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Research Article

The Weibull Claim Model: Bivariate Extension, Bayesian, and Maximum Likelihood Estimations

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Using a class of claim distributions, we introduce the Weibull claim distribution, which is a new extension of the Weibull distribution with three parameters. The maximum likelihood estimation method is used to estimate the three unknown parameters, and the asymptotic confidence intervals and bootstrap confidence intervals are constructed. In addition, we obtained the Bayesian estimates of the unknown parameters of the Weibull claim distribution under the squared error and linear exponential function (LINEX) and the general entropy loss function. Since the Bayes estimators cannot be obtained in closed form, we compute the approximate Bayes estimates via the Markov Chain Monte Carlo (MCMC) procedure. By analyzing the two data sets, the applicability and capabilities of the Weibull claim model are illustrated. The fatigue life of a particular type of Kevlar epoxy strand subjected to a fixed continuous load at a pressure level of 90% until the strand fails data set was analyzed.

1. Introduction

The use of statistical distributions to model life phenomena has attracted considerable research interest. Recent articles have demonstrated the potential of statistical distributions in modelling life data. The Weibull distribution with two parameters is a well-known model that can be effectively used for data modelling in lifetime analysis. The Weibull distribution was introduced by Frechet [1] and first applied by Rosin and Rammler [2] to describe the distribution of a particle size. The Weibull distribution has many applications in most fields. Let X be a random variable (R.V.) that follows the two-parameter Weibull distribution (λ, γ), then its cumulative distribution function (CDF), denoted by $F(x; \lambda, \gamma)$, is given by

$$F(x; \lambda, \gamma) = 1 - \exp\left(-\left(\frac{x}{\lambda}\right)^{\gamma}\right), x \geq 0; \lambda, \gamma > 0. \tag{1}$$

The corresponding probability density function (PDF), survival function (SF), and hazard rate function (HRF) of the Weibull R.V. are given, respectively, by

$$\begin{aligned} f(x; \lambda, \gamma) &= \frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1} \exp\left(-\left(\frac{x}{\lambda}\right)^{\gamma}\right), x > 0, \\ S(x; \lambda, \gamma) &= \exp\left(-\left(\frac{x}{\lambda}\right)^{\gamma}\right), x > 0, \end{aligned} \tag{2}$$

and

$$h(x; \lambda, \gamma) = \frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}, x > 0. \tag{3}$$

In this article, we focus on modelling a new three-parameter modification of the Weibull distribution, the Weibull claim distribution. In 1997, Marshall and Olkin [3] developed a new family by adding a shape parameter to the basic distribution, which many researchers used to find and study new distributions. The Weibull claim distribution is introduced using the class of claim distributions introduced by Ahmad et al. [4] based on the Marshall and Olkin mechanism. The CDF and the PDF of a class of claim distributions are, respectively, given by

$$G(x;\sigma,\lambda,\gamma)=\frac{\sigma(F(x;\lambda,\gamma))^2}{1-(1-\sigma)F(x;\lambda,\gamma)},x\in\mathbb{R}, \quad (4)$$

and

$$g(x;\sigma,\lambda,\gamma)=\frac{\sigma f(x;\lambda,\gamma)F(x;\lambda,\gamma)[2-(1-\sigma)F(x;\lambda,\gamma)]}{[1-(1-\sigma)F(x;\lambda,\gamma)]^2},x\in\mathbb{R}. \quad (5)$$

In the field of Big Data science and other related fields, the best possible description of real-world phenomena is an important research topic (see [5–8] for details). Recent studies have shown the potential of statistical models in various areas of applied science. The aim of this paper is to deepen this research area of distribution theory and propose a new statistical model that provides a better fit to survival time data.

The structure of this paper is as follows. Section 2 introduces the Weibull claim distribution. The bivariate extension of the Weibull claim distribution is discussed in Section 3. In Section 4, we first discuss the likelihood estimation of the Weibull claim distribution and the asymptotic and bootstrap confidence intervals based on the observed Fisher information matrix. We then discuss the Bayesian estimation of the unknown parameters under the squared error, LINEX, and the general entropy loss function and perform a Monte Carlo simulation study. In Section 5, simulations and two real data sets are performed to evaluate the efficiency of the proposed model. Section 6 provides a brief conclusion.

2. The Weibull Claim Distribution

The CDF and the PDF of the Weibull claim distribution can be constructed directly by substituting the formulas (1)–(3) into (4) and (5), respectively. Now, X is a Weibull claim R.V. if its CDF is given by

$$G(x;\sigma,\lambda,\gamma)=\frac{\sigma\left(1-\exp\left(-(x/\lambda)^{\gamma}\right)\right)^2}{1-(1-\sigma)\left(1-\exp\left(-(x/\lambda)^{\gamma}\right)\right)},x\in\mathbb{R}. \quad (6)$$

The corresponding PDF is given by

$$g(x;\sigma,\lambda,\gamma)=\frac{\sigma\left(\gamma/\lambda\,(x/\lambda)^{\gamma-1}\exp\left(-(x/\lambda)^{\gamma}\right)\right)\left(1-\exp\left(-(x/\lambda)^{\gamma}\right)\right)\left[2-(1-\sigma)\left(1-\exp\left(-(x/\lambda)^{\gamma}\right)\right)\right]}{\left[1-(1-\sigma)\left(1-\exp\left(-(x/\lambda)^{\gamma}\right)\right)\right]^2}, \quad x\in\mathbb{R}. \quad (7)$$

The main motivations for using the Weibull claim in practice are as follows:

(1) A very simple and convenient way to modify the Weibull distribution.
(2) The improvement of the functions and flexibility of the Weibull distribution.
(3) The introduction of an extended version of the Weibull distribution with a closed form of the distribution function.
(4) The best fit to data in many sciences and fields.
(5) Another important motivation for the proposed approach is the introduction of new distributions by adding only one additional parameter instead of two or more parameters.

3. Bivariate Weibull Claim Distribution

Morgenstern [9] introduced the copula model. Copula models are used to represent the common CDF of the two marginal univariate distributions. If $G(x_j)$ is the CDF of X_1 and X_2 and ρ is the dependence measure between X_1 and X_2, the joint CDF defined by the copula model, denoted by $G(x_1,x_2)$, is given by

$$G(x_1,x_2)=C_{\rho}\left(G(x_1),G(x_2)\right). \quad (8)$$

Conway [10] introduced the joint CDF and PDF of the copula model, respectively, by

$$G(x_1,x_2)=G(x_1)G(x_2)\cdot\left[1+\rho\left(1-G(x_1)\right)\left(1-G(x_2)\right)\right],-1<\rho<1, \quad (9)$$

and

$$g(x_1,x_2)=g(x_1)g(x_2)\left[1+\rho\left(1-2G(x_1)\right)\left(1-2G(x_2)\right)\right]. \quad (10)$$

If the R.V.s X_1 and X_2 follow the Bivariate Weibull claim distribution, then its CDF is given by

$$\begin{aligned}G(x_1,x_2)=&\left(\frac{\sigma_1\left(1-\exp\left(-(x_1/\lambda_1)^{\gamma_1}\right)\right)^2}{1-(1-\sigma_1)\left(1-\exp\left(-(x_1/\lambda_1)^{\gamma_1}\right)\right)}\right)\\&\times\left(\frac{\sigma_2\left(1-\exp\left(-(x_2/\lambda_2)^{\gamma_2}\right)\right)^2}{1-(1-\sigma_2)\left(1-\exp\left(-(x_2/\lambda_2)^{\gamma_2}\right)\right)}\right)\\&\times\left[1+\rho\left(1-\left(\frac{\sigma_1\left(1-\exp\left(-(x_1/\lambda_1)^{\gamma_1}\right)\right)^2}{1-(1-\sigma_1)\left(1-\exp\left(-(x_1/\lambda_1)^{\gamma_1}\right)\right)}\right)\right)\right.\\&\left.\times\left(1-\left(\frac{\sigma_2\left(1-\exp\left(-(x_2/\lambda_2)^{\gamma_2}\right)\right)^2}{1-(1-\sigma_2)\left(1-\exp\left(-(x_2/\lambda_2)^{\gamma_2}\right)\right)}\right)\right)\right],\\&-1<\rho<1,\end{aligned} \quad (11)$$

where $x_1,x_2,\lambda_1,\gamma_1,\sigma_1,\lambda_2,\gamma_2,\sigma_2>0$. The corresponding PDF be

$$g(x_1, x_2) = \left(\frac{\sigma_1\left(\gamma_1/\lambda_1 (x_1/\lambda_1)^{\gamma_1 - 1} \exp(-(x_1/\lambda_1)^{\gamma_1})\right)(1 - \exp(-(x_1/\lambda_1)^{\gamma_1}))[2 - (1 - \sigma_1)(1 - \exp(-(x_1/\lambda_1)^{\gamma_1}))]}{[1 - (1 - \sigma_1)(1 - \exp(-(x_1/\lambda_1)^{\gamma_1}))]^2}\right)$$
$$\times \left[\begin{array}{l} 1 + \rho\left(1 - 2\left(\dfrac{\sigma_1(1 - \exp(-(x_1/\lambda_1)^{\gamma_1}))^2}{1 - (1 - \sigma_1)(1 - \exp(-(x_1/\lambda_1)^{\gamma_1}))}\right)\right) \\ \left(1 - 2\left(\dfrac{\sigma_2(1 - \exp(-(x_2/\lambda_2)^{\gamma_2}))^2}{1 - (1 - \sigma_2)(1 - \exp(-(x_2/\lambda_2)^{\gamma_2}))}\right)\right) \end{array}\right]$$
$$\times \left(\frac{\sigma_2\left(\gamma_2/\lambda_2 (x_2/\lambda_2)^{\gamma_2 - 1} \exp(-(x_2/\lambda_2)^{\gamma_2})\right)(1 - \exp(-(x_2/\lambda_2)^{\gamma_2}))[2 - (1 - \sigma_2)(1 - \exp(-(x_2/\lambda_2)^{\gamma_2}))]}{[1 - (1 - \sigma_2)(1 - \exp(-(x_2/\lambda_2)^{\gamma_2}))]^2}\right). \tag{12}$$

Figures 1 and 2 show different PDF and CDF values of the bivariate Weibull claim model for $\rho = 1$. Figure 3, on the other hand, shows the SF and HF of the bivariate Weibull distribution model.

4. Parameter Estimation

Many studies were examined in the articles to estimate the three unknown parameters. Let $\mathbf{x} = x_1, x_2, \ldots, x_n$ be a sample from the Weibull claim distribution.

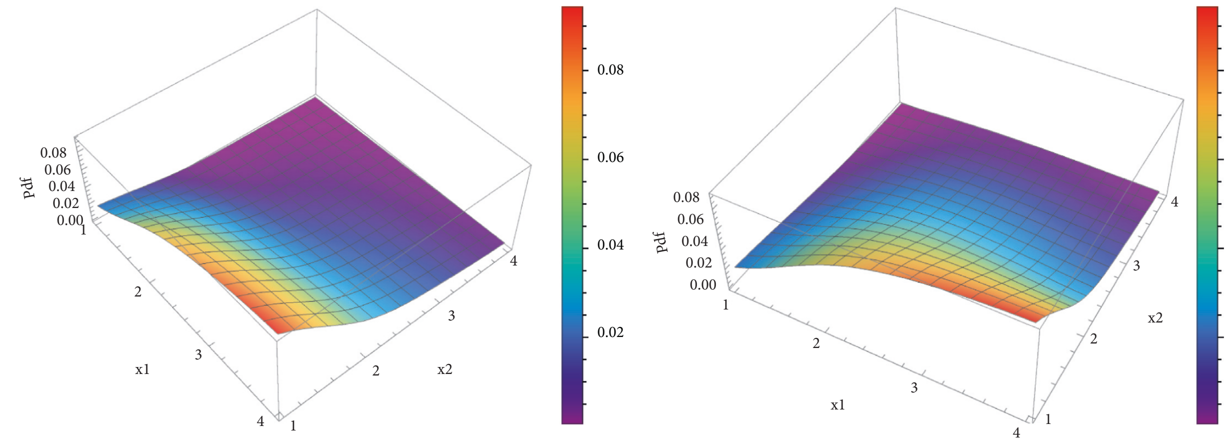
4.1. Maximum Likelihood Method. The likelihood function corresponding to equation (7) is given by

$$L(\sigma, \lambda, \gamma) = n!\left(\frac{\sigma\gamma}{\lambda^\gamma}\right)^n \left(\prod_{i=1}^n x_i\right)^{\gamma - 1} \exp\left(-\sum_{i=1}^n \left(\frac{x_i}{\lambda}\right)^\gamma\right)$$
$$\times \prod_{i=1}^n \left(\frac{(1 - \exp(-(x_i/\lambda)^\gamma))[2 - (1 - \sigma)(1 - \exp(-(x_i/\lambda)^\gamma))]}{[1 - (1 - \sigma)(1 - \exp(-(x_i/\lambda)^\gamma))]^2}\right), x \in \mathbb{R}. \tag{13}$$

The log-likelihood function corresponding to equation (13) is given by

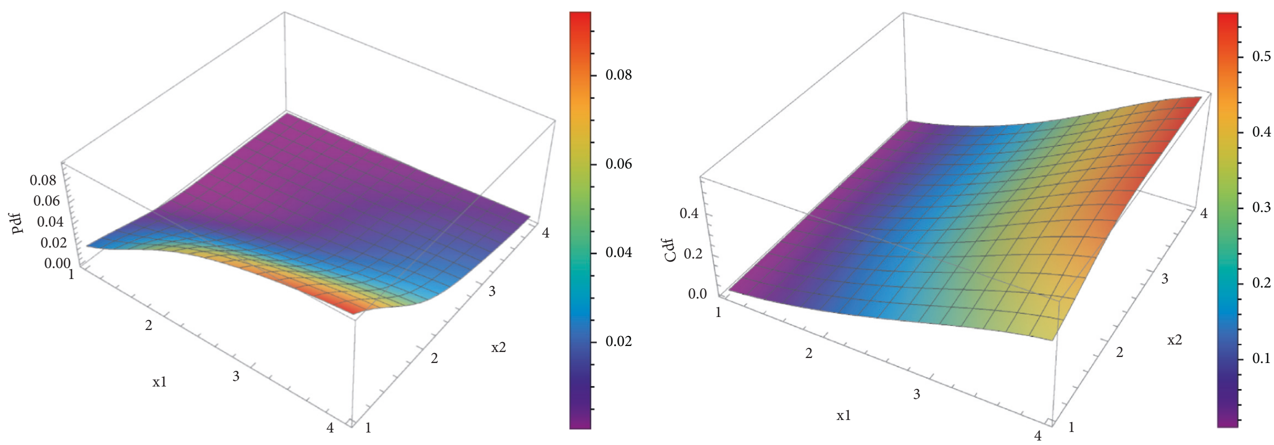
$$\ell = \log n! + n \log \sigma + n \log \gamma - n\gamma \log \lambda + n(\gamma - 1)\sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \left(\frac{x_i}{\lambda}\right)^\gamma$$
$$+ \sum_{i=1}^n \log\left(1 - \exp\left(-\left(\frac{x_i}{\lambda}\right)^\gamma\right)\right) + \sum_{i=1}^n \log\left[2 - (1 - \sigma)\left(1 - \exp\left(-\left(\frac{x_i}{\lambda}\right)^\gamma\right)\right)\right]$$
$$- 2\sum_{i=1}^n \log\left[1 - (1 - \sigma)\left(1 - \exp\left(-\left(\frac{x_i}{\lambda}\right)^\gamma\right)\right)\right]. \tag{14}$$

The equation in equation (14) can be solved by using Newton–Raphson method. The corresponding partial derivatives are



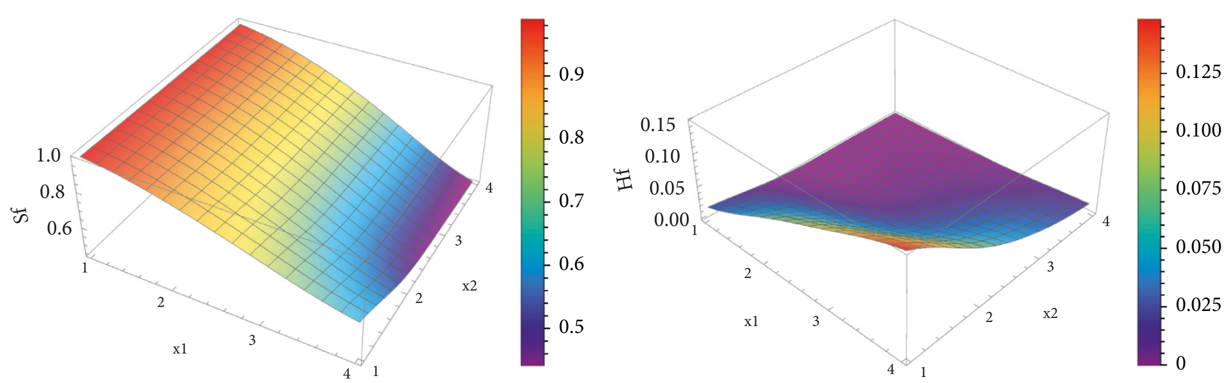
(a) (b)

Figure 1: Different plot for the PDF of the Bivariate Weibull claim distribution for $\sigma_1 = 3, \lambda_1 = 1, \gamma_1 = 1, \sigma_2 = 5, \lambda_2 = 4, \gamma_2 = 2$, and $\rho = -0.9$ (a) and $\rho = -1$ (b).



(a) (b)

Figure 2: Plots of the PDF (a) and CDF (b) of the Bivariate Weibull claim model for $\sigma_1 = 3, \lambda_1 = 1, \gamma_1 = 1, \sigma_2 = 5, \lambda_2 = 4$, and $\gamma_2 = 2$.



(a) (b)

Figure 3: Plots of the SF (a) and HF (b) of the Bivariate Weibull claim model for $\sigma_1 = 3, \lambda_1 = 1, \gamma_1 = 1, \sigma_2 = 5, \lambda_2 = 4, \gamma_2 = 2$, and $\rho = 1$.

$$\frac{\partial \ell}{\partial \lambda} = \frac{\gamma}{\lambda}\left\{ n + \frac{1}{\lambda}\left(\sum_{i=1}^{n} x_i\left(\frac{x_i}{\lambda}\right)^{-1+\gamma} - \sum_{i=1}^{n} \frac{e^{-(x_i/\lambda)^{\gamma}} x_i (x_i/\lambda)^{-1+\gamma}}{\left(1-e^{-(x_i/\lambda)^{\gamma}}\right)} \right.\right.$$
$$\left.\left. + (1-\sigma)\left\{ \sum_{i=1}^{n} \frac{e^{-(x_i/\lambda)^{\gamma}} x_i (x_i/\lambda)^{-1+\gamma}}{\left(2-\left(1-e^{-(x_i/\lambda)^{\gamma}}\right)(1-\sigma)\right)} - 2\sum_{i=1}^{n} \frac{e^{-(x_i/\lambda)^{\gamma}} x_i (x_i/\lambda)^{-1+\gamma}}{\left(1-\left(1-e^{-(x_i/\lambda)^{\gamma}}\right)(1-\sigma)\right)} \right\} \right) \right\},$$
$$\frac{\partial \ell}{\partial \gamma} = n\left(\frac{1}{\gamma} - \log[\lambda] + \sum_{i=1}^{n}\log[x_i]\right) - \sum_{i=1}^{n}\log\left[\frac{x_i}{\lambda}\right]\left(\frac{x_i}{\lambda}\right)^{\gamma}$$
$$+ \sum_{i=1}^{n} \frac{e^{-(x_i/\lambda)^{\gamma}}\log[x_i/\lambda](x_i/\lambda)^{\gamma}}{1-e^{-(x_i/\lambda)^{\gamma}}} + (1-\sigma)\left\{ 2\sum_{i=1}^{n} \frac{e^{-(x_i/\lambda)^{\gamma}}\log[x_i/\lambda](x_i/\lambda)^{\gamma}}{1-\left(1-e^{-(x_i/\lambda)^{\gamma}}\right)(1-\sigma)} - \sum_{i=1}^{n} \frac{e^{-(x_i/\lambda)^{\gamma}}\log[x_i/\lambda](x_i/\lambda)^{\gamma}}{2-\left(1-e^{-(x_i/\lambda)^{\gamma}}\right)(1-\sigma)} \right\}, \tag{15}$$

And,

$$\frac{\partial \ell}{\partial \sigma} = \frac{n}{\sigma} - 2\sum_{i=1}^{n} \frac{1-e^{-(x_i/\lambda)^{\gamma}}}{1-\left(1-e^{-(x_i/\lambda)^{\gamma}}\right)(1-\sigma)} + \sum_{i=1}^{n} \frac{1-e^{-(x_i/\lambda)^{\gamma}}}{2-\left(1-e^{-(x_i/\lambda)^{\gamma}}\right)(1-\sigma)}. \tag{16}$$

The maximum likelihood estimators for the parameters of the Weibull distribution λ, γ, and σ are the solutions of the above nonlinear equations.

Then, we can calculate the asymptotic confidence intervals of the parameters σ, λ, and γ. The covariance matrix of the observed variance for the MLEs of the parameters $\widehat{V} = [\sigma_{i,j}], i, j = 1, 2, 3$ was assumed as follows:

$$\widehat{V} = -\begin{bmatrix} \frac{\partial^2 l}{\partial \sigma^2} & \frac{\partial^2 l}{\partial \sigma\, \partial \lambda} & \frac{\partial^2 l}{\partial \sigma\, \partial \gamma} \\ \frac{\partial^2 l}{\partial \lambda\, \partial \sigma} & \frac{\partial^2 l}{\partial \lambda^2} & \frac{\partial^2 l}{\partial \lambda\, \partial \gamma} \\ \frac{\partial^2 l}{\partial \gamma\, \partial \sigma} & \frac{\partial^2 l}{\partial \gamma\, \partial \lambda} & \frac{\partial^2 l}{\partial \gamma^2} \end{bmatrix}^{-1}_{\left(\sigma=\widehat{\sigma}_{ML}, \lambda=\widehat{\lambda}_{ML}, \gamma=\widehat{\gamma}_{ML}\right)}. \tag{17}$$

$100(1-\alpha)\%$ two-sided approximate confidence intervals for the parameters σ, λ, and γ are then given by

$$\begin{aligned} &\widehat{\sigma} \pm z_{\alpha/2}\sqrt{V(\widehat{\sigma})}, \\ &\widehat{\lambda} \pm z_{\alpha/2}\sqrt{V(\widehat{\lambda})}, \end{aligned} \tag{18}$$

and

$$\widehat{\gamma} \pm z_{\alpha/2}\sqrt{V(\widehat{\gamma})}, \tag{19}$$

respectively, where $V(\widehat{\sigma})$, $V(\widehat{\lambda})$, and $V(\widehat{\gamma})$ are the estimated variances of $\widehat{\sigma}_{ML}$, $\widehat{\lambda}_{ML}$, and $\widehat{\gamma}_{ML}$, which are given by the diagonal elements of $\widehat{V}$, and $z_{\alpha/2}$ is the upper $(\alpha/2)$ percentile of the standard normal distribution.

Next, obtaining the bootstrap C.I. for boot-p for the unknown parameters $\phi = (\sigma, \lambda, \gamma)$, we apply the following algorithms (for more details, see [11, 12]).

Boot-p interval's algorithm is as follows:

Step 1. Generate $x_{1:\,n}, x_{2:\,n}, \ldots, x_{n:\,n}$ from the Weibull claim distribution and derive an estimate $\widehat{\phi}$ of ϕ.

Step 2. Generate another sample $x^*_{1:\,n}, x^*_{2:\,n}, \ldots, x^*_{n:\,n}$ using $\widehat{\phi}$. Then, derive the updated bootstrap estimate $\widehat{\phi}^*$ from ϕ.

Step 3. Repeat step 2 with a given number B of repetitions.

Step 4. Using $\widehat{F}(x) = P(\widehat{\phi}^* \leq x)$, i.e., the CDF of $\widehat{\phi}^*$, the $100(1-t)\%$ C.I. of ϕ is given by

$$\left(\widehat{\phi}_{\text{Boot}-p}\left(\frac{\alpha}{2}\right), \widehat{\phi}_{\text{Boot}-p}\left(1-\frac{\alpha}{2}\right)\right), \tag{20}$$

where $\widehat{\phi}_{\text{Boot}-p}(x) = \widehat{F}^{-1}(x)$ and x is prefixed.

4.2. Bayesian Estimation. Bayesian inference is an appropriate method to work with the full samples of the Weibull claim distribution. Given that Weibull claim distributions are so rare, prior information is very useful. We assume that σ, λ, and γ are R.V.s following prior DFs: Uniform $(\sigma; 0, a_\sigma)$, Uniform $(\lambda; 0, a_\lambda)$, and Uniform $(\gamma; 0, a_\gamma)$, respectively. The posterior DF of σ, λ, and γ and the data under the uniform priors can take the form as follows:

$$\begin{aligned}\pi_u^*(\sigma,\lambda,\gamma|\mathbf{x}) &\propto L(\sigma,\lambda,\gamma|\mathbf{x})\pi_1(\sigma,\lambda,\gamma),\\ &= \frac{n!}{J_u a_\sigma a_\lambda a_\gamma}\left(\frac{\sigma\gamma}{\lambda^\gamma}\right)^n\left(\prod_{i=1}^n x_i\right)^{\gamma-1}\exp\left(-\sum_{i=1}^n\left(\frac{x_i}{\lambda}\right)^\gamma\right)\\ &\times\prod_{i=1}^n\left(\frac{(1-\exp(-(x_i/\lambda)^\gamma))[2-(1-\sigma)(1-\exp(-(x_i/\lambda)^\gamma))]}{[1-(1-\sigma)(1-\exp(-(x_i/\lambda)^\gamma))]^2}\right),\end{aligned} \tag{21}$$

where J_1 is the normalizing constant. Next, we suppose that σ,λ, and γ are R.V.s that follow the prior DF Gamma $(\sigma; a_1, b_1)$, Gamma $(\lambda; a_2, b_2)$, and Gamma $(\gamma; a_3, b_3)$, respectively, where a_i and b_i are positive constants and $i = 1, 2, 3$. The posterior DF of σ,λ,γ and the data under the Gamma priors can take the forms as follows:

$$\begin{aligned}\pi_g^*(\sigma,\lambda,\gamma|\mathbf{x}) &\propto L(\sigma,\lambda,\gamma|\mathbf{x})\pi_1(\sigma,\lambda,\gamma),\\ &= \frac{n!\sigma^{a_1-1}\lambda^{a_2-1}\gamma^{a_3-1}}{J_g}\left(\frac{\sigma\gamma}{\lambda^\gamma}\right)^n\left(\prod_{i=1}^n x_i\right)^{\gamma-1}\\ &\times\exp\left(-\left(b_1\sigma+b_2\lambda+b_3\gamma+\sum_{i=1}^n\left(\frac{x_i}{\lambda}\right)^\gamma\right)\right)\\ &\times\prod_{i=1}^n\left(\frac{(1-\exp(-(x_i/\lambda)^\gamma))[2-(1-\sigma)(1-\exp(-(x_i/\lambda)^\gamma))]}{[1-(1-\sigma)(1-\exp(-(x_i/\lambda)^\gamma))]^2}\right),\end{aligned} \tag{22}$$

where J_g is the marginal probability DF of $\mathbf{x}$.

4.3. MCMC Method. In the following algorithm, we use Metropolis Hastings (M-H) procedure with normal DF to simulate samples from the distributions:

(1) Set the initial values $\sigma^{(0)},\lambda^{(0)}$, and $\gamma^{(0)}$. Then, simulate sample of size n from Weibull claim distribution, next let $l = 1$.
(2) Simulate $\sigma^{(*)},\lambda^{(*)}$, and $\gamma^{(*)}$. using the proposal distributions $N(\sigma^{(l-1)},\mathrm{Var}(\hat{\sigma}))$, $N(\lambda^{(l-1)},\mathrm{Var}(\hat{\lambda}))$, and $N(\gamma^{(l-1)},\mathrm{Var}(\hat{\gamma}))$.
(3) Obtain the probability $r = \min(\pi^*(\sigma^{(*)},\lambda^{(*)},\gamma^{(*)})/\pi^*(\sigma^{(l-1)},\lambda^{(l-1)},\gamma^{(l-1)}),1)$.
(4) Simulate U from Uniform (0, 1).
(5) If $U < r$, then $(\sigma^{(l)},\lambda^{(l)},\gamma^{(l)}) = (\sigma^{(*)},\lambda^{(*)},\gamma^{(*)})$. If $U \ge r$, then $(\sigma^{(l)},\lambda^{(l)},\gamma^{(l)}) = (\sigma^{(l-1)},\lambda^{(l-1)},\gamma^{(l-1)})$.
(6) Set $l = l + 1$.
(7) Iterate Steps 2–6, M repetitions, and get $\sigma^{(l)},\lambda^{(l)}$, and $\gamma^{(l)}$ for $l = 1,\ldots,M$.

In order to conduct a Bayesian analysis, usually quadratic loss function is considered. A very popular quadratic loss is the squared error loss function given by $L_{\mathrm{SE}}(\vartheta,\hat{\vartheta}) = (\vartheta-\hat{\vartheta})^2$, where $\hat{\vartheta}$ is an estimate of the unknown parameter ϑ against SE loss function. By using the generated random samples from the above Gibbs sampling technique and for N is the n burn, then the Bayes estimator of ϑ, say $\hat{\vartheta}_{\mathrm{SE}}$, can be obtained as

$$\hat{\vartheta}_{\mathrm{SE}} = E_\vartheta[\vartheta|\mathbf{x}] = \frac{1}{M-N}\sum_{l=N+1}^M \vartheta^{(l)}. \tag{23}$$

The second loss function is the LINEX loss function, given by

$$L_{LE}(\vartheta,\hat{\vartheta}) = \exp[\rho(\vartheta-\hat{\vartheta})]-\rho(\vartheta-\hat{\vartheta})-1, \rho\neq 0. \tag{24}$$

The approximate Bayes estimate of $\vartheta = \sigma,\lambda,\gamma$ under LE loss function based on the Gibbs sampling technique becomes

$$\hat{\vartheta}_{LE} = \frac{-1}{\rho}\log(E_\vartheta[\exp(-\rho\vartheta)|\mathbf{x}]) = \frac{-1}{\rho}\log\left(\frac{\sum_{l=N+1}^M\exp\left(-\rho\vartheta^{(l)}\right)}{M-N}\right). \tag{25}$$

Finally, the general entropy (GE) loss function is given by

$$L_{GE}(\vartheta,\hat{\vartheta}) = \left(\frac{\hat{\vartheta}}{\vartheta}\right)^\varepsilon - \varepsilon\log\left(\frac{\hat{\vartheta}}{\vartheta}\right) - 1. \tag{26}$$

The approximate Bayes estimate of the parameters is given by

$$\hat{\vartheta}_{GE} = \left(E_\vartheta[\vartheta^{-\varepsilon}|\mathbf{x}]\right)^{-1/\varepsilon} = \left(\frac{1}{M-N}\sum_{l=N+1}^M\left(\vartheta^{(l)}\right)^{-\varepsilon}\right)^{-1/\varepsilon}. \tag{27}$$

MCMC HPD credible interval algorithm is as follows:

(1) Sort $\sigma^{(*)}, \lambda^{(*)}$, and $\gamma^{(*)}$ in rising values.

(2) The lower bounds of σ, λ, and γ in the rank $(M-N)*\alpha/2$.

(3) The lower bounds of σ, λ, and γ in the rank $(M-N)*(1-\alpha/2)$.

(4) Iterate the previous steps M times. Get the average value of the lower and upper bounds of σ, λ, and γ.

5. Simulation Study

The simulation study is conducted as follows:

(1) Random samples of size $n = 25, 30, \ldots, 100$ from the Weibull claim distribution are simulated.

(2) Parameters were estimated using maximum likelihood and Bayesian methods.

(3) Two LINEX loss functions are used: $LE1$ when $\rho = -0.1$ and $LE2$ when $\rho = 0.5$.

(4) M iterations are performed to obtain the absolute biases and expected risk (ER) of these estimators.

(5) The point estimates of the parameters using MLE and MCMC methods are obtained.

(6) We obtain 95% and 90% HPD credibility intervals using MCMC methods.

(7) The biases and ERs are, respectively, given by

$$\text{Bias}(\widehat{\vartheta}) = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{\vartheta}_i - \vartheta\right). \tag{28}$$

and

$$ER(\widehat{\vartheta}) = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{\vartheta}_i - \vartheta\right)^2. \tag{29}$$

The MLE and Bayes simulation results of the Weibull claim distribution parameters λ, γ, and σ are presented in Tables 1 and 2.

Table 1: Point estimation of the parameters $\lambda = 0.25, \gamma = 2.5$, and $\sigma = 1.77$.

		Point				
n	Par.	ML	SE	LE1	LE2	GE
		0.7731	0.4948	0.499	0.4717	0.4004
	σ	0.5189	0.2407	0.2448	0.2176	0.1462
		1.1062	0.1774	0.1825	0.1498	0.1106
		1.1854	2.3515	2.3562	2.3252	2.3245
25	β	−1.3166	−0.1505	−0.1459	−0.1769	−0.1776
		4.5458	0.2117	0.2104	0.2194	0.2243
		1.2235	1.6291	1.6316	1.6153	1.6099
	γ	−0.5508	−0.1453	−0.1428	−0.159	−0.1645
		1.3222	0.0591	0.0587	0.0617	0.0635
		0.7668	0.3421	0.3444	0.3296	0.2842
	σ	0.5127	0.0879	0.0903	0.0755	0.0301
		1.0357	0.0844	0.0865	0.0735	0.0598
		1.2194	2.4733	2.4775	2.4498	2.4511
50	β	−1.2826	−0.0288	−0.0246	−0.0523	−0.051
		4.9297	0.1642	0.1642	0.1643	0.1667
		1.1609	1.7314	1.7333	1.7208	1.7177
	γ	−0.6135	−0.043	−0.041	−0.0535	−0.0567
		1.0077	0.0429	0.0431	0.0423	0.0428
		0.7788	0.2381	0.2387	0.235	0.2185
	σ	0.5246	−0.016	−0.0154	−0.0191	−0.0356
		1.0116	0.0235	0.0239	0.0216	0.0198
		1.1982	2.5796	2.5813	2.5705	2.5715
200	β	−1.3039	0.0776	0.0792	0.0684	0.0695
		4.6889	0.0899	0.0904	0.0876	0.0884
		1.2423	1.7735	1.7739	1.7706	1.7699
	γ	−0.5321	−0.0009	−0.0004	−0.0038	−0.0045
		1.2619	0.0187	0.0188	0.0186	0.0186

First line represents estimate, second line represents bias, and third line represents ER.

5.1. Application of the Weibull Claim Model. The data set represents the lifetimes of Kevlar 49/epoxy strands subjected to constant sustained pressure at 90% stress level until strand failure. For previous studies on data, see [13–16]. The data are as follows: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, and 9.0960.

Table 3 presents the point estimation of the Kevlar 49/epoxy strands parameters while Table 4 presents the corresponding interval estimation. Table 5 compares the Weibull claim distribution based on some detection criteria, such as the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), and consistent Akaike information criterion (CAIC). The goodness-of-fit results of the Weibull claim model (W-claim) are compared with those of some other models, including the Exponential claim distribution (Exp-claim), the Weibull distribution (W-D), and the exponential distribution (EXP-D). Table 6 compares the W-claim with the Kolmogorov–Smirnov test for one sample.

6. Discussion and Future Framework

The experimental side showed, depending on the two statistical criteria, the mean square errors and the bias in estimating the parameters, that the Bayes method ranks first in computing the parameter estimates and that the MLE method ranks second, especially for small samples. As the sample size increases, the MSE values for each experiment decrease. The values of the cumulative distribution are between zero and one, and they increase and are directly proportional to time. Based on the test results, it was shown that the proposed distribution best represents the data compared with competing distributions based on some AIC, BIC, HQIC, and CAIC. The goodness-of-fit results of the Weibull claim model are compared with Exp-claim, W-D, and EXP-D. The results in Tables 5 and 6 suggest that the

Table 2: Interval estimation of the parameters $\lambda = 0.25$, $\gamma = 2.5$, and $\sigma = 1.77$.

Par.		Interval					
		ML	Boot	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
25	σ	0.4904 1.0557	0.032 4.419	0.044 1.185	0.0439 1.1964	0.0434 1.1168	0.0308 1.0389
		0.5654	4.387	1.141	1.1525	1.0734	1.0081
		0.5351 1.011	0.06 2.393	0.1 1.099	0.0999 1.1164	0.098 1.0685	0.077 0.9656
		0.476	2.333	0.999	1.0165	0.9705	0.8886
	β	0.9039 1.4669	0.321 7.427	1.286 3.036	1.2912 3.0424	1.2611 3.0142	1.246 3.0178
		0.563	7.106	1.75	1.7511	1.7531	1.7717
		0.9484 1.4224	0.339 5.861	1.562 2.966	1.5691 2.9687	1.5198 2.9512	1.4956 2.9544
		0.474	5.522	1.404	1.3997	1.4313	1.4588
	γ	0.9431 1.504	0.632 4.698	1.225 1.921	1.2262 1.9241	1.2189 1.9026	1.214 1.899
		0.561	4.066	0.696	0.6979	0.6837	0.685
		0.9874 1.4597	0.672 2.948	1.294 1.916	1.2955 1.9189	1.2844 1.8904	1.277 1.8838
		0.4723	2.276	0.622	0.6234	0.606	0.6068
50	σ	0.476 1.0575	0.027 3.965	0.055 1.134	0.055 1.1445	0.0548 1.0708	0.0473 0.9557
		0.5815	3.938	1.079	1.0894	1.016	0.9084
		0.522 1.0116	0.06 2.621	0.094 1.016	0.0941 1.0293	0.0936 0.9373	0.0733 0.7855
		0.4895	2.561	0.922	0.9352	0.8437	0.7122
	β	0.8653 1.5736	0.327 7.597	1.585 3.293	1.5862 3.2973	1.5808 3.2636	1.5788 3.2719
		0.7083	7.27	1.708	1.711	1.6828	1.6931
		0.9213 1.5176	0.348 6.879	1.769 3.216	1.7706 3.2208	1.7526 3.1854	1.7449 3.1936
		0.5963	6.531	1.447	1.4502	1.4327	1.4487
	γ	0.8316 1.4902	0.639 3.469	1.331 2.124	1.3318 2.1294	1.3276 2.098	1.3251 2.0963
		0.6585	2.83	0.793	0.7976	0.7704	0.7712
		0.8837 1.4381	0.674 2.24	1.367 2.078	1.3683 2.0805	1.3587 2.061	1.3567 2.059
		0.5544	1.566	0.711	0.7123	0.7022	0.7023
200	σ	0.494 1.0635	0.031 3.946	0.078 0.538	0.0781 0.5402	0.0779 0.5244	0.0701 0.4669
		0.5695	3.915	0.46	0.4621	0.4464	0.3968
		0.539 1.0185	0.056 2.561	0.096 0.404	0.0965 0.4056	0.0958 0.3958	0.083 0.3698
		0.4795	2.505	0.308	0.3091	0.3	0.2868
	β	0.9225 1.4739	0.324 7.627	1.929 3.129	1.9309 3.1324	1.9188 3.1103	1.9171 3.1152
		0.5515	7.303	1.2	1.2015	1.1916	1.1981
		0.9661 1.4303	0.354 6.49	2.138 2.983	2.138 2.9868	2.1262 2.9604	2.1257 2.9653
		0.4642	6.136	0.845	0.8488	0.8341	0.8395
	γ	0.9646 1.5201	0.633 4.604	1.499 2.037	1.4991 2.0382	1.4971 2.0323	1.4962 2.0316
		0.5555	3.971	0.538	0.539	0.5352	0.5354
		1.0085 1.4762	0.67 2.883	1.548 1.967	1.5486 1.9677	1.5467 1.9624	1.546 1.9616
		0.4677	2.213	0.419	0.4191	0.4157	0.4157

The first, second, third, and fourth lines show a 95% credible HPD interval, the corresponding 95% width, 90% credible HPD interval, and the corresponding 95% width of the parameter, respectively.

Table 3: Point estimation of the Kevlar 49/epoxy strands parameters.

		Point				
N	Par.	ML	SE	LE1	LE2	GE
76	σ	0.3541	0.4319	0.4319	0.4319	0.4318
		−0.0107	0.0778	0.0778	0.0778	0.0776
		0.0076	0.0061	0.0061	0.0061	0.0061
	β	5.8049	5.7685	5.7686	5.7681	5.7684
		0.3423	−0.0364	−0.0363	−0.0368	−0.0365
		0.0855	0.002	0.002	0.0021	0.002
	γ	1.0619	0.9862	0.9863	0.9862	0.9862
		0.0597	−0.0757	−0.0757	−0.0757	−0.0758
		0.0082	0.0058	0.0058	0.0058	0.0058

First line represents estimate, second line represents bias, and third line represents ER.

Table 4: Interval estimation of the Kevlar 49/epoxy strands parameters.

Par.		Interval											
		ML		Boot		HPD_S		HPD_{LE1}		HPD_{LE2}		HPD_{GE}	
76	σ	0.1831	0.5251	0.213	0.5	0.418	0.442	0.4176	0.4415	0.4176	0.4415	0.4174	0.4415
		0.342		0.287		0.024		0.0239		0.024		0.0241	
		0.2102	0.4981	0.222	0.491	0.422	0.441	0.4217	0.4412	0.4217	0.4412	0.4214	0.4412
		0.2879		0.269		0.019		0.0195		0.0196		0.0198	
	β	5.2317	6.3781	5.337	6.284	5.719	5.824	5.7186	5.8241	5.7186	5.8236	5.7186	5.8239
		1.1464		0.947		0.105		0.1055		0.105		0.1053	
		5.3224	6.2874	5.363	6.263	5.729	5.812	5.7287	5.8119	5.7287	5.8112	5.7287	5.8117
		0.9651		0.9		0.083		0.0832		0.0825		0.083	
	γ	0.8839	1.2399	0.909	1.212	0.973	1.009	0.9732	1.0086	0.9732	1.0083	0.9732	1.0081
		0.356		0.303		0.036		0.0353		0.0351		0.0349	
		0.9121	1.2118	0.916	1.204	0.975	1.002	0.9753	1.0023	0.9753	1.0023	0.9753	1.0023
		0.2997		0.288		0.027		0.027		0.027		0.027	

The first, second, third, and fourth lines show a 95% credible HPD interval, the corresponding 95% width, 90% credible HPD interval, and the corresponding 95% width of the parameter, respectively.

Table 5: Relative quality of the W-claim vs. competing models.

Model	AIC	CAIC	BIC	HQIC
W-claim (0.0087, 0.4494, 0.0497)	246.5292	246.8625	253.5214	249.3236
Exp-claim (1.1915, 1.3556)	249.3394	249.5038	254.0009	251.2024
W-D (1.3348, 2.1229)	249.0626	249.227	253.7241	250.9255
EXP-D (1.9587)	256.2287	256.2827	258.5594	257.1601

Table 6: One-sample Kolmogorov–Smirnov test.

Model	KS	p value
W-claim (0.0087, 0.4494, 0.0497)	0.078999	0.700
Exp-claim (1.1915, 1.3556)	0.091496	0.518
W-D (1.3348, 2.1229)	0.10734	0.322
EXP-D (1.9587)	0.1664	0.026

W-claim distribution provides a better fit than other competing models and could be chosen as a suitable model for analyzing heavy-tail engineering data.

We recommend the use of the Bayes method for parameter estimation. In the future, it is possible to expand the scientific aspects associated with the application, such as the medical, engineering, and industrial aspects. Other estimation methods can also be used to estimate the parameters of the proposed new Weibull claim model and compare them with the methods used in this study.

7. Concluding Remarks

In this paper, we introduced a new extension of the Weibull distribution, the Weibull claim model. We determined the maximum likelihood estimators for the parameters of the Weibull claim distribution and performed a Bayesian Monte Carlo simulation study. The performances of the Bayesian estimators are better than those of the corresponding ML estimators. The new Weibull claim model outperformed many other old models. The performance of the Bayesian estimates is more excellent than that of the corresponding ML estimators. We found that the proposed model provides a better characterization of industry events by analyzing the Kevlar 49/epoxy strands data. Therefore, it can be a perfect model for predicting future cases.

Data Availability

All the datasets used in this paper are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] M. Frechet, "On the law of probability of the maximum deviation, Annals of the Polish Mathematical Society," *Cracovie*, vol. 6, pp. 93–116, 1927.

[2] P. Rosin and E. Rammler, "The laws governing the fineness of powdered coal," *Journal of the Institute of Fuel*, vol. 7, pp. 29–36, 1933.

[3] A. Marshall and I. Olkin, "A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families," *Biometrika*, vol. 84, no. 3, pp. 641–652, 1997.

[4] Z. Ahmad, E. Mahmoudi, and G. G. Hamedani, "A class of claim distributions: properties, characterizations and applications to insurance claim data," *Communications in Statistics: Theory and Methods*, vol. 49, pp. 1–26, 2020.

[5] E. Gomez-Deniz and E. Calder in-Ojeda, "Financial and actuarial properties of the Beta-Pareto as a long-tail distribution," *Spanish Journal of Statistics*, vol. 2, no. 1, pp. 7–21, 2020.

[6] E. Gomez-Deniz and E. Calder in-Ojeda, "On the usefulness of the logarithmic skew normal distribution for describing claims size data," *Math- Ematical Problems in Engineering*, vol. 1-9, 2020.

[7] Z. Ahmad, E. Mahmoudi, and S. Dey, "A new family of heavy tailed distributions with an application to the heavy tailed insurance loss data," *Communications in Statistics - Simulation and Computation*, vol. 49, pp. 1–24, 2020.

[8] Z. Ahmad, E. Mahmoudi, and G. G. Hamedani, "A new extended alpha power transformed family of distributions: properties, characterizations and an application to a data set in the insurance sciences," *Communications for Statistical Applications and Methods*, vol. 28, no. 1, pp. 1–19, 2021.

[9] D. Morgenstern, "Einfache beispiele zweidimensionaler verteilungen," *Mitteilings-blatt Fur Mathematische Statistik*, vol. 8, pp. 234-235, 1956.

[10] D. A. Conway, "Farlie-Gumbel-Morgenstern distributions," in *Encyclopedia of Statistical Sciences*, Kotz and N. L. Johnson, Eds., vol. 3, pp. 28–31, Wiley, New York, 1983.

[11] D. Kundu and A. Joarder, "Analysis of Type-II progressively hybrid censored data," *Computational Statistics & Data Analysis*, vol. 50, no. 10, pp. 2509–2528, 2006.

[12] M. Dube, R. Garg, and H. Krishna, "On progressively first failure censored Lindley distribution," *Computational Statistics*, vol. 31, no. 1, pp. 139–163, 2016.

[13] I. Pobočíková1, Z. Sedliačková1, and M. Michalková, "Transmuted Weibull distribution and its applications," *MATEC Web of Conferences*, vol. 157, Article ID 08007, 2018.

[14] W. Emam and K. S. Sultan, "Bayesian and maximum likelihood estimations of the Dagum parameters under combined-unified hybrid censoring," *MBE*, vol. 8, no. 3, p. 2930—2951, 2021.

[15] R. E. Barlow, R. H. Toland, and T. Freeman, "A Bayesian analysis of stress rupture life of Kevlar 49/epoxy spherical pressure vessels," in *Proceedings of the Conference on Applications of Statistics*Marcel dekker, New York, NY, USA, 1984.

[16] D. F. Andrews and A. M. Herzberg, *Data: A Collection of Problem from Many Fields for the Student and Researcher Worker*, Springer Series in Statistics, New York, NY, USA, 1985.

[17] I. B. Abdul-Moniem, "Transmuted burr type III distribution," *Journal of Statistics: Advances in Theory and Applications*, vol. 14, no. 1, pp. 37–47, 2015.

[18] E. A. Owoloko, P. E. Oguntunde, and A. O. Adejumo, "Performance rating of the transmuted exponential distribution: an analytical approach," *SpringerPlus*, vol. 4, no. 1, p. 818, 2015.

[19] T. Bjerkedal, "Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercile bacilli," *American Journal of Hygiene*, vol. 72, pp. 100–148, 1960.

[20] E. A. Owoloko, P. E. Oguntunde, and A. O. Adejumo, "A comparative analysis on the performance of the convoluted exponencial distribution and the exponential distribution in terms of flexibility," *Journal of Mathematics and Statistics*, vol. 12, 2016.