The Solutions of Legendre’s and Chebyshev’s Differential Equations by Using the Differential Transform Method

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1.Introduction

Pukhov [1] first proposed the differential transform method (DTM) in 1982. In the meantime, Zhou [2] introduced the DTM theory in 1986, which was utilized to solve nonlinear and linear initial value problems (IVP) in electric circuit analysis. The Taylor series [2] was used to develop the DTM, which was touted as a new method. It resembles a semi-analytical technique that employs the Taylor series to generate power series representations of differential equation solutions. The method is effective for obtaining approximate and exact solutions to a linear [3] and nonlinear [4] ordinary differential equations’ (ODEs) system. It is an iterative procedure for gaining differential equations’ analytic series solutions. In several previous works, the DTM was created to solve many types of integral and differential equations. Ali [5] created the DTM to solve partial differential equations (PDE), whereas Ayaz [6, 7] implemented it to differential algebraic equations.

Arikoglu and Ozkol [8] also used the DTM to solve integro-differential equations with boundary values’ conditions (BVCs). Furthermore, Odibat et al. [9] employed the DTM to solve separable kernel Volterra integral equations. The DTM was used by Tari and Ziyace [3] in solving a system of 2D nonlinear Volterra integrodifferential equations. Here, the DTM was used to solve systems of integral and integro-differential equations, fractional differential equations, multi-order fractional differential equations, time-fractional diffusion equation and the singularly perturbed Volterra integral equations [5, 9–15]. Nonlinear parabolic-hyperbolic PDEs have also been studied using the DTM. In works by Biazar and Abdul Halim-Haasan [10] and also Bervillier [11] solved the two-dimensional Fredholm integral equations. Meanwhile, Abdewahaid [16] proposed a new 1D differential transform basic formula. Meanwhile, based on the DTM [17], El-moneam, Badr, and Ahmed Msmali proposed a general scheme to solve linear first-order DE systems. Some types of Euler–Cauchy ODE have been solved using DTM by El-moneam et al. [18]. Moreover, Esmail Hesameddini and Amir Peyrovi solved Chebyshev and Legendre equations using the homotopy perturbation method (HPM) [19].

For two special nonconstant coefficient ODES, namely, the Chebyshev and Legendre equations, this study uses the PSM and DTM.

Chebyshev and Legendre polynomials have several applications in physics and mathematics. Chebyshev polynomials are essential in approximation theory, particularly, in a polynomial interpolation general theory that yields an
approximate polynomial for continuous functions having a minimum error norm. Note that Chebyshev polynomials are obtained by solving the Chebyshev DE:

\[ (1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0. \] (1)

The equation above is known as the first kind of polynomials denoted as \( T_n(x) \); meanwhile,

\[ (1 - x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + n(n + 2)y = 0, \] (2)

resembles the polynomial of the second kind expressed as \( U_n(x) \), in which \( n \) refers to a constant. The usual PSM can be used to solve Chebyshev’s DE. At \( x = \pm 1 \), these equations exhibit singular points. As a result, the series solution of equations converges only around the origin for \( |x| \leq 1 \). \( T_n(x) \) denotes a polynomial in \( x \) of degree \( n \) expressed by the relation, where \( n \) denotes a positive integer of Chebyshev’s polynomial:

\[ T_n(x) = \cos n \theta, \quad x = \cos \theta, \quad n = 0, 1, 2, \ldots \] (3)

Conversely, the second kind polynomial \( U_n(x) \) is a degree \( n \) polynomial in \( x \) defined by

\[ U_n(x) = \frac{\sin (n + 1) \theta}{\sin \theta}, \quad x = \cos \theta, \quad n = 0, 1, 2, \ldots \] (4)

The first- and second-kind polynomials are the result of certain simple calculations:

\[ T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \ldots, \]
\[ U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \ldots \] (5)

Furthermore, via (3) and (4), the equations given below for \( T_n(x) \) as well as \( U_n(x) \) are obtained as below:

\[ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots, \]
\[ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \ldots. \] (6)

Refaat and Mason and Handscomb [20, 21] include information on relations between first and second kinds of polynomials, Chebyshev polynomial zeros, integrals, orthogonality, and derivatives, as well as many other features.

The Legendre DE is the other equation that will be explored in this work,

\[ (1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0, \] (7)

in which \( n \) represents a constant. Legendre polynomials are those that have a nonnegative integer solution to these equations. The PSM, such as Chebyshev, can be used to find equations over a range of values \((-1, 1)\). The Newtonian potential theory relies heavily on the solutions to these equations. Legendre polynomials also appear in PDEs when the Laplace equation \( \Delta u = 0 \).

Legendre polynomials are defined in another way by Rodrigues’ formula:

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \bigg[(x^2 - 1)^n\bigg], \quad n = 0, 1, 2, \ldots. \] (8)

The Legendre polynomials can be obtained immediately from this definition:

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2} (3x^2 - 1)P_3(x) \]
\[ = \frac{1}{2} (5x^3 - 3x), \ldots. \] (9)

Legendre’s function of fractional order, recursive definition, orthogonality, and other Legendre polynomial features can be found in [20, 21].

The DTM’s fundamental concepts will be explained in Section 2. Next, in Section 3, we solve Legendre’s equation using the DTM and compare the result to the solution obtained using PSM in Section 4. Finally, in Section 5, we use the DTM to solve Chebyshev’s equation and compare it to the solution obtained using the PSM in Section 6.


In [1, 2, 22, 23], the DTM’s fundamental concept and theorems, as well as its applicability to numerous types of DE, are given. We shall give a review of the DTM for the reader’s convenience. In doing so, we assume that function \( f(x) \in C^{\infty}(I) \), in which \( x_0 \) denotes any point in interval \( I \). Here, Taylor’s series of \( f(x) \) about \( x_0 \) can, therefore, be represented by

\[ f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i. \] (10)

**Definition 1.** If \( f(x) \) is an analytic function on \( x_0 \), then \( f(x) \)’s \( k^{th} \) order differential transform is expressed as

\[ F(k) = D_x^k\{f(x)\} \]
\[ = \bigg[ \frac{f^{(k)}(x)}{k!} \bigg]_{x=x_0}. \] (11)

Notice that, \( F(k) \)’s inverse differential transform is expressed as
\[ D_T^{-1} [F(k)] = f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k. \] (12)

**Theorem 1.** Let \( f(x) \) and \( g(x) \) refer to analytic functions having differential transforms \( F(k) \) and \( G(k) \) accordingly; thus,

\[ D_T[\alpha f(x) + \beta g(x)] = \alpha F(k) + \beta G(k), \] (13)
in which \( \alpha \) and \( \beta \) represent constants.

**Theorem 2.** Let \( f(x) \) denote an analytic function, with differential transform \( F(k) \); then,

\[ D_T\left\{ \frac{d^m f(x)}{dx^m} \right\} = \frac{(k + m)!}{k!} F(k + m). \] (14)

**Theorem 3.** Let \( f(x) = x^m; \) then, \( F(k) = \delta_{k,m} \), where \( \delta_{k,m} \) is the Kronecker delta.

**Theorem 4.** Let \( f(x) \) and \( g(x) \) express analytic functions in which \( f(x) = \int_0^x g(x)dx \).

Thus, the function \( f(x) \)'s differential transform is expressed as

\[ F(k) = \frac{G(k-1)}{k} \] (15)

The proves of the following results can be found in [4].

**Theorem 5.** Let \( f_1(x) \) as well as \( f_2(x) \) represent analytic functions, in which \( f(x) = f_1(x). f_2(x) \); then,

\[ F(k) = \sum_{n=0}^{k} F_1(n)F_2(k-n). \] (16)

**Theorem 6.** Let \( f(x) \) resembles an analytic function with \( D_T[f(x)] = F(k) \); thus,

\[ D_T\{x^m f^{(n)}(x)\} = \sum_{i=0}^{k} \frac{(k + n - i)!}{(k - i)!} F(k + n - i), \] (17)

and if \( m = n \), then

\[ D_T\{x^m f^{(n)}(x)\} = \prod_{i=0}^{n-1} (k - i)F(k). \] (18)

**Theorem 7.** Let \( f(x) \) represent an analytic function, in which \( D_T[f(x)] = F(k) \); then,

\[ D_T\{e^{ax} f^{(n)}(x)\} = \sum_{i=0}^{k} \frac{a^i}{i!} (k + n - i)! F(k + n - i), \]

\[ D_T\{\cos(\alpha x) f^{(n)}(x)\} = \sum_{i=0}^{k} \frac{\alpha^i}{i!} \cos\left(\frac{i\pi}{2}\right) (k + n - i)! F(k + n - i), \] (19)

and

\[ D_T\{\sin(\alpha x) f^{(n)}(x)\} = \sum_{i=0}^{k} \frac{\alpha^i}{i!} \sin\left(\frac{i\pi}{2}\right) (k + n - i)! F(k + n - i). \] (20)

**Theorem 8.** Let \( f(x) \) become an analytic function, where \( D_T[f(x)] = F(k) \); then,

\[ D_T\left\{ \frac{d}{dx} (xf^{(1)}(x)) \right\} = \frac{(k + 1)(k + n - m + 1)!}{(k - m + 1)!} F(k + n - m + 1). \] (21)

Notice that, for \( n = 1 \), (21) reduces to the formula given below:

\[ D_T\left\{ \frac{d}{dx} (x f^{(1)}(x)) \right\} = (k + 1)^2 F(k + 1). \] (22)

**Theorem 9.** Let \( f(x) \) be an analytic function, while \( D_T[f(x)] = F(k) \); then,

\[ D_T\left\{ \frac{d}{dx} (x^m f^{(n)}(x)) \right\} = \frac{(k + 1)(k + n - m + 1)!}{(k - m + 1)!} F(k + n - m + 1). \] (23)

### 3. The Solutions of Legendre’s Differential Equation

Now, consider the initial value problem given by

\[ (1 - x^2) \frac{d^2 y(x)}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0, \] (24)

\[ y(0) = C_0, \]

\[ y'(0) = C_1, \] (25)

referred to as the Legendre differential equation, in which \( C_0 \) and \( C_1 \) are constants, whereas \( n \) refers to an integer.

We use the differential transform to solve the IVP (24) and (25):

\[ D_T\left\{ \frac{d^2 y(x)}{dx^2} \right\} - D_T\left\{ \frac{d}{dx} \left( x^2 \frac{dy(x)}{dx} \right) \right\} + D_T\{ (n(n + 1)y(x) \} = 0. \] (26)

Next, using the differential transform properties (14) and (17) enables us to find
\[(k + 2)(k + 1)Y'(k + 2) - k(k + 1)Y'(k) + n(n + 1)Y(k) = 0. \quad (27)\]

This leads to the recurrence relation:
\[Y'(k + 2) = \frac{1}{(k + 2)(k + 1)} \{k(k + 1) - n(n + 1)\}Y(k), \quad (28)\]
\[Y(0) = C_0, \quad Y(1) = C_1.\]

The recurrence relation (28) gives
\[Y(4) = \frac{(n - 2)(n + 3)}{4!} Y(2), \quad (n - 2)n(n + 1)(n + 3) Y(0), \quad (29)\]
\[Y(5) = \frac{(n - 3)(n + 4)}{5!} Y(3), \quad (30)\]
\[Y(6) = \frac{(n - 4)(n + 5)}{6!} Y(4), \quad (31)\]
\[\vdots \quad \text{and so on. If we choose } Y(0) = 1 \text{ and } Y(1) = 0, \text{ we obtain the solution}\]
\[y_1(x) = 1 - \frac{n(n + 1)}{2!} x^2 + \frac{(n - 2)n(n + 1)(n + 3)}{4!} x^4 + \frac{(n - 4)(n - 2)n(n + 1)(n + 3)(n + 5)}{6!} x^6 + \ldots \quad (32)\]

Meanwhile, choosing \(Y(0) = 0\) and \(Y(1) = 1\), we determine a second solution given by
\[y_2(x) = x - \frac{(n - 1)(n + 2)}{3!} x^3 + \frac{(n - 3)(n - 1)t(n + 2)n(n + 4)}{5!} x^5 + \frac{(n - 5)(n - 3)(n - 1)(n + 2)(n + 4)(n + 6)}{7!} x^7 + \ldots \quad (33)\]

Because both series converge for \(|x| < 1\), their convergence radius is unity. Since (32) contains only even power of \(x\) while (33) contains only odd powers, (24) general solution may be expressed as follows:

\[Y(2) = \frac{-n(n + 1)}{2!} Y(0), \quad (29)\]
\[Y(3) = \frac{-(n - 1)(n + 2)}{3!} Y(1), \quad (30)\]
\[Y(j + 2) = \frac{(n - j)(n + j + 1)}{(j + 2)(j + 1)} Y(j), \quad j = 0, 1, 2, 3, 4, \ldots \]

Let \(j\) take on the values \(0, 1, 2, 3, 4, \ldots \); then, the recurrence relation (30) yields
\[ y(x) = C_0 y_1(x) + C_1 y_2(x), \text{ for } |x| < 1. \]  
(34)

Note that if \( n \) refers to an even integer, then the first series vanishes; meanwhile, \( y_2(x) \) denotes an infinite series. For instance, if \( n = 4 \), hence

\[ y_4(x) = C_0 \left[ 1 - 10x^2 + \frac{35}{3}x^4 \right]. \]  
(35)

Likewise, the series for \( y_2(x) \) vanishes with \( x^n \) when \( n \) is an odd integer. To put it in another way, we get an \( n^{th} \)-degree polynomial solution of the Legendre equation when \( n \) is a non-negative integer.

A solution is a constant multiple of a Legendre equation solution. Note that it is advisable to use specific values for \( C_0 \) or \( C_1 \), which depends on whether \( n \) denotes an odd or even positive integer.

For \( n = 0 \), we select \( C_0 = 1 \); meanwhile, for \( n = 2, 4, 6, \ldots \),

\[ C_0 = (-1)^{n/2} \frac{1.3 \ldots (n - 1)}{2.4 \ldots n}, \]  
(36)

whereas, for \( n = 1 \), we choose \( C_1 = 1 \), and for \( n = 3, 5, 7, \ldots \),

\[ C_1 = (-1)^{(n - 1)/2} \frac{21.3 \ldots (n - 2)}{2.4 \ldots (n - 1)}. \]  
(37)

For example, when \( n = 4 \), we have

\[ y_1(x) = (-1)^{3/2} \frac{3}{2} \left[ 1 - 10x^2 + \frac{35}{3}x^4 \right]. \]  
(38)

Legendre polynomials, written as \( P_n(x) \), are specified \( n^{th} \)-degree polynomial solutions.

We may deduce that the first several Legendre polynomials are represented by the series for \( y_1(x) \) and \( y_2(x) \) and from the above choices of \( C_0 \) and \( C_1 \) as shown below:

\[ P_0(x) = 1, \]

\[ P_1(x) = x, \]

\[ P_2(x) = \frac{1}{2} (3x^2 - 1), \]

\[ P_3(x) = \frac{1}{2} (5x^3 - 3x), \]

\[ P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3). \]  
(39)

We can demonstrate that these polynomials are particular solutions of the DE:

\[ n = 0: \left( 1 - x^2 \right) \frac{d^2y(x)}{dx^2} - 2x \frac{dy(x)}{dx} = 0, \]

\[ n = 1: \left( 1 - x^2 \right) \frac{d^2y(x)}{dx^2} - 2x \frac{dy(x)}{dx} + 2y = 0, \]

\[ n = 2: \left( 1 - x^2 \right) \frac{d^2y(x)}{dx^2} - 2x \frac{dy(x)}{dx} + 6y = 0, \]  
(40)

\[ n = 3: \left( 1 - x^2 \right) \frac{d^2y(x)}{dx^2} - 2x \frac{dy(x)}{dx} + 12y = 0, \]

\[ \vdots \]

\section*{4. The Solution of Legendre’s Equation Based on the Power Series Method}

We solve the Legendre (24) with the PSM and compare our results to the power series results.

There exist two linearly independent solutions of the form \( y(x) = \sum_{n=0}^{\infty} a_n x^n \) since \( x = 0 \) resembles an ordinary point of (24). We discover that when we substitute

\[ \sum_{n=0}^{\infty} \left[ (n + 1)(n + 2) \right] a_n x^n = 0. \]  
(41)

Upon collecting terms gives

\[ \sum_{n=0}^{\infty} \left[ (n + 2)(n + 1) a_{n+2} - n(n-1) a_n \right] x^n = 0. \]  
(42)

The recurrence relation is, therefore, given by

\[ a_{n+2} = \frac{[(n+1)(n+2) \mu_2(n+1)]}{(n+1)(n+2)} a_n \]  
(43)

If we choose a \( a_0 = 1 \) and \( a_1 = 0 \), for \( n = 0, 1, 2, \ldots \), we get the solution:

\[ y_1(x) = 1 - \frac{\mu_2(n+1)}{2!} x^2 + \frac{\mu_2(n+1)(n+2)(n+3)}{4!} x^4 - \ldots. \]  
(44)

Alternatively, we can find a second solution by setting \( a_0 = 0 \) and \( a_1 = 1 \):

\[ y_2(x) = x - \frac{\mu_2(n+1)(n+2)}{3!} x^3 + \frac{\mu_2(n+1)(n+2)(n+3)(n+4)}{5!} x^5 - \ldots. \]  
(45)

Since (44) contains only the even power of \( x \) and (45) has only odd powers, equation (24) general solution can be expressed as

\[ y(x) = C_1 y_1(x) + C_2 y_2(x), \text{ for } |x| < 1. \]  
(46)
The Solution of Chebyshev’s Differential Equation (DE)

Chebyshev’s DE is given by

\[ (1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0, \]

\[ y(0) = A, \]

\[ y'(0) = nB. \tag{47} \]

The Chebyshev DE possess regular singular points at -1, 1, as well as ∞ in the case of |x| < 1. By employing the \( k^{th} \) differential transform to both sides of (47), it can be solved utilizing the DTM:

\[ (k + 1)(k + 2)Y(k + 2) - k(k - 1)Y(k) + kY(k) - n^2Y(k) = 0. \tag{48} \]

Then,

\[ Y(k + 2) = \frac{k^2 - n^2}{(k + 1)(k + 2)} Y(k). \tag{49} \]

From this, we obtain, for the even coefficients,

\[ Y(2) = \frac{n^2}{2} Y(0), \]

\[ Y(2) = \frac{2^2 - n^2}{3.4} Y(2) = \frac{(2^2 - n^2)(-n^2)}{1.2.3.4} Y(0), \tag{50} \]

\[ Y(2k) = \frac{(2k)^2 - n^2}{2k} \left[ \frac{(2k - 1)^2 - n^2}{2(k - 1)} \right] \ldots \left( -n^2 \right)^{2k} \left( 2k + 2 \right) Y(0). \]

And for the odd coefficients,

\[ Y(3) = \frac{1 - n^2}{6} Y(1), \]

\[ Y(5) = \frac{(3^2 - n^2)(1 - n^2)}{5!} Y(1), \]

\[ Y(2k - 1) = \frac{(2k - 1)^2 - n^2}{2k - 1} \left[ \frac{(2k - 3)^2 - n^2}{2(k - 1)} \right] \ldots \left( -n^2 \right)^{2k - 1} \left( 2k + 1 \right) Y(1). \tag{51} \]

Another way to solve (47) is by changing variables that give the equivalent forms of (47) as follows.

By letting \( x = \cos(t) \), we get the following differential equation:

\[ \frac{d^2y}{dt^2} + n^2 y(t) = 0. \tag{58} \]

The coefficients that are even expressed as

\[ Y(k_{even}) = \frac{Y(0) \prod_{j=1}^{k/2} (k - 2j)^2 - n^2}{\Gamma(1 - 1/2k - 1/2n) \Gamma(1 - 1/2k + 1/2n)} = \frac{2^{k-1} \pi \csc(1/2\pi n)}{\Gamma(1 - 1/2k - 1/2n) \Gamma(1 - 1/2k + 1/2n)} Y(0). \tag{52} \]

And for odd coefficients given as

\[ Y(k_{odd}) = \frac{Y(1) \prod_{j=1}^{(k-1)/2} (k - 2j)^2 - n^2}{\Gamma(1 - 1/2k - 1/2n) \Gamma(1 - 1/2k + 1/2n)} = \frac{2^{k-1} \pi \csc(1/2\pi n)}{\Gamma(1 - 1/2k - 1/2n) \Gamma(1 - 1/2k + 1/2n)} Y(1). \tag{53} \]

The general solution is then given by summing over all indices:

\[ y(x) = Y(0) \left[ 1 + \sum_{k=3,5,7,\ldots} \frac{Y(k_{even}) x^k}{k!} \right] + Y(1) \left[ x + \sum_{k=3,5,7,\ldots} \frac{Y(k_{odd}) x^k}{k!} \right]. \tag{54} \]

This may be accomplished in the closed form as follows:

\[ y(x) = Y(0) \cos(n \sin^{-1}(x)) + \frac{Y(1)}{n} \sin(n \sin^{-1}(x)). \tag{55} \]

Changing the variables yields the solution’s equivalent form:

\[ y(x) = b_1 \cos(n \cos^{-1}(x)) + b_2 \sin(n \sin^{-1}(x)), \tag{56} \]

where \( U_n(x) \) refers to a Chebyshev polynomial of the second kind whereas \( T_n(x) \) refers to a Chebyshev polynomial of the first kind.

The solution can also be written in another way:

\[ y(x) = C_1 \cosh \left( n \ln \left( x + \sqrt{x^2 - 1} \right) \right) + C_2 \sinh \left( n \ln \left( x + \sqrt{x^2 - 1} \right) \right). \tag{57} \]
\[
\frac{d^2 y(t)}{dt^2} - n^2 y(t) = 0.
\]  

(59)

When both sides of (58) and (59) are applied with the \(k^{th}\) differential transform, the result yields

\[
Y(k + 2) = -\frac{n^2}{(k + 2)(k + 1)} Y(k),
\]

(60)

and

\[
Y(k + 2) = \frac{n^2}{(k + 2)(k + 1)} Y(k).
\]

(61)

Let \(k = 0, 1, 2, 3, 4, \ldots\) be the recurrence relation (60); it then gives

\[
Y(0) = A,
\]

\[
Y(1) = nB,
\]

\[
Y(2) = \frac{n^2}{2!} A,
\]

\[
Y(3) = \frac{n^3}{3!} B,
\]

\[
Y(4) = \frac{n^4}{4!} A,
\]

\[
Y(5) = \frac{n^5}{5!} B,
\]

\[
Y(6) = \frac{n^6}{6!} A,
\]

\[
Y(7) = -\frac{n^7}{7!} B,
\]

\[
\vdots
\]

\[
\vdots
\]

Then, the solution of (44) is given by

\[
y(t) = \sum_{k=0}^{\infty} Y(k)x^k = A \left(1 - \frac{(nt)^2}{2!} + \frac{(nt)^4}{4!} - \frac{(nt)^6}{6!} + \ldots\right)
\]

\[
+ B \left(\frac{nt}{3!} - \frac{(nt)^3}{5!} + \frac{(nt)^5}{7!} - \frac{(nt)^7}{9!} + \ldots\right),
\]

\[
= A \cos(nt) + B \sin(nt).
\]

(62)

or

\[
y(x) = A \cos(n \cos^{-1} x) + B \sin(n \cos^{-1} x).
\]

(64)

Equivalently,

\[
y(x) = AT_n(x) + BU_n(x), |x| < 1,
\]

(65)

in which \(T_n(x)\) and \(U_n(x)\) denote Chebyshev polynomials of the first and second kinds, correspondingly, of degree \(n\).

Similarly, from (61), we obtain

\[
y(t) = A \cosh(nt) + B \sinh(nt),
\]

(66)

or

\[
y(t) = A \cosh(n \cosh^{-1} x) + B \sinh(n \cosh^{-1} x), |x| > 1.
\]

(67)

Equivalently,

\[
y(x) = AT_n(x) + BU_n(x), |x| > 1.
\]

(68)

6. The Solution of Chebyshev’s Equation by Using the Power Series Method (PSM)

Consider Chebyshev’s equation given by

\[
(1 - x^2) \frac{d^2 y(x)}{dx^2} - x \frac{dy}{dx} + \mu^2 y = 0.
\]

(69)

Regular singular points in the Chebyshev differential equation for \(|x| < 1\) are -1, 1, and \(\infty\). The PSM can solve it via the expansions:

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n,
\]

(70)

\[
y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}
\]

(71)

\[
y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.
\]

(72)

Now, we plug (70)–(72) into (69) to obtain
\[
(1 - x^2) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \mu^2 \sum_{n=0}^{\infty} a_n x^n = 0,
\]
\[
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} na_n x^n + \mu^2 \sum_{n=0}^{\infty} a_n x^n = 0,
\]
\[
(2a_2 + \mu^2 a_0) + \left( (\mu^2 - 1) a_1 + 6a_3 \right) x + \sum_{n=2}^{\infty} \left( (n+2)(n+1)a_{n+2} + (\mu^2 - n^2) \right) x^n = 0.
\]

So,
\[
2a_2 + \mu^2 a_0 = 0, \quad (74)
\]
\[
(\mu^2 - 1) a_1 + 6a_3 = 0, \quad (75)
\]
and by induction,
\[
a_{n+2} = \frac{n^2 - \mu^2}{(n+1)(n+2)} a_n, \quad \text{for } n = 2, 3, \ldots
\]

Since (74) and (75) are special cases of (76), the general recurrence relation can be written as
\[
a_{m+2} = \frac{n^2 - \mu^2}{(n+1)(n+2)} a_n, \quad \text{for } n = 0, 1, 2, 3, \ldots
\]

From this, we obtain, for the even coefficients,
\[
a_2 = \frac{\mu^2}{2!} a_0,
\]
\[
a_4 = \frac{(2^2 - \mu^2)(-\mu^2)}{1.2.3.4} a_0,
\]
\[
a_{2n} = \frac{\left( (2n)^2 - \mu^2 \right) \left( (2n-2)^2 - \mu^2 \right) \ldots (\mu^2)}{(2n)!} a_0,
\]
\[
\quad \text{and for the odd coefficients,}
\]
\[
a_3 = \frac{1 - \mu^2}{3!} a_1,
\]
\[
a_5 = \frac{\left( 3^2 - \mu^2 \right) \left( 1^2 - \mu^2 \right) }{5!} a_1,
\]
\[
a_{2n-1} = \frac{\left( (2n-1)^2 - \mu^2 \right) \left( (2n-3)^2 - \mu^2 \right) \ldots (1^2 - \mu^2)}{(2n+1)!} a_1.
\]

The even coefficients can be written as
\[
a_{k,\text{even}} = \frac{a_0 \prod_{j=1}^{k/2} (k-2j)^2 - \mu^2}{\Gamma(1-1/2k-1/2\mu) \Gamma(1-1/2k+1/2\mu)}
\]
\[
= \frac{2^{k-1} \pi \csc (1/2\pi n)}{\Gamma(1-1/2k-1/2\mu) \Gamma(1-1/2k+1/2\mu)} a_0.
\]

And for odd coefficients given as
\[
a_{k,\text{odd}} = \frac{1}{\Gamma(1-1/2k-1/2\mu) \Gamma(1-1/2k+1/2\mu)} a_0.
\]

The general solution is then given by summing over all indices:
\[
y(x) = a_0 \left[ 1 + \sum_{k=2,4,6,\ldots} a_{k,\text{even}} x^k \right] + a_1 \left[ x + \sum_{k=3,5,7,\ldots} a_{k,\text{odd}} x^k \right].
\]

This may be accomplished in the closed form as follows:
\[
y(x) = a_0 \cos \left( n \sin^{-1}(x) \right) + \frac{a_1}{n} \sin \left( n \sin^{-1}(x) \right).
\]

Changing the variables yields the solution’s equivalent form.
\[
y(x) = b_1 \cos \left( n \cos^{-1}(x) \right) + b_2 \sin \left( n \sin^{-1}(x) \right),
\]
\[
= b_1 T_n(x) + b_2 \sqrt{1 - x^2} U_{n-1}(x),
\]
where \( U_n(x) \) denotes Chebyshev’s polynomial of the second kind, whereas \( T_n(x) \) refers to Chebyshev polynomial of the first kind.

The findings produced using the DTM are identical.

7. Conclusion

We successfully used the DTM to solve Chebyshev’s and Legendre’s DE in this study. The current method eliminates the computing challenges associated with previous traditional methods by allowing all computations to be computed using simple manipulations. As a result, this method can be used to solve a variety of DE without the need for discretization, linearization, or perturbation. Also, the solution of Chebyshev’s equation is obtained using the power series method. In comparison between these two methods, the results are identical to the results obtained via the differential transform method.

Data Availability

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


