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Research Article

A Novel Generalized-*M* Family: Heavy-Tailed Characteristics with Applications in the Engineering Sector

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The Weibull distribution has prominent applications in the engineering sector. However, due to its monotonic behavior of the hazard function, the Weibull model does not provide the best fit for data in many cases. This paper introduces a new family of distributions to obtain new flexible distributions. The proposed family is called a novel generalized-*M* family. Based on this approach, an updated version of the Weibull distribution is introduced. The updated version of the Weibull distribution is called a novel generalized Weibull distribution. The proposed distribution is able to capture four different patterns of the hazard function. Some mathematical properties of the proposed method are obtained. Furthermore, the maximum likelihood estimators of the proposed family are also obtained. Moreover, a simulation study is conducted for evaluating these estimators. For illustrating the proposed model, two data sets from the engineering sector are analyzed. Based on some well-known analytical measures, it is shown that the novel generalized Weibull distribution is the best competing distribution for analyzing the engineering data sets.

1. Introduction

The two-parameter Weibull distribution is a famous statistical distribution defined on $\mathbb{R}^+$. It can be used as a suitable alternative choice for predicting, modeling, and analyzing lifetime data sets in healthcare, engineering, and other related areas. For data modeling in the engineering sector, numerous modifications of the Weibull distribution have been introduced and implemented, refer to Pedrosa et al. [1]; Bala and Napiah [2]; Huo et al. [3]; Liao et al. [4]; Shu et al. [5]; Li et al. [6]; and Bilal et al. [7].

On one side, the Weibull distribution generalizes the exponential and Rayleigh distributions whereas, on the other side, it can provide the characteristics of the other distributions. For some review studies based on the applications of the Weibull distribution, we refer to Nadarajah et al. [8]; Almalki and Nadarajah [9]; and Wais [10].

Let the random variable V has the Weibull distribution with scale parameter $\sigma > 0$ and shape parameter $\alpha > 0$, if its HF (hazard function) $h(v; \Xi)$ has the following form:

$$h(v; \Xi) = \alpha\sigma v^{\alpha-1}, v > 0. \quad (1)$$

From the expression of $h(v; \Xi)$ in (1), we can see that the Weibull distribution offers data modeling with (i) decreasing HF shape, if $\alpha < 1$, (ii) increasing HF shape, if $\alpha > 1$, or (iii) constant HF shape, if $\alpha = 1$.

In many cases, the data sets in the engineering sector follow the unimodal, bathtub, or modified unimodal HF shapes. In such cases, the Weibull distribution is not a suitable choice to implement for modeling the engineering data sets. In order to provide a close fit to such type of engineering data sets, numerous updated versions of the Weibull distribution have been produced and implemented. For example, (i) Ahmad et al. [11] studied a new Weibull–Weibull distribution for analyzing the strength of the alumina material; (ii) Zhao et al. [12] introduced TIHT–Weibull distribution for modeling the failure time of coating machines; (iii) Wang et al. [13] implemented the NG–Weibull distribution for analyzing the failure time of the electronic machine. For more recent modifications of the Weibull distribution, we refer to Klakattawi [14]; Wang et al. [15]; Tianshuai et al. [16]; Al-Sobhi [17]; Gonzalez et al. [18]; Emam and Tashkandy [19]; Prataviera [20]; and Rehman et al. [21].

To overcome the above deficiency of the Weibull distribution, we introduce a new updated form of the Weibull distribution. The new updated form of the Weibull distribution is introduced by proposing a new statistical family of distributions. The new family is called a novel generalized-M (NGen-M) family of distributions. The NGen-M family can be used to update/increase the fitting power of the classical/traditional or other modified and generalized distributions.

Definition: a random variable V has the NGen-M distributions, if its DF (distribution function) $K(v;\lambda,\Xi)$ is given by

$$K(v;\lambda,\mathbf{\Gamma}) = 1 - \frac{[1-M(v;\mathbf{\Gamma})]}{\lambda}\left(\lambda - [M(v;\mathbf{\Gamma})]^2\right), v \in \mathbb{R}, \quad (2)$$

where $\lambda \geq 1, \lambda \leq -1$, and $M(v;\Xi)$ represents the baseline DF. In order to show that $K(v;\lambda,\Xi)$ is a valid DF, a complete proof is provided in Proposition 1 and Proposition 2.

Proposition 1. *Let's consider the DF $K(v;\lambda,\Xi)$ in Eq. (2), first, we have to show that*

$$\lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) = 0, \quad (3)$$

$$\lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) = 1. \quad (4)$$

Proof.

$$\begin{aligned}\lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= \lim_{v\longrightarrow -\infty}\left\{1 - \frac{[1-M(v;\mathbf{\Gamma})]}{\lambda}\left(\lambda - [M(v;\mathbf{\Gamma})]^2\right)\right\},\\ \lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 1 - \frac{[1-M(-\infty;\mathbf{\Gamma})]}{\lambda}\left(\lambda - [M(-\infty;\mathbf{\Gamma})]^2\right).\end{aligned} \quad (5)$$

Since $M(v;\Xi)$ is a DF, we have

$$\lim_{v\longrightarrow -\infty} M(v;\mathbf{\Gamma}) = 0. \quad (6)$$

Hence, from (5), we have

$$\begin{aligned}\lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 1 - \frac{[1-0]}{\lambda}[\lambda - 0],\\ \lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 1 - \frac{1}{\lambda}\lambda,\\ \lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 0.\end{aligned} \quad (7)$$

Also, we have

$$\begin{aligned}\lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= \lim_{v\longrightarrow -\infty}\left\{1 - \frac{[1-M(v;\mathbf{\Gamma})]}{\lambda}\left(\lambda - [M(v;\mathbf{\Gamma})]^2\right)\right\},\\ \lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 1 - \frac{[1-M(\infty;\mathbf{\Gamma})]}{\lambda}\left(\lambda - [M(\infty;\mathbf{\Gamma})]^2\right).\end{aligned} \quad (8)$$

As we stated above that $M(v;\Xi)$ is a DF, we have

$$\lim_{v\longrightarrow -\infty} M(v;\mathbf{\Gamma}) = 1. \quad (9)$$

Therefore, from (8), we have

$$\begin{aligned}\lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 1 - \frac{[1-1]}{\lambda}[\lambda - 1],\\ \lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 1 - \frac{0}{\lambda}\lambda,\\ \lim_{v\longrightarrow -\infty} K(v;\lambda,\mathbf{\Gamma}) &= 1.\end{aligned} \quad (10)$$

□

Proposition 2. *The DF $K(v;\lambda,\Xi)$ presented in Eq. (2) is a differentiable as well as a right continuous function.*

Proof.

$$\frac{d}{dv}K(v;\lambda,\mathbf{\Gamma}) = k(v;\lambda,\mathbf{\Gamma}). \quad (11)$$

The proof of Proposition 2 is very straightforward, and hence, omitted. Based on the mathematical derivations in Proposition 1 and Proposition 2, we can see that $K(v;\lambda,\Xi)$ is a valid DF.

For $v \in \mathbb{R}$, in link to $K(v;\lambda,\Xi)$, the PDF (probability density function) $k(v;\lambda,\Xi)$ is given by

$$k(v;\lambda,\mathbf{\Gamma}) = \frac{m(v;\mathbf{\Gamma})}{\lambda}\left(\lambda + 2M(v;\mathbf{\Gamma}) - 3[M(v;\mathbf{\Gamma})]^2\right), \quad (12)$$

where $d/dvM(v;\Xi) = m(v;\Xi)$.

Corresponding to DF $K(v;\lambda,\Xi)$ and PDF $k(v;\lambda,\Xi)$, the SF $S(v;\lambda,\Xi) = 1 - K(v;\lambda,\Xi)$, HF $h(v;\lambda,\Xi) = k(v;\lambda,\Xi)/1 - K(v;\lambda,\Xi)$, and cumulative HF (CHF) $H(v;\lambda,\Xi) = -\log[1 - K(v;\lambda,\Xi)]$ are given by

$$\begin{aligned}S(v;\lambda,\mathbf{\Gamma}) &= \frac{[1-M(v;\mathbf{\Gamma})]}{\lambda}\left(\lambda - [M(v;\mathbf{\Gamma})]^2\right),\\ h(v;\lambda,\mathbf{\Gamma}) &= \frac{m(v;\mathbf{\Gamma})\left(\lambda + 2M(v;\mathbf{\Gamma}) - 3[M(v;\mathbf{\Gamma})]^2\right)}{[1-M(v;\mathbf{\Gamma})]\left(\lambda - [M(v;\mathbf{\Gamma})]^2\right)},\end{aligned} \quad (13)$$

$$S(v;\lambda,\mathbf{\Gamma}) = -\log\left(\frac{[1-M(v;\mathbf{\Gamma})]}{\lambda}\left(\lambda-[M(v;\mathbf{\Gamma})]^2\right)\right), \quad (14)$$

respectively.

By using the DF $M(v;\Xi)$ in (2) as a base model, we can introduce an updated version of any base model. In this paper, we use $M(v;\Xi)$ as a DF of the Weibull model, to introduce a new version of the Weibull distribution. The updated version of the Weibull model is called a novel generalized-Weibull (NGen-Weibull) distribution. Some basic expressions of the NGen-Weibull distribution along with different plots of PDF and HF are obtained in Section 2. □

2. A NGen-Weibull Distribution

Consider the DF $M(v;\Xi)$ of the Weibull distribution given by

$$M(v;\mathbf{\Gamma}) = 1-e^{-\sigma v^{\alpha}}, v\geq 0, \alpha>0, \sigma>0, \quad (15)$$

with PDF $m(v;\Xi)$ given by

$$m(v;\mathbf{\Gamma}) = \alpha\sigma v^{\alpha-1}e^{-\sigma v^{\alpha}}, v>0, \alpha>0, \delta>0, \quad (16)$$

where $\Xi = (\alpha,\sigma)$.

By incorporating (15) in (1), we obtain the DF of the NGen-Weibull distribution. Let V has the NGen-Weibull distribution, if its DF $K(v;\lambda,\Xi)$ is given by

$$K(v;\lambda,\mathbf{\Gamma}) = 1-\frac{e^{-\sigma v^{\alpha}}}{\lambda}\left(\lambda-\left[1-e^{-\sigma v^{\alpha}}\right]^2\right). \quad (17)$$

For $V>0$, the PDF $k(v;\lambda,\Xi)$ of the NGen-Weibull distribution is given by

$$k(v;\lambda,\mathbf{\Gamma}) = \frac{\alpha\sigma v^{\alpha-1}e^{-\sigma v^{\alpha}}}{\lambda}\left(\lambda+2\left(1-e^{-\sigma v^{\alpha}}\right)-3\left(1-e^{-\sigma v^{\alpha}}\right)^2\right). \quad (18)$$

Visual display of different plots of $k(v;\lambda,\Xi)$ and $K(v;\lambda,\Xi)$ is provided in Figure 1. From the plots of $k(v;\lambda,\Xi)$, we can see that the NGen-Weibull has four different patterns of PDF such as (i) reverse-J shaped (blue curve), (ii) right-skewed (red curve), (iii) left-skewed (green curve), and (vi) symmetrical (black curve).

The plots of $k(v;\lambda,\Xi)$ in Figure 1 are obtained for (i) $\alpha=1.2, \sigma=1.800, \lambda=1.2$ (blue curve), (ii) $\alpha=5.9$, $\sigma=0.001, \lambda=2.4$ (red curve), (iii) $\alpha=3.2, \sigma=0.080, \lambda=5.2$ (green curve), and (iv) $\alpha=0.5, \sigma=1.300, \lambda=3.2$ (black curve).

Furthermore, for $V>0$, the SF, HF, and CHF of the NGen-Weibull distribution are presented as follows:

$$S(v;\lambda,\mathbf{\Gamma}) = \frac{e^{-\sigma v^{\alpha}}}{\lambda}\left(\lambda-\left[1-e^{-\sigma v^{\alpha}}\right]^2\right),$$

$$h(v;\lambda,\mathbf{\Gamma}) = \frac{\alpha\sigma v^{\alpha-1}\left(\lambda+2\left(1-e^{-\sigma v^{\alpha}}\right)-3\left(1-e^{-\sigma v^{\alpha}}\right)^2\right)}{\left(\lambda-\left[1-e^{-\sigma v^{\alpha}}\right]^2\right)}, \quad (19)$$

$$H(v;\lambda,\mathbf{\Gamma}) = -\log\left(\frac{e^{-\sigma v^{\alpha}}}{\lambda}\left(\lambda-\left[1-e^{-\sigma v^{\alpha}}\right]^2\right)\right), \quad (20)$$

respectively.

A graphical illustration of $h(v;\lambda,\Xi)$ and $S(v;\lambda,\Xi)$ is presented in Figure 2. From the plots of $h(v;\lambda,\Xi)$, we can see that the NGen-Weibull distribution has the ability to model data with four different shapes of failure function. The HF shapes of NGen-Weibull distribution include (i) increasing shaped (blue curve), (ii) unimodal (green curve), (iii) modified unimodal (red curve), and (vi) decreasing (black curve).

The plots of $h(v;\lambda,\Xi)$ in Figure 2 are obtained for (i) $\alpha=0.9, \sigma=1.1, \lambda=1.2$ (blue curve), (ii) $\alpha=1.0, \sigma=1.1, \lambda=1.8$ (red curve), (iii) $\alpha=0.8, \sigma=1.5, \lambda=1.3$ (green curve), and (iv) $\alpha=1.2, \sigma=1.2, \lambda=1.9$ (black curve).

The proposed model has certain advantages over the other classical and modified distributions, for example.

(i) The proposed NGen-Weibull distribution is a simple extension of the Weibull model by adding one additional parameter. Most of the updated versions of the Weibull distribution have more than one additional parameter.

(ii) The proposed NGen-Weibull distribution has four different shapes of the HF. There are only a few distributions that can model real-life data sets with unimodal and decreasing-increasing-decreasing shapes of HF.

(iii) The proposed NGen-Weibull distribution obeys the HT properties. This fact indicates that the NGen-Weibull distribution can be a suitable candidate distribution for modeling the financial data sets. Due to the HT characteristics, the NGen-Weibull distribution can be used quite effectively for dealing with the extreme value data sets.

(iv) The NGen-Weibull distribution provides the fit best to real-life engineering data sets as compared to other competing models, see Section 5.

3. Some Distributional Properties

This section offers the derivation of some distributional properties of the NGen-M distributions. These properties include identifiability property, heavy-tailed property, quantile function, and r^{th} moment.

3.1. The Identiability Property.

3.1. The Identiability Property. In this subsection, we derive the identifiability property of the NGen-M distributions. The parameter λ is called identifiable, if

$$\lambda_1 = \lambda_2. \quad (21)$$

Suppose λ_1 and λ_2 be the two parameters with DFs $K(v;\lambda_1,\Xi)$ and $K(v;\lambda_2,\Xi)$, respectively. From the mathematical definition of the identifiability property, we have

$$K(v;\lambda_1,\mathbf{\Gamma}) = K(v;\lambda_2,\mathbf{\Gamma}). \quad (22)$$

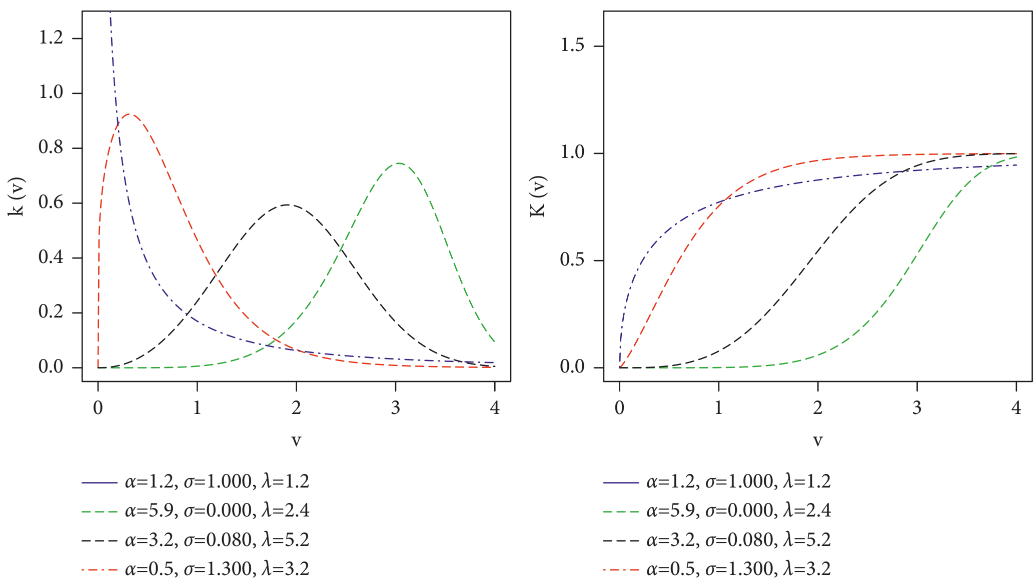

Figure 1: Visual display of different plots of $k(v;\lambda,\Xi)$ and $K(v;\lambda,\Xi)$.

Using (2) in (22), we get

$$\begin{aligned}
&1-\frac{[1-M(v;\boldsymbol{\Gamma})]}{\lambda_1}\left(\lambda_1-[M(v;\boldsymbol{\Gamma})]^2\right)\\
&=1-\frac{[1-M(v;\boldsymbol{\Gamma})]}{\lambda_2}\left(\lambda_2-[M(v;\boldsymbol{\Gamma})]^2\right),\\
&\frac{[1-M(v;\boldsymbol{\Gamma})]}{\lambda_1}\left(\lambda_1-[M(v;\boldsymbol{\Gamma})]^2\right)\\
&=\frac{[1-M(v;\boldsymbol{\Gamma})]}{\lambda_2}\left(\lambda_2-[M(v;\boldsymbol{\Gamma})]^2\right),\\
&\lambda_2[1-M(v;\boldsymbol{\Gamma})]\left(\lambda_1-[M(v;\boldsymbol{\Gamma})]^2\right)\\
&=\lambda_1[1-M(v;\boldsymbol{\Gamma})]\left(\lambda_2-[M(v;\boldsymbol{\Gamma})]^2\right),\\
&\lambda_2[1-M(v;\boldsymbol{\Gamma})][M(v;\boldsymbol{\Gamma})]^2\\
&=\lambda_1[1-M(v;\boldsymbol{\Gamma})][M(v;\boldsymbol{\Gamma})]^2,\\
&\lambda_1=\lambda_2.
\end{aligned}\tag{23}$$

3.2. The HT Property. A distribution with DF $K(v;\lambda,\Xi)$ is called a HT model, if its SF satisfies

$$\lim_{v\longrightarrow\infty}e^{pv}[1-K(v;\lambda,\boldsymbol{\Gamma})]=\infty,\tag{24}$$

where $p>0$.

The HT distributions obey an important property called the regular variational property. A probability distribution is called regularly varying if it satisfies

$$\lim_{v\longrightarrow\infty}\frac{1-K(tv;\lambda,\boldsymbol{\Gamma})}{1-K(v;\lambda,\boldsymbol{\Gamma})}=t^a,a>0,t>0,\tag{25}$$

where the term a is called the index of regular variation.

Here, we mathematical prove the regular variational property of the NGen-M distributions. Based on the findings of Seneta [22], in terms of SF $[1-M(v;\Xi)]$, we have the following:

Theorem 1. *If a distribution with SF $[1-M(v;\Xi)]$ is regular varying distribution, then $[1-K(v;\lambda,\Xi)]$ is a regular varying distribution.*

Proof. Suppose $\lim_{v\longrightarrow-\infty}[1-M(tv;\Xi)]/[1-M(v;\Xi)]=g(v)$ is finite but nonzero $\forall v>0$. Then, incorporating (2), we get

$$\begin{aligned}
&\lim_{v\longrightarrow\infty}\frac{K(av;\lambda,\boldsymbol{\Gamma})}{K(v;\lambda,\boldsymbol{\Gamma})}=\lim_{v\longrightarrow\infty}\frac{[1-M(av;\boldsymbol{\Gamma})]/\lambda\left(\lambda-[M(av;\boldsymbol{\Gamma})]^2\right)}{[1-M(v;\boldsymbol{\Gamma})]/\lambda\left(\lambda-[M(v;\boldsymbol{\Gamma})]^2\right)},\\
&\lim_{v\longrightarrow\infty}\frac{K(av;\lambda,\boldsymbol{\Gamma})}{K(v;\lambda,\boldsymbol{\Gamma})}=\lim_{v\longrightarrow\infty}\frac{[1-M(av;\boldsymbol{\Gamma})]}{[1-M(v;\boldsymbol{\Gamma})]}\times\frac{\left(\lambda-[M(av;\boldsymbol{\Gamma})]^2\right)}{\left(\lambda-[M(v;\boldsymbol{\Gamma})]^2\right)},\\
&\lim_{v\longrightarrow\infty}\frac{K(av;\lambda,\boldsymbol{\Gamma})}{K(v;\lambda,\boldsymbol{\Gamma})}=g(v)\times\lim_{v\longrightarrow\infty}\frac{\left(\lambda-[M(av;\boldsymbol{\Gamma})]^2\right)}{\left(\lambda-[M(v;\boldsymbol{\Gamma})]^2\right)},\\
&\lim_{v\longrightarrow\infty}\frac{K(av;\lambda,\boldsymbol{\Gamma})}{K(v;\lambda,\boldsymbol{\Gamma})}=g(v)\frac{\left(\lambda-[M(a.\infty;\boldsymbol{\Gamma})]^2\right)}{\left(\lambda-[M(\infty;\boldsymbol{\Gamma})]^2\right)},\\
&\lim_{v\longrightarrow\infty}\frac{K(av;\lambda,\boldsymbol{\Gamma})}{K(v;\lambda,\boldsymbol{\Gamma})}=g(v)\frac{(\lambda--1)}{(\lambda-1)},\\
&\lim_{v\longrightarrow\infty}\frac{K(av;\lambda,\boldsymbol{\Gamma})}{K(v;\lambda,\boldsymbol{\Gamma})}=g(v).
\end{aligned}\tag{26}$$

Since (26) is nonzero, $\forall v>0$. Therefore, $[1-K(av;\lambda,\Xi)]$ is the SF of the regular varying distribution. □

3.3. The Quantile Function. The QF (quantile function) of the NGen-M distributions with DF $K(v;\lambda,\Xi)$ is derived as

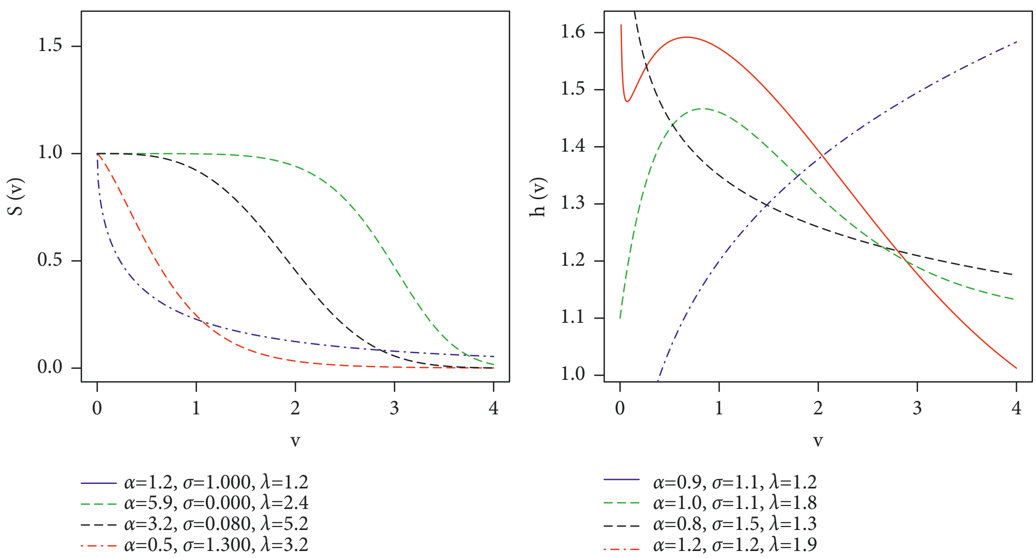

FIGURE 2: Graphical illustration of $h(v;\lambda,\Xi)$ and $S(v;\lambda,\Xi)$.

$$v = Q(u) = K^{-1}(u) = M^{-1}(t), \quad (27)$$

where $u \in (0,1)$ and t is the solution of $(1-t)[\lambda - t^2] - \lambda(1-u)$. The expression in (27) can be implemented to generate random numbers from any special model of the NGen-M family.

3.4. The r^{th} Moment. Some basic distributional properties of any probability distribution can be derived and studied with the help of moments. This subsection offers the derivation of the r^{th} moment of NGen-M distributions. Let V have the NGen-M distributions with PDF $k(v;\lambda,\Xi)$, then, its r^{th} moment can be obtained as

$$\mu_r' = \int_\Omega v^r k(v;\lambda,\mathbf{\Gamma})dv. \quad (28)$$

Using (12) in (28), we get

$$\begin{aligned}
\mu_r' &= \int_\Omega v^r \frac{m(v;\mathbf{\Gamma})}{\lambda}\left(\lambda + 2M(v;\mathbf{\Gamma}) - 3[M(v;\mathbf{\Gamma})]^2\right)dv,\\
\mu_r' &= \int_\Omega v^r m(v;\mathbf{\Gamma})dv + \frac{2}{\lambda}\int_\Omega v^r m(v;\mathbf{\Gamma})M(v;\mathbf{\Gamma})dv\\
&\quad - \frac{3}{\lambda}\int_\Omega v^r m(v;\mathbf{\Gamma})[M(v;\mathbf{\Gamma})]^2 dv,\\
\mu_r' &= \int_\Omega v^r m(v;\mathbf{\Gamma})dv + \frac{1}{\lambda}\int_\Omega 2v^r m(v;\mathbf{\Gamma})[M(v;\mathbf{\Gamma})]^{2-1}dv\\
&\quad - \frac{1}{\lambda}\int_\Omega 3v^r m(v;\mathbf{\Gamma})[M(v;\mathbf{\Gamma})]^{3-1}dv,\\
\mu_r' &= \int_\Omega v^r m(v;\mathbf{\Gamma})dv + \frac{1}{\lambda}\int_\Omega v^r f_1(v;\mathbf{\Gamma})dv\\
&\quad - \frac{1}{\lambda}\int_\Omega v^r f_2(v;\mathbf{\Gamma})dv,
\end{aligned} \quad (29)$$

where $f_1(v;\Xi) = 2m(v;\Xi)[M(v;\Xi)]^{2-1}$ is the PDF of the exponentiated version of the NGen-M distributions with the exponentiated parameter 2 whereas $f_2(v;\Xi) = 3m(v;\Xi)[M(v;\Xi)]^{3-1}$ is the PDF of the exponentiated version of the NGen-M distributions with the exponentiated parameter 3.

4. Estimation and Simulation

This part of the paper offers the derivation of the maximum likelihood estimators (MLEs) of the NGen-M distributions with parameters λ and Ξ. Let $V_1, V_2, \ldots, V_n$ be a set of sample taken from $k(v;\lambda,\Xi)$. The corresponding LF (likelihood function) $\Upsilon(v;\lambda,\Xi)$ is

$$\begin{aligned}
\Upsilon(v;\lambda,\mathbf{\Gamma}) &= \prod_{i=1}^n k(v_i;\lambda,\mathbf{\Gamma}),\\
\Upsilon(v;\lambda,\mathbf{\Gamma}) &= \prod_{i=1}^n \frac{m(v_i;\mathbf{\Gamma})}{\lambda}\left(\lambda + 2M(v_i;\mathbf{\Gamma}) - 3[M(v_i;\mathbf{\Gamma})]^2\right),\\
\Upsilon(v;\lambda,\mathbf{\Gamma}) &= \left(\frac{1}{\lambda}\right)^n \prod_{i=1}^n m(v_i;\mathbf{\Gamma}) \prod_{i=1}^n\left(\lambda + 2M(v_i;\mathbf{\Gamma}) - 3[M(v_i;\mathbf{\Gamma})]^2\right).
\end{aligned} \quad (30)$$

Corresponding to (30), the log LF $\Delta(\lambda,\Xi)$ is

$$\begin{aligned}
\Delta(\lambda,\mathbf{\Gamma}) &= n\log\left(\frac{1}{\lambda}\right) + \sum_{i=1}^n \log m(v_i;\mathbf{\Gamma})\\
&\quad + \sum_{i=1}^n \log\left(\lambda + 2M(v_i;\mathbf{\Gamma}) - 3[M(v_i;\mathbf{\Gamma})]^2\right).
\end{aligned} \quad (31)$$

Corresponding to $\Delta(\lambda,\Xi)$, the partial derivatives are

$$\frac{\partial}{\partial\lambda}\Delta(\lambda,\mathbf{\Gamma}) = -\frac{n}{\lambda} + \sum_{i=1}^n \frac{1}{\left(\lambda + 2M(v_i;\mathbf{\Gamma}) - 3[M(v_i;\mathbf{\Gamma})]^2\right)}, \quad (32)$$

Table 1: The results of the simulation study of the NGen-Weibull model for $\alpha = 1.4, \sigma = 1, \lambda = 1.5$.

n	Parameters	MLEs	MSEs	Biases
25	α	1.57272700	0.10565145	0.17272731
	σ	1.15407200	0.06573026	0.15407212
	λ	3.230664 00	6.47732440	2.03066430
50	α	1.52288400	0.05360254	0.12288441
	σ	1.13838900	0.04414655	0.13838869
	λ	2.69624200	4.51638700	1.49624160
75	α	1.49738600	0.03260828	0.09738595
	σ	1.12120700	0.03170536	0.12120685
	λ	2.50034400	3.78230730	1.30034420
100	α	1.48435200	0.02587331	0.08435211
	σ	1.10939400	0.02752766	0.10939399
	λ	2.24930400	2.82126370	1.04930410
150	α	1.46361500	0.01655346	0.06361484
	σ	1.09583800	0.02307459	0.09583788
	λ	2.16424400	2.76536040	0.96424440
200	α	1.45604900	0.01408698	0.05604902
	σ	1.08893700	0.02016455	0.08893705
	λ	2.02291800	2.18558790	0.82291830
300	α	1.44233200	0.00948109	0.04233205
	σ	1.07478400	0.01469160	0.07478449
	λ	1.80914600	1.47998310	0.60914590
400	α	1.43438700	0.00710442	0.03438667
	σ	1.06524200	0.01251653	0.06524163
	λ	1.67936500	1.08931920	0.47936480
500	α	1.43820200	0.00648074	0.03820168
	σ	1.06577600	0.01241515	0.06577620
	λ	1.67539100	1.02479190	0.47539100
600	α	1.43061800	0.00528021	0.03061809
	σ	1.06079900	0.01133082	0.06079897
	λ	1.64763900	1.01068440	0.44763910
700	α	1.43231000	0.00516572	0.03231025
	σ	1.05961300	0.01092145	0.05961322
	λ	1.62284000	0.91456200	0.42284040
750	α	1.41498100	0.00400002	0.02498083
	σ	1.04965000	0.00870671	0.04964955
	λ	1.51043600	0.63627510	0.32043650

Table 2: The results of simulation study of the NGen-Weibull model for $\alpha = 1.2, \sigma = 1, \lambda = 1.9$.

n	Parameters	MLEs	MSEs	Biases
25	α	1.31593100	0.06018998	0.11593107
	σ	1.15609700	0.06972226	0.15609747
	λ	3.521391 00	6.81451700	2.12139120
50	α	1.27756200	0.02981195	0.07756164
	σ	1.11790400	0.03426620	0.11790373
	λ	3.09305000	5.33528900	1.69305020
75	α	1.25680900	0.01970635	0.05680943
	σ	1.11059700	0.02987176	0.11059712
	λ	2.82767500	4.37315800	1.42767490
100	α	1.24857200	0.01439279	0.04857249
	σ	1.09752400	0.02442831	0.09752428
	λ	2.70442400	3.93945800	1.30442440
150	α	1.24033500	0.00977599	0.04033521
	σ	1.08714200	0.01868892	0.08714193
	λ	2.54872100	3.39428500	1.14872050
200	α	1.23262200	0.00796338	0.03262186
	σ	1.08038100	0.01673055	0.08038127
	λ	2.371564 00	2.86718800	0.97156410
300	α	1.22736100	0.00530184	0.02736133
	σ	1.07003400	0.01288601	0.07003359
	λ	2.22198800	2.33619000	0.82198780
400	α	1.22436800	0.00405492	0.02436773
	σ	1.06347700	0.01088123	0.06347681
	λ	2.12821500	1.93233700	0.72821510
500	α	1.22567700	0.00368187	0.02567661
	σ	1.06176400	0.01014781	0.06176444
	λ	2.08797300	1.74341300	0.68797340
600	α	1.22408200	0.00313356	0.02408230
	σ	1.06221500	0.01017182	0.06221497
	λ	2.07237900	1.67631700	0.67237860
700	α	1.22470600	0.00298151	0.02470586
	σ	1.05719500	0.00926426	0.05719467
	λ	2.01288900	1.09605800	0.61288930
750	α	1.21170200	0.00269140	0.02170158
	σ	1.05450700	0.00855028	0.05450734
	λ	1.91506500	0.99343000	0.57506490

$$\frac{\partial}{\partial \mathbf{\Gamma}} \Delta(\lambda, \mathbf{\Gamma}) = \sum_{i=1}^{n} \frac{\partial/\partial \mathbf{\Gamma} m(v_i; \mathbf{\Gamma})}{m(v_i; \mathbf{\Gamma})} + \sum_{i=1}^{n} \frac{2\partial/\partial \mathbf{\Gamma} M(v_i; \mathbf{\Gamma}) - 6M(v_i; \mathbf{\Gamma})\partial/\partial \mathbf{\Gamma} M(v_i; \mathbf{\Gamma})}{\left(\lambda + 2M(v_i; \mathbf{\Gamma}) - 3[M(v_i; \mathbf{\Gamma})]^2\right)}. \tag{33}$$

Equating $\partial/\partial\lambda\Delta(\lambda, \Xi)$ and $\partial/\partial\Xi\Delta(\lambda, \Xi)$ to zero, and solving, we obtain the MLEs $(\widehat{\lambda}, \widehat{\Xi})$ of (λ, Ξ).

After obtaining the expressions of the MLEs, now we provide a simulation study to assess performances of $\widehat{\lambda}$ and $\widehat{\Xi}$. To carry out the evaluation of $\widehat{\lambda}$ and $\widehat{\Xi}$ through a simulation study, we generate RNs (random numbers) from the PDF $g(u; \lambda, \Xi)$ using the inverse DF method.

The simulation results are obtained for (i) $\alpha = 1.4, \sigma = 1, \lambda = 1.5$ and (ii) $\alpha = 1.2, \sigma = 1, \lambda = 1.9$. The simulation results are obtained using *R*-function with ``*L-BFGS-B*'' algorithm and *Adequacy Model* library. To check the performances of $\widehat{\lambda}$ and $\widehat{\Xi}$, two statistical evaluation criteria such as

$$\text{MSE}(\widehat{\lambda}) = \sum_{i=1}^{n} \left(\widehat{\lambda}_i - \lambda\right)^2, \tag{34}$$

$$\text{Bias}(\widehat{\lambda}) = \sum_{i=1}^{n} \left(\widehat{\lambda}_i - \lambda\right), \tag{35}$$

were chosen. These two evaluation criteria were also calculated for $\widehat{\Xi}$.

In link to $\alpha = 1.4, \sigma = 1, \lambda = 1.5$, the simulation results of the NGen-Weibull distribution are presented in Table 1 whereas the simulation results for $\alpha = 1.2, \sigma = 1, \lambda = 1.9$ are provided in Table 2.

From the simulation results obtained in Tables 1 and 2, it is obvious that as the size of n increases.

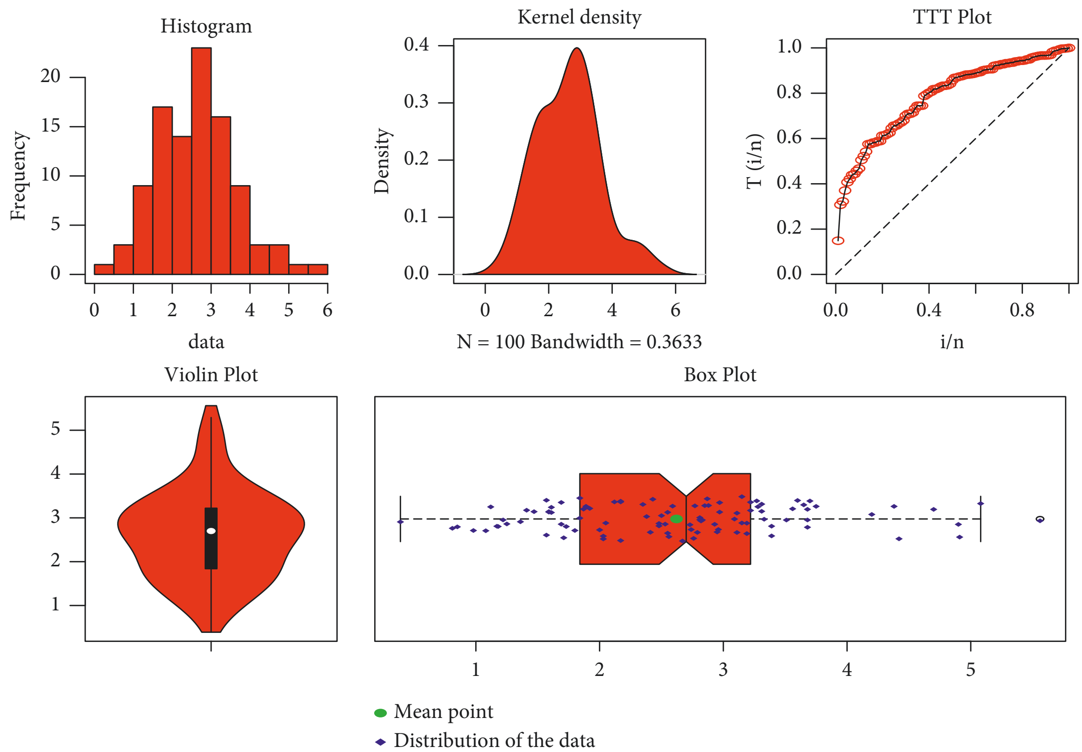

Figure 3: Some basic plots using data 1.

Table 3: The values of $\hat{\alpha}_{\mathrm{MLE}}, \hat{\sigma}_{\mathrm{MLE}}, \hat{\lambda}_{\mathrm{MLE}}, \hat{\beta}_{\mathrm{MLE}}, \hat{a}_{\mathrm{MLE}}$, and $\hat{b}_{MLE}$ of the fitted models using data 1.

Models	$\hat{\alpha}_{\mathrm{MLE}}$	$\hat{\sigma}_{\mathrm{MLE}}$	$\hat{\lambda}_{\mathrm{MLE}}$	$\hat{a}_{\mathrm{MLE}}$	$\hat{b}_{\mathrm{MLE}}$	$\hat{\beta}_{\mathrm{MLE}}$
NGen-Weibull	2.74215	0.04383	1.96167	—	—	—
Weibull	2.79613	0.04843	—	—	—	—
Exp-Weibull	2.29181	0.11245	—	1.41472	—	—
MO-Weibull	2.41759	0.10374	—	—	—	1.73356
Kum-Weibull	2.12695	0.09729	—	1.52838	1.58933	—

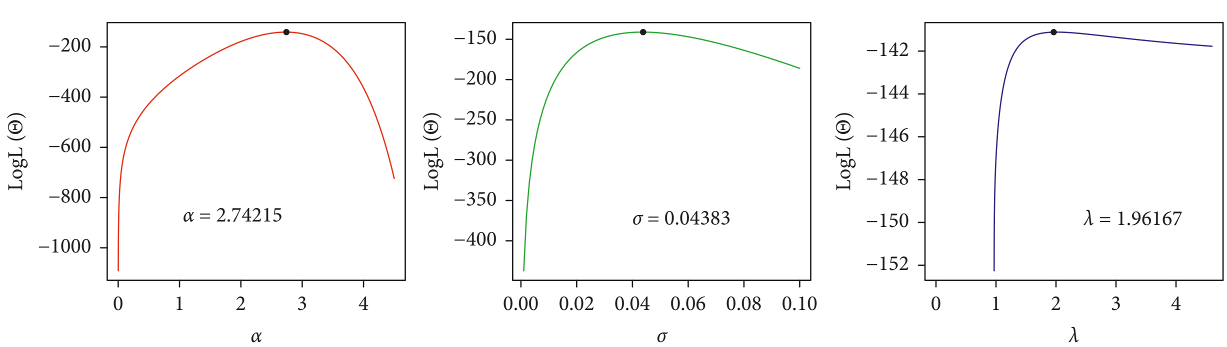

Figure 4: The profiles of the log-likelihood function of the MLEs using data 1.

(i) The estimated values of $\hat{\lambda}_{MLE}$ and $\hat{\Xi}_{MLE}$ tend to be stable

(ii) The MSEs of $\hat{\lambda}_{MLE}$ and $\hat{\Xi}_{MLE}$ decrease

(iii) The biases of $\hat{\lambda}_{MLE}$ and $\hat{\Xi}_{MLE}$ tend to zero

5. Data Analyses

This section offers the illustration of the NGen-Weibull distribution using two data sets taken from the engineering sector. We apply the NGen-Weibull distribution to these

Table 4: The values of the analytical tests and P-value of the fitted models for data 1.

Models	CM	AD	KS	P-value
NGen-Weibull	0.05880	0.36448	0.06099	0.85080
Weibull	0.06230	0.41584	0.06271	0.82640
Exp-Weibull	0.07454	0.42356	0.07332	0.65530
MO-Weibull	0.05945	0.41085	0.06389	0.80890
Kum-Weibull	0.07583	0.42602	0.07185	0.68010

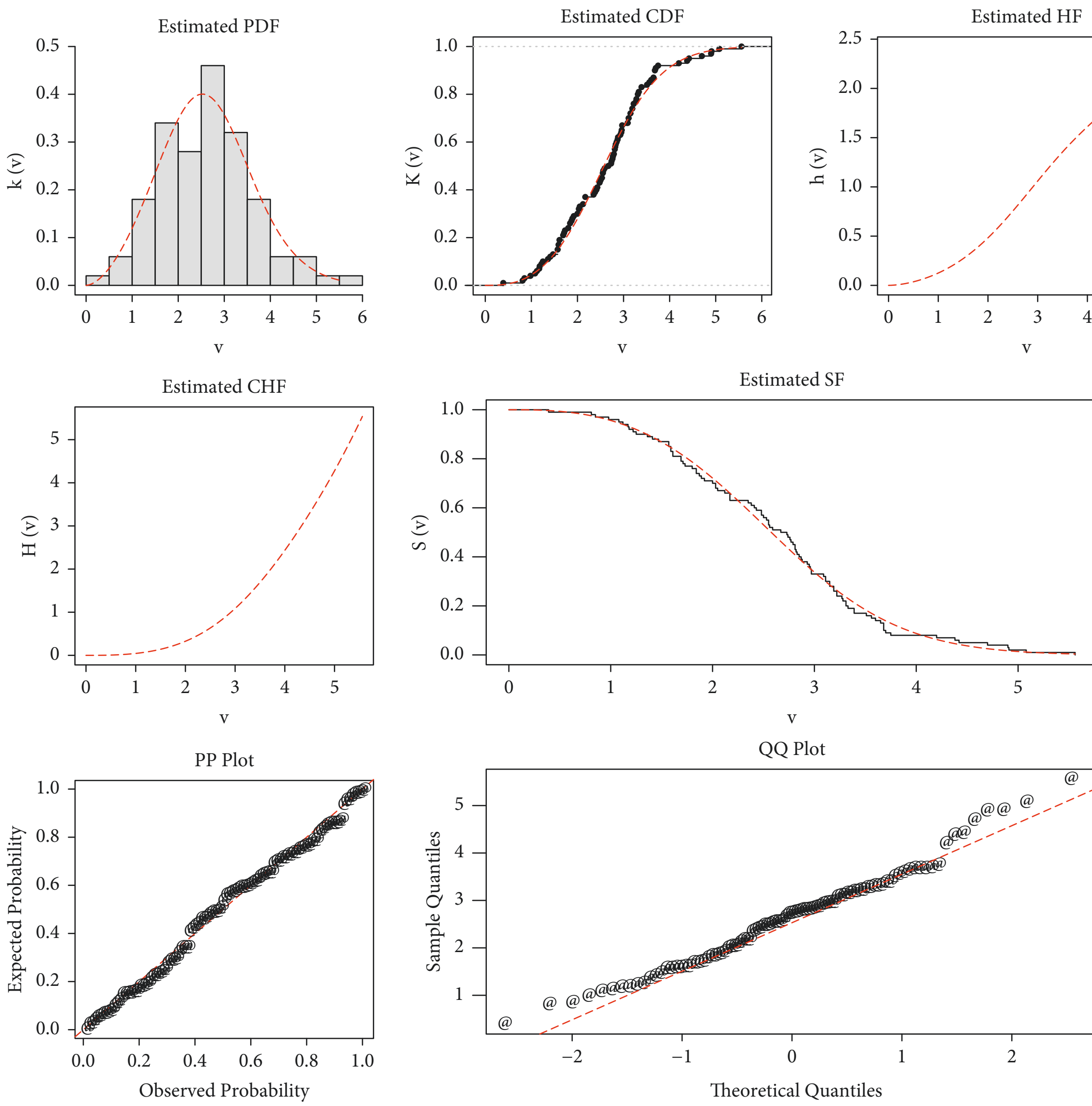

Figure 5: The estimated PDF, CDF, SF, HF, CHF, PP, and QQ plots of the proposed modeling using data 1.

data sets and compare its fitting results with the Weibull distribution, and its other versions include (*i*) Marshall–Olkin–Weibull (MO-Weibull), (*ii*) exponentiated Weibull (Exp-Weibull), and (*iii*) Kumaraswamy–Weibull (Kum-Weibull) distributions.

The DFs of the MO-Weibull, Exp-Weibull, and Kum-Weibull distributions are given, respectively, by

$$M(v;\beta,\mathbf{\Gamma}) = \frac{1-e^{-\sigma v^{\alpha}}}{1-(1-\beta)e^{-\sigma v^{\alpha}}}, \quad v \geq 0, \beta > 0, \alpha > 0, \sigma > 0,$$

$$M(v;a,\mathbf{\Gamma}) = \left(1-e^{-\sigma v^{\alpha}}\right)^{a}, \quad v \geq 0, a > 0, \alpha > 0, \sigma > 0, \tag{36}$$

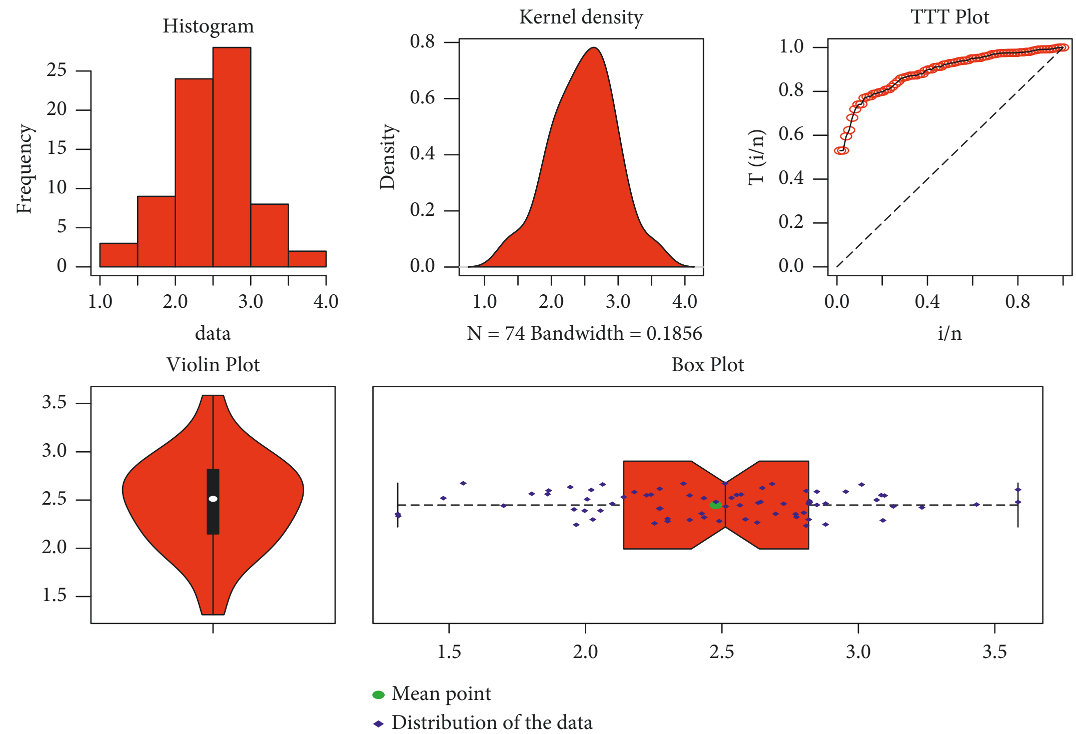

Figure 6: Some basic plots for data 2.

Table 5: The values of $\hat{\alpha}_{\mathrm{MLE}}, \hat{\sigma}_{\mathrm{MLE}}, \hat{\lambda}_{\mathrm{MLE}}, \hat{\beta}_{\mathrm{MLE}}, \hat{a}_{\mathrm{MLE}}$, and $\hat{b}_{MLE}$ of the fitted models using data 2.

Models	$\hat{\alpha}_{\mathrm{MLE}}$	$\hat{\sigma}_{\mathrm{MLE}}$	$\hat{\lambda}_{\mathrm{MLE}}$	$\hat{a}_{\mathrm{MLE}}$	$\hat{b}_{\mathrm{MLE}}$	$\hat{\beta}_{\mathrm{MLE}}$
NGen-Weibull	5.67621	0.00341	3.33031	—	—	—
Weibull	5.49996	0.00452	—	—	—	—
Exp-Weibull	4.27654	0.02298	—	1.86440	—	—
MO-Weibull	5.10675	0.00839	—	—	—	1.54105
Kum-Weibull	4.15566	0.01751	—	1.72203	1.62877	—

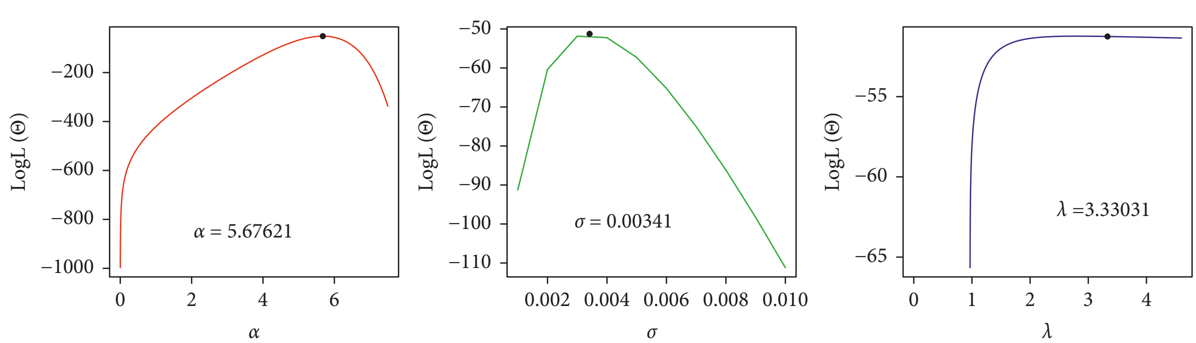

Figure 7: The profiles of the log-likelihood function of the MLEs of the proposed model using data 2.

$$M(v; a, b, \Gamma) = 1 - \left[1 - \left(1 - e^{-\sigma v^{\alpha}}\right)^{a}\right]^{b}, \quad v \geq 0, a > 0, b > 0, \alpha > 0, \sigma > 0. \tag{37}$$

Furthermore, we consider some analytical measures to compare the fitting results of the NGen-Weibull and other competing distributions. These measures include (i) AD (Anderson Darling) test, (ii) CM (Cramér-von Mises) test, and (iii) KS (Kolmogorov-Smirnov) test with p-value. The numerical values of these measures are computed as

$$-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log M(v_i) + \log\{1 - M(v_{n-i+1})\}\right],$$
$$\frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{2i-1}{2n} - M(v_i)\right]^{2}, \tag{38}$$

Table 6: The values of the analytical tests and P-value of the fitted models using data 2.

Models	CM	AD	KS	P-value
NGen-Weibull	0.02435	0.20724	0.05987	0.95350
Weibull	0.02619	0.23118	0.07003	0.86110
Exp-Weibull	0.02907	0.21912	0.06376	0.92430
MO-Weibull	0.02933	0.25807	0.07222	0.83480
Kum-Weibull	0.02663	0.20815	0.06117	0.94890

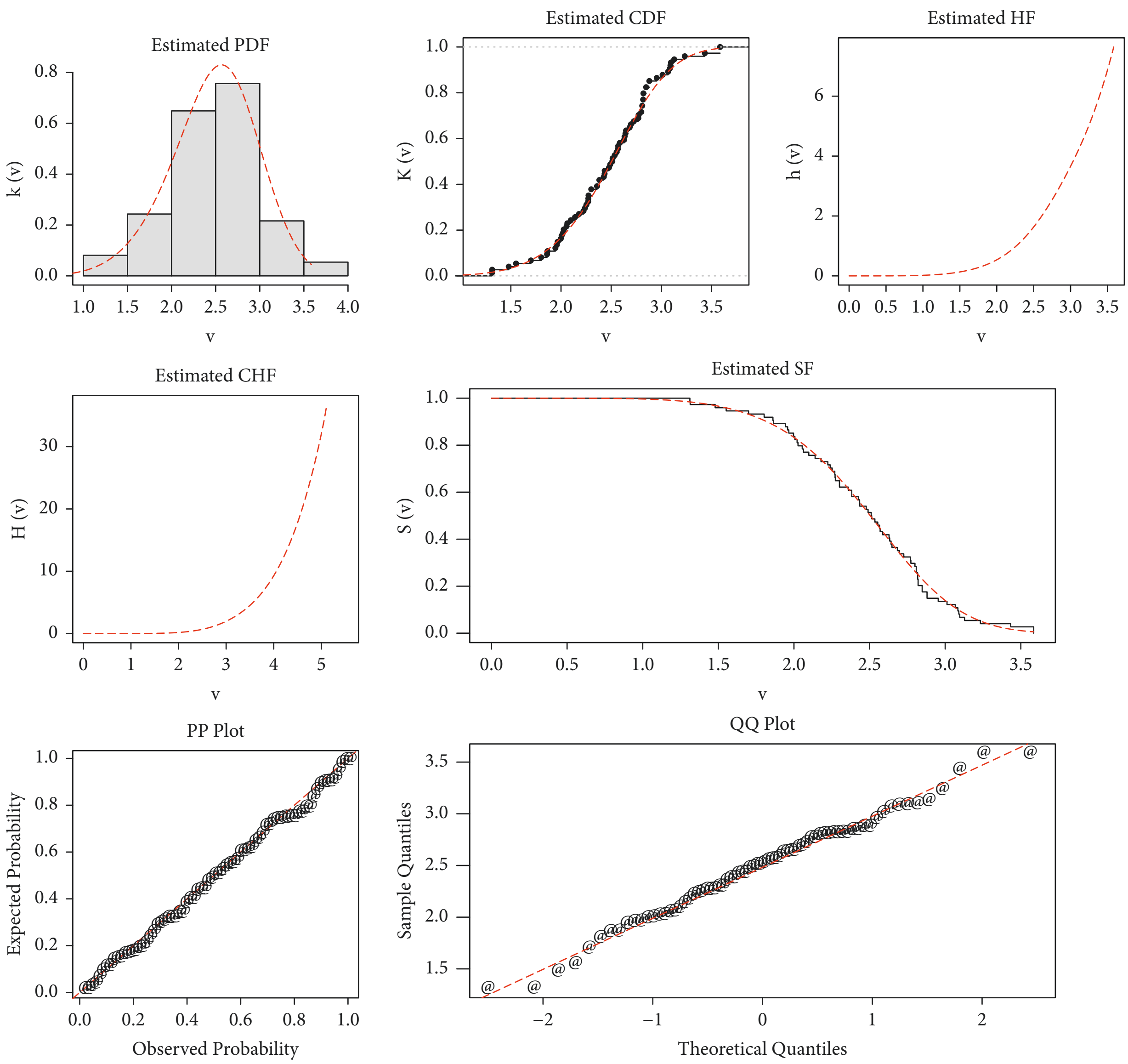

Figure 8: The estimated PDF, CDF, SF, HF, CHF, PP, and QQ plots of the proposed modeling for data 2.

$$\sup_v \left[M_n(v) - M(v) \right], \tag{39}$$

respectively.

5.1. Data 1. This subsection offers the first illustration of the NGen-Weibull distribution by analyzing a data set representing the breaking stress of carbon fibres (in Gba). Previously, several authors have also studied this data set. For example, (i) Barreto-Souza et al. [23] considered this data set using the Frechet, beta Frechet (B-Frechet), and exponentiated Frechet (Exp-Frechet) distributions, (ii) Ogunde et al. [24] analyzed this data set using the Nadarajah Haghighi Gompertz (NH-Gompertz) distribution, and (iii) Eghwerido et al. [25] studied this data set using the inverse odd Weibull (IO-Weibull) and a different variant of the Weibull distribution.

Some key measures of Data 1 are given by minimum = 0.390, 1st quartile = 1.840, median = 2.700, mean = 2.621, 3rd

quartile = 3.220, maximum = 5.560, skewness = 0.3681541, kurtosis = 3.104939, variance = 1.027964, and range = 5.17. Corresponding to Data 1, some useful plots are obtained in Figure 3.

Corresponding to Data 1, the values of the MLEs $\widehat{\alpha}_{\text{MLE}}, \widehat{\sigma}_{\text{MLE}}, \widehat{\lambda}_{\text{MLE}}, \widehat{\beta}_{\text{MLE}}$, and $\widehat{a}_{\text{MLE}}, \widehat{b}_{\text{MLE}}$ of the fitted models are obtained in Table 3. The profiles of the Log-LF of the MLEs of the NGen-Weibull model are plotted in Figure 4. Furthermore, the values of the analytical measures of the fitted models are presented in Table 4. Based on the numerical illustration presented in Table 4, we can see that the NGen-Weibull distribution is the best competing model.

Moreover, in support of the best fitting of the NGen-Weibull distribution using Data 1, a visual illustration of the NGen-Weibull distribution is presented in Figure 5. The plots in Figure 5 reveal that the NGen-Weibull distribution closely fits the fitted PDF, CDF, SF, HF, CFH, probability-probability (PP), and quantile-quantile (QQ) functions.

5.2. Data 2. This subsection offers the second illustration of the NGen-Weibull distribution by considering a data set representing the tensile strength, measured in GPa, of 69 carbon fibres. Previously, numerous authors have also analyzed this data set. For example, (i) Aldahlan [26] considered this data set using the Burr-X exponentiated exponential (B X EExp) distribution, (ii) Eyob et al. [27] studied this data set using the weighted quasi Akash (WQA) distribution, and (iii) Shukla and Shanker [28] analyzed this data set using the truncated Akash (TA) distribution.

The basic measures of Data 2 are given by minimum = 1.312, 1^{st} quartile = 2.150, median = 2.513, mean = 2.477, 3^{rd} quartile = 2.816, maximum = 3.585, skewness = -0.1541874, kurtosis = 2.95123, variance = 0.2378321, and range = 2.273. Corresponding to Data 2, some basic plots are presented in Figure 6.

Corresponding to data 2, the numerical values of the MLEs $\widehat{\alpha}_{\text{MLE}}, \widehat{\sigma}_{\text{MLE}}, \widehat{\lambda}_{\text{MLE}}, \widehat{a}_{MLE}$, and $\widehat{\beta}_{\text{MLE}}, \widehat{b}_{\text{MLE}}$ of the competing distributions are presented in Table 5. The profiles of the Log-LF of $\widehat{\alpha}_{\text{MLE}}, \widehat{\sigma}_{\text{MLE}}$, and $\widehat{\lambda}_{\text{MLE}}$ of the NGen-Weibull distribution are plotted in Figure 7. The values of the analytical measures of the competing distributions are obtained in Table 6. Based on the numerical findings in Table 6, we can see that the NGen-Weibull distribution has the smallest values of the analytical measures. This fact reveals that the NGen-Weibull model is the best model among the competing distributions.

After showing the best fitting of the NGen-Weibull distribution using Data 2 (see, Table 6), a graphical illustration is also provided in Figure 8. The plots of the fitted PDF, CDF, SF, HF, CFH, PP, and QQ functions are presented in Figure 8. These plots confirm the best fitting of the NGen-Weibull distribution.

6. Concluding Remarks

In this article, a new statistical approach for generating new probability distributions was introduced. The new statistical approach was named as NGen-M distributions. Based on the proposed NGen-M distributions method, an updated form of the Weibull distribution was introduced. The updated extension of the Weibull model was named as NGen-Weibull distribution. Certain distributional properties of the NGen-M distributions were derived. The parameters of the NGen-M distributions were estimated via implementing the maximum likelihood approach. A simulation study was also conducted to evaluate the estimators of the NGen-M distributions. In the end, the NGen-M distributions approach was illustrated by analyzing two data sets considered from the engineering sector. To both the engineering data sets, the NGen-Weibull distribution was applied in comparison with other competing models. Using certain statistical test (analytical measures), it is shown that the NGen-Weibull distribution was the best model.

In the future, we are motivated to implement the NGen-Weibull distribution for modeling real-life data sets in other fields such as the healthcare sector, hydrology, agriculture, metrology, geology, finance, banking, civil engineering, and education. We are also motivated to introduce the bivariate extension of the NGen-Weibull distribution for modeling the bivariate financial data sets such as import and export, among others.

Data Availability

The data sets are given in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] B. Pedrosa, J. A. F. O. Correia, C. A. S. Rebelo, and M. Veljkovic, “Reliability of fatigue strength curves for riveted connections using normal and weibull distribution functions,” *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering*, vol. 6, no. 3, Article ID 04020034, 2020.

[2] N. Bala and M. Napiah, “Fatigue life and rutting performance modelling of nanosilica/polymer composite modified asphalt mixtures using Weibull distribution,” *International Journal of Pavement Engineering*, vol. 21, no. 4, pp. 497–506, 2020.

[3] X. Huo, S. K. Khosa, Z. Ahmad, Z. Almaspoor, M. Ilyas, and M. Aamir, “A new lifetime exponential-X family of distributions with applications to reliability data,” *Mathematical Problems in Engineering*, vol. 2020, Article ID 1316345, 1 page, 2020.

[4] Q. Liao, Z. Ahmad, E. Mahmoudi, and G. G. Hamedani, “A new flexible bathtub-shaped modification of the Weibull model: properties and applications,” *Mathematical Problems in Engineering*, vol. 202011 pages, Article ID 3206257, 2020.

[5] M. H. Shu, C. W. Wu, B. M. Hsu, and T. C. Wang, “Lifetime performance-qualified sampling system under a Weibull distribution with failure-censoring,” *Quality Engineering*, vol. 33, no. 3, pp. 404–416, 2021.

[6] L. Li, J. Guan, P. Yuan, Y. Yin, and Y. Li, “A Weibull distribution-based method for the analysis of concrete fracture,” *Engineering Fracture Mechanics*, vol. 256, Article ID 107964, 2021.

[7] M. Bilal, M. Mohsin, and M. Aslam, "Weibull-exponential distribution and its application in monitoring industrial process," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6650237, 1 page, 2021.

[8] S. Nadarajah, G. M. Cordeiro, and E. M. M. Ortega, "The exponentiated Weibull distribution: a survey," *Statistical Papers*, vol. 54, no. 3, pp. 839–877, 2013.

[9] S. J. Almalki and S. Nadarajah, "Modifications of the Weibull distribution: a review," *Reliability Engineering & System Safety*, vol. 124, pp. 32–55, 2014.

[10] P. Wais, "A review of Weibull functions in wind sector," *Renewable and Sustainable Energy Reviews*, vol. 70, pp. 1099–1107, 2017.

[11] Z. Ahmad, M. Elgarhy, and G. G. Hamedani, "A new Weibull-X family of distributions: properties, characterizations and applications," *Journal of Statistical Distributions and Applications*, vol. 5, no. 1, pp. 5–18, 2018.

[12] W. Zhao, S. K. Khosa, Z. Ahmad, M. Aslam, and A. Z. Afify, "Type-I heavy tailed family with applications in medicine, engineering and insurance," *PLoS One*, vol. 15, no. 8, Article ID e0237462, 2020.

[13] W. Wang, Z. Ahmad, O. Kharazmi, C. B. Ampadu, E. H. Hafez, and M. M. Mohie El-Din, "New generalized-x family: modeling the reliability engineering applications," *PLoS One*, vol. 16, no. 3, Article ID e0248312, 2021.

[14] H. S. Klakattawi, "Survival analysis of cancer patients using a new extended Weibull distribution," *PLoS One*, vol. 17, no. 2, Article ID e0264229, 2022.

[15] Y. Wang, Q. Cheng, M. Wang, and Z. Liu, "Weibull distribution-based neural network for stochastic capacity estimation," *Journal of Transportation Engineering, Part A: Systems*, vol. 148, no. 4, Article ID 04022009, 2022.

[16] T. Wang, X. He, J. Wang, and Y. Li, "Detail fatigue rating method based on bimodal weibull distribution for DED Ti-6.5 Al-2Zr-1Mo-1V titanium alloy," *Chinese Journal of Aeronautics*, vol. 35, no. 4, pp. 281–291, 2022.

[17] M. M. Al-Sobhi, "The extended Weibull distribution with its properties, estimation and modeling skewed data," *Journal of King Saud University Science*, vol. 34, no. 2, Article ID 101801, 2022.

[18] F. J. Gonzalez, J. I. Gonzalez, S. Soler, C. E. Repetto, B. J. Gómez, and D. B. Berdichevsky, "New procedure to estimate plasma parameters through the q-Weibull distribution by using a Langmuir probe in a cold plasma," *Plasma Research Express*, vol. 4, no. 1, Article ID 015003, 2022.

[19] W. Emam and Y. A. Tashkandy, "Khalil new generalized weibull distribution based on ranked samples: estimation, mathematical properties, and application to COVID-19 data," *Symmetry*, vol. 14, no. 5, pp. 853, 2022.

[20] F. Prataviera, "Reparameterized flexible Weibull distribution with some applications," *American Journal of Mathematical and Management Sciences*, vol. 41, no. 3, pp. 259–277, 2022.

[21] H. Rehman, N. Chandra, F. S. Hosseini-Baharanchi, A. R. Baghestani, and M. A. Pourhoseingholi, "Cause-specific hazard regression estimation for modified Weibull distribution under a class of non-informative priors," *Journal of Applied Statistics*, vol. 49, no. 7, pp. 1784–1801, 2022.

[22] E. Seneta, "Karamata's characterization theorem, feller and regular variation in probability theory," *Publications de l'Institut Mathématique*, vol. 71, no. 85, pp. 79–89, 2002.

[23] W. Barreto-Souza, G. M. Cordeiro, and A. B. Simas, "Some results for beta Fréchet distribution," *Communications in Statistics - Theory and Methods*, vol. 40, no. 5, pp. 798–811, 2011.

[24] A. A. Ogunde, I. O. Ajao, and G. A. Olalude, "On the application of Nadarajah Haghighi Gompertz distribution as a life time distribution," *Open Journal of Statistics*, vol. 10, no. 5, pp. 850–862, 2020.

[25] J. T. Eghwerido, J. D. Ikwuoche, and O. D. Adubisi, "Inverse odd Weibull generated family of distribution," *Pakistan Journal of Statistics and Operation Research*, vol. 16, no. 3, pp. 617–633, 2020.

[26] M. A. Aldahlan, "A new three-parameter lifetime distribution: properties and applications," *International Journal of Innovation in Science and Mathematics*, vol. 7, no. 2, pp. 54–66, 2019.

[27] T. Eyob, R. Shanker, K. K. Shukla, and T. A. Leonida, "Weighted quasi Akash distribution: properties and applications," *American Journal of Mathematics and Statistics*, vol. 9, no. 1, pp. 30–43, 2019.

[28] K. K. Shukla and R. Shanker, "Truncated Akash distribution: properties and applications," *Biometrics & Biostatistics International Journal*, vol. 9, no. 5, pp. 179–184, 2020.