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Research Article

On the Solution of Fractional Order KdV Equation and Its Periodicity on Bounded Domain Using Radial Basis Functions

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The Coimbra concept of fractional order derivative is used to build a numerical approach using radial functions in this paper. The Coimbra derivative is capable of modelling a dynamic system with varying fractional order behaviour over time. The proposed scheme's stability and convergence are investigated. In one and two space dimensions, the developed approach is validated for the given model. By applying a periodic boundary condition on a bounded domain, the model's periodicity is shown statistically. The acquired findings demonstrate the new numerical scheme's potency and, as a result, its high order accuracy.

1. Introduction

The Korteweg-De Vries (KdV) equation is first derived by Boussinesq in the year 1870. Later on in 1895, the same model was retrieved by Korteweg and de Vries [1] with the presumption of compact amplitude and huge wave length. In many nonlinear dispersive physical systems, the evolution of long wave can be expressed by the KdV type equation (see [2–5] and the references therein). In mathematical sciences and engineering, evolutionary nonlinear equations play a major role to model physical phenomena [2, 6]. In the theory of shallow water waves, the KdV equation is one of the most essential equations in nonlinear evolution developed in [4] and the references therein. Some of the important aspects of solutions of these dispersive equations discovered through observations are their long-time behaviour and known periodicity in time [7]. The important event of eventual periodicity has been presented previously in [8], and in more recent work [9, 10], a new solution is reestablished corresponding to the KdV equation. In addition, the forced oscillations and the stability of the KdV equation have been carried out in a very recent work [11–14]. In applied mathematics, physics, and other related fields, a rich filed of research has evolved within the last century because of computational and analytic research on fractional and classical KdV equation [15–19].

Both the theory and application of fractional calculus have advanced dramatically in the previous two decades. The nonlocal quality of fractional calculus and its effectiveness in reproducing anomalous diffusion that happens in transport dynamics in complex systems, such as fluid motion in viscoelastic medium, are the most important advantages [20], anomalous transfer in biology [21] and porous materials [22], etc. Control theory, entropy theory, image processing, and wave propagation phenomena all employing fractional calculus can be found in [23–26]. The creation of tools to offer a mathematical structure for sophisticated physical systems and processes has been aided by breakthroughs in current variable order (VO) fractional calculus [27]. As a result of its appropriateness for modelling in a wide range of subjects, including science, engineering, and a variety of other disciplines, variable order fractional differential equations (VO-FDEs) have gained prominence [28–31]. Physical modelling utilizing VO-FDE models has been the subject of a large-scale investigation. For example, Kobelev et al. [32] highlighted the dynamical and statistical systems with varying memory difficulties where the fractal dimension changes with coordinate and time. Coimbra et al.

[33] used VO-fractional operators to investigate the visco-elasticity oscillator. Al-Mekhlafi and Sweilam [34] proposed a new multistrain TB model based on the VO-fractional derivative as a nonlinear ordinary differential equation extension. Due to the enormous number of applications, analytical and numerical techniques for solving variable order fractional order differential equations (VO-FDEs) have increased substantially in the last year. The analytical solution of VO-FDEs, on the other hand, is frequently difficult to obtain. Therefore, numerical approaches are used as sophisticated methods for numerical approximation of VO-FDEs in general [29, 35–38].

The Caputo, the Liouville, the Marchaud, the Grunwald, and the Coimbra definitions are some of the recent variable order operator definitions suggested in the literature [33, 39]. Samko et al. [39] analyzed that the Riemann variable order definitions lost some features, meaning that the Marchaud operator is better than the Riemann-Liouville type operator. Ramirez et al. [40] also compared the variable order operators such as operators due to Riemann-Liouville, Marchaud, Caputo, and Coimbra using a simple criterion: the variable order operator must return the correct fractional derivative that corresponds to the argument of the functional order. Only the operator due to Coimbra and the Marchaud satisfy the aforementioned elementary condition [40], as well as the Coimbra variable order operator is more efficient numerically. Soon et al. [41] also demonstrated that the Coimbra variable order operator satisfies a mapping requirement and that it is the only formulation that returns the necessary derivatives as a function of $x(t)$ for transitions between elastic and viscous regimes. Ramirez [40] demonstrated that the Coimbra concept is the most appropriate for physical modelling since it has essential properties that are desirable.

The numerical solution of the KdV problem of order $0 < \tau(t) < 1$ and its eventual periodicity over confined domain is achieved using RBF with Coimbra variable order derivative. The following equations represent the proposed models in both one-dimensional and two-dimensional space:

$$D_t^{\tau(t)} w(x,t) + \varepsilon w(x,t) w_x(x,t) + \nu w_{xxx}(x,t) = f(x,t),\ t > 0,\ x \in \Omega, \tag{1}$$

with the following initial condition

$$w(x,t) = g(x),\ t = 0,\ x \in \Omega, \tag{2}$$

and the boundary conditions given by

$$w(x,t) = h(x,t),\ t \ge 0,\ x \in \partial\Omega, \tag{3}$$

where $0 < \tau(t) < 1$.

$$\begin{aligned} D_t^{\tau(t)} w(x,y,t) &+ \varepsilon w^2(x,y,t) w_x(x,y,t) \\ &+ w_{xxx}(x,y,t) + w_{xyy}(x,y,t) = f(x,y,t), \end{aligned} \tag{4}$$

where $(x, y) \in \Omega$, with the following boundary and initial conditions

$$\begin{aligned} w(x,y,t) &= h(x,y,t) \in \partial\Omega\ , t \ge 0,\ w(x,y,0), \\ &= g(x,y),\ (x,y) \in \Omega. \end{aligned} \tag{5}$$

The models in the above form are selected for the sake of comparison given in [42]. The Coimbra variable order derivative is defined by (7) in the next section.

1.1. The KdV Equation. Suppose that a wave propagates along a horizontal channel of the unform width along the positive direction of x-axis alone. Let the depth of the channel be d, t be the time, and x be the horizontal coordinate and let $w(x,t)$ be the vertical distance of the fluid surface in equilibrium position. Let the amplitude of the wave be small enough, then the irrotational wave propagation can be modelled by the following equation known as the KdV equation

$$w(x,y)_t + \varepsilon w(x,y) w_x(x,y) + \nu w_{xxx}(x,y) = 0, \tag{6}$$

where the first term u_t represents the unform wave translation, and the other two terms $\varepsilon w(x,t) w_x(x,t)$ and $\nu w_{xxx}(x,t)$ serve for the modification of the wave under the influences of nonlinear term ww_x and dispersive term w_{xxx}, respectively.

1.2. Coimbra Variable Order Derivative. Modelling physical problems is better using the Coimbra concept. Variable order differentials are a useful tool for studying systems where the order changes with regard to one or more parameters, such as the management of a nonlinear visco-elasticity oscillator.

$$D_t^{\tau(t)} w(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_{0_+}^{t} \frac{w'(s)ds}{(t-s)^{\tau(t)}} + \frac{\beta t^{-\tau(t)}}{\Gamma(1-\tau(t))}. \tag{7}$$

$0 < \tau(t) < 1$, $\beta = w(0_+) - w(0_-)$, and the above operator require only one initial condition $w(0_+)$. The integer order derivative with respect to the variable t is denoted by $w'(t)$ [33].

2. Analysis of RBF Approximation Method for Fractional Order KdV Equations

In the theory of multivariate approximation, the radial basis functions (RBF) method is the most extensively used tool. A generalized refinement of the multiquadric approach is RBF approximation methods. The MQ has a long history of application and theoretical research can be found in [43, 44]. The MQ approach is widely used in geology, geodesy, geophysics, and other domains, see [44]. Franke [45] conducted a comparative study in the field of MQ. Meanwhile, a key period in RBF history occurred, see for example [46], when Charles Micchelli refined the theory of the MQ method by establishing requirements that guarantee the system matrix nonsingularity for MQ methods. Schoenberg [47] is attribute with the results that generate the invertibility of the system matrix. Micchelli went on to say that Schoenberg's constraints could be relaxed to allow many more functions to be included and that adequate conditions for functions could be applied to make the system matrix nonsingular. In 1990, physicist Kansa [48] discovery quickly disseminated the study, and RBF is used in a systematic approach for numerically solving partial differential equations and is meshless [49]. In numerous branches of applied areas [50, 51], a huge amount of mathematical applications of RBF are employed. In numerical techniques for solving PDEs with reasonable accuracy in multidimensions, Madych exposed the convergence rate of spectral order for MQ interpolation in [52]. In comparison to other state-of-the-art methodologies, these findings propelled RBF research forward swiftly, and the RBF methods drew appreciable attention in the literature as mesh-free approaches and their capacity to attain spectral accuracy for PDE numerical solutions on irregular domains [53]. In this work, a numerical scheme based on RBF and Coimbra derivative is constructed for fractional order KdV equations (1)–(5) defined in the following form:

$$D_t^{\tau(t)} w(t,x) = f(t,x) + \mathscr{L} w(t,x),\ 0 \le t \le T,\ x \in \Omega \subset \mathbb{R}^d,\ d \ge 1, \tag{8}$$

with the following boundary and initial conditions

$$\mathscr{B} w(t,x) = g(t,x),\ x \in \partial\Omega\ ,\ w(t=0,x) = w_0(x), x \in \Omega, \tag{9}$$

with $0 < \tau(t) < 1$.

3. Variable Order Differential Operator Approximation

There are numerous definitions of varying order operators in the literature [54], but in the current study, we are using the definition due to Coimbra [33]. Because this variable order derivative has a great capability to model many complicated mechanical problems with accuracy, the Coimbra variable order operator has the capability to investigate and analyze the dynamics behaviour of many physical models, for example, the fractional forces which cannot be approximated accurately with constant order fractional operator or some other variable order derivatives. In the work [40], the authors performed a comparative study for solving a dynamical system and demonstrated that the Coimbra variable order derivative produced better results in many aspects than the nine definitions of variable order derivatives used in this study.

Now, for the numerical approximation of the Coimbra variable order derivative, we consider for $t \in [0, T]$, and let $t_n = (n-1)\delta t$, where $n = 1, \ldots, M+1$, then at time level t_n, the Coimbra variable order derivative defined in (7) can be given by the following equation:

$$D_{t_n}^{\tau(t_n)} w = \frac{1}{\Gamma(1-\tau(t_n))} \int_0^{t_n} (t_n - s)^{-\tau(t_n)} w'(x,s) ds + \frac{\beta t^{-\tau(t_n)}}{\Gamma(1-\tau(t_n))}. \tag{10}$$

Let us denote the last term of this equation by h_n, then we have

$$h_n = t^{-\tau(t_n)} \frac{\beta}{\Gamma(1-\tau(t_n))}. \tag{11}$$

By using (11) in (10), we get the following form:

$$D_n^{\tau(t_n)} w(x,t_n) = C_n \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\tau(t_n)} ds \left[\frac{w(x,t_{k+1}) - w(x,t_k)}{\delta t} \right] + h_n, \tag{12}$$

where $C_n = (1/\Gamma(1-\tau(t_n)))$; after further simplification, we get

$$D_n^{\tau(t_n)} w(x,t_n) = C_n \sum_{k=1}^{n-1} \left[\frac{w(x,t_{k+1}) - w(x,t_k)}{\delta t} \right] \int_{t_k}^{t_{k+1}} (t_n - s)^{-\tau(t_n)} ds + h_n, \tag{13}$$

and by simplifying the integral involved, we have

$$D_n^{\tau(t_n)} w(x,t_n) = C_n \sum_{k=1}^{n-1} \left[\frac{(t_n - t_k)^{1-\tau(t_n)} - (t_n - t_{k+1})^{1-\tau(t_n)}}{(1-\tau(t_n))} \right] \left[\frac{w(x,t_{k+1}) - w(x,t_k)}{\delta t} \right] + h_n. \tag{14}$$

Denoting the quantity $(t_n - t_k)^{1-\tau(t_n)} - (t_n - t_{k+1})^{1-\tau(t_n)}$ by the b_{k+1}, we get

$$D_n^{\tau(t_n)} w(x, t_n) = \frac{(\delta t)^{-1} C_n}{\Gamma(1-\tau(t_n))} \sum_{k=1}^{n-1} b_{k+1}[w(x, t_{k+1}) - w(x, t_k)] + h_n, \quad (15)$$

and assuming the value $(\delta t)^{-1} C_n / \Gamma(1-\tau(t_n))$ be denoted by a_n, we get

$$D_n^{\tau(t_n)} w(x, t_n) = a_n \sum_{k=1}^{n-1} b_{k+1}[w(x, t_{k+1}) - w(x, t_k)] + h_n. \quad (16)$$

Splitting the first term of this series and rewriting in the form, we have

$$D_n^{\tau(t_n)} w(x, t_n) = a_n \sum_{k=1}^{n-2} b_k[w(x, t_{k+1}) - w(x, t_k)] + a_n b_n[w(x, t_n) - w(x, t_{n-1})] + h_n. \quad (17)$$

Assuming $S_n = a_n \sum_{k=1}^{n-2} b_k[w(x, t_{k+1}) - (x, t_k)] + h_n$, the approximation of Coimbra variable order derivative can be represented in the more simplified form

$$D_n^{\alpha(t_n)} w(x, t_n) = a_n b_n\left[w^n(x) - w^{n-1}(x)\right] + S_n, \quad (18)$$

which is the Coimbra variable order differential operator's finite-difference approximation.

4. RBF Approximation Scheme

The RBF interpolant can be characterized as a linear combination of radial basis functions, as seen in the equation below. For a set of N scattered nodes $x_i \in \Omega \subset \mathbb{R}^d$, $d \geq 1$,

$$w(x, t_n) = \sum_{x_j \in \Omega} \lambda^n \kappa_j\left(\left\|x - x_j\right\|\right), x \in \Omega, \quad (19)$$

where λ^n denotes the expansion coefficients at any time t_n, κ_j denotes an RBF centred at $x_j \in \Omega$, and $\|.\|$ denotes a distance norm in $\mathbb{R}^d, d \geq 1$. It is possible to obtain the matrix form of (19) by

$$w^n = A\lambda^n, \quad (20)$$

where A is a square matrix termed a system matrix, and the entries are $A_{ij} = \kappa_j(\|x_i - x_j\|)$. If $\mathscr{L}$ is a spatial operator and $\mathscr{B}$ is a boundary operator, then (19) is obtained.

$$\begin{aligned} \mathscr{L} w(x, t_n) &= \sum_{x_j \in \Omega} \lambda^n \mathscr{L} \kappa_j\left(\left\|x - x_j\right\|\right), x \in \Omega, \\ \mathscr{B} w(x, t_n) &= \sum_{x_j \in \Omega} \lambda^n \mathscr{B} \kappa_j\left(\left\|x - x_j\right\|\right), x \in \partial\Omega. \end{aligned} \quad (21)$$

The above two equations can be expressed in matrix form by

$$\begin{pmatrix} \mathscr{L} w^n(x) \\ \mathscr{B} w^n(x) \end{pmatrix} = \begin{pmatrix} \mathscr{L} \kappa_j\left(\left\|x - x_j\right\|\right), & x \in \Omega \\ \mathscr{B} \kappa_j\left(\left\|x - x_j\right\|\right), & x \in \partial\Omega \end{pmatrix} \lambda^n. \quad (22)$$

We obtain it in a more compressed form

$$M_{LB} w^n(x) = D\lambda^n, \quad (23)$$

In case of identity operators $\mathscr{L}$, $\mathscr{B}$, the equation above can be written as

$$w^n(x) = B\lambda^n. \quad (24)$$

Model equations (1)–(5) can be approximated in the following way employing the θ– weighted scheme, VODO finite-difference approximation, and RBF spatial operator approximation

$$\begin{aligned} a_n b_n\left[w^n(x) - w^{n-1}(x)\right] + S_n &= \theta M_{LB} w^n(x) \\ &\quad + (1-\theta) M_{LB} w^{n-1}(x) + f^n(x). \end{aligned} \quad (25)$$

Substituting the values of from (19) and (20), we get

$$\begin{aligned} \left[a_n b_n B\lambda^n - a_n b_n B\lambda^{n-1}\right] + S_n &= (1-\theta)) D\lambda^{n-1} + \theta D\lambda^n + f^n(x), \\ [b_n a_n B - \theta D]\lambda^n &= [(1-\theta)D + a_n b_n B]\lambda^{n-1} \\ &\quad + f^n(x) - S_n. \end{aligned} \quad (26)$$

This numerical strategy based on RBF can be solved at any point in time t_n to acquire the value of λ^n, for which we can use (20). Using equation (20) to remove the value of λ^n, we get

$$\begin{aligned} [b_n a_n B - \theta D] A^{-1} w^n &= [(1-\theta) D + b_n a_n B] A^{-1} w^{n-1} \\ &\quad + f^n(x) - S_n. \end{aligned} \quad (27)$$

The amplification matrix of the numerical scheme (27) is the following matrix E:

$$E = A[b_n a_n B - \theta D]^{-1}[(1-\theta)D + b_n a_n B]A^{-1}. \quad (28)$$

In fact, the matrices A and B are identical, as shown by (20) and (24), because matrix B is a special instance of matrix D for identity operators. It is evident from the definitions of a_n and b_n that they are positive real integers, hence $\eta = a_n b_n > 0$. As a result, (28) amplification matrix can be represented in a more basic form as follows:

$$E = \left[a_n b_n A A^{-1} - \theta D A^{-1}\right]^{-1}\left[a_n b_n A A^{-1} + (1-\theta) D A^{-1}\right]. \quad (29)$$

Now, for $\theta = (1/2)$, and denoting $(1/2) D A^{-1}$ by Q, we obtain

$$E = [\eta I - Q]^{-1}[\eta I + Q]. \quad (30)$$

Lemma 1. *If Q is a square matrix of rank $N \times N$ with negative eigenvalues, then the estimate for any $\eta > 0$ is*

$$\left\| (\eta I - Q)^{-1} (\eta I + Q) \right\| \leq 1. \tag{31}$$

This is also true for the Euclidean norm.

Proof: Let the eigenvectors of the matrix Q be $\{u^i\}_{i=1}^N$ with the corresponding eigenvalues ν_i, thus we have the following equation:

$$(\eta I - Q)^{-1} (\eta I - Q) u^i = \frac{\eta + \nu_i}{\eta - \nu_i} u^i, \; i = 1, \ldots N. \tag{32}$$

Suppose the vectors be orthonormal eigensystem for the matrix $(\eta I - Q)^{-1} (\eta I - Q)$, since $\eta > 0$ and all the eigenvalues of the matrix Q are negative, so we obtain

$$\left\| (\eta I - Q)^{-1} (\eta I - Q) \right\| = \left\| \frac{\eta + \nu_i}{\eta - \nu_i} \right\| \leq 1, \forall i = 1, \ldots N. \tag{33}$$

□

5. The Numerical Scheme's Error Analysis

The VODO numerical scheme of order in time is $\mathcal{O}((\delta t)^{2-\tau})$, whereas RBF numerical scheme is mostly dependent on the RBF utilized for the derivation of other differentiation matrices and the RBF system matrix, as demonstrated in the previous work discussion. The order of convergence of several forms of RBF has been determined in [51]. Let the spatial numerical approximation corresponding to the present numerical scheme for a given RBF be of order $\mathcal{O}(h^q)$, $q \geq 0$, and h be the separation distance between the scattered nodes utilized for RBF interpolant. For the numerical scheme specified in (22), let w^n be the approximate solution, w be the precise solution, and $\Theta_n = w^n - w$ be the error at time t_n:

$$\Theta_n = E\Theta_{n-1} + \mathcal{O}\left((\delta t)^{2-\tau} + h^q\right). \tag{34}$$

The above numerical technique's amplification matrix, E, is mostly determined by the type of RBF and the scale factor used. Assume that when the condition of Lemma 1 holds for a given optimal shape parameter value and optimal RBF option, then

$$\|E\| \leq 1, \tag{35}$$

is a criterion for the numerical scheme's stability in (27). Assuming that both the initial solution value and the solution are sufficiently smooth along with $\delta t \longrightarrow 0$,

$$\|\Theta_n\| \leq \|E\| \|\Theta_{n-1}\| + C_1\left(h^q + (\delta t)^{2-\tau}\right), n = 1{:}\ M + 1, \tag{36}$$

where C_1 stands for a constant. At time $t = 0$, the error $\|\Theta_n\|$ always fulfils the initial as well as the boundary condition via mathematical induction.

$$\|\Theta_n\| \leq C_1 \left(1 + \sum_{i=1}^{n-1} \|E\|^i \right)\left((\delta t)^{h^q + 2 - \tau}\right), n = 1{:}\ M + 1, \tag{37}$$

when the condition (35) holds, then

$$\|\Theta_n\| \leq n C_1\left((\delta t)^{2-\tau} + h^q\right), n = 1{:}\ M + 1, \tag{38}$$

This demonstrates that the current numerical method for VODO is convergent.

6. The Numerical Methodology for Variable Order Diffusion Models

Problem 1. We consider the KdV equation defined in (1)–(3) for the following value of function f:

$$f(x,t) = \frac{t^5 e^{-x^2}}{25}\left[\frac{t^{-\tau}}{\Gamma(6-\tau)} - \frac{t^2 x e^{-x^2}}{180000} + \frac{x}{10} - \frac{x^3}{15}\right], \tag{39}$$

with the following initial condition

$$w(x,0) = 0, \; a \leq x \leq b, \tag{40}$$

where the boundary conditions can be extracted from the exact solution

$$w(x,t) = \frac{t^5 e^{-x^2}}{3000}, \tag{41}$$

where $0 < \tau < 1$ is the fractional order. The compactly supported radial basis function defined as $\kappa(r,\varepsilon) = (35(\varepsilon r)^2 + 18\varepsilon r + 3)(1 - \varepsilon r)_+^6$, with a support size of $\varepsilon = 0.5$, is used to solve this issue across the spatial domain $[-4, 4]$ and the number of points $N = 100$ are used. For various settings of order τ, step size δt, and collocation points N, the results are presented in Figures 1 and 2 and Table 1, where the accuracy is quantified in terms of maximum error norm. When we gave a periodic boundary condition at $x = -1$, like $w(a,t) = \sin(20\pi t)\tanh(5t)$ with $f = 0$, the solution at each point of the domain is periodic in time for 1D fractional order KdV equation.

Problem 2. In this last example, we consider the following 2D KdV equation defined in (3)–(5) with the following value of the function f

$$f(x,y,t) = t^6 \text{sech(x))sexh}(y)\left[\frac{720}{\Gamma(7-\tau)} t^{-\tau(t)} + \tanh(x)\left(-6\left(t^6 \text{sech}(x)\text{sech}(y)\right)^2 - \tanh^2(x) + 5\text{sech}(x)^2 - \tanh^2(y) + \text{sech}(y)^2\right)\right], \tag{42}$$

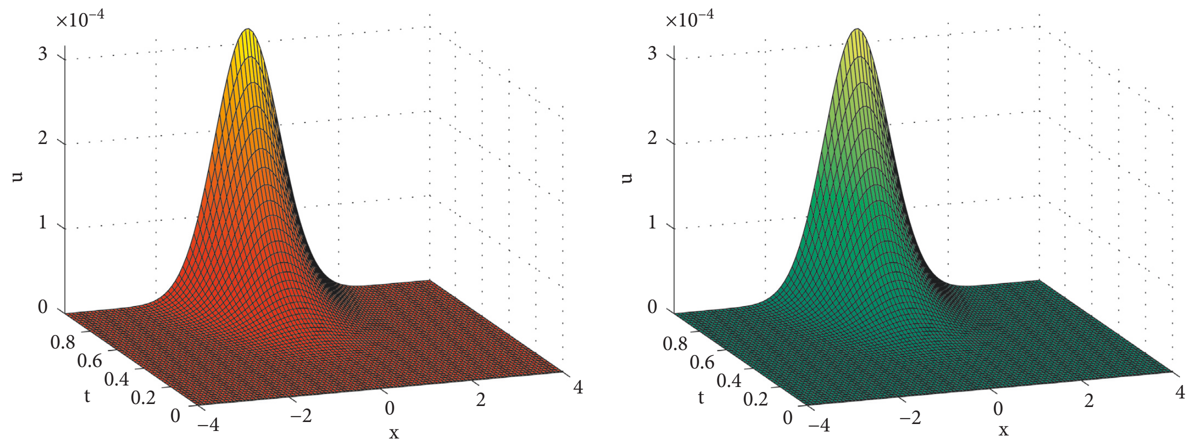

Figure 1: Approximate solution of RBF-based method (red) and exact solution (green) to Problem 1 at time $t \in [0, 1]$, $x \in [-4, 4]$, $\alpha = 0.5$, and $\delta t = 0.01$, with CS-RBF, $\varepsilon = 0.5$.

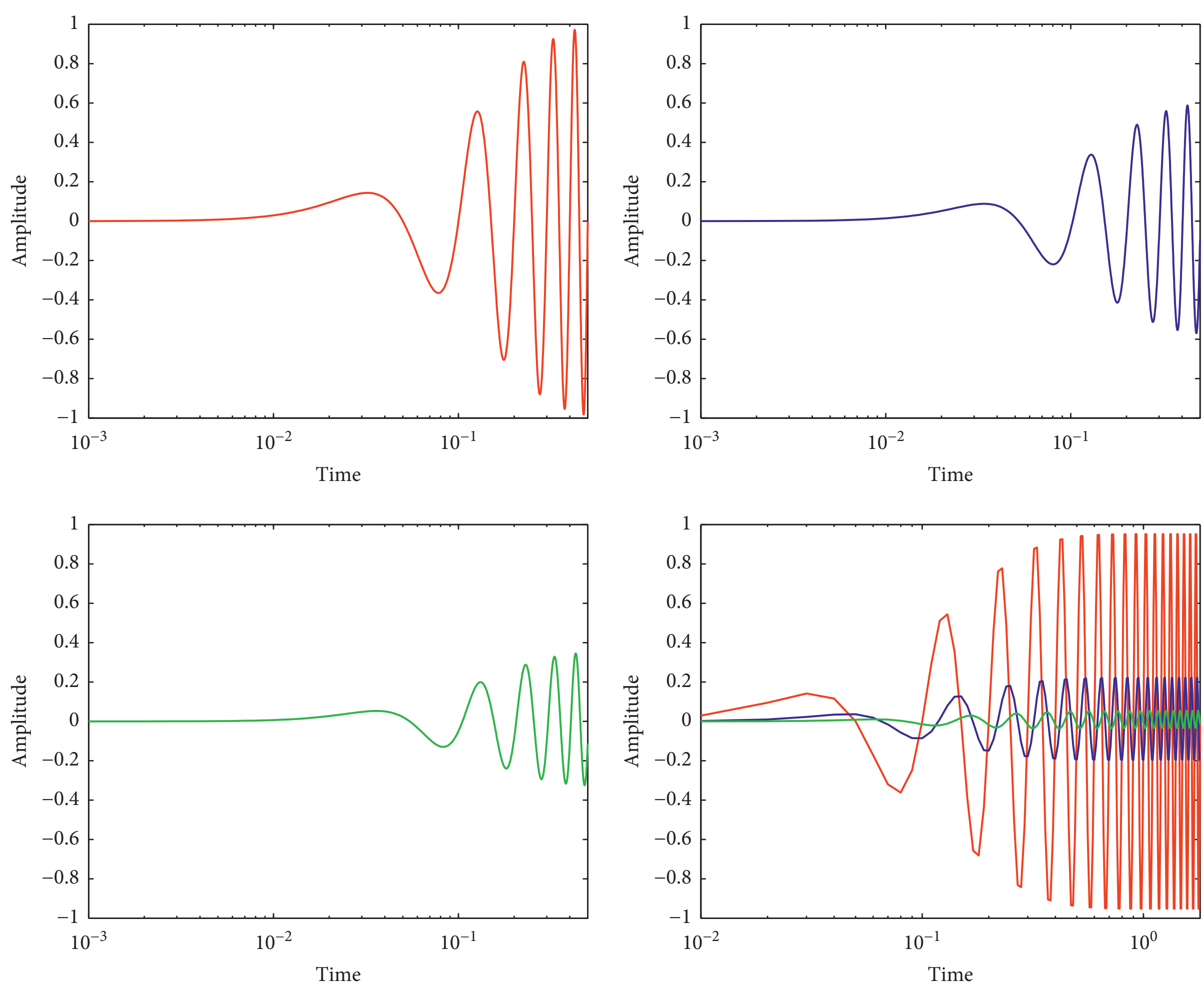

Figure 2: Approximate solution: periodicity of solution at $x = -1$ (red), $x = -0.7588$ (blue), and at $x = -0.5075$ (green) at time $t \in [0, 0.5]$, $x \in [-1, 1]$, $\tau = 0.2$, and $\delta t = 0.001$, with CS-RBF, $\varepsilon = 0.5$, corresponding to Problem 1.

Table 1: An approximate solution to Problem 1 based on RBF, when $t = 1$, error = max $(|w_{ap} - w_{ex}|)$.

(τ, N)	$\delta t = 0.1$	$\delta t = 0.01$	$\delta t = 0.001$
$(0.1,\ 10^2)$	1.3572e-006	3.6674e-008	1.8149e-008
$(0.2,\ 10^2)$	3.2295e-006	7.7781e-008	1.6031e-008
$(0.3,\ 10^2)$	5.7785e-006	1.5589e-007	1.6168e-008
$(0.5,\ 10^2)$	1.3801e-005	5.2623e-007	2.4744e-008

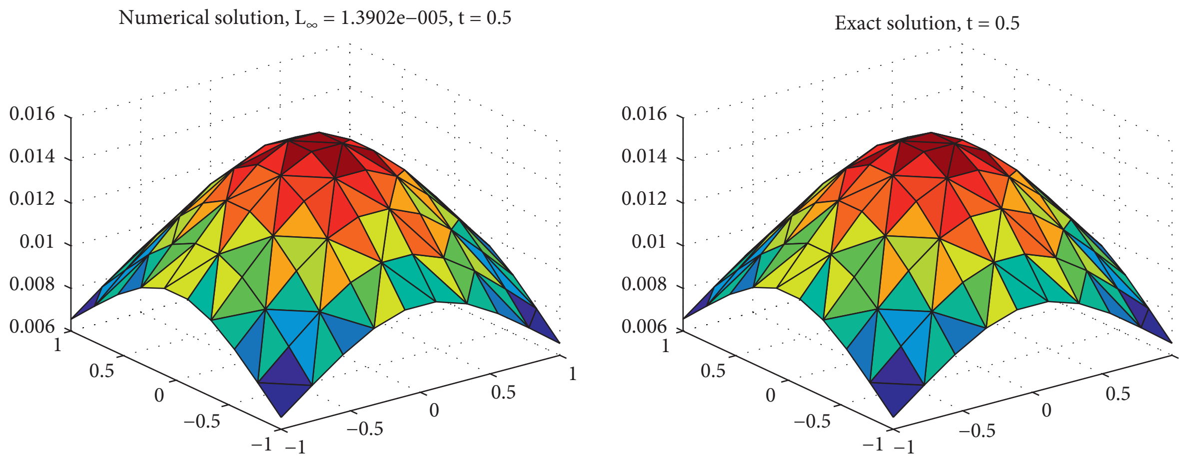

Figure 3: A comparison of RBF-based numerical method and exact solution of Problem 2, for $N = 10^2$, $t = 0.5$, $\tau(t) = 0.4 + 0.2\ \sin(0.5\pi t/T)$.

Table 2: An approximate solution to Problem 2 based on RBF, when $t = 0.5$, error = max $(|w_{ap} - w_{ex}|)$.

$(\varepsilon,\ N)$	$\delta t = 0.1$	$\delta t = 0.01$	$\delta t = 0.001$	$\delta t = 0.0001$
$(1,\ 5^2)$	0.0104	0.0016	1.7022e-004	1.8747e-005
$(1,\ 6^2)$	0.0111	0.0017	1.7335e-004	2.2170e-005
$(1,\ 7^2)$	0.0102	0.0016	1.7993e-004	1.8235e-005
$(1,\ 10^2)$	0.0114	0.0017	1.2957e-004	4.3734e-005

with the following initial:

$$w(x, y, 0) = 0,\ \ -1 \le x, y \le 1, \tag{43}$$

and the boundary conditions can be taken from the following exact solution:

$$w(x, y, t) = t^6 \text{sech}(x)\text{sech}(y),\ t > 0,\ (x, y) \in \partial\Omega, \tag{44}$$

$0 < \tau(t) < 1$, $\tau(t) = 0.4 + 0.2\ \sin(0.5\pi t/T)$. The current RBF-based solution solves this problem, and the results are displayed in Figure 3 and Table 2, respectively. The current numerical technique appears to be convergent and stability attained when $\delta t \longrightarrow 0$. This backs up the prior sections' convergence and stability analysis of the current numerical system is achieved.

Problem 3. In the last example, we consider the irregular domain within the regular domain $[-1, 1]^2$. The variable order which is used in this computation is defined by the function $\alpha(t) = 0.4 + 0.2\ \sin(0.5\pi t/T)$, $T = 1$. We used different number of nodes $N = 56, 196, 400, 676$ in the irregular domain. The radial basis function defined by $\kappa(r, \varepsilon) = \sqrt{r^2 + \varepsilon}$ is implemented in this problem and its corresponding shape parameter value changes solution accuracy, which is calculated using the formula $\varepsilon = (1/\log(N))$ [55]. The results are shown in graphical form and can be seen in Figure 4.

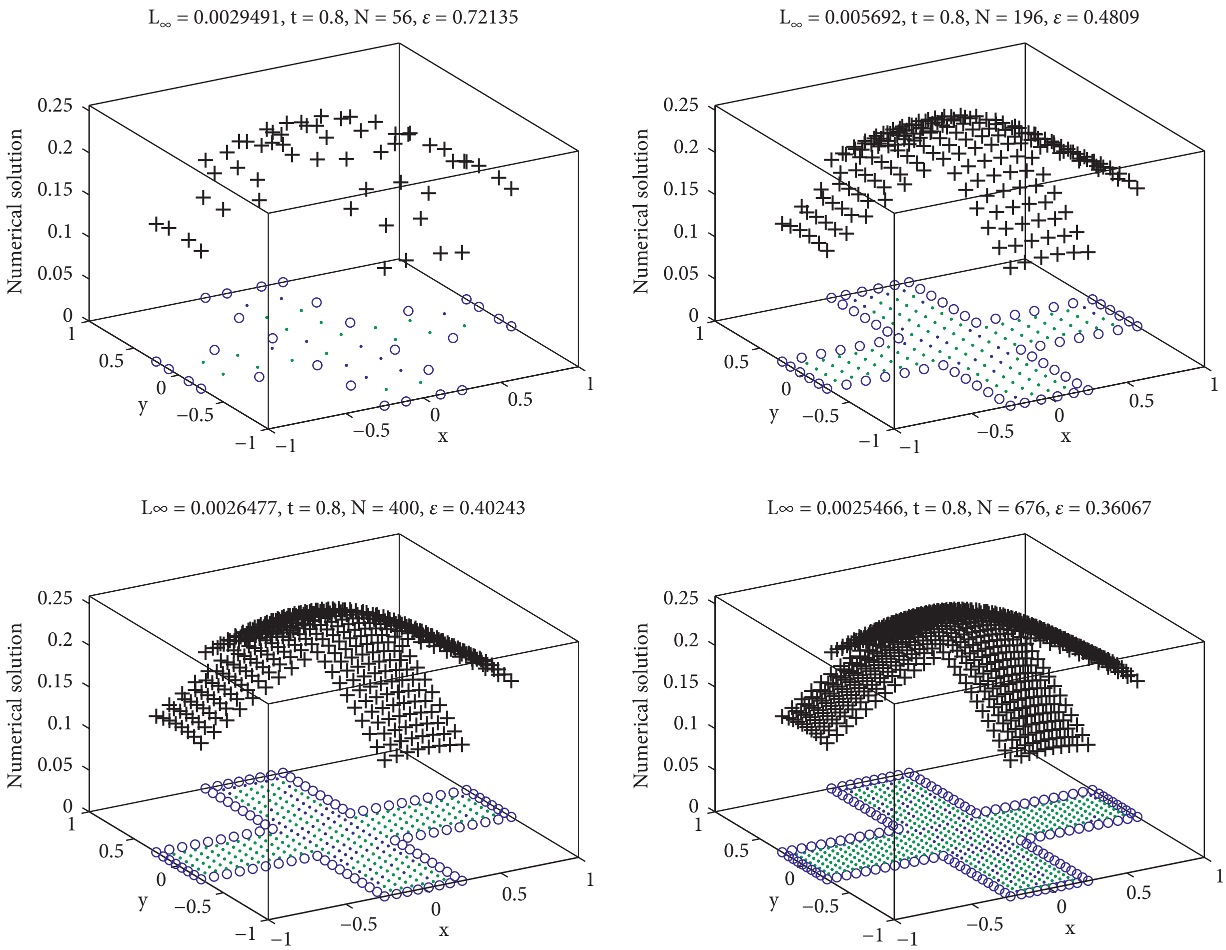

Figure 4: RBF-based numerical simulations of Problem 3, in irregular domain, $\tau(t) = 0.4 + 0.2\sin(0.5\pi t/T)$.

7. Conclusion

The numerical solution of the variable order KdV models in 1D and 2D is achieved using an RBF-based numerical approach. The RBF is used to approximate the spatial derivative, whereas the Coimbra derivative is used to approximate the variable order time differential operator. The numerical scheme's stability and convergence are established. The current numerical technique is found to have a sensitivity in temporal integration. The periodicity of the KdV equation in 1D is explored, and it is demonstrated that the solution is periodic in time at each point of the domain for the fractional order KdV in 1D. The suggested numerical system provides the capacity to numerically approximate numerous complex mechanical problems with ease and precision. The Coimbra variable order operator can be used to examine and analyze the dynamics of a variety of physical models, such as fractional forces, which cannot be accurately modelled with a constant order fractional operator.

Data Availability

The date used to support this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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