambda_1\left(v_1\|x_1-x_2\| + v_2\|y_1-y_2\|\right),\\ &\quad\le \lambda_f\Lambda_1(v_1+v_2)\left(\|x_1-x_2\| + \|y_1-y_2\|\right).\end{aligned} \tag{24}$$

Similarly, we can find

$$\left\|\aleph_2(x_1,y_1)-\aleph_2(x_2,y_2)\right\| \le \lambda_g\Lambda_2(\tau_1+\tau_2)\left(\|x_1-x_2\| + \|y_1-y_2\|\right). \tag{25}$$

Combining (24) and (25) yields

$$\left\|\aleph_2(x_1,y_1)-\aleph_2(x_2,y_2)\right\| \le \left[\lambda_f\Lambda_1(v_1+v_2) + \lambda_g\Lambda_2(\tau_1+\tau_2)\right]\times\left(\|x_1-x_2\| + \|y_1-y_2\|\right). \tag{26}$$

□

Theorem 2. . *Assume (C1) and (C3) and (C4) hold if* $(\lambda_f\Lambda_1\omega_1 + \lambda_g\Lambda_2\theta_1) < 1$ *and* $(\lambda_f\Lambda_1\omega_2 + \lambda_g\Lambda_2\theta_2) < 1$. *Then, problem (1) has at least one solution.*

Proof. We first prove that the operator $\aleph\colon H\longrightarrow H$ is completely continuous; obviously, the operator is continuous as a result that f, g, ψ, and φ are all assumed to be continuous.

By $(C4)$, $\forall (x, y) \in S$, we have

$$\left|\aleph_1 (x, y)(t)\right| \leq \lambda_f \times \sup_{a\leq t\leq T}\left\{t^{-\lambda}\int_a^t s^{\lambda-1H}I_{a^+}^{p-1}|\psi(s, x(s), y(s))|\mathrm{d}s\right\} \leq \lambda_f \Lambda_1 \sigma_1, \tag{27}$$

$$\left\|\aleph_2 (x, y)\right\| < \lambda_g \Lambda_2 \sigma_2. \tag{28}$$

Combining inequalities (27) and (28) yields $\|\aleph_2 (x, y)\| \leq \lambda_f \Lambda_1 \sigma_1 + \lambda_g \Lambda_2 \sigma_2$; that is, the operator $\aleph$ is uniformly bounded.

Next, we prove equicontinuity for the operator $\aleph$; for this, we let $t_1, t_2 \in [a, T]$, $(t_1 < t_2)$.

Then,

$$\begin{aligned}&\left|\aleph_1 (x, y)(t_2) - \aleph_1 (x, y)(t_1)\right| \leq \lambda_f \\ &\times\left(\begin{array}{l}\sup\limits_{a\leq t\leq T}\left\{t_2^{-\lambda}\int_a^{t_2} s^{\lambda-1H}I_{a^+}^{p-1}|\psi(s, x(s), y(s))|\mathrm{d}s\right\}\\ -\sup\limits_{a\leq t\leq T}\left\{t_1^{-\lambda}\int_a^{t_2} s^{\lambda-1H}I_{a^+}^{p-1}|\psi(s, x(s), y(s))|\mathrm{d}s\right\}\end{array}\right) \\ &\leq \frac{\lambda_f \sigma_1}{|\lambda|\Gamma(p)}\left(\left(t_1^{\lambda} - a^{\lambda}\right)\left(\ln\frac{t_1}{a}\right)^{p-1} + t_2^{\lambda}\left(\ln\frac{t_2}{a}\right)^{p-1}\right)\cdot\left|t_2^{\lambda} - t_1^{\lambda}\right|,\end{aligned} \tag{29}$$

$$\begin{aligned}&\left|\aleph_2 (x, y)(t_2) - \aleph_2 (x, y)(t_1)\right| \leq \lambda_g \\ &\times\left(\begin{array}{l}\sup\limits_{a\leq t\leq T}\left\{t_2^{-\mu}\int_a^{t_2} s^{\mu-1H}I_{a^+}^{q-1}|\psi(s, x(s), y(s))|\right\}\mathrm{d}s\\ -\sup\limits_{a\leq t\leq T}\left\{t_1^{-\mu}\int_a^{t_1} s^{\mu-1H}I_{a^+}^{q-1}|\psi(s, x(s), y(s))|\mathrm{d}s\right\}\end{array}\right) \\ &\leq \frac{\lambda_g \sigma_2}{|\mu|\Gamma(q)}\left(\left(t_1^{\mu} - a^{\mu}\right)\left(\ln\frac{t_1}{a}\right)^{q-1} + t_2^{\mu}\left(\ln\frac{t_2}{a}\right)^{q-1}\right)\cdot\left|t_2^{\mu} - t_1^{\mu}\right|.\end{aligned} \tag{30}$$

R.H.Ss are both independent on (x, y); in addition, R.H.Ss of both (29) and (30) approach to zero when $t_1 \longrightarrow t_2$ and they imply that the operator $\aleph(x, y)$ is equicontinuous; consequently, the operator $\aleph(x, y)$ is completely continuous.

To finish, we establish the bounded set given by $\Omega = \{(x, y) \in H\colon (x, y) = \beta\aleph(x, y), \beta \in [0, 1]\}$, and then $\forall t \in [0, 1]$, with $(x, y) = \beta\aleph(x, y)$, we obtain

$$\begin{aligned}x(t) &= \beta\aleph_1 (x, y)(t),\\ y(t) &= \beta\aleph_2 (x, y)(t).\end{aligned} \tag{31}$$

By $(C3)$, we get

$$\|x\| \leq \lambda_f \Lambda_1 \left(\omega_0 + \omega_1\|x\| + \omega_2\|y\|\right), \tag{32}$$

$$\|y\| \leq \lambda_g \Lambda_2 \left(\theta_0 + \theta_1\|x\| + \theta_2\|y\|\right). \tag{33}$$

Consequently, we have

$$\|x\| + \|y\| \leq \left(\lambda_f \Lambda_1 \omega_0 + \lambda_g \Lambda_2 \theta_0\right) + \left(\lambda_f \Lambda_1 \omega_1 + \lambda_g \Lambda_2 \theta_1\right)\|x\| + \left(\lambda_f \Lambda_1 \omega_2 + \lambda_g \Lambda_2 \theta_2\right)\|y\|. \tag{34}$$

Inequality (34) can be written as follows:

$$\|(x, y)\| \leq \left(\lambda_f \Lambda_1 \omega_0 + \lambda_g \Lambda_2 \theta_0\right)/\Lambda_0, \tag{35}$$

where $\Lambda_0 = \min\{1 - (\lambda_f \quad \Lambda_1 \omega_1 + \lambda_g \Lambda_2 \theta_1),\ 1 - (\lambda_f \Lambda_1 \omega_2 + \lambda_g \Lambda_2 \theta_2)\}$. By (35), we conclude that Ω is bounded. Hence, Leray–Schauder alternative applies; that is, problem (1) has at least one solution. This completes the proof. ☐

4. Example

Consider the following initial value problem:

$$\begin{cases} \left({}^{H}D_{a^+}^{7/4} + {}^{H}D_{a^+}^{3/4}\right)\left(\dfrac{x(t)}{1/3|\sin\ x(t)| + 2}\right) = 3\ \ln\ t + \dfrac{1}{11\sqrt{t^3 + 15}}\dfrac{|x|}{1 + |x|} + \dfrac{1}{44}\tan^{-1}\ y, \quad 0 < 1 \le t \le e, \\ \left({}^{H}D_{a^+}^{5/4} + \mu^{H}D_{a^+}^{1/4}\right)\left(\dfrac{y(t)}{1/3|\sin\ y(t)| + 1}\right) = e^{-t}\cos\ t + \dfrac{1}{20}\left(\tan^{-1}\ y + \ \tan^{-1}\ x\right), \\ x(1) = x'(1) = 0, \\ y(1) = y'(1) = 0. \end{cases} \tag{36}$$

Here, $1 = \lambda = \mu$, $p = 7/4, q = 5/4$, $f(t, x, y) = 1/3(|\sin\ x| + 2)$, $g(t, x, y) = 1/3(|\cos\ y| + 1)$,

$$\psi(t, x(t), y(t)) = 3\ \ln\ t + \frac{1}{11\sqrt{t^3 + 15}}\frac{|x|}{1 + |x|} + \frac{1}{44}\tan^{-1} y,$$
$$\varphi(t, x(t), y(t)) = e^{-t}\cos\ t + \frac{1}{20}\left(\tan^{-1}\ y + \ \tan^{-1}\ x\right). \tag{37}$$

Observe that

$$\left|\psi(t, x_1, y_1) - \psi(t, x_2, y_2)\right| \le \frac{1}{44}\left|x_2 - x_1\right| + \frac{1}{44}\left|y_2 - y_1\right|,$$
$$\left|\varphi(t, x_1, y_1) - \varphi(t, x_2, y_2)\right| \le \frac{1}{20}\left|x_2 - x_1\right| + \frac{1}{20}\left|y_2 - y_1\right|,$$
$$\left[\lambda_f \Lambda_1\left(v_1 + v_2\right) + \lambda_g \Lambda_2\left(\tau_1 + \tau_2\right)\right] \le 0.1363483 < 1. \tag{38}$$

Thus, problem (36) satisfies all the conditions of Theorem 1; accordingly, we conclude that the B.V.P has a unique solution on $[1, e]$.

5. Conclusion and Future Work

In this article, the existence and uniqueness theory of solutions for sequential fractional differential system involving Hadamard fractional derivatives of order $1 < p,\ q < 2$ with initial conditions were investigated. For the future work, the researcher may generalize our system by taking an $n \times 1$ system of sequential fractional differential equations and may apply another type of fractional derivatives such as Psi-Hilfer and Psi-Caputo fractional derivatives.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

M.A and KA contributed to each part of this work equally and read and approved the final version of the manuscript.

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