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Research Article

On Characterization of Graphs Structures Connected with Some Algebraic Properties

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In this paper, we have characterized graph structures connected with some algebraic properties. Also, this paper is actually the concatenation of graph theory and algebra. We have introduced left and right inverse graphs of antiautomorphic inverse property loops. Also, there is a connection between bipartite graphs and mathematical structures, commutator subloop, associator subloop, and associative part, the nucleus of the loop, through edge labeling.

1. Introduction and Definitions

Ruth Moufang, a German geometer, invented the quasigroup, an extension of the nonassociative algebraic structure loop. In her Ph.D. thesis, she linked this algebraic structure to the nondesarguesian plane. Moufang was the first mathematician to propose nonassociative mathematical structures, following David Hilbert's work on projective geometry [1]. She established that the octonion coordinates of the Cayley plane did not meet associative law using Desargues' theorem. The Moufang loop is an algebraic structure that is connected to the Moufang plane. The tabular depiction of a loop or a quasigroup is known as a Latin square. We can solve issues in a variety of fields of mathematics using quasigroups and loops [2]. Mutually orthogonal quasigroups enable the creation of finite projective planes, which is one of the fundamental combinatorial problems associated with quasigroups, and algebraic nets are strongly related to loop structures in geometry via partial planes, projective planes, and affine planes [3, 4].

In probabilistic reasoning of the experimental setup, blocking is the structuring of experimental material in clusters (blocks) that are analogous to one another. Blocking can be employed to solve the pseudoreplication problem. Row-column configurations for two limiting components in the research setup are known as Latin squares [5, 6]. An error correction code, in coding theory, information theory, telecommunications, and computing to control data errors over noisy communication or unreliable channels [7, 8] in cases where communication is disrupted by more types of sets of Latin squares that are orthogonal to each other have found application as error-correcting codes [9–11]. Determining if a substantially filled square can be completed to generate a Latin square is NP-complete [12].

Through the use of combinatorics and group theory, Sudoku puzzles can be examined mathematically to solve several well-known problems. Latin squares can also be found in the more contemporary KenKen puzzles. Kamisado is a two-player abstract strategy board game played on a multicoloured board of order 8. Each player is in charge of an eight-piece octagonal dragon tower. Latin squares have been used in a number of board games, notably the widely used theoretical authentic puzzle game Kamisado [13]. In statistical sampling, a square grid comprising sample sites is considered as a Latin square if each row and column has only one sample [14]. A Latin hypercube is a multidimensional generalization of this concept, for each example being the only one in each coordinate axes hyperplane that incorporates it [15, 16]. According to graph theory, a rook's graph is a mathematical structure that reflects all valid actions of the rook chess set on a chessboard. Each node of a rook's graph indicates a face on a chessboard, but each edge depicts a permissible move from one face to the next. Two primary ways to describe these very same graphs mathematically are the Cartesian products of two simple graphs and the line graphs of bipartite graphs. In other words, we can say a graph with Latin squares as its colorings are known as Rook's graph. In [17], the author proved that all three nuclei of antiautomorphic inverse property loops are coincident.

The work on the seven bridges of Konigsberg, written by Leonhard Euler and published in 1736, is considered the very first article in graph theory's history. This bough of mathematics plays an important role to solve many certifiable applications. Data structures are an area of computer science that employs graphs to represent data organization, computation flow, communication networks, and computing devices. Problems in social media, travel, biology, computer chip design, mapping the evolution of neurodegenerative disorders, and many other areas can all benefit from a similar approach (see [18, 19]). Additionally, the Feynman graphs and mathematical procedures embody particle physics concepts in a manner that is intimately related to the experimental values being investigated [20].

Algebraic graph theory and group theory are strongly interconnected. The theory of algebraic graphs has been applied to a variety of fields, including dynamic systems and complexity. In molecular biology and genomics, graphs are frequently used to model and analyse datasets with complicated interactions. Graph-based techniques are extensively used in solitary transcriptomic research to group cells into cell types. Modeling the connections among genetic variants in a process, also including biosynthetic processes and genetic regulation systems, is another application [21]. Many graph theory problems and theorems have revolved around different techniques of colouring graphs. Four color problems are an example of this great effort. Modern coding theory makes substantial use of bipartite graphs, particularly to decipher coded language received from the network. This can be seen in Tanner graphs, and a set of vertices is partitioned into codeword digits and combinations of digits that should total to zero, and factor graphs, for example [22]. Petri nets make use of the qualities of directed bipartite graphs and other capabilities to allow mathematical demonstrations of system behaviour while also making simulations of the system simple to build [23]. According to the mathematical property that every two lines have at most one common point and every two points are collinear, Levi graphs need not have any cycles of length 4; thus, their girth has to be six or higher [24]. A bipartite graph's biadjacency matrix is a matrix with 1 for each pair of nodes that are contiguous and 0 for nonadjacent nodes. Equivalences among hypergraphs, bipartite graphs, and directed graphs can be described using adjacency matrices.

A hypergraph is a mathematical construction that has vertices and edges, very much like an undirected graph, but the edges can be any collection of vertices instead of needing to have exactly two ends. The bipartite graph can be considered as the directed graph's bipartite double cover [25].

A nonempty set $\odot$ is called *groupoid* or *magma* if the binary operation $\diamond\colon \odot \times \odot \longrightarrow \odot$ is closed or internal. This algebraic structure can be denoted by $(\odot, \diamond)$. The unique solution of two equations $\zeta_1 \diamond \zeta_2 = \zeta_3$ and $\zeta_4 \diamond \zeta_1 = \zeta_3$ for ζ_2, ζ_4 implies magma $(\odot, \diamond)$ is a quasigroup $\forall \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \odot$. In the presence of identity law, quasigroup can be taken as a *loop* with same left and right neutral element. Two self-bijections $J_l\colon \odot \longrightarrow \odot$ defined by $(\zeta)J_l = \zeta^l$ and $J_r\colon \longrightarrow \odot$ defined by $(\zeta)J_r = \zeta^r$ are known as left inverse permutation and right inverse permutation, respectively, of the loop $\odot$ where ζ^l, ζ^r are said to be left inverse and right inverse of ζ, respectively, $\forall \zeta \in \odot$. A loop $\odot$ is called antiautomorphic inverse property loop (AAIPL) if and only if one of the following two equations are satisfied.

$$\begin{aligned} (\zeta_1\zeta_2)^l &= \zeta_2^l\zeta_1^l, \\ (\zeta_1\zeta_2)^r &= \zeta_2^r\zeta_1^r, \end{aligned} \tag{1}$$

$\forall \zeta_1, \zeta_2 \in \odot$. Also, we define some very important algebraic substructures of $\odot$. Sets

$$\begin{aligned}
C(\odot) &= \{[\zeta_1, \zeta_2] \in \odot \,|\, \zeta_1\diamond\zeta_2 = (\zeta_2\diamond\zeta_1)\diamond[\zeta_1, \zeta_2] \forall \zeta_1, \zeta_2 \in \odot\}, \\
A(\odot) &= \{[\zeta_1, \zeta_2, \zeta_3] \in \odot \,|\, (\zeta_1\diamond\zeta_2)\diamond\zeta_3 = (\zeta_1\diamond(\zeta_2\diamond\zeta_3))\diamond[\zeta_1, \zeta_2, \zeta_3] \forall \zeta_1, \zeta_2, \zeta_3 \in \odot\}, \\
N_l(\odot) &= \{\zeta_1 \in \odot \,|\, \zeta_1\diamond(\zeta_2\diamond\zeta_3) = (\zeta_1\diamond\zeta_2)\diamond\zeta_3 \forall \zeta_2, \zeta_3 \in \odot\}, \\
N_m(\odot) &= \{\zeta_2 \in \odot \,|\, \zeta_1\diamond(\zeta_2\diamond\zeta_3) = (\zeta_1\diamond\zeta_2)\diamond\zeta_3 \forall \zeta_1, \zeta_3 \in \odot\}, \\
N_r(\odot) &= \{\zeta_3 \in \odot \,|\, \zeta_1\diamond(\zeta_2\diamond\zeta_3) = (\zeta_1\diamond\zeta_2)\diamond\zeta_3 \forall \zeta_1, \zeta_2 \in \odot\}, \\
N(\odot) &= N_l(\odot) \cap N_m(\odot) \cap N_r(\odot), \\
Z(\odot) &= \{\zeta_1 \in \odot \,|\, \zeta_1\diamond\zeta_2 = \zeta_2\diamond\zeta_1 \forall \zeta_2 \in \odot\} \cap N(\odot),
\end{aligned} \tag{2}$$

are known as commutator subloop, associator subloop, left nucleus, middle nucleus, right nucleus, nucleus, and centre of the loop $\odot$, respectively.

A *graph* $\Gamma = (\Sigma_1, \Sigma_2)$ is a concatenation of Σ_2, an edge set, and Σ_1, a vertex set. In a graph Γ, lines represent edges or links. Distance between two vertices ζ_1 and ζ_2 is the length of the shortest path from ζ_1 to ζ_2 denoted by $d(\zeta_1, \zeta_2)$, and degree $\mu(\zeta)$ of a vertex ζ is the number of shared edges [26, 27]. We can consider Γ as a finite graph if both sets Σ_1 and Σ_2 have finite number of elements. A graph, without loops and multiple edges, Γ is taken as a simple graph and an undirected graph if the edges do not indicate directions; otherwise, it can be taken as directed graph. If there is an edge between nodes ζ_1 and ζ_2 in set Σ_2 of graph Γ, they are called neighbours (adjacent), and the edge is called incident in this case. In the undirected graph Γ, vertices of degree 0 and 1 are called isolated and pendant, respectively. If there is an edge between every two different vertices, a simple graph K_n on n vertices is taken to be *complete graph*. Due to the presence of an edge between any two distinct vertices, graph Γ is taken as *connected graph*. A connected graph, without multiple edges and loops, is known as a *network*. The graph Γ is said to be *bipartite* if there is a partition $\{\Sigma', \Sigma''\}$ of Σ_1 such that every edge in Γ connects a vertex in Σ' and a vertex in Σ''. If there must be an edge between the vertices, a bipartite graph is said to be *complete bipartite graph*, and this graph is denoted by K_{m_1,m_2} where $|\Sigma'| = m_1$, $|\Sigma''| = m_2$. Graph Γ is considered to be *star graph*, $K_{m_1,1}$, if any one of the sets Σ', Σ'' is singleton and *balanced bipartite graph*, K_{m_1,m_1}, if $m_1 = m_2$. A simple graph's *edge labeling* is a mapping $\eta\colon G \longrightarrow \Sigma_2$, from G, set of integers or symbols, to Σ_2. Let $\odot$ be any loop of finite order p with Latin square $\{a_{ij}\}_{p\times p}$ and let $X = \{X_1, X_2, \ldots, X_p\}$, $Y = \{Y_1, Y_2, \ldots, Y_p\}$ be any two disjoint sets of symbols, then the mappings $\eta_1\colon X \times Y \longrightarrow \{a_{ij}\}_{p\times p}$ defined by $(X_i, Y_j) \mapsto a_{ij}$, $\eta_2\colon \{a_{ij}\}_{p\times p} \longrightarrow \Sigma_2$ are known as P-edge labeling. Table 1 and Figure 1 represent P-edge labeling.

Now, let $\odot$ be any loop of finite-order p with Latin square $\{(a_i, a_j)\}_{p\times p}$, then the mapping $\eta\colon \{(a_i, a_j)\}_{p\times p} \longrightarrow \Sigma_2$ is known as V-edge labeling (see [28, 29]) for more details. Table 2 and Figure 2 indicate V-edge labeling.

Moreover, a directed graph $\Gamma^{(\zeta,\zeta^l)}_{(\odot,\diamond)}$ $(\Gamma^{(\zeta,\zeta^r)}_{(\odot,\diamond)})$ is called a left inverse graph (right inverse graph) if $\Sigma_1 = \odot$ and $\Sigma_2 = \{(\zeta, \zeta^l)\colon \forall \zeta \in \odot\}$ $(\Sigma_2 = \{(\zeta, \zeta^r)\colon \forall \zeta \in \odot\})$ where the edge $(\zeta, \zeta^l)((\zeta, \zeta^r))$ indicates the initial vertex ζ and final vertex ζ^l (ζ^r) in $\odot$.

2. Main Results

Let $\Omega_1, \Omega_2 = \mathbb{Z}_2$ be any nonabelian group and finite Boolean ring, respectively, with $\{\beta_1^{(\vartheta_1,\vartheta_2)}, \beta_2^{(\vartheta_1,\vartheta_2)}, \beta_3^{(\vartheta_1,\vartheta_2)}, \beta_4, \beta_5, \beta_6\}$ be the nonempty set of mappings $\beta_1^{(\vartheta_1,\vartheta_2)}\colon \Omega_1 \longrightarrow \Omega_1$, $\beta_2^{(\vartheta_1,\vartheta_2)}\colon \Omega_1 \longrightarrow \Omega_1$, $\beta_3^{(\vartheta_1,\vartheta_2)}\colon \Omega_1 \times \Omega_1 \longrightarrow \Omega_1 \times \Omega_1$, $\beta_4\colon \Omega_1 \longrightarrow \Omega_1$, $\beta_5\colon \Omega_1 \times \Omega_1 \longrightarrow \Omega_1 \times \Omega_1$, $\beta_6\colon \Omega_1 \times \Omega_1 \longrightarrow \Omega_1$ where $(\varpi)\beta_4 = \varpi^{-1}$, inverse function, $(\varpi_1, \varpi_2)\beta_5 = (\varpi_2, \varpi_1)$, reverse function, $(\varpi_1, \varpi_2)\beta_6 = \varpi_2\varpi_1 \forall \varpi, \varpi_1, \varpi_2 \in \Omega_1$, and $\forall\, \vartheta_1, \vartheta_2 \in \Omega_2$.

Table 1: Latin square of order 4.

(η_1, η_2)	Y_1	Y_2	Y_3	Y_4
X_1	a	b	c	d
X_2	b	c	d	a
X_3	c	d	a	b
X_4	d	a	b	c

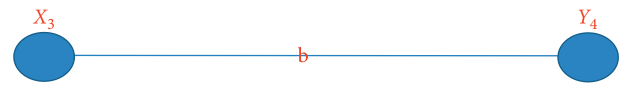

Figure 1: P-edge labeling.

Table 2: Loop of order 4.

(a, 0)	(a, 1)	(b, 0)	(b, 1)
(a, 1)	(a, 0)	(b, 1)	(b, 0)
(b, 0)	(b, 1)	(a, 1)	(a, 0)
(b, 1)	(b, 0)	(a, 0)	(a, 1)

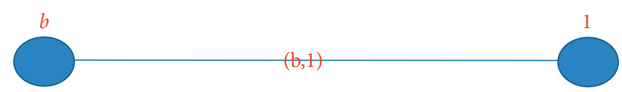

Figure 2: V-edge labeling.

Theorem 1. *Let $\diamond$ be the binary operation on the set $\odot = \Omega_1 \times \Omega_2$ defined by*

$$(\varpi_1, \vartheta_1) \diamond (\varpi_2, \vartheta_2) = \left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1 + \vartheta_2\right), \tag{3}$$

with

$$\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2 = \left(\varpi_1 \beta_1^{(\vartheta_1,\vartheta_2)}, \varpi_2 \beta_2^{(\vartheta_1,\vartheta_2)}\right) \beta_3^{(\vartheta_1,\vartheta_2)} \beta_6, \tag{4}$$

where $\beta_1^{(\vartheta_1,\vartheta_2)} = \beta_4^{\vartheta_2}, \beta_2^{(\vartheta_1,\vartheta_2)} = \beta_4^{\vartheta_1}, \beta_3^{(\vartheta_1,\vartheta_2)} = \beta_5^{\vartheta_1\vartheta_2}$ $\forall \varpi_1, \varpi_2 \in \Omega_1$, and $\forall\, \vartheta_1, \vartheta_2 \in \Omega_2$. Then, $(\odot, \diamond)$ is the loop. Moreover, $\Gamma^P_{(\odot,\diamond)} = K_{2\Delta,2\Delta}$ for the cardinality Δ of Ω_1.

Proof. Clearly by the definition of $\diamond$ algebraic structure $(\odot, \diamond)$ is groupoid or magma. And, $\forall (\varpi, \vartheta) \in \odot$, then there exists $(1, 0) \in \odot$ such that

$$\begin{aligned}
(\varpi, \vartheta) \diamond (1, 0) &= (1, 0) \diamond (\varpi, \vartheta) = (\varpi, \vartheta),\\
(\varpi, \vartheta) \diamond (1, 0) &= \left(\varpi \underset{\vartheta,0}{\wedge} 1, \vartheta + 0\right),\\
&\quad \left(\left(\varpi \beta_4^0, 1\beta_4^\vartheta\right)\beta_5^0 \beta_6, \vartheta\right)\\
&= ((\varpi, 1)\beta_6, \vartheta)\\
&= (\varpi, \vartheta).
\end{aligned} \tag{5}$$

And,

$$(1,0)\diamond(\varpi,\vartheta)=\left(1\underset{0,\vartheta}{\wedge}\varpi,0+\vartheta\right)$$
$$\left(\left(1\beta_4^{\vartheta},\varpi\beta_4^{0}\right)\beta_5^{0}\beta_6,\vartheta\right)$$
$$\left((1,\varpi)\beta_6,\vartheta\right)$$
$$(\varpi,\vartheta). \tag{6}$$

Now, $\forall(\varpi,\vartheta)\in\odot$, then there exists $(\varpi^{-1},-\vartheta)\in\odot$ such that

$$\begin{aligned}(\varpi,\vartheta)\diamond\left(\varpi^{-1},-\vartheta\right)&=\left(\varpi^{-1},-\vartheta\right)\diamond(\varpi,\vartheta)=(1,0),\\(\varpi,\vartheta)\diamond(\varpi,\vartheta)^{r}&=(\varpi,\vartheta)\diamond\left(\varpi^{-1},-\vartheta\right)\\&=\left(\varpi\underset{\vartheta,-\vartheta}{\wedge}\varpi^{-1},\vartheta-\vartheta\right)\\(\varpi,\vartheta)\diamond(\varpi,\vartheta)^{r}&=\left(\varpi\underset{\vartheta,-\vartheta}{\wedge}\varpi^{-1},0\right).\end{aligned}\tag{7}$$

Case 1: If $\vartheta=0$,

$$\begin{aligned}\varpi\underset{\vartheta,-\vartheta}{\wedge}\varpi^{-1}&=\left(\varpi\beta_4^{-\vartheta},\varpi^{-1}\beta_4^{\vartheta}\right)\beta_5^{(\vartheta)(-\vartheta)}\beta_6\\&=\left(\varpi,\varpi^{-1}\right)\beta_6\\&=\varpi^{-1}\varpi\\&=1.\end{aligned}\tag{8}$$

Case 2: If $\vartheta=1$,

$$\begin{aligned}\varpi\underset{\vartheta,-\vartheta}{\wedge}\varpi^{-1}&=\left(\varpi\beta_4^{-\vartheta},\varpi^{-1}\beta_4^{\vartheta}\right)\beta_5\beta_6\\&=\left(\varpi^{-1},\varpi\right)\beta_5\beta_6\\&=\left(\varpi,\varpi^{-1}\right)\beta_6\\&=\varpi^{-1}\varpi\\&=1\\(\varpi,\vartheta)^{l}\diamond(\varpi,\vartheta)&=\left(\varpi^{-1},-\vartheta\right)\diamond(\varpi,\vartheta)\\&=\left(\varpi^{-1}\underset{-\vartheta,\vartheta}{\wedge}\varpi,-\vartheta+\vartheta\right)\\&=\left(\varpi^{-1}\underset{-\vartheta,\vartheta}{\wedge}\varpi,0\right)\\(\varpi,\vartheta)^{l}\diamond(\varpi,\vartheta)&=\left(\varpi^{-1}\underset{-\vartheta,\vartheta}{\wedge}\varpi,0\right).\end{aligned}\tag{9}$$

Case 3: If $\vartheta=0$,

$$\begin{aligned}\varpi^{-1}\underset{-\vartheta,\vartheta}{\wedge}\varpi&=\left(\varpi^{-1}\beta_4^{\vartheta},\varpi\beta_4^{-\vartheta}\right)\beta_5\beta_6\\&=\left(\varpi^{-1},\varpi\right)\beta_6\\&=\varpi\varpi^{-1}\\&=1.\end{aligned}\tag{10}$$

Case 4: If $\vartheta=1$,

$$\begin{aligned}\varpi^{-1}\underset{-\vartheta,\vartheta}{\wedge}\varpi&=\left(\varpi^{-1}\beta_4^{\vartheta},\varpi\beta_4^{-\vartheta}\right)\beta_5\beta_6\\&=\left(\varpi,\varpi^{-1}\right)\beta_5\beta_6\\&=\left(\varpi^{-1},\varpi\right)\beta_6\\&=\varpi.\varpi^{-1}\\&=1.\end{aligned}\tag{11}$$

We have the result by using Case 1, Case 2 in (7) and Case 3, Case 4 in Equation 2. Since $|\Omega_1|=\Delta$ so $|(\odot,\diamond)|=2\Delta$; thus, by P-edge labeling and without taking care of $\diamond$, the associated graph of $(\odot,\diamond)$ is $K_{2\Delta,2\Delta}$. □

Theorem 2. *With the usual notations, set* $(\odot,\diamond)$ *is an antiautomorphic inverse property loop. Also,* $\Gamma_{(\odot,\diamond)}^{(\zeta,\zeta^{l})}=\Gamma_{(\odot,\diamond)}^{(\zeta,\zeta^{r})}\forall\zeta\in(\odot,\diamond)$.

Proof. $\forall(\varpi_1,\vartheta_1),(\varpi_2,\vartheta_2)\in(\odot,\diamond)$

$$\begin{aligned}&[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]^{r}=\left(\varpi_1\wedge_{\vartheta_1,\vartheta_2}\varpi_2,\vartheta_1+\vartheta_2\right)^{r},\\&[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]^{r}=\left(\left(\varpi_1\beta_4^{\vartheta_2},\varpi_2\beta_4^{\vartheta_1}\right)\beta_5^{\vartheta_1\vartheta_2}\beta_6,\vartheta_1+\vartheta_2\right)^{r},\\&(\varpi_2,\vartheta_2)^{r}\diamond(\varpi_1,\vartheta_1)^{r}=\left(\varpi_2^{-1},-\vartheta_2\right)\diamond\left(\varpi_1^{-1},-\vartheta_1\right),\\&\left(\varpi_2^{-1}\underset{-\vartheta_2,-\vartheta_1}{\wedge}\varpi_1^{-1},-\vartheta_2-\vartheta_1\right)\\&\left(\left(\varpi_2^{-1}\beta_4^{-\vartheta_1},\varpi_1^{-1}\beta_4^{-\vartheta_2}\right)\beta_5^{(-\vartheta_2)(-\vartheta_1)}\beta_6,-\vartheta_2-\vartheta_1\right)\\&(\varpi_2,\vartheta_2)^{r}\diamond(\varpi_1,\vartheta_1)^{r}=\left(\left(\varpi_2^{-1}\beta_4^{-\vartheta_1},\varpi_1^{-1}\beta_4^{-\vartheta_2}\right)\beta_5^{(-\vartheta_2)(-\vartheta_1)}\beta_6,-\vartheta_2-\vartheta_1\right).\end{aligned}\tag{12}$$

Case 1: If $\vartheta_1=0$ and $\vartheta_2=0$,

$$\begin{aligned}[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]^{r}&=((\varpi_1,\varpi_2)\beta_6,\vartheta_1+\vartheta_2)^{r}\\&=(\varpi_2\varpi_1,\vartheta_1+\vartheta_2)^{r}\\&=\left(\varpi_1^{-1}\varpi_2^{-1},-\vartheta_1-\vartheta_2\right)\\(\varpi_2,\vartheta_2)^{r}\diamond(\varpi_1,\vartheta_1)^{r}&=\left(\varpi_2^{-1},\varpi_1^{-1}\right)\beta_6,-\vartheta_2-\vartheta_1)\\&\left(\varpi_1^{-1}\varpi_2^{-1},-\vartheta_2-\vartheta_1\right).\end{aligned}\tag{13}$$

Case 2: If $\vartheta_1=1$ and $\vartheta_2=0$,

$$\begin{aligned}[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]^{r}&=\left(\left(\varpi_1,\varpi_2^{-1}\right)\beta_6,\vartheta_1+\vartheta_2\right)^{r}\\&=\left(\varpi_2^{-1}\varpi_1,\vartheta_1+\vartheta_2\right)^{r}\\&=\left(\varpi_1^{-1}\varpi_2,-\vartheta_1-\vartheta_2\right)\\(\varpi_2,\vartheta_2)^{r}\diamond(\varpi_1,\vartheta_1)^{r}&=(\varpi_2,\vartheta_2)^{r}\diamond(\varpi_1,\vartheta_1)^{r}\\&=\left(\varpi_2^{-1},-\vartheta_2\right)\left(\varpi_1^{-1},-\vartheta_1\right)\\&=\left(\varpi_2^{-1}\underset{-\vartheta_2,-\vartheta_1}{\wedge}\varpi_1^{-1},-\vartheta_2-\vartheta_1\right)\\&=\left(\left(\varpi_2^{-1}\beta_4^{-\vartheta_1},\varpi_1^{-1}\beta_4^{-\vartheta_2}\right)\beta_5^{(-\vartheta_2)(-\vartheta_1)}\beta_6,-\vartheta_2-\vartheta_1\right)\\&=\left(\left(\varpi_2,\varpi_1^{-1}\right)\beta_6,-\vartheta_2-\vartheta_1\right)\\&=\left(\varpi_1^{-1}\varpi_2,-\vartheta_2-\vartheta_1\right).\end{aligned}\tag{14}$$

Case 3: If $\vartheta_1 = 0$ and $\vartheta_2 = 1$,

$$\begin{aligned}[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]^r &= \left(\left(\varpi_1^{-1},\varpi_2\right)\beta_6,\vartheta_1+\vartheta_2\right)^r\\ &= \left(\varpi_2\varpi_1^{-1},\vartheta_1+\vartheta_2\right)^r\\ &= \left(\varpi_1\varpi_2^{-1},-\vartheta_1-\vartheta_2\right)\\ (\varpi_2,\vartheta_2)^r\diamond(\varpi_1,\vartheta_1)^r &= \left(\left(\varpi_2^{-1},\varpi_1\right)\beta_6,-\vartheta_2-\vartheta_1\right)\\ &\quad \left(\varpi_1\varpi_2^{-1},-\vartheta_2-\vartheta_1\right).\end{aligned} \tag{15}$$

Case 4: If $\vartheta_1 = 1$ and $\vartheta_2 = 1$,

$$\begin{aligned}[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]^r &= \left(\left(\varpi_1^{-1},\varpi_2^{-1}\right)\beta_5\beta_6,\vartheta_1+\vartheta_2\right)^r\\ &= \left(\varpi_1^{-1}\varpi_2^{-1},\vartheta_1+\vartheta_2\right)^r\\ &= (\varpi_2\varpi_1,-\vartheta_1-\vartheta_2)\\ (\varpi_2,\vartheta_2)^r\diamond(\varpi_1,\vartheta_1)^r &= (\varpi_2,\varpi_1)\beta_5\beta_6,-\vartheta_2-\vartheta_1)\\ &= (\varpi_2\varpi_1,-\vartheta_2-\vartheta_1).\end{aligned} \tag{16}$$

Therefore, the loop $(\odot,\diamond)$ is AAIPL by Equations (12) and (13) and these four cases. Moreover, we have seen that the left and right inverse graphs of any arbitrary element ζ of $(\odot,\diamond)$ are the same. So, the left and right inverse graphs of this AAIPL are the same, $\Gamma^{(\zeta,\zeta')}_{(\odot,\diamond)} = \Gamma^{(\zeta,\zeta')}_{(\odot,\diamond)} \forall \zeta \in (\odot,\diamond)$. □

Theorem 3. *Let $\odot$ be the AAIPL. Then, the set $dr(\Omega_1)\times\{0\}$ is a commutator subloop of $\odot$. Moreover, $\Gamma^V_{C(\odot)} = K_{1,\Delta'}$ where $\Delta' = |dr(\Omega_1)|$ and $dr(\Omega_1)$ are the derived subgroup of Ω_1.*

Proof. Let (ϖ,ϑ) be the commutator of any two elements (ϖ_1,ϑ_1) and (ϖ_2,ϑ_2) in the loop $(\odot,\diamond)$.

Case 1: If $\vartheta_1 = 0$ and $\vartheta_2 = 0$,

$$\begin{aligned}(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2) &= [(\varpi_2,\vartheta_2)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi,\vartheta),\\ \left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2,\vartheta_1+\vartheta_2\right) &= \left(\varpi_2 \underset{\vartheta_2,\vartheta_1}{\wedge} \varpi_1,\vartheta_2+\vartheta_1\right)\diamond(\varpi,\vartheta),\\ \left(\left(\varpi_1\beta_4^{\vartheta_2},\varpi_2\beta_4^{\vartheta_1}\right)\beta_5^{\vartheta_1\vartheta_2}\beta_6,\vartheta_1+\vartheta_2\right) &= \left(\left(\varpi_2\beta_4^{\vartheta_1},\varpi_1\beta_4^{\vartheta_2}\right)\beta_5^{\vartheta_2\vartheta_1}\beta_6,0\right)\diamond(\varpi,\vartheta),\\ ((\varpi_1,\varpi_2)\beta_6,0) &= ((\varpi_2,\varpi_1)\beta_6,0)\diamond(\varpi,\vartheta),\\ (\varpi_2\varpi_1,0) &= (\varpi_1\varpi_2,0)\diamond(\varpi,\vartheta),\\ &= \left(\varpi_1\varpi_2 \underset{0,\vartheta}{\wedge} \varpi,\vartheta\right)\\ &= \left(\left(\varpi_1\varpi_2\beta_4^{\vartheta},\varpi\beta_4^{0}\right)\beta_5^{(0)(\vartheta)}\beta_6,\vartheta\right)\\ &= \left(\left(\varpi_1\varpi_2\beta_4^{\vartheta},\varpi\right)\beta_6,\vartheta\right)\\ &= \left(\varpi\left(\varpi_1\varpi_2\beta_4^{\vartheta}\right),\vartheta\right)\\ &= (\varpi(\varpi_1\varpi_2),\vartheta)\\ (\varpi_2\varpi_1,0) &= (\varpi(\varpi_1\varpi_2),\vartheta).\end{aligned} \tag{17}$$

Equation (17) is true only when $\varpi = \varpi_2\varpi_1\varpi_2^{-1}\varpi_1^{-1}$ and $\vartheta = 0$.

Case 2: If $\vartheta_1 = 1$ and $\vartheta_2 = 0$,

$$\begin{aligned}(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2) &= [(\varpi_2,\vartheta_2)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi,\vartheta),\\ \left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2,\vartheta_1+\vartheta_2\right) &= \left(\varpi_2 \underset{\vartheta_2,\vartheta_1}{\wedge} \varpi_1,\vartheta_2+\vartheta_1\right)\diamond(\varpi,\vartheta),\\ \left(\left(\varpi_1\beta_4^{\vartheta_2},\varpi_2\beta_4^{\vartheta_1}\right)\beta_5^{\vartheta_1\vartheta_2}\beta_6,1\right) &= \left(\varpi_2 \underset{\vartheta_2,\vartheta_1}{\wedge} \varpi_1,1\right)\diamond(\varpi,\vartheta),\\ \left(\left(\varpi_1,\varpi_2^{-1}\right)\beta_6,1\right) &= \left(\left(\varpi_2\beta_4^{\vartheta_1},\varpi_1\beta_4^{\vartheta_2}\right)\beta_5^{(\vartheta_2)(\vartheta_1)}\beta_6,1\right)\diamond(\varpi,\vartheta),\\ \left(\varpi_2^{-1}\varpi_1,1\right) &= \left(\left(\varpi_2^{-1},\varpi_1\right)\beta_6,1\right)\diamond(\varpi,\vartheta)\\ &= \left(\varpi_1\varpi_2^{-1},1\right)\diamond(\varpi,\vartheta)\\ &= \left(\varpi_1\varpi_2^{-1} \underset{1,\vartheta}{\wedge} \varpi,1+\vartheta\right)\\ &= \left(\left(\varpi_1\varpi_2^{-1}\beta_4^{\vartheta},\varpi\beta_4^{1}\right)\beta_5^{(1)(\vartheta)}\beta_6,1+\vartheta\right)\\ &= \left(\left(\varpi_1\varpi_2^{-1},\varpi^{-1}\right)\beta_6,1+\vartheta\right)\\ &= \left(\varpi^{-1}\left(\varpi_1\varpi_2^{-1}\right),1+\vartheta\right)\\ \left(\varpi_2^{-1}\varpi_1,1\right) &= \left(\varpi^{-1}\left(\varpi_1\varpi_2^{-1}\right),1+\vartheta\right).\end{aligned} \tag{18}$$

Equation (18) holds when $\varpi = \varpi_1\varpi_2^{-1}\varpi_1^{-1}\varpi_2$ and $\vartheta = 0$.

Case 3: If $\vartheta_1 = 0$ and $\vartheta_2 = 1$,

$$
\begin{aligned}
(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2) &= [(\varpi_2,\vartheta_2)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right) &= \left(\varpi_2 \underset{\vartheta_2,\vartheta_1}{\wedge} \varpi_1, \vartheta_2+\vartheta_1\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, 1\right) &= \left(\left(\varpi_2\beta_4^{\vartheta_1},\varpi_1\beta_4^{\vartheta_2}\right)\zeta^{\vartheta_2\vartheta_1}\beta_6, 1\right)\diamond(\varpi,\vartheta),\\
\left(\left(\varpi_1\beta_4^{\vartheta_2},\varpi_2\beta_4^{\vartheta_1}\right)\zeta^{\vartheta_1\vartheta_2}\beta_6, 1\right) &= \left(\left(\varpi_2,\varpi_1^{-1}\right)\beta_6, 1\right)\diamond(\varpi,\vartheta),\\
\left(\left(\varpi_1^{-1},\varpi_2\right)\beta_6, 1\right) &= \left(\varpi_1^{-1}\varpi_2, 1\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_2\varpi_1^{-1}, 1\right) &= \left(\varpi_1^{-1}\varpi_2 \underset{1,\vartheta}{\wedge} \varpi, 1+\vartheta\right)\\
&= \left(\left(\varpi_1^{-1}\varpi_2\beta_4^{\vartheta},\varpi\beta_4\right)\zeta^{\vartheta}\beta_6, 1+\vartheta\right)\\
&= \left(\varpi^{-1}\left(\varpi_1^{-1}\varpi_2\right), 1+\vartheta\right)\\
\left(\varpi_2\varpi_1^{-1}, 1\right) &= \left(\varpi^{-1}\left(\varpi_1^{-1}\varpi_2\right), 1+\vartheta\right).
\end{aligned}
\tag{19}
$$

Equation (19) holds when $\varpi = \varpi_1^{-1}\varpi_2\varpi_1\varpi_2^{-1}$ and $\vartheta = 0$.

Case 4: If $\vartheta_1 = 1$ and $\vartheta_2 = 1$,

$$
\begin{aligned}
(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2) &= [(\varpi_2,\vartheta_2)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right) &= \left(\varpi_2 \underset{\vartheta_2,\vartheta_1}{\wedge} \varpi_1, \vartheta_2+\vartheta_1\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, 0\right) &= \left(\left(\varpi_2\beta_4^{\vartheta_1},\varpi_1\beta_4^{\vartheta_2}\right)\beta_5^{\vartheta_1\vartheta_2}\beta_6, 0\right)\diamond(\varpi,\vartheta),\\
\left(\left(\varpi_1\beta_4^{\vartheta_2},\varpi_2\beta_4^{\vartheta_1}\right)\beta_5^{\vartheta_1\vartheta_2}\beta_6, 0\right) &= \left(\left(\varpi_2^{-1},\varpi_1^{-1}\right)\beta_5\beta_6, 0\right)\diamond(\varpi,\vartheta),\\
\left(\left(\varpi_1^{-1},\varpi_2^{-1}\right)\beta_5\beta_6, 0\right) &= \left(\varpi_2^{-1}\varpi_1^{-1}, 0\right)\diamond(\varpi,\vartheta),\\
\left(\left(\varpi_2^{-1},\varpi_1^{-1}\right)\beta_6, 0\right) &= \left(\varpi_2^{-1}\varpi_1^{-1} \underset{0,\vartheta}{\wedge} \varpi, \vartheta\right),\\
\left(\varpi_1^{-1}\varpi_2^{-1}, 0\right) &= \left(\left(\varpi_2^{-1}\varpi_1^{-1}\beta_4^{\vartheta},\varpi\beta_4^{0}\right)\beta_5^{0}\beta_6, \vartheta\right),\\
&= \left(\varpi\left(\varpi_2^{-1}\varpi_1^{-1}\right), \vartheta\right)\\
\left(\varpi_1^{-1}\varpi_2^{-1}, 0\right) &= \left(\varpi\left(\varpi_2^{-1}\varpi_1^{-1}\right), \vartheta\right).
\end{aligned}
\tag{20}
$$

Equation (20) is true only when $\varpi = \varpi_1^{-1}\varpi_2^{-1}\varpi_1\varpi_2$ and $\vartheta = 0$. Thus, by Equations (19)–(22), the commutator subloop is $dr(\Omega_1)\times\{0\}$. And, the associated graph through V-edge labeling is $K_{1,\Delta'}$. □ □

Theorem 4. *Let $\odot$ be the AAIPL. Then, the nontrivial set $dr(\Omega_1)\times\{0\}$ is the associator subloop of $\odot$. Moreover, $\Gamma^{V}_{A(\odot)} = K_{1,\Delta'}$ where $\Delta' = |dr(\Omega_1)|$ and $dr(\Omega_1)$ are the derived subgroup of Ω_1.*

Proof. Let (ϖ,ϑ) be the associator of any three elements (ϖ_1,ϑ_1), (ϖ_2,ϑ_2), (ϖ_3,ϑ_3) in the loop $(\odot,\diamond)$.

Case 1: If $\vartheta_1 = 0$, $\vartheta_2 = 1$, $\vartheta_3 = 0$,

$$\begin{aligned}
(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [((\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2))\diamond(\varpi_3,\vartheta_3)]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{\vartheta_2,\vartheta_3}{\wedge} \varpi_3, \vartheta_2+\vartheta_3\right) &= \left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^0, \varpi_3\beta_4^1\right)\beta_5^{\vartheta_2\vartheta_3}\beta_6, 1) &= \left(\varpi_1\beta_4^{\vartheta_2}, \varpi_2\beta_4^0\right)\beta_5^{0.1}\beta_6, 1)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2,\varpi_3^{-1}\right)\beta_6, 1) &= \left(\varpi_2\varpi_1^{-1}, 1\right)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_3^{-1}\varpi_2, 1\right) &= \left(\varpi_2\varpi_1^{-1} \underset{1,\vartheta_3}{\wedge} \varpi_3, 1+\vartheta_3\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,(\vartheta_2+\vartheta_3)}{\wedge} \varpi_3^{-1}\varpi_2, 1\right) &= \left(\left(\varpi_2\varpi_1^{-1}\beta_4^0, \varpi_3\beta_4^1\right)\beta_5^{1.0}\beta_6\diamond(\varpi,\vartheta), \left(\varpi_1\beta_4^1, \varpi_3^{-1}\varpi_2\beta_4^0\right)\beta_5^{0.1}\beta_6, 1\right))\\
&= \left[\left(\varpi_3^{-1}\left(\varpi_2\varpi_1^{-1}\right), 1\right)\diamond(\varpi,\vartheta), \left(\left(\varpi_3^{-1}\varpi_2\right)\varpi_1^{-1}, \vartheta_1+1\right)\right) = \left[\varpi_3^{-1}\left(\varpi_2\varpi_1^{-1}\right) \underset{1,\vartheta}{\wedge} \varpi, 1+\vartheta\right)]\\
&= \left[\left(\varpi_3^{-1}\left(\varpi_2\varpi_1^{-1}\right)\beta_4^0, \varpi\beta_4^1\right)\beta_5^{0.1}\beta_6, 1\right)]\\
&= \left(\varpi^{-1}\varpi_3^{-1}\left(\varpi_2\varpi_1^{-1}\right), 1\right)\\
\left(\left(\varpi_3^{-1}\varpi_2\right)\varpi_1^{-1}, 1\right) &= \left(\varpi^{-1}\varpi_3^{-1}\left(\varpi_2\varpi_1^{-1}\right), 1\right).
\end{aligned} \tag{21}$$

Equation (21) holds when $\varpi = 1$ and $\vartheta = 0$.

Case 2: If $\vartheta_1 = 0$, $\vartheta_2 = 0$, $\vartheta_3 = 0$,

$$\begin{aligned}
(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [((\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2))\diamond(\varpi_3,\vartheta_3)]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{\vartheta_2,\vartheta_3}{\wedge} \varpi_3, \vartheta_2+\vartheta_3\right) &= \left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^0, \varpi_3\beta_4^0\right)\beta_6, 0) &= \left(\varpi_1\beta_4^0, \varpi_2\beta_4^0\right)\beta_6, 0)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond(\varpi_3,\varpi_2), 0) &= (\varpi_2\varpi_1, 0)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond(\varpi_3\varpi_2, 0) &= \left(\varpi_2\varpi_1 \underset{0,\vartheta_3}{\wedge} \varpi_3, 0+\vartheta_3\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{0,0}{\wedge}\right)\varpi_3\varpi_2, \vartheta_1+0)) &= \left(\left(\varpi_2\varpi_1\beta_4^0, \varpi_3\beta_4^0\right)\beta_6, 0\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1\beta_4^0, \varpi_3\varpi_2\beta_4^0\right)\beta_6, 0)) &= (\varpi_3(\varpi_2\varpi_1), 0)\diamond(\varpi,\vartheta),\\
((\varpi_3\varpi_2)\varpi_1, 0)) &= \left(\varpi_3(\varpi_2\varpi_1) \underset{0,\vartheta}{\wedge} \varpi, 0+\vartheta\right),\\
&= \left(\left(\varpi_3(\varpi_2\varpi_1)\beta_4^0, \varpi\beta_4^0\right)\beta_6, \vartheta\right)),\\
&= (\varpi\varpi_3(\varpi_2\varpi_1), \vartheta),\\
((\varpi_3\varpi_2)\varpi_1, 0) &= (\varpi\varpi_3(\varpi_2\varpi_1), \vartheta).
\end{aligned} \tag{22}$$

Equation (22) is valid if $\varpi = 1$ and $\vartheta = 0$.

Case 3: If $\vartheta_1 = 1$, $\vartheta_2 = 0$, $\vartheta_3 = 1$,

$$\begin{aligned}
(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [((\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2))\diamond(\varpi_3,\vartheta_3)]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{\vartheta_2,\vartheta_3}{\wedge} \varpi_3,\vartheta_2+\vartheta_3\right) &= \left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2,\vartheta_1+\vartheta_2\right)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^1,\varpi_3\beta_4^0\right)\beta_6,1) &= \left(\varpi_1\beta_4^0,\varpi_2\beta_4^1\right)\beta_6,1)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_3\varpi_2^{-1},1\right) &= \left(\varpi_2^{-1}\varpi_1,1\right)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_3\varpi_2^{-1},1\right) &= \left(\varpi_2^{-1}\varpi_1 \underset{1,\vartheta_3}{\wedge} \varpi_3,1+\vartheta_3\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{1,1}{\wedge} \varpi_3\varpi_2^{-1},1+1\right) &= \left(\left(\varpi_2^{-1}\varpi_1\beta_4^1,\varpi_3\beta_4^1\right)\beta_5^1\beta_6,0\right)(\varpi,\vartheta),\\
\left(\varpi_1\beta_4^1,\varpi_3\varpi_2^{-1}\beta_4^1\right)\beta^1\beta_6,0)) &= \left(\left(\varpi_1^{-1}\varpi_2\right)\varpi_3^{-1}),0\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1^{-1}\left(\varpi_2\varpi_3^{-1}\right),0)\right) &= \left(\left(\varpi_1^{-1}\varpi_2\right)\varpi_3^{-1}) \underset{0,\vartheta}{\wedge} \varpi,0+\vartheta\right))\\
&= \left(\left(\varpi_1^{-1}\varpi_2\right)\varpi_3^{-1})\beta_4^0,\varpi\beta_4^0\right)\beta_6,\vartheta))\\
&= \left(\varpi\left(\varpi_1^{-1}\varpi_2\right)\varpi_3^{-1},\vartheta\right)\\
&= \left(\left(\varpi_1^{-1}\left(\varpi_2\varpi_3^{-1}\right),0\right)\right) = \left(\varpi\left(\varpi_1^{-1}\varpi_2\right)\varpi_3^{-1},\vartheta\right).
\end{aligned} \tag{23}$$

The (23) is true for $\varpi = 1$ and $\vartheta = 0$.

Case 4: If $\vartheta_1 = 1$, $\vartheta_2 = 0$, $\vartheta_3 = 0$,

$$\begin{aligned}
(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [((\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2))\diamond(\varpi_3,\vartheta_3)]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{\vartheta_2,\vartheta_3}{\wedge} \varpi_3,\vartheta_2+\vartheta_3\right) &= \left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2,\vartheta_1+\vartheta_2\right)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^0,\varpi_3\beta_4^0\right)\beta_6,0) &= \left(\varpi_1\beta_4^0,\varpi_2\beta_4^1\right)\beta_6,1))\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond(\varpi_3,\varpi_2),1) &= \left(\varpi_2^{-1}\varpi_1,1\right)\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond(\varpi_3\varpi_2,1) &= \left(\varpi_2^{-1}\varpi_1 \underset{1,\vartheta_3}{\wedge} \varpi_3,1+\vartheta_3\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{1,\vartheta_3}{\wedge} \varpi_3\varpi_2,\vartheta_1+0\right)) &= \left(\left(\varpi_2^{-1}\varpi_1\beta_4^0,\varpi_3\beta_4^1\right)\beta_6,1\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1\beta_4^0,\varpi_3\varpi_2\beta_4^1\right)\beta_6,1)) &= \left(\varpi_3^{-1}\left(\varpi_2^{-1}\varpi_1\right),1\right)\diamond(\varpi,\vartheta),\\
\left(\left(\varpi_2^{-1}\varpi_3^{-1}\right)\varpi_1,1\right)) &= \left(\varpi_3^{-1}\left(\varpi_2^{-1}\varpi_1\right) \underset{1,\vartheta}{\wedge} \varpi,1+\vartheta\right))\\
&= \left(\left(\varpi_3^{-1}\left(\varpi_2^{-1}\varpi_1\right)\beta_4^0,\varpi\beta_4^1\right)\beta_6,1)\right)\\
\left(\left(\varpi_2^{-1}\varpi_3^{-1}\right)\varpi_1,1)\right) &= \left(\varpi\varpi_3^{-1}\left(\varpi_2^{-1}\varpi_1\right),1\right).
\end{aligned} \tag{24}$$

Case 5: If $\vartheta_1 = 0$, $\vartheta_2 = 0$, $\vartheta_3 = 1$,

$$\begin{aligned}
(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{0,1}{\wedge} \varpi_3, 0+1\right) &= \left(\left[\left(\varpi_1 \underset{0,0}{\wedge} \varpi_2, 0+0\right)\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^1,\varpi_3\beta_4^0\right)\beta_6,1) &= \left[\left(\left(\varpi_1\beta_4^0,\varpi_2\beta_4^0\right)\beta_6,0\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2^{-1},\varpi_3\right)\beta_6,1) &= [(\varpi_2\varpi_1,0)\diamond(\varpi_3,\vartheta_3)]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_3\varpi_2^{-1},1\right) &= \left(\left(\varpi_2\varpi_1 \underset{0,1}{\wedge} \varpi_3, 0+1\right)\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,1}{\wedge} \varpi_3\varpi_2^{-1}, \vartheta_1+1\right)) &= \left(\left(\varpi_2\varpi_1\beta_4^1,\varpi_3\beta_4^0\right)\beta_6,1\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1\beta_4^1,\varpi_3\varpi_2^{-1}\beta_4^0\right)\beta_6,1)) &= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right),1\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1-1,\varpi_3\varpi_2^{-1}\right)\beta_6,1)) &= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right) \underset{1,\vartheta}{\wedge} \varpi, 1+\vartheta\right),\\
\left(\left(\varpi_3\varpi_2^{-1}\right)\varpi_1^{-1},1\right)) &= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right)\beta_4^0,\varpi\beta_4^1,1\right)\beta_6\\
&= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right),\varpi^{-1}\right)\beta_6,1)\\
&= \left(\varpi^{-1}\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right),1\right)\\
\left(\left(\varpi_3\varpi_2^{-1}\right)\varpi_1^{-1},1\right)) &= \left(\varpi^{-1}\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right),1\right).
\end{aligned} \tag{25}$$

Case 6: If $\vartheta_1 = 1,\ \vartheta_2 = 1,\ \vartheta_3 = 1,$

$$\begin{aligned}
(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{1,1}{\wedge} \varpi_3, 1+1\right) &= \left(\left[\left(\varpi_1 \underset{1,1}{\wedge} \varpi_2, 1+1\right)\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^1,\varpi_3\beta_4^1\right)\beta_5\beta_6,0) &= \left[\left(\left(\varpi_1\beta_4^1,\varpi_2\beta_4^1\right)\beta_5\beta_6,0\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2^{-1},\varpi_3^{-1}\right)\beta_5\beta_6,0) &= \left[\left(\varpi_1^{-1}\varpi_2^{-1},0\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2^{-1}\varpi_3^{-1},0\right) &= \left(\left(\varpi_1^{-1}\varpi_2^{-1} \underset{0,1}{\wedge} \varpi_3, 0+1\right)\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,0}{\wedge} \varpi_2^{-1}\varpi_3^{-1}, \vartheta_1+0\right)) &= \left(\left(\varpi_1^{-1}\varpi_2^{-1}\beta_4^1,\varpi_3\beta_4^0\right)\beta_6,1\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1\beta_4^0,\varpi_2^{-1}\varpi_3^{-1}\beta_4^1\right)\beta_6,1)) &= (\varpi_3(\varpi_2\varpi_1),1)\diamond(\varpi,\vartheta),\\
((\varpi_3\varpi_2)\varpi_1)1) &= \left(\varpi_3(\varpi_2\varpi_1) \underset{1,\vartheta}{\wedge} \varpi, 1+\vartheta\right),\\
((\varpi_3\varpi_2)\varpi_1,1)) &= \left(\varpi_3(\varpi_2\varpi_1)\beta_4^0,\varpi\beta_4^1\right)\beta_6,1+\vartheta)\\
&= \left(\varpi_3(\varpi_2\varpi_1),\varpi^{-1}\right)\beta_6,1+\vartheta\right)\\
&= \left(\varpi^{-1}\varpi_3(\varpi_2\varpi_1),1+\vartheta\right)\\
((\varpi_3\varpi_2)\varpi_1,1)) &= \left(\varpi^{-1}\varpi_3(\varpi_2\varpi_1),1+\vartheta\right).
\end{aligned} \tag{26}$$

The (26) is true for $\varpi = 1$ and $\vartheta = 0$.

Case 7: If $\vartheta_1 = 1,\ \vartheta_2 = 1,\ \vartheta_3 = 0,$

$$\begin{aligned}(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{1,0}{\wedge} \varpi_3, 1+0\right) &= \left(\left[\left(\varpi_1 \underset{1,1}{\wedge} \varpi_2, 1+1\right)\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^0, \varpi_3\beta_4^1\right)\beta_6, 1) &= \left[\left(\left(\varpi_1\beta_4^1, \varpi_2\beta_4^1\right)\beta_5\beta_6, 0\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2, \varpi_3^{-1}\right)\beta_6, 1) &= \left[\left(\varpi_1^{-1}\varpi_2^{-1}, 0\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_3^{-1}\varpi_2, 1\right) &= \left(\left(\varpi_1^{-1}\varpi_2^{-1} \underset{0,0}{\wedge} \varpi_3, 0+0\right)\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,0}{\wedge} \varpi_2^{-1}\varpi_3^{-1}, \vartheta_1+1\right)) &= \left(\left(\varpi_1^{-1}\varpi_2^{-1}\beta^0, \varpi_3\beta^0\right)\beta_5, 0\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1\beta_4^1, \varpi_3^{-1}\varpi_2\beta_4^1\right)\beta_5\beta_6, 0)) &= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right), 0\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1^{-1}\left(\varpi_2^{-1}\varpi_3\right), 0\right)) &= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right) \underset{0,\vartheta}{\wedge} \varpi, 0+\vartheta\right),\\
\left(\varpi_1^{-1}\left(\varpi_2^{-1}\varpi_3\right), 0\right)) &= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}\right)\beta_4^0, \varpi\beta_4^0\right)\beta_6, \vartheta)\\
&= \left(\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}, \varpi\right)\beta_6, \vartheta\right)\\
&= \left(\varpi\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}, \vartheta\right)\left(\varpi_1^{-1}\left(\varpi_2^{-1}\varpi_3\right), 0\right)\right) = \left(\varpi\varpi_3\left(\varpi_1^{-1}\varpi_2^{-1}, \vartheta\right).\end{aligned} \tag{27}$$

Case 8: If $\vartheta_1 = 0,\ \vartheta_2 = 1,\ \vartheta_3 = 1$,

$$\begin{aligned}(\varpi_1,\vartheta_1)\diamond[(\varpi_2,\vartheta_2)\diamond(\varpi_3,\vartheta_3)] &= [(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)]\diamond(\varpi_3,\vartheta_3)\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2 \underset{1,1}{\wedge} \varpi_3, 1+1\right) &= \left(\left[\left(\varpi_1 \underset{0,1}{\wedge} \varpi_2, 0+1\right)\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2\beta_4^1, \varpi_3\beta_4^1\right)\beta_6, 0) &= \left[\left(\left(\varpi_1\beta_4^1, \varpi_2\beta_4^0\right)\beta_5\beta_6, 0\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2^{-1}\varpi_3^{-1}\right), 0) &= \left[\left(\varpi_2\varpi_1^{-1}, 1\right)\diamond(\varpi_3,\vartheta_3)\right]\diamond(\varpi,\vartheta),\\
(\varpi_1,\vartheta_1)\diamond\left(\varpi_2^{-1}\varpi_3^{-1}, 0\right) &= \left(\left(\varpi_2\varpi_1^{-1} \underset{1,1}{\wedge} \varpi_3, 0\right)\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1 \underset{\vartheta_1,0}{\wedge} \varpi_2^{-1}\varpi_3^{-1}, \vartheta_1+0\right)) &= \left(\left(\varpi_2\varpi_1^{-1}\beta^1, \varpi_3\beta^1\right)\beta_5\beta_6, 0\right)\diamond(\varpi,\vartheta),\\
\left(\varpi_1\beta_4^0, \varpi_2^{-1}\varpi_3^{-1}\beta_4^0\right)\beta_6, 0)) &= \left(\left(\varpi_1\varpi_2^{-1}\right)\varpi_3^{-1}), 0)\right)\diamond(\varpi,\vartheta),\\
\left(\left(\varpi_2^{-1}\varpi_3^{-1}\right)\varpi_1, 0\right)) &= \left(\left(\varpi_1\varpi_2^{-1}\right)\varpi_3^{-1}) \underset{0,\vartheta}{\wedge} \varpi, 0+\vartheta\right),\\
\left(\left(\varpi_2^{-1}\varpi_3^{-1}\right)\varpi_1, 0\right)) &= \left(\varpi\left(\varpi_1\varpi_2^{-1}\right)\varpi_3^{-1})\beta_4^0, \varpi\beta_4^0\right)\beta_6, 0),\\
\left(\left(\varpi_2^{-1}\varpi_3^{-1}\right)\varpi_1, 0\right)) &= \left(\varpi\left(\varpi_1\varpi_2^{-1}\right)\varpi_3^{-1}), 0\right).\end{aligned} \tag{28}$$

Finally, by Equations (25)–(28), we have ϖ is any commutator of the nonabelian group Ω_1 and $\vartheta = 0$. So, the subloop $dr(\Omega_1)\times\{0\}$ is the associator subloop of $\odot$, and through V-edge labeling, $\Gamma^V_{A(\odot)} = K_{1,\Delta'}$ is the associated graph with this algebraic structure. □ □

Theorem 5. *Let $\odot$ be the AAIPL. Then, set $N(\odot)$ is the trivial subloop of $\odot$. And, $\Gamma^V_{N(\odot)} = K_{1,1}$.*

Proof. Let (ϖ,ϑ) be in the left nucleus of the loop $\odot$, then we can consider the following cases:

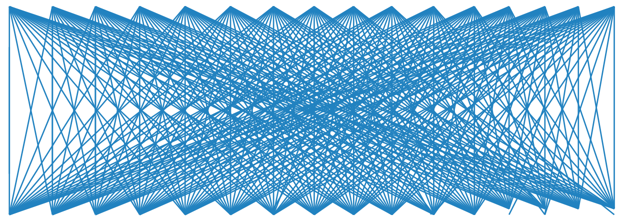

Figure 3: Associated graph of a loop $K_{16,16}$.

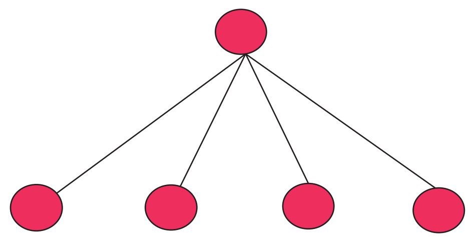

Figure 4: Star graph of the substructure $K_{1,4}$.

Figure 5: Associated graph of the nucleus $K_{1,1}$.

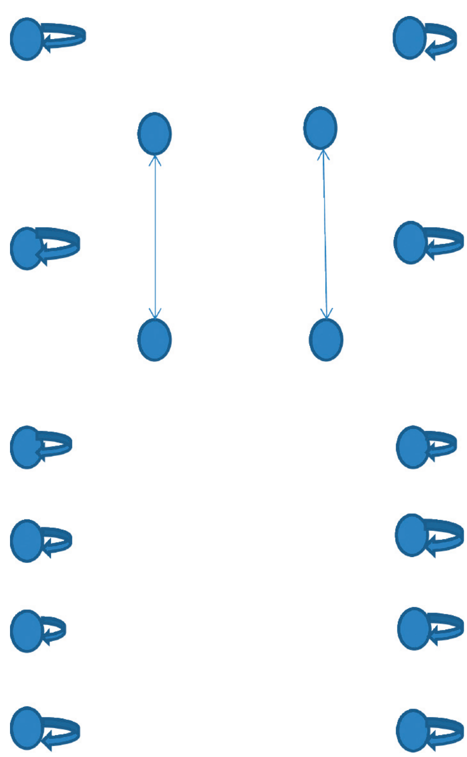

Figure 6: Inverse graph of the loop.

Table 3: AAIPL of order 16.

◇	(1, 0)	(1, 1)	$(\varpi_1,0)$	$(\varpi_1,1)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$	$(\varpi_1^3,0)$	$(\varpi_1^3,1)$	$(\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$
(1, 0)	(1, 0)	(1, 1)	$(\varpi_1,0)$	$(\varpi_1,1)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$	$(\varpi_1^3,0)$	$(\varpi_1^3,1)$	$(\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$
(1, 1)	(1, 1)	(1, 0)	$(\varpi_1^3,1)$	$(\varpi_1^3,0)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$	$(\varpi_1,1)$	$(\varpi_1,0)$	$(\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,0)$
$(\varpi_1,0)$	$(\varpi_1,0)$	$(\varpi_1^3,1)$	$(\varpi_1^2,0)$	(1, 1)	$(\varpi_1^3,0)$	$(\varpi_1,1)$	(1, 0)	$(\varpi_1^2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_2,1)$
$(\varpi_1,1)$	$(\varpi_1,1)$	$(\varpi_1^3,0)$	(1, 1)	$(\varpi_1^2,0)$	$(\varpi_1^3,1)$	$(\varpi_1,0)$	$(\varpi_1^2,1)$	(1, 0)	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$
$(\varpi_1^2,0)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$	$(\varpi_1^3,0)$	$(\varpi_1^3,1)$	(1, 0)	(1, 1)	$(\varpi_1,0)$	$(\varpi_1,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,1)$
$(\varpi_1^2,1)$	$(\varpi_1^2,1)$	$(\varpi_1^2,1)$	$(\varpi_1,1)$	$(\varpi_1,0)$	(1, 1)	(1, 0)	$(\varpi_1^3,1)$	$(\varpi_1^3,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1\varpi_2,0)$
$(\varpi_1^3,0)$	$(\varpi_1^3,0)$	$(\varpi_1,1)$	(1, 0)	$(\varpi_1^2,1)$	$(\varpi_1,0)$	$(\varpi_1^3,1)$	$(\varpi_1^2,0)$	(1, 1)	$(\varpi_1\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$
$(\varpi_1^3,1)$	$(\varpi_1^3,1)$	$(\varpi_1,0)$	$(\varpi_1^2,1)$	(1, 0)	$(\varpi_1,1)$	$(\varpi_1^3,0)$	(1, 1)	$(\varpi_1^2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_2,0)$
$(\varpi_2,0)$	$(\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	(1, 0)	(1, 1)	$(\varpi_1,0)$	$(\varpi_1,1)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$	$(\varpi_1^3,0)$	$(\varpi_1^3,1)$
$(\varpi_2,1)$	$(\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	(1, 1)	(1, 0)	$(\varpi_1,1)$	$(\varpi_1^3,1)$	$(\varpi_1^2,1)$	$(\varpi_1^2,0)$	$(\varpi_1^3,1)$	$(\varpi_1,0)$
$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1^3,0)$	$(\varpi_1^3,1)$	(1, 0)	(1, 1)	$(\varpi_1,0)$	$(\varpi_1,1)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$
$(\varpi_1\varpi_2,1)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1^3,1)$	$(\varpi_1,0)$	(1, 1)	(1, 0)	$(\varpi_1,1)$	$(\varpi_1^3,0)$	$(\varpi_1^2,1)$	$(\varpi_1^2,0)$
$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$	$(\varpi_1^3,0)$	$(\varpi_1^3,1)$	(1, 0)	(1, 1)	$(\varpi_1,0)$	$(\varpi_1,1)$
$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1^2\varpi_2,)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1^2,1)$	$(\varpi_1^2,0)$	$(\varpi_1^3,1)$	$(\varpi_1,0)$	(1, 1)	(1, 0)	$(\varpi_1,1)$	$(\varpi_1^3,0)$
$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_1,0)$	$(\varpi_1,1)$	$(\varpi_1^2,0)$	$(\varpi_1^2,1)$	$(\varpi_1^3,0)$	$(\varpi_1^3,1)$	(1, 0)	(1, 1)
$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^3\varpi_2,1)$	$(\varpi_1^3\varpi_2,0)$	$(\varpi_1^2\varpi_2,1)$	$(\varpi_2,0)$	$(\varpi_1\varpi_2,1)$	$(\varpi_1\varpi_2,0)$	$(\varpi_2,1)$	$(\varpi_1^2\varpi_2,0)$	$(\varpi_1,1)$	$(\varpi_1^3,0)$	$(\varpi_1^2,1)$	$(\varpi_1^2,0)$	$(\varpi_1^3,1)$	$(\varpi_1,0)$	(1, 1)	(1, 0)

Table 4: Graphs with algebraic structures.

Ω_1	AAIPL	$C(\odot)$	$A(\odot)$	$N(\odot)$	$\Gamma^P_{(\odot,\diamond)}$	$\Gamma^V_{C(\odot)}$	$\Gamma^V_{A(\odot)}$	$\Gamma^V_{N(\odot)}$
D_4	$D_4\times\mathbb{Z}_2$	$\mathrm{dr}(D_4)\times\{0\}$	$\mathrm{dr}(D_4)\times\{0\}$	$\{1\}\times\{0\}$	$K_{8,8}$	$K_{2,1}$	$K_{2,1}$	$K_{1,1}$
D_6	$D_6\times\mathbb{Z}_2$	$\mathrm{dr}(D_6)\times\{0\}$	$\mathrm{dr}(D_6)\times\{0\}$	$\{1\}\times\{0\}$	$K_{12,12}$	$K_{3,1}$	$K_{3,1}$	$K_{1,1}$
D_8	$D_8\times\mathbb{Z}_2$	$\mathrm{dr}(D_8)\times\{0\}$	$\mathrm{dr}(D_8)\times\{0\}$	$\{1\}\times\{0\}$	$K_{16,16}$	$K_{4,1}$	$K_{4,1}$	$K_{1,1}$
D_{2n}	$D_{2n}\times\mathbb{Z}_2$	$\mathrm{dr}(D_{2n})\times\{0\}$	$\mathrm{dr}(D_{2n})\times\{0\}$	$\{1\}\times\{0\}$	$K_{4n,4n}$	$K_{n,1}$	$K_{n,1}$	$K_{1,1}$

Case 1: If $\vartheta_1=0$, $\vartheta_2=0$,

$$\begin{aligned}
(\varpi,\vartheta)\diamond[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)] &= [(\varpi,\vartheta)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right) &= \left(\varpi \underset{\vartheta,\vartheta_1}{\wedge} \varpi_1, \vartheta+\vartheta_1\right)(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond(\varpi_1\varpi_2, \vartheta_1+\vartheta_2) &= (\varpi\varpi_1, \vartheta+\vartheta_1)\diamond(\varpi_2,\vartheta_2),\\
\left(\varpi \underset{\vartheta,\vartheta_1+\vartheta_2}{\wedge} \varpi_1\varpi_2, \vartheta+(\vartheta_1+\vartheta_2)\right) &= \left(\varpi\varpi_1 \underset{\vartheta+\vartheta_1,\vartheta_2}{\wedge} \varpi_2, (\vartheta+\vartheta_1)+\vartheta_2\right),\\
((\varpi_2\varpi_1)\varpi, \vartheta+(\vartheta_1+\vartheta_2)) &= (\varpi_2(\varpi_1\varpi), (\vartheta+\vartheta_1)+\vartheta_2).
\end{aligned} \tag{29}$$

Case 2: If $\vartheta_1=0$, $\vartheta_2=1$,

$$\begin{aligned}
(\varpi,\vartheta)\diamond[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)] &= [(\varpi,\vartheta)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right) &= \left(\varpi \underset{\vartheta,\vartheta_1}{\wedge} \varpi_1, \vartheta+\vartheta_1\right)(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond\left(\varpi_1^{-1}\varpi_2, 1\right) &= (\varpi\varpi_1, 0)\diamond(\varpi_2,\vartheta_2),\\
\left(\varpi \underset{\vartheta,1}{\wedge} \varpi_1^{-1}\varpi_2, \vartheta+1\right)) &= \left(\varpi\varpi_1 \underset{0,1}{\wedge} \varpi_2, 0+\vartheta_2\right),\\
\left(\left(\varpi_2\varpi_1^{-1}\right)\varpi^{-1}, 1\right)) &= \left(\varpi_2\left(\varpi^{-1}\varpi_1^{-1}\right), 1\right).
\end{aligned} \tag{30}$$

Case 3: If $\vartheta_1=1$, $\vartheta_2=0$,

$$\begin{aligned}
(\varpi,\vartheta)\diamond[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)] &= [(\varpi,\vartheta)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right) &= \left(\varpi \underset{\vartheta,\vartheta_1}{\wedge} \varpi_1, \vartheta+\vartheta_1\right)(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond(\varpi_1\varpi_2, \vartheta_1+\vartheta_2) &= (\varpi\varpi_1, \vartheta+\vartheta_1)\diamond(\varpi_2,\vartheta_2),\\
\left(\varpi \underset{\vartheta,\vartheta_1+\vartheta_2}{\wedge} \varpi_1\varpi_2, \vartheta+(\vartheta_1+\vartheta_2)\right) &= \left(\varpi\varpi_1 \underset{\vartheta+\vartheta_1,\vartheta_2}{\wedge} \varpi_2, (\vartheta+\vartheta_1)+\vartheta_2\right),\\
\left(\left(\varpi_2^{-1}\varpi_1\right)\varpi^{-1}, \vartheta+(\vartheta_1+\vartheta_2)\right) &= \left(\varpi_2^{-1}\left(\varpi_1\varpi^{-1}\right), (\vartheta+\vartheta_1)+\vartheta_2\right).
\end{aligned} \tag{31}$$

Case 4: If $\vartheta_1=1$, $\vartheta_2=1$,

$$\begin{aligned}
(\varpi,\vartheta)\diamond[(\varpi_1,\vartheta_1)\diamond(\varpi_2,\vartheta_2)] &= [(\varpi,\vartheta)\diamond(\varpi_1,\vartheta_1)]\diamond(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond\left(\varpi_1 \underset{\vartheta_1,\vartheta_2}{\wedge} \varpi_2, \vartheta_1+\vartheta_2\right) &= \left(\varpi \underset{\vartheta,\vartheta_1}{\wedge} \varpi_1, \vartheta+\vartheta_1\right)(\varpi_2,\vartheta_2),\\
(\varpi,\vartheta)\diamond(\varpi_1\varpi_2, \vartheta_1+\vartheta_2) &= (\varpi\varpi_1, \vartheta+\vartheta_1)\diamond(\varpi_2,\vartheta_2),\\
\left(\varpi \underset{\vartheta,\vartheta_1+\vartheta_2}{\wedge} \varpi_1\varpi_2, \vartheta+(\vartheta_1+\vartheta_2)\right) &= \left(\varpi\varpi_1 \underset{\vartheta+\vartheta_1,\vartheta_2}{\wedge} \varpi_2, (\vartheta+\vartheta_1)+\vartheta_2\right),\\
\left(\left(\varpi_1^{-1}\varpi_2^{-1}\right)\varpi, \vartheta+(\vartheta_1+\vartheta_2)\right) &= \left(\varpi\varpi_1^{-1}\right)\varpi_2^{-1}, (\vartheta+\vartheta_1)+\vartheta_2).
\end{aligned} \tag{32}$$

By Equations (29)–(32), we have a trivial subloop $N(\odot)$ of $\odot$, and the associated graph through V-edge labeling is $\Gamma^V_{N(\odot)}=K_{1,1}$.

Let $\Omega_1=D_8$ and $\Omega_2=\mathbb{Z}_2$, then Figures 3–6 are the associated graphs of the loop, commutator subloop, associator subloop, the nucleus of the loop, and inverse graph of the loop, respectively, and Table 3 shows antiautomorphic inverse property loop.

Let Ω_1 be the dihedral group of order $2n$ with $dr(\Omega_1) = \{1, \varpi, \varpi^2, \ldots, \varpi^n\}$. Then, we can sum up all the above results in Table 4. □

3. Conclusion and Future Applications

In this paper, we have shown, with the help of finite Boolean ring and nonabelian groups, the construction of non-associative structures, antiautomorphic inverse property loops, and a graphical overview of loops, commutator subloop, nontrivial associator subloop, and trivial nucleus. It will be very interesting to see some other well-known mathematical structures that construct AAIPL and their associated graphs. This combination of algebra and graphs can be used in Simulink, and the adjacency matrices of left and right inverse graphs, which were presented here, can be employed in data structures. In computer systems that manage graphs, the adjacency matrix is a data structure and can be used to describe graphs. A commonly preferred data structure that is also utilised in this application is the adjacency list.

Data Availability

The data used to support the findings of this study are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

This work was equally contributed by all authors.

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