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Research Article

Blow-Up Solution in Logarithmic Sensitivity and Indirect Signal Generation Keller–Segel System

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In this paper, we consider the following indirect signal generation and logarithmic sensitivity $\begin{cases} \varepsilon n_t = \Delta n - \chi \nabla \cdot (n \nabla \ln c) & x \in \Omega, t > 0 \\ c_t = \Delta c - c + w & x \in \Omega, t > 0 \\ w_t = \Delta w - w + n & x \in \Omega, t > 0 \end{cases}$ under homogeneous Neumann boundary conditions in a ball domain $\Omega \subset R^N$ $(N \geq 4)$ with smooth boundary $\partial\Omega$. This paper considers in the singular limit $\varepsilon \longrightarrow 0$; the result comes from the finite time blow-up of arbitrary large values of n in the corresponding nonlocal scalar parabolic equation case when $N \geq 4$ and $\chi > 2N/N - 2$.

1. Introduction

Keller and Segel in [1] proposed the following fully parabolic equations to describe the aggregation of certain types of bacteria.

$$\begin{cases} \tau_n n_t = \Delta n - \nabla \cdot (n \chi(n, c) \nabla c), & x \in \Omega, t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega, t > 0, \end{cases} \tag{1}$$

where the unknowns $n = n(t, x)$ and $c(t, x)$ denote the cell density and the concentration of chemical substances, respectively. The given function $\chi(n, c)$ represents the chemosensitivity function and physical domain $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ is a bounded domain with a smooth boundary. This model describes a biological process in which cells move towards their preferred environment and the signal is produced by the cells themselves. When the diffusion of chemical signals is much faster than that of cells, the system can be simplified as follows:

$$\begin{cases} \tau_n n_t = \Delta n - \nabla \cdot (n \chi(n, c) \nabla c), & x \in \Omega, t > 0, \\ 0 = \Delta c - c + n, & x \in \Omega, t > 0. \end{cases} \tag{2}$$

For its rigorous mathematical proof, we can see in [2]. Recently, Li et al. in [3] have considered the stability analysis of the Keller–Segel model under fluid action in the two-dimensional case and has given the corresponding numerical experiments. For more references about the chemotaxis-fluid system, the corresponding global solvability of classical solutions has been investigated by [4–12] in two or three-dimensional situations. We also mention complicated variants, e.g., involving rotational flux [13–20] and logistic source terms [21–26] as well as nonlinear diffusion [4, 9, 16, 27–32].

Another important chemotaxis model is formed with a singular sensitivity function, such as $\chi(n, c) = \chi/c$. This model is proposed by the Weber-Fechner law of stimulus perception [33] and supported by experimental [34] and theoretical evidence [35]. This fully parabolic logarithmic Keller–Segel system evidently lacks some good structures, which weakens the corresponding analysis skills. It is worth noting that this knowledge seems very fragmented, but it is essentially reduced to the relevant initial boundary value problems, and the assumptions allowing global solvability are based on $\tau_n > 0$. When the dimension $N = 1$, there is a globally bounded smooth solution for any initial data [36]. The same conclusion is $N = 2$ and $\chi \leq \chi_0$ with some $\chi_0 > 1.015$[37] or $N \geq 2$ and $\chi < \sqrt{2/N}$ ([38–42]). In addition, some globally generalized solutions involved in general geometry [43] with some $\chi < \sqrt{N+2/3N-4}$ and in radially symmetric settings [44] with some $\chi < N/N-2$. Accordingly, the integrable global solutions of nonradial symmetry under the assumption when $N=3$ and $\chi < \sqrt{8}$ or $2 \leq N \neq 3$ and $\chi < N/N-2$[45]. In the similar parabolic-elliptic case, removing the technical assumption under the three-dimensional condition can also prove the global existence and integrability of the solution in the nonradial case when $N \geq 2$ and $\chi < N/N-2$[46], the corresponding classical solution is obtained when $N=2$ and $\chi > 0$ or $N \geq 3, \chi < 2/N-2$[47]. For the quasilinear chemotaxis-Navier–Stokes system of this problem, there are lots of good results in [48–53].

On the other hand, based on the simplification of the scalar parabolic equation, it can be shown in [54] that the system (1) of parabolic-elliptic ($\tau_n = 0$) allows the radial solution to blow-up in a finite time if $N \geq 3$ and $\chi > 2N/N-2$. And, through the result of the global measure expansion of the radial solution of the classical Keller–Segel system beyond blow-up in [55], it can be inferred that there is no global L^1-solution in this parameter region. The research on blow-up model has a strong physical background, such as gash healing, expansion, and collapse of geometric flow and energy released by stars in the universe.

An indirect signal generation without sensitivity function is also a very important Keller–Segel types model. Lin et al. [56] established the global existence and large-time behavior in $\Omega \subset \mathbb{R}^N, N = 2, 3$. After Wu et al. in [57] added the singular term, investigated the global boundedness and large-time behavior of the above-given problem. The global existence for $N = 2, 3$ and blow-up solutions for $N \geq 4$ were studied by Fujie and Senba in [58]. Tao and Wang [59] considered the global solvability, boundedness, blow-up, existence of nontrivial stationary solutions, and asymptotic behavior. Stinner et al. [60] gave the global existence and some basic boundedness of weak solutions for a PDE-ODE system. Li and Li [61] considered the blow-up of nonradial solutions of the parabolic-elliptic-elliptic model in two dimensions. Recently, Viglialoro [62] has investigated explicit low bounded of blow-up time for a chemotaxis system. Chiyo et al. [63] studied the blow-up phenomena of a chemotaxis system with superlinear logistic degradation in $\Omega \subset \mathbb{R}^N, N \geq 3$.

Because the more delicate analytical technique of limit in the fully parabolic framework with the logarithmic term when τ_n suitably small, Winkler [64] considered how far the condition of the chemosensitivity χ plays a role in the limit process of the system (1) for $\tau_n \longrightarrow 0$. To motivate this idea, we study the following fully parabolic equations of the indirect signal.

$$\begin{cases} \varepsilon n_{\varepsilon t} = \Delta n_\varepsilon - \chi \nabla \cdot \left(n_\varepsilon \nabla \ln c_\varepsilon\right), & x \in \Omega, t > 0, \\ c_{\varepsilon t} = \Delta c_\varepsilon - c_\varepsilon + w_\varepsilon, & x \in \Omega, t > 0, \\ w_{\varepsilon t} = \Delta w - w_\varepsilon + n_\varepsilon, & x \in \Omega, t > 0, \end{cases} \tag{3}$$

where the parameter χ is a positive constant, $\Omega \subset \mathbb{R}^N \, (N \geq 4)$ is a ball, under the assumption of the no-flux Neumann boundary condition for n, c and w, i.e.,

$$\frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = \frac{\partial w_\varepsilon}{\partial \nu} = 0, \quad x \in \partial\,\Omega,\ t > 0, \tag{4}$$

where ν is the unit outward normal vector on $\partial\Omega$ and of the initial conditions.

$$n(x,0) = n_0(x), c(x,0) = c_0(x), w(x,0) = w_0(x), \quad x \in \Omega. \tag{5}$$

satisfying

$$\begin{cases} 0 \leq n_0(x) \in C^0(\overline{\Omega}), & n_0(x) \equiv 0, x \in \overline{\Omega}, \\ c_0(x) \in W^{1,\infty}(\Omega)\text{snonnegative}, & \inf\limits_{x \in \overline{\Omega}} c_0(x) > 0, \\ w_0 \in W^{1,\infty}(\Omega)\text{isnonnegative}, & \inf\limits_{x \in \overline{\Omega}} w_0(x) > 0. \end{cases} \tag{6}$$

Let $m := \|n_0\|_{L^1(\Omega)}$. Then, m is a positive constant. The goal is to establish the identity of system (3) under the limit version in Section 3.

$$n = \frac{mc^\chi}{\|c^\chi\|_{L^1(\Omega)}}. \tag{7}$$

Using the obtained identity (7) and the variation-of-constants formula, we can obtain the following equation:

$$w(\cdot,t) = e^{t(\Delta-1)} w_0 + m \int_0^t e^{(t-s)(\Delta-1)} \frac{c^\chi(\cdot,s)}{\|c^\chi(\cdot,s)\|_{L^1(\Omega)}} \mathrm{d}s. \tag{8}$$

and

$$c_t = \Delta c - c + e^{t(\Delta-1)} w_0 + m \int_0^t e^{(t-s)(\Delta-1)} \frac{c^\chi(\cdot,s)}{\|c^\chi(\cdot,s)\|_{L^1(\Omega)}} \mathrm{d}s. \tag{9}$$

Applying the scalar parabolic problems of the type obtained in (9), the analysis method of a well-known nonlocal parabolic problems for suitable chosen radial initial data c_0 the respective limit function c should be blow-up after some finite time whenever $\chi > 2N/N-2$ in Section . Next, we give the following theorem.

Theorem 1. *Let $N \geq 4$ and $\Omega := B_R(0) \subset \mathbb{R}^N$ with some $R > 0$, and assume $\chi > 2N/N-2$ and $m > 0$. Then, there exist $c_0, w_0 \in W^{1,\infty}$ and $T > 0$ such that $c_0 > 0, w_0 > 0$ in $\overline{\Omega}$, and such that for any nonnegative $n_0 \in C^0(\overline{\Omega})$ with $\int_\Omega n_0 = m$ and each $\varepsilon \in (0,1)$), it is possible to choose $T_\varepsilon \in (0,T]$ and functions $n_\varepsilon, c_\varepsilon$ and w_ε belonging to $C^0(\overline{\Omega} \times [0,T_\varepsilon)) \cap C^{2,1}(\overline{\Omega} \times [0,T_\varepsilon)))$ such that $n_\varepsilon \geq 0$ and $c_\varepsilon > 0, w_\varepsilon > 0$ in $\overline{\Omega} \times [0,T_\varepsilon)$, that $(n_\varepsilon, c_\varepsilon, w_\varepsilon)$ solves (3) classically in $\overline{\Omega} \times [0,T_\varepsilon)$, and that*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{t \in (0,T_\varepsilon)} \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} = \infty \text{ for all } p > \frac{N}{4}. \tag{10}$$

and especially, giving any $M > 0$ and $\varepsilon > 0$ we can find $\varepsilon \in (0,\varepsilon_0), x_\star \in \Omega$ and $t_\star \in (0,T_\varepsilon)$ such that

$$n(x_\star, t_\star) \geq M. \tag{11}$$

2. The Limit Procedure $\varepsilon \longrightarrow 0$ in (3)

2.1. Local Existence and Conditional ε-Independent Estimates. Firstly, we give the well-established the local existence of a classical solution to (3) for each fixed $\varepsilon > 0$, along with a convenient extensibility criterion.

Lemma 1. *Let $N \geq 4$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and let $\chi > 0$ and $\varepsilon > 0$. Then, for any choice of n_0, c_0 and w_0 satisfying (6), there are $T_{\max,\varepsilon} \in (0,\infty]$ and a uniquely determined pair $(n_\varepsilon, c_\varepsilon, w_\varepsilon)$ of functions.*

$$\begin{cases} n_\varepsilon \in C^0\left(\overline{\Omega} \times \left[0, T_{\max,\varepsilon}\right)\right) \cap C^{2,1}\left(\overline{\Omega} \times \left(0, T_{\max,\varepsilon}\right)\right), \\ c_\varepsilon \in \bigcap_{r>N} C^0\left(\left[0, T_{\max,\varepsilon}; W^{1,r}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times \left(0, T_{\max,\varepsilon}\right)\right), , \\ w \in C^0\left(\overline{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,\infty)\right) \end{cases} \tag{12}$$

such that $n_\varepsilon \geq 0, c_\varepsilon > 0$ and $w_\varepsilon > 0$ in $\overline{\Omega} \times [0, T_{\max,\varepsilon})$, that $(n_\varepsilon, c_\varepsilon, w_\varepsilon)$ solves (3) classically in $\Omega \times (0, T_{\max,\varepsilon})$, and that

$$\text{if } T_{\max,\varepsilon} < \infty, \text{then } \limsup_{t \longrightarrow T_{\max,\varepsilon}} \left\| n_\varepsilon(\cdot,t) \right\|_{L^p(\Omega)} = \infty \text{ for all } p > \frac{N}{4}. \tag{13}$$

Proof. We can use the local existence and extensibility to complete the proof of Lemma 1. We can refer to literature ([36], Lemma 10, Lemma 3.1 and 3.2) for relevant details.

The following lemma is helpful to prove the upper bounded of w_ε. □

Lemma 2. *([60], Lemma 3.4) Let $T > 0$, and suppose that z is a nonnegative absolutely continuous function on $[0,T)$ satisfying*

$$z'(t) + az(t) \leq f(t) \text{ for } a.e. t \in (0,T). \tag{14}$$

With some $a > 0$ and a nonnegative function $f \in L^1_{loc}([0,T))$ for which there exists $b > 0$ such that

$$\int_t^{t+1} f(s)\mathrm{d}s \leq b \text{ for all } t \in [0, T-1). \tag{15}$$

Then,

$$z(t) \leq \max\left\{z(0) + b, \frac{b}{a} + 2b\right\} \quad \text{for all } t \in (0,T). \tag{16}$$

In what follows, we let $(T_{\max,\varepsilon})_{\varepsilon\in(0,1)}$ and $((n_\varepsilon, c_\varepsilon, w_\varepsilon))_{\varepsilon\in(0,1)}$ be as obtained in Lemma 1. Next, we assume that $p > N/4$ and $T > 0$ have the following properties.

$$\begin{aligned} &T_{\max,\varepsilon} \geq T \text{ for all } \varepsilon \in \left(0, \varepsilon_*\right), \\ &\sup_{\varepsilon\in\left(0,\varepsilon_*\right) t\in(0,T)} \left\| n_\varepsilon(\cdot,t) \right\|_{L^p(\Omega)} < \infty. \end{aligned} \tag{17}$$

We will give the pointwise lower estimates of c_ε and w_ε, which plays an important role in the full text.

Lemma 3. *If (A) holds with some $T > 0, p > N/4$ and $\varepsilon_* \in (0,1)$, then there exist $C > 0$ and $\delta_0 > 0$ such that*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \text{ for all } t \in \left(0, T_{\max,\varepsilon}\right). \tag{18}$$

and

$$\|w(\cdot,t)\|_{L^1(\Omega)} \leq C. \tag{19}$$

as well as

$$\min\left\{c_\varepsilon(x,t), w_\varepsilon(x,t)\right\} \geq \delta_0 \quad \text{for all } x \in \Omega, t \in (0,T), \varepsilon \in \left(0, \varepsilon_*\right). \tag{20}$$

Proof. Integrating the first equation of (3), we can obtain (18). Then, integrating the third equation of (3), we have the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w_\varepsilon + \int_\Omega w_\varepsilon = \int_\Omega n_\varepsilon. \tag{21}$$

Next, we apply Lemma 2 and (18) to establish (19). Therefore, using the convexity of Ω and comparison argument ([65], Lemma 4), the following Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ has properties.

$$e^{t\Delta}\psi \geq c_1 \int_\Omega \psi \text{ in } \Omega \quad \text{for all } t > 1. \tag{22}$$

In order to get the pointwise lower estimate appropriately, we employ a variation-of-constants representation of c_ε and w_ε to see that

$$\begin{aligned} w_\varepsilon(\cdot,t) &= e^{-t}e^{t\Delta}w_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}n_\varepsilon(\cdot,s)\mathrm{d}s, \\ &\geq e^{-t}\cdot\inf_\Omega w_0 + \int_{(t-2)_+}^{(t-1)_+} e^{-(t-s)}\cdot\left\{c_1\int_\Omega n_0\right\}\mathrm{d}s, \\ &\geq \max\left\{e^{-2}\inf_\Omega w_0, c_1 e^{-2}\int_\Omega n_0\right\}, \\ &\coloneqq \delta_1 \text{ for all } t \in (0,T), \varepsilon \in \left(0, \varepsilon_*\right), \end{aligned} \tag{23}$$

where $\delta_1 > 0$ because $\inf_\Omega w_0$ and $\int_\Omega n_0$ are positive by (6). Similarly, we have the following equation:

$$\begin{aligned} c_\varepsilon(\cdot,t) &= e^{-t}e^{t\Delta}c_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}w_\varepsilon(\cdot,s)\mathrm{d}s, \\ &\geq e^{-t}\cdot\inf_\Omega c_0 + \int_{(t-2)_+}^{(t-1)_+} e^{-(t-s)}\cdot\left\{c_1\int_\Omega w_\varepsilon\right\}\mathrm{d}s, \\ &\geq \max\left\{e^{-2}\inf_\Omega c_0, c_1 e^{-2}\delta_1|\Omega|\right\}, \\ &\coloneqq \delta_2 \text{ for all } t \in (0,T), \varepsilon \in \left(0, \varepsilon_*\right), \delta_2 > 0. \end{aligned} \tag{24}$$

Taking $\delta_0 \coloneqq \min\{\delta_1, \delta_2\}$, this entails (20). □

Lemma 4. *If (A) holds with some $T > 0, p > N/4$ and $\varepsilon \in (0,1)$, then there exist $q > N/2$ and $C > 0$ such that*

$$\left\|w_\varepsilon(\cdot,t)\right\|_{L^q(\Omega)} \leq C \quad \text{for all } t \in (0,T), \text{each } \varepsilon \in \left(0, \varepsilon_*\right). \tag{25}$$

Proof. Without loss of generality, we may assume that $p < N/2$. Using the variation-of-constants formula and the estimate of the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ on Ω [66] give the following equation:

$$\begin{aligned}\|w_\varepsilon(\cdot,t)\|_{L^q(\Omega)} &= \left\|e^{t(\Delta-1)}w_0 + \int_0^t e^{(t-s)(\Delta-1)}n_\varepsilon(\cdot,s)\mathrm{d}s\right\|_{L^q(\Omega)},\\ &\leq c_1 + c_1\int_0^t\left(1+(t-s)^{-N/2(1/p-1/q)}\right)e^{-\lambda(t-s)},\\ &\quad \|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)}\mathrm{d}s.\end{aligned} \tag{26}$$

Since $1/p - 2/N < 2/N$, we have $N/2(1/p - 1/N/2) < 1$. So we can take $q > N/2$ such that $N/2(1/p - 1/q) < 1$. Therefore $c_2 := \int_0^\infty (1+\sigma^{-N/2(1/p-1/q)})e^{-\lambda\sigma}\mathrm{d}\sigma < \infty$, this ensures that

$$\begin{aligned}&\|w_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq c_1 + c_1c_2\sup_{t\in(0,T)}\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)},\\ &\quad\text{for all } t\in(0,T), \varepsilon\in(0,\varepsilon_*).\end{aligned} \tag{27}$$

This completes the proof of Lemma 4. □

Lemma 5. *Suppose that* (A) *holds with some* $T>0$, $p>N/4$ *and* $\varepsilon\in(0,1)$. *Then there exist* $r>N$ *and* $C>0$ *such that*

$$\|c_\varepsilon(\cdot,t)\|_{W^{1,r}(\Omega)} \leq C \text{ for all } t\in(0,T), \textit{each } \varepsilon\in(0,\varepsilon_*). \tag{28}$$

Proof. For simplicity of expression from Lemma 4, we assume that $N/2<q<n$. Then we can fix $N<r<Nq/N-q$ to find $c_3>0$ such that for all $\varepsilon\in(0,\varepsilon_*)$,

$$\begin{aligned}\|c_\varepsilon(\cdot,t)\|_{W^{1,r}(\Omega)} &= \left\|e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-s)(\Delta-1)}w_\varepsilon(\cdot,s)\mathrm{d}s\right\|_{W^{1,r}(\Omega)},\\ &\leq c_3 + c_3\int_0^t\left(1+(t-s)^{-1/2-N/2(1/q-1/r)}\right),\\ &\quad e^{-\lambda(t-s)}\|w_\varepsilon(\cdot,s)\|_{L^q(\Omega)}\mathrm{d}s.\end{aligned} \tag{29}$$

where $1/2 + N/2(1/q-1/r) < 1/2 + N/2(1/q - N - q/Nq) = 1$. This entails that

$$\begin{aligned}&\|c_\varepsilon(\cdot,t)\|_{W^{1,r}(\Omega)} \leq c_3 + c_2c_3\sup_{t\in(0,T)}\|w_\varepsilon(\cdot,t)\|_{L^q(\Omega)},\\ &\quad\text{for all } t\in(0,T), \varepsilon\in(0,\varepsilon_*).\end{aligned} \tag{30}$$

Therefore, applying the above inequality and Lemma 4, we can obtain Lemma 5. □

Lemma 6. *Suppose that (A) with some* $T>0$, $p>N/4$ *and* $\varepsilon_*\in(0,1)$. *Then there exists* $C>0$ *such that*

$$\int_0^T\int_\Omega |\nabla n_\varepsilon|^2 \leq C \quad \text{for all } \varepsilon\in(0,\varepsilon_*). \tag{31}$$

Proof. We multiply the first equation of the system (4), integrate by parts and use Hölder's inequality to deduce that

$$\begin{aligned}\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^2 + \int_\Omega|\nabla n_\varepsilon|^2 &= \chi\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla n_\varepsilon\cdot\nabla c_\varepsilon,\\ &\leq \frac{1}{2}\int_\Omega|\nabla n_\varepsilon|^2 + \frac{\chi^2}{2}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon^2}|\nabla c_\varepsilon^2|.\end{aligned} \tag{32}$$

Using Gagliardo–Nirenberg inequality and Young's inequality to the second term at the right end of formula (32), we can obtain the following equation:

$$\begin{aligned}\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^2 + \int_\Omega|\nabla n_\varepsilon|^2 &\leq c_5\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^2,\\ &\leq c_5\left\|n_\varepsilon^2\right\|_{L^{r/r-2}(\Omega)}\left\||\nabla c_\varepsilon|^2\right\|_{L^{r/2}(\Omega)},\\ &\leq c_6\|\nabla n_\varepsilon\|_{L^2(\Omega)}^{2N(r+2)/(N+2)r}\|n_\varepsilon\|_{L^1(\Omega)}^{4(r-N)/(N+2)r}\\ &\quad + c_6\|n_\varepsilon\|_{L^1(\Omega)}^2,\\ &\leq \frac{1}{2}\int_\Omega|\nabla n_\varepsilon|^2 + c_7.\end{aligned} \tag{33}$$

That is,

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\varepsilon^2 + \frac{1}{2}\int_\Omega|\nabla n_\varepsilon|^2 \leq c_7. \tag{34}$$

Therefore, integrating the two sides of the above-given inequality, we have the following equation:

$$\begin{aligned}&\varepsilon\int_\Omega n_\varepsilon^2 + \frac{1}{2}\int_0^t\int_\Omega|\nabla n_\varepsilon|^2 \leq \varepsilon\int_\Omega n_0^2 + c_7t,\\ &\quad\text{for all } t\in(0,T), \varepsilon\in(0,\varepsilon_*).\end{aligned} \tag{35}$$

which together with (6) establishes Lemma 6. □

Lemma 7. *If (A) is valid with some* $T>0$, $p>N/4$ *and* $\varepsilon_*\in(0,1)$, *then there exists* $C>0$ *such that*

$$\int_0^T\int_\Omega|\Delta c_\varepsilon|^2 \leq C \quad \text{for all } \varepsilon_*\in(0,\varepsilon_*). \tag{36}$$

and

$$\int_0^T\int_\Omega c_{\varepsilon t}^2 \leq C \quad \text{for all } \varepsilon_*\in(0,\varepsilon_*). \tag{37}$$

Proof. Multiplying the third equation of (3) with w and making use of the integration by parts, we have the following equation:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla w_\varepsilon|^2 + \int_\Omega|\nabla w_\varepsilon|^2 + \int_\Omega w_\varepsilon^2 &\leq \int_\Omega n_\varepsilon w_\varepsilon,\\ &\leq \|n_\varepsilon\|_{L^2(\Omega)}\|w_\varepsilon\|_{L^2(\Omega)},\\ &\leq \frac{3}{4}\int_\Omega n_\varepsilon^2 + \frac{1}{3}\int_\Omega w_\varepsilon^2.\end{aligned} \tag{38}$$

Taking the L^2 inner product of $(4)_2$ with $-\Delta c_\varepsilon$ and using the integration by parts, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|\nabla c_{\varepsilon}|^{2}+\int_{\Omega}|\Delta c_{\varepsilon}|^{2}=-\int_{\Omega}|\nabla c_{\varepsilon}|^{2}-\int_{\Omega}w_{\varepsilon}\Delta c_{\varepsilon},$$
$$\leq -\int_{\Omega}|\nabla c_{\varepsilon}|^{2}+\frac{3}{4}\int_{\Omega}|\Delta c_{\varepsilon}|^{2}+\frac{1}{3}\int_{\Omega}w_{\varepsilon}^{2}. \quad (39)$$

We multiply the second equation of (3) with c_ε and $c_{\varepsilon t}$, respectively, then add them together and use Young's inequality to get the following equation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega}c_{\varepsilon}^{2}+\int_{\Omega}|\nabla c_{\varepsilon}|^{2}\right\}+\int_{\Omega}c_{\varepsilon t}^{2}=\int_{\Omega}c_{\varepsilon t}w_{\varepsilon},$$
$$\leq\frac{3}{4}\int_{\Omega}c_{\varepsilon t}^{2}+\frac{1}{3}\int_{\Omega}w_{\varepsilon}^{2}. \quad (40)$$

Combining with (39)–(41), we have the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega}c_{\varepsilon}^{2}+2\int_{\Omega}|\nabla c_{\varepsilon}|^{2}+\int_{\Omega}|\nabla w_{\varepsilon}|^{2}\right\},$$
$$+\frac{1}{2}\int_{\Omega}c_{\varepsilon t}^{2}+2\left\{\int_{\Omega}|\nabla c_{\varepsilon}|^{2}+|\nabla w_{\varepsilon}|^{2}\right\}, \quad (41)$$
$$+\frac{1}{2}\int_{\Omega}|\Delta c_{\varepsilon}|^{2}\leq\frac{3}{2}\int_{\Omega}n_{\varepsilon}^{2}\quad \text{for all } \varepsilon\in(0,\varepsilon_{*}).$$

We can employ the Gagliardo-Nirenberg inequality together with $m := \int_\Omega n_0$ to obtain the following equation:

$$\frac{3}{2}\int_{\Omega}n_{\varepsilon}^{2}\leq c_{8}\|\nabla n_{\varepsilon}\|_{L^{2}(\Omega)}^{2N/N+2}\|n_{\varepsilon}\|_{L^{1}(\Omega)}^{4/N+2}+c_{4}\|n_{\varepsilon}\|_{L^{1}(\Omega)}^{2},$$
$$\leq\frac{1}{2}\int_{\Omega}|\nabla n_{\varepsilon}|^{2}+\left(c_{8}+\frac{1}{2}c_{8}^{N+2/2}\right)m^{2}, \quad (42)$$

This ensures that

$$\int_{\Omega}c_{\varepsilon}^{2}+2\int_{\Omega}|\nabla c_{\varepsilon}|^{2}+\int_{\Omega}|\nabla w_{\varepsilon}|^{2}+\frac{1}{2}\int_{0}^{t}\int_{\Omega}c_{\varepsilon t}^{2},$$
$$+2\int_{0}^{t}\left\{\int_{\Omega}|\nabla c_{\varepsilon}|^{2}+|\nabla w_{\varepsilon}|^{2}\right\}+\frac{1}{2}\int_{0}^{t}\int_{\Omega}|\Delta c_{\varepsilon}|^{2}, \quad (43)$$
$$\leq\int_{\Omega}c_{0}^{2}+2\int_{\Omega}|\nabla c_{0}|^{2}+2\int_{\Omega}|\nabla w_{0}|^{2}+\frac{1}{2}\int_{0}^{T}\int_{\Omega}|\nabla n_{\varepsilon}|^{2},$$
$$+\left(c_{8}+\frac{1}{2}c_{8}^{N+2/2}\right)m^{2}T\quad\text{for all } \varepsilon\in(0,\varepsilon_{*}).$$

Thus, we can complete the proof of Lemma 7 from Lemma 6. □

2.2. Passing to the Singular Limit. With the above-given important prior estimation, we can carry out the following limit process. The purpose of this part is to take the limit of $n_\varepsilon, c_\varepsilon, w_\varepsilon$, which is to hope that the obtained limit can meet the limit version solution of the system (3). This idea comes from Winkler in [64].

Lemma 8. *Assume (A) with some $T>0, p>N/4$ and $\varepsilon_* \in (0,1)$. Then, there exists $(\varepsilon_j)_{j\in\mathbb{N}\subset(0,\varepsilon_*)}$ and functions.*

$$\begin{cases} n\in L^{2}\left((0,T);H^{1}(\Omega)\right),\\ c\in L^{2}\left((0,T);H^{1}(\Omega)\right)\cap L^{\infty}(\Omega\times(0,T)),\\ w\in L^{2}\left((0,T);H^{1}(\Omega)\right)\cap L^{q}(\Omega\times(0,T)),\end{cases} \quad (44)$$

Such that $\varepsilon_j \longrightarrow 0$ as $j\longrightarrow\infty$, that $n\geq 0, c>0$ and $w>0$ a.e., in $\Omega\times(0,T)$, that

$$\frac{1}{c}\in L^{\infty}(\Omega\times(0,T)). \quad (45)$$

and that as $\varepsilon=\varepsilon_j\longrightarrow 0$ we have the following equation:

$$n_{\varepsilon}\rightharpoonup n\,\text{in}\,L^{2}(\Omega\times(0,T)), \quad (46)$$

$$\nabla n_{\varepsilon}\rightharpoonup\nabla n\,\text{in}\,L^{2}(\Omega\times(0,T)), \quad (47)$$

$$c_{\varepsilon}\longrightarrow c\,\text{in}\,L^{2}(\Omega\times(0,T)),\,a.e.\text{in}\,\Omega\times(0,T), \quad (48)$$

$$\nabla c_{\varepsilon}\longrightarrow\nabla c\,\text{in}\,L^{2}(\Omega\times(0,T)), \quad (49)$$

$$w_{\varepsilon}\rightharpoonup w\,\text{in}\,L^{2}(\Omega\times(0,T)), \quad (50)$$

$$\nabla w_{\varepsilon}\rightharpoonup\nabla w\,\text{in}\,L^{2}(\Omega\times(0,T)), \quad (51)$$

$$\frac{\nabla c_{\varepsilon}}{c_{\varepsilon}}\longrightarrow\frac{\nabla c}{c}\,\text{in}\,L^{2}(\Omega\times(0,T)). \quad (52)$$

Moreover, we have the following identities:

$$\int_{0}^{T}\int_{\Omega}\nabla n\cdot\nabla\varphi=\chi\int_{0}^{T}\int_{\Omega}n\nabla\ln c\cdot\nabla\varphi, \quad (53)$$

and

$$-\int_{0}^{T}\int_{\Omega}c\varphi_{t}-\int_{\Omega}c_{0}\varphi(\cdot,0)+\int_{0}^{T}\int_{\Omega}\nabla c\cdot\nabla\varphi,$$
$$+\int_{0}^{T}\int_{\Omega}c\varphi=\int_{0}^{T}\int_{\Omega}w\varphi. \quad (54)$$

as well as

$$-\int_{0}^{T}\int_{\Omega}w\varphi_{t}-\int_{\Omega}w_{0}\varphi(\cdot,0)+\int_{0}^{T}\int_{\Omega}\nabla w\cdot\nabla\varphi,$$
$$+\int_{0}^{T}\int_{\Omega}w\varphi=\int_{0}^{T}\int_{\Omega}n\varphi. \quad (55)$$

For each $\varphi\in C_0^\infty(\overline{\Omega}\times[0,T))$.

Proof. In light of Lemma 4, Lemma 6, and (42) and together with the embedding $W^{1,r}(\Omega)\hookleftarrow L^\infty(\Omega), r>N$, we have the following equation:

$$
\begin{aligned}
&(n_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)} \in L^2\big((0,T);H^1(0,T)\big),\\
&(c_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)} \in L^2\big((0,T);H^2(0,T)\big)\cap L^\infty,\\
&\qquad\big((0,T);W^{1,N}(\Omega)\big)\cap L^\infty(\Omega\times(0,T)),\\
&(c_{\varepsilon t})_{\varepsilon\in(0,\varepsilon_*)} \in L^2(\Omega\times(0,T)),\\
&(w_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)} \in L^2\big((0,T);H^1(\Omega)\big)\cap L^\infty\big((0,T);H^1(\Omega)\big).
\end{aligned}
\tag{56}
$$

According to Aubin–Lions lemma [67] and the standard compactness arguments, we can extract a sequence $(\varepsilon_j)_{j\in\mathbb{R}}\subset(0,\varepsilon_*)$ along which (46)–(51) hold with some nonnegative functions $n\in L^2((0,T);H^1(\Omega)), c\in L^2((0,T);H^1(\Omega))$, and $w\in L^2((0,T);H^1(\Omega))$ as $\varepsilon=\varepsilon_j\longrightarrow 0$. By $(c_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}\in L^\infty((0,T);W^{1,N}(\Omega))$ and (49), we may employ Fatou's lemma to obtain $c\in L^\infty((0,T);W^{1,N}(\Omega))$. Similarly, $(c_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}\in L^\infty(\Omega\times(0,T))$ and (48) entails that $c\in L^\infty(\Omega\times(0,T))$. Furthermore, (48), the weak closedness of convex sets in $L^\infty(\Omega\times(0,T))$, and Lemma 3 warrants that (45) and (52).

Then, testing the respective equations of system (3) with φ and using of the integration by parts, we have the following equation:

$$
\begin{aligned}
&-\varepsilon\int_0^T\int_\Omega n_\varepsilon\varphi_t-\varepsilon\int_\Omega n_0\varphi(\cdot,0)+\int_0^T\int_\Omega\nabla n_\varepsilon,\\
&\cdot\nabla\varphi=\chi\int_0^T\int_\Omega n_\varepsilon\nabla\ln c_\varepsilon\cdot\nabla\varphi.
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
&-\int_0^T\int_\Omega c_\varepsilon\varphi_t-\int_\Omega c_0\varphi(\cdot,0)+\int_0^T\int_\Omega\nabla c_\varepsilon\cdot\nabla\varphi,\\
&+\int_0^T\int_\Omega c_\varepsilon\varphi=\int_0^T\int_\Omega w_\varepsilon\varphi.
\end{aligned}
\tag{58}
$$

as well as

$$
\begin{aligned}
&-\int_0^T\int_\Omega w_\varepsilon\varphi_t-\int_\Omega w_0\varphi(\cdot,0)+\int_0^T\int_\Omega\nabla w_\varepsilon\cdot\nabla\varphi,\\
&+\int_0^T\int_\Omega w_\varepsilon\varphi=\int_0^T\int_\Omega n_\varepsilon\varphi.
\end{aligned}
\tag{59}
$$

For all $\varphi\in C_0^\infty(\overline{\Omega}\times[0,T))$. We apply (57), (46)–(47), and (52) to obtain (53). Similarly, using (48)–(50) and (58), we derive (54). And, thanks to (46) and (50)-(51), we deduce that (55). □

2.3. Identical Equation. Next, we give an important identity equation under the regular time. These techniques and methods are similar to the literature [64], thus we ignore the corresponding proof.

Lemma 9. *Suppose that A holds with some $T>0, p>N/4$ and $\varepsilon_*\in(0,1)$, and let (n,c,w) be a solution of (3) and Z_* be a non-Lebesgue point set of times. Then, we have the following equation:*

$$
n(\cdot,t)=\frac{mc^\chi(\cdot,t)}{\int_\Omega c^\chi(\cdot,t)}\ \text{for all}\ \frac{t\in(0,T)}{Z_\star}.
\tag{60}
$$

3. Blow-Up in the Nonlocal Limit Problem

Lemma 10. *Let $N\ge 4$ and $\Omega=B_R(0)\subset\mathbb{R}^N$ with some $R>0$, and assume $\chi>2N/N-2$ and $m>0$. Then, there exists $c_0,w_0\in W^{1,\infty}(\Omega),T>0$ and a uniquely determined $\hat{c}\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}\Omega\times(0,T)$ such that $c_0>0,w_0>0$ in $\overline{\Omega}$, and $\hat{f}(\cdot,t){:}=e^{t(\Delta-1)}w_0+m\int_0^t e^{(t-s)(\Delta-1)}\hat{c}^\chi(\cdot,s)/\int_\Omega\hat{c}^\chi(\cdot,s)dxds$ in $\overline{\Omega}\times(0,T)$, that $\hat{c}$ solves*

$$
\begin{cases}
\hat{c}_t=\Delta\hat{c}-\hat{c}+f(\cdot,t), & x\in\Omega,t\in(0,T),\\[2ex]
\dfrac{\partial\hat{c}}{\partial\nu}=0, & x\in\partial\Omega\ ,t\in(0,T),\\[2ex]
\hat{c}(x,0)=c_0(x), & x\in\Omega.
\end{cases}
\tag{61}
$$

in the classical sense, and that

$$
\limsup_{t\longrightarrow T}\|\hat{c}(\cdot,t)\|_{L^\infty(\Omega)}=\infty.
\tag{62}
$$

Proof. A straightforward adaptation of the reasoning from ([68], Section 44.2, also [69]) makes positive and radially symmetric $c_0,w_0\in W^{1,\infty}$ such that with some $T>0$ and a unique function $0<v\in C^0(\overline{\Omega})\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$, we have the following equation:

$$
\begin{cases}
v_t=\Delta v+g(\cdot,t), & x\in\Omega,t\in(0,T),\\[2ex]
\dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega\ ,t\in(0,T),\\[2ex]
v(x,0)=c_0(x), & x\in\Omega,
\end{cases}
\tag{63}
$$

where $g(\cdot,t)=e^t\hat{f}(\cdot,t)=e^{t\Delta}w_0+m\int_0^t e^{(t-s)\Delta}e^s v^\chi(\cdot,s)/\int_\Omega v^\chi(\cdot,s)ds$ and $\limsup\|v(\cdot,t)\|_{L^\infty(\Omega)}=\infty$. Let $v(x,t)=e^t\hat{c}(x,t)$ for $(x,t)t\in^T n\overline{\Omega}q\times h[0,T)$. As a simple calculation shows that the defined $\hat{c}$ satisfies the conclusion of the proposition. □

Proof of Theorem 1. Let $\chi>2N/N-2$ and $m>0$, we fix $c_0,\hat{c}$ and T as given by Lemma 10, and let $n_0\in C^0(\overline{\Omega})$ be a nonnegative function with $\int_\Omega n_0=m$. We now take $(T_{\max,\varepsilon})_{\varepsilon\in(0,1)}$ and $(n_\varepsilon,c_\varepsilon,w_\varepsilon)_{\varepsilon\in(0,1)}$ from Lemma 1, and let $T_\varepsilon{:}=\min\{T_{\max,\varepsilon},T\}$ for $\varepsilon\in(0,1)$. Next, we claim that (11) holds. We use the proof by contradiction to prove it.

If (11) is false, then we have the following equation:

$$
\limsup_{\varepsilon\longrightarrow 0}\ \sup_{t\in(0,T_\varepsilon)}\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}<\infty\text{for some } p>\frac{p}{4}.
\tag{64}
$$

First, we claim that there exists a $\hat{\varepsilon}_*\in(0,1)$ such that $T\le T_{\max,\varepsilon}$ for all $\varepsilon\in(0,\varepsilon_*)$. Otherwise, we can find $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ satisfying $\varepsilon_j\longrightarrow 0$ as $j\longrightarrow\infty$ and $T_{\max,\varepsilon_j}<T$, then we can obtain from (13) that

$$\begin{aligned}\sup_{t\in(0,T_\varepsilon)}\left\|n_{\varepsilon_j}(\cdot,t)\right\|_{L^p(\Omega)} &= \sup_{t\in(0,T_{\max,\varepsilon})}\left\|n_{\varepsilon_j}(\cdot,t)\right\|_{L^p(\Omega)},\\ &\geq \lim_{t\longrightarrow T_{\max,\varepsilon j}}\sup\left\|n_{\varepsilon_j}(\cdot,t)\right\|_{L^p(\Omega)},\\ &= \infty \text{for all } j\in\mathbb{N},\end{aligned} \tag{65}$$

which is contradictory about (64). Therefore, we deduce that $T_\varepsilon = T$ for all $\varepsilon \in (0,\hat{\varepsilon}_*)$. This ensures that (A) is valid. Thus, we can employ Lemma 8 to see that

$$c \in L^2\Big((0,T); H^1(\Omega)\cap L^\infty(\Omega\times(0,T))\Big), \tag{66}$$

such that $c>0$ a.e. in $\Omega\times(0,T)$, that

$$\frac{1}{c}\in L^\infty(\Omega\times(0,T)), \tag{67}$$

and that (54) holds with some $w\in L^q(\Omega\times(0,T))$ for all $q>N/2$. Applying the Lemma 9, we have the following equation:

$$n(\cdot,t) = \frac{mc^\chi(\cdot,t)}{\int_\Omega c^\chi(\cdot,t)}\text{a.e.in}\,\Omega\times(0,T). \tag{68}$$

Then, by the third equation of the system (3), (68), and using the variation-of-constants formula gives the following equation:

$$w(\cdot,t) = e^{t(\Delta-1)}w_0 + m\int_0^t e^{(t-s)(\Delta-1)}\frac{c^\chi(\cdot,s)}{\left\|c^\chi(\cdot,s)\right\|_{L^1(\Omega)}}\mathrm{d}s. \tag{69}$$

In consequence, c would form a bounded generalized solution, in the standard sense specified in [70], of

$$\begin{cases} c_t = \Delta c - c + f(\cdot,t), & x\in\Omega, t\in(0,T),\\ \dfrac{\partial c}{\partial \nu} = 0, & x\in\partial\Omega, t\in(0,T),\\ c(x,0) = c_0(x), & x\in\Omega,\end{cases} \tag{70}$$

with $f(x,t) := e^{t(\Delta-1)}w_0(x) + m\int_0^t e^{(t-s)(\Delta-1)}c^\chi(x,s)/\|c^\chi(x,s)\|_{L^1(\Omega)}\mathrm{d}s$. Using the Neumann heat semigroup estimate, $(45)_2$ and (66)-(67), there exists $C_5>0$ such that

$$\begin{aligned}\|f(x,t)\|_{L^\infty(\Omega)} &\leq \left\|w_0\right\|_{L^\infty(\Omega)} + m|\Omega|\left\|\frac{1}{c}\right\|^\chi_{L^\infty(\Omega)},\\ &\int_0^t\left\|e^{(t-s)(\Delta-1)}c^\chi(x,s)\right\|_{L^\infty(\Omega)} \leq C_5.\end{aligned} \tag{71}$$

Therefore, the standard results on Hölder regularity in scalar parabolic equations [71] is used to warrant that $c\in C^{\theta,\theta/2}(\overline{\Omega}\times[0,T])$ for some $\theta\in(0,1)$, whereupon classical Schauder theory [70] would ensure that f is Hölder continuous in $\overline{\Omega}\times[0,T]$, thus the function $c\in C^{2,1}$, and furthermore actually solve (70) in the classical sense. Therefore, the uniqueness feature in Proposition 3.1 enforces that c must be coincide with $\hat{c}$. This is contradictory with the boundedness claim in (66). Hence (10) must be valid. Thanks to $L^\infty(\Omega)\subset\cap_{p>N/4}L^p(\Omega)$, we could deduce that thus also the growth property statement in (11) result. □

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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