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Research Article

Decision-Making Approach Based on Generalized Aggregation Operators with Complex Single-Valued Neutrosophic Hesitant Fuzzy Set Information

Harish Garg,[1] **Zeeshan Ali**,[2] **Ibrahim M. Hezam**,[3] **and Jeonghwan Gwak**[4,5,6,7]

[1]*School of Mathematics, Thapar Institute of Engineering & Technology, Deemed University, Patiala 147004, Punjab, India*
[2]*Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad, Pakistan*
[3]*Department of Statistics & Operations Research, College of Sciences, King Saud University, Riyadh, Saudi Arabia*
[4]*Department of Software, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[5]*Department of Biomedical Engineering, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[6]*Department of AI Robotics Engineering, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[7]*Department of IT & Energy Convergence (BK21 FOUR), Korea National University of Transportation, Chungju 27469, Republic of Korea*

Correspondence should be addressed to Harish Garg; harishg58iitr@gmail.com

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A strategic decision-making technique can help the decision maker to accomplish and analyze the information in an efficient manner. However, in our real life, an uncertainty will play a dominant role during the information collection phase. To handle such uncertainties in the data, we present a decision-making algorithm under the single-valued neutrosophic (SVN) environment. The SVN is a powerful way to deal the information in terms of three degrees, namely, "truth," "falsity," and "indeterminacy," which all are considered independent. The main objective of this study is divided into three folds. In the first fold, we state the novel concept of complex SVN hesitant fuzzy (CSVNHF) set by incorporating the features of the SVN, complex numbers, and the hesitant element. The various fundamental and algebraic laws of the proposed CSVNHF set are described in details. The second fold is to state the various aggregation operators to obtain the aggregated values of the considered CSVNHF information. For this, we stated several generalized averaging operators, namely, CSVNHF generalized weighted averaging, ordered weighted average, and hybrid average. The various properties of these operators are also stated. Finally, we discuss a multiattribute decision-making (MADM) algorithm based on the proposed operators to address the problems under the CSVNHF environment. A numerical example is given to illustrate the work and compare the results with the existing studies' results. Also, the sensitivity analysis and advantages of the stated algorithm are given in the work to verify and strengthen the study.

1. Introduction

The multiattribute decision-making (MADM) method is one of the efficient methods to solve the decision-making problems by considering the different experts, their preferences, and alternatives. The chief objective of this problem is to address the best alternatives, when the information related to them is accessed under the vague and imprecise information. In other words, the decision-making strategy aims to grow the chance of the benefits and reduce the chance of the cost during the decision-making procedure for simplifying genuine life dilemmas. Since its appearance, a huge number of people have worked on decision-making strategies under the presence of a crisp set. However, in several situations, it is very complicated to provide the information related to the objects in terms of precise number, due to the involvement of the uncertainties in the data. To reduce the loss of data during the process, in 1965, Zadeh [1] firstly put forward the theory of fuzzy set (FS), by extending the range of the crisp set (which is $\{0, 1\}$) to the unit interval.

Due to this beneficial work, a lot of space was created for a decision maker to make a beneficial decision from the family of alternatives. After the successful presentation of the FS theory, a huge number of individuals have described it in the circumstance of different places [2]. As ambiguity and complexity are involved in every region of life, in the presence of these dilemmas, it is very complicated for FS to survive with the old mathematical structure (covered only truth grade (TG)). In several cases, several experts have faced a lot of data in the arrangement of "yes" or "no," which is very complex for FS to resolve. To reduce the level of the deficiencies and worries, Atanassov [3] changed the shape of FS and put forward the well-known shape, called intuitionistic FS (IFS). IFS is the modified technique of FS, which includes two different terms, called TG $\mp(u)$ and falsity grade (FG) $F(u)$ with a satiable and strong character in the shape of $0 \leq \mp(u) + F(u) \leq 1$. IFS is a different structure from the mathematical structure of FS to switch uncertain data. By taking advantage of the IFSs, several studies have been conducted by various scholars such as interval-valued IFSs [4], distance measures [5], circular IFS [6], and so on.

In the IFSs, each element is characterized with two degrees, truth and falsity, to access the information. However, in several real-life situations, very complex ambiguity is encountered during processing the information, and hence under the consideration of these dilemmas, it is very complicated for IFS to survive with the old mathematical structure (in terms of TG and FG only). In other words, sometimes several experts have faced a lot of data in the arrangement of "yes," "abstinence," and "no," which is very complex for IFS to resolve. To reduce the level of such deficiencies, the fundamental mathematical structure of the neutrosophic set (NS) was put forward by Smarandache [7]. NS is one of the massive dominant and reliable techniques which can easily determine the solution to every complicated problem that occurs in genuine life dilemmas The concept of NS is extended to the single-valued NS (SVNS) and its corresponding operators [8] by the researchers. Since its appearance, scholars have studied it under different environments. For instance, in [9], the authors have defined the Dombi weighted aggregation operators for the collections of SVNSs. In [10], the scholars put forward the Bonferroni mean operators for SVNS. In [11], the authors put forward the COPRAS method for SVNS. For more details about the study on NSs, we refer the readers to [12–17] and their corresponding references.

In all the studies listed above, almost all the studies were conducted by considering only the real component of the grades of the element. However, the periodic nature of the rating of the expert is not considered in the decision-making process. To address it completely, there is a need to express the rating of the expert from real interval [0, 1] to the unit disc in the complex plane. This idea was highlighted by Ramot et al. [18] in 2002 who presented the concept of complex FS (CFS). In CFS, each object is identified with two degrees TG and FG under complex domain such as $t' e^{i2\pi\theta_{t'}}$ where $t', \theta_{t'} \in [0, 1]$ represent the real and amplitude terms of the expert rating. It is clearly seen that CFS can handle the vague information with one or two sorts of data in the shape of singleton terms. Some application of the CFS towards the decision-making process is summarized in [19]. Again, the scope of the CFS is limited as it considers only the truth degree and fails to consider the falsity degree at the time of the execution. For instance, if some expert diagnosed data like "yes" or "no" and each has two possibilities, then CFS is very complicated for diagnosing the solution of the above scenario. To reduce the above complications, Alkouri and Salleh [20] proposed the complex IFS (CIFS), which includes the two different terms, called TG ($t' e^{i2\pi\theta_{t'}}$) and FG ($f' e^{i2\pi\theta_{f'}}$) in the shape of complex numbers with proficient and well-known characteristics $0 \leq \mathbf{t}' + \mathbf{f}' \leq 1$ and $0 \leq \theta_{t'} + \theta_{f'} \leq 1$. To handle problematic and unseen situations, a huge number of people have employed the above theory in different regions, for illustration, the study in [21] includes the distance measures constructed under the CIFSs, while the study in [22] includes the information measures constructed under the CIFS. Further, CIFS theory has been widely applied in different categories such as aggregation operators [23], group theory [24], and generalized geometric operators [25].

Since CIFS theory is able to deal only with "yes" or "no" decision in the form of degrees TG and FG, it is unable to deal with the term "abstinence." For this, a structure of complex NS (CNS) was proposed by Ali and Smarandache [26] by considering the independent membership grades of "yes," "abstinence," and "no" over the unit disc of complex plane. The structure of CNS is easily implemented in every region of life which includes ambiguity and awkward sort of data. In order to flexibly share preferences, Torra [27] came up with the idea of hesitant fuzzy set (HFS), which allowed agents to provide multiple membership grades for a specific alternative-criterion pair. By this, the issue of hesitation was handled effectively. Related to MADM problems, several researchers have addressed the problem by using HFS features. For instance, Rodriguez et al. [28] investigated an interesting review on HFS models and its usage in MADM models. Xu and Zhou [29] identified a problem with HFS and designed a consensus building model by considering multiple experts for a specific alternative-criterion pair. In [30], the authors defined the similarity measures based on complex HFS and stated their application to pattern recognition.

From the above listed literature, we noted that the several researchers have utilized the advantages of CIFS, HFS, NS, and CNS to address the problems related to the MADM. However, it is noted that all these theories are unable to handle some uncertain cases which occur during accessing the decision-making problems. For instance, if a person made committee, for laptop enterprise, which consists of ten members, the head of this committee would like to choose the suitable laptop according to the feasibility and suitability. To get the best one, each committee member provides their opinions about different laptops in terms of their prices and name of the model. As the model and price of the laptop change frequently over time, there exist a lot of uncertainties during the execution. Under such circumstances, it is difficult to access the information using several existing sets. To address it completely, in this article, we have presented an extension of the NSs by keeping the features of hesitant set and complex membership degree and defined the novel set named as complex single-valued neutrosophic hesitant fuzzy set (CSVNHFS). The idea behind this set is to address the ambiguity in the data when it is arranged in the form of "yes," "abstinence," and "no" under the complex domain. In the presented set, each element is characterized with three independent hesitant degrees, namely, TG ($t' e^{i\theta_{t'}}$), abstinence ($a' e^{i\theta_{a'}}$), and FG ($f' e^{i\theta_{f'}}$), over the unit disc of complex plane with the conditions $0 \leq t' + a' + f' \leq 3$ and + where $0 \leq t', a', f' \leq 1$ and $0 \leq \theta_{t'}, \theta_{a'}, \theta_{f'} \leq 2$. After managing the information under such features and to state more information about it, we define various operational laws and study their characteristics. To explore about the laws, we stated several weighted averaging operators to aggregate the collective information into a single one. Additionally, we state a MADM algorithm to explain the working of the proposed work and demonstrate it with the help of numerical examples. The major advantages of the proposed set are that several existing theories are considered as a special case of the proposed one. For instance, by removing the components $\theta_{t'}, \theta_{a'}, \theta_{f'}$ during the information phase, the proposed set reduces to SVNS. On the other hand, when we set $\theta_{a'} = \theta_{f'} = 0$, then the set reduces to CHFS. Similarly, when we set $\theta_{a'} = 0$ and all other degrees as a single number, then it reduces to CIFS. Finally, when we consider all the degrees in the form of singleton set, then the proposed CSVNHFS reduces to CSVNS, while when we set $\theta_{t'} = \theta_{a'} = \theta_{f'} = 0$, then the set reduces to SVN hesitant fuzzy set.

In this paper, the main contribution of the present work is summarized as follows:

(1) To present a new concept named as CSVNHFS to address the uncertainties in the data and hence describe their algebraic and operational laws.

(2) To initiate several generalized averaging operators, namely, CSVNHF generalized weighted averaging, ordered weighted average, and hybrid average, denoted by CSVNHFGWA, CSVNHFGOWA, and CSVNHFGHWA, respectively

(3) To discuss the MADM technique under the presence of stated work. Also, to show the flexibility of the stated operators, several important results and their properties are also elaborated.

(4) A numerical example is given to illustrate the work and compare the results with the existing studies' results. Also, the sensitivity analysis and advantages of the stated algorithm are given in the work to verify and strengthen the study.

The rest of the work is organized as follows. In Section 2, we revise various prevailing concepts like FSs, CFSs, NSs, SVNSs, CNSs, HFSs, generalized weighted averaging (GWA), generalized ordered weighted averaging (GOWA), generalized hybrid averaging (GHA) operators, and their operational laws. In Section 3, we analyze the fundamental theory of the CSVNHF setting and described its algebraic laws. In Section 4, we define the various generalized operators, namely, CSVNHFGWA, CSVNHFGOWA, and CSVNHFGHWA. To show the flexibility of the diagnosed operators, several important results and their properties are also elaborated. In Section 5, a MADM algorithm is stated and illustrated with numerical example. Sensitivity analysis and advantages of the work are also presented to verify and feasibility of the theory. Section 6 draws the conclusion of our study.

2. Preliminaries

In this section, some prevailing concepts are revised. Let X, $Ŧ(u)$, $Ą(u)$, and $F(u)$, be fixed set, TG, abstinence, and FG, respectively.

Definition 1 (see [1]). The FS is initiated by

$$F = \left\{ \frac{(u,\, Ŧ(u))}{u \in X} \right\}, \tag{1}$$

where $0 \leq Ŧ(u) \leq 1$.

Definition 2 (see [18]). The CFS is initiated by

$$N = \left\{ \frac{(u,\, Ŧ(u))}{u \in X} \right\}, \tag{2}$$

where $Ŧ(u) = t' e^{i\theta_{t'}}$ with the conditions $0 \leq t' \leq 1$ and $0 \leq \theta_{t'} \leq 2$.

Definition 3 (see [7]). The NS is initiated by

$$N = \left\{ \frac{(u,\, Ŧ(u), Ą(u), F(u))}{u \in X} \right\}, \tag{3}$$

with the conditions $0^{-} EŦ(u) + Ą(u) + F(u)E3^{+}$ and $0^{-} EŦ(u), Ą(u), F(u)E1^{+}$. Further, $n = \{Ŧ(u), Ą(u), F(u)\}$ represents the NN (neutrosophic number).

Definition 4 (see [8]). The SVNS is initiated by

$$N = \left\{ \frac{(u, Ŧ(u), Ą(u), F(u))}{u \in X} \right\}, \tag{4}$$

with the conditions $0 \leq Ŧ(u) + Ą(u) + F(u) \leq 3$ and $0 \leq Ŧ(u), Ą(u), F(u) \leq 1$. Further, $n = \{Ŧ(u), Ą(u), F(u)\}$ represents the single-valued neutrosophic number (SVNN); simply, we write $n = (Ŧ, Ą, F)$.

Definition 5 (see [26]). The CNS is initiated by

$$N = \left\{ \frac{(u, Ŧ(u), Ą(u), F(u))}{u \in X} \right\}, \tag{5}$$

where $Ŧ(u) = t' e^{i\theta_{t'}}$, $Ą(u) = a' e^{i\theta_{a'}}$, and $F(u) = f' e^{i\theta_{f'}}$ with the conditions $0^{-} \leq t' + a' + f' \leq 3^{+}$ and $0^{-} \leq \theta_{t'} + \theta_{a'} + \theta_{f'} \leq 6^{+}$, where $0^{-} \leq t', a', f' \leq 1^{+}$ and $0^{-} \leq \theta_{t'}, \theta_{a'}, \theta_{f'} \leq 2^{+}$. Further, $n = \{Ŧ(u), Ą(u), F(u)\}$ represents the complex neutrosophic number (CNN); simply, we write $n = (Ŧ, Ą, F) = (t' e^{i\theta_{t'}}, a' e^{i\theta_{a'}}, f' e^{i\theta_{f'}})$.

Definition 6. (see [27]). A HFS is initiated by

$$E = \{(u, h_E(u))\colon \text{ where } h_E(u) \text{ is a finite subset of } [0,1]\}, \quad (6)$$

is called HFS, where $h = h_E(u)$ is called hesitant fuzzy element (HFE).

Definition 7 (see [27]). Let h, h_1, and h_2 be three HFEs with $\gamma > 0$. Then,

(1) $h_1 \oplus h_2 = \coprod_{t_1 \in h_1, t_2 \in h_2} \{t_1 + t_2 - t_1 t_2\}$.
(2) $h_1 \otimes h_2 = \coprod_{t_1 \in h_1, t_2 \in h_2} \{t_1 t_2\}$.
(3) $h^\gamma = \prod_{t \in h} \{t^\gamma\}$.
(4) $\gamma h = \coprod_{t \in h} \{1 - (1-t)^\gamma\}$.

Definition 8 (see [8]). The generalized weighted average (GWA) operator is given by $GWA\colon \Omega^n \longrightarrow \Omega$:

$$\mathrm{GWA}(n_1, n_2, \ldots, n_n) = \left(\sum_{i=1}^{n} \omega_i n_i^\gamma\right)^{1/\gamma}, \quad (7)$$

where Ω represents the family of all positive integers with $\gamma > 0$. Further, the weighted vector is denoted and defined by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\text{Ŧ}}$, $\omega_i \in [0,1]$, where $\sum_{i=1}^n \omega_i = 1$.

Definition 9 (see [8]). The generalized ordered weighted average (GOWA) operator is given by GOWA: $\Omega^n \longrightarrow \Omega$:

$$\mathrm{GOWA}(n_1, n_2, \ldots, n_n) = \left(\sum_{i=1}^{n} \omega_i n_{o(i)}^{'\gamma}\right)^{1/\gamma}, \quad (8)$$

where Ω represents the family of all positive integers with $\gamma > 0$ and $n_{o(i)}'$ is the ith largest term of n_i, i.e., $n_{o(i)}' \le n_{o(i-1)}'$. Further, the weighted vector is denoted and defined by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\text{Ŧ}}$, $\omega_i \in [0,1]$, where $\sum_{i=1}^n \omega_i = 1$.

Definition 10 (see [8]). The generalized hybrid weighted average (GHWA) operator is given by GHWA: $\Omega^n \longrightarrow \Omega$:

$$\mathrm{GHWA}(n_1, n_2, \ldots, n_n) = \left(\sum_{i=1}^{n} \omega_i n_{o(i)}^{'\gamma}\right)^{1/\gamma}, \quad (9)$$

where Ω represents the family of all positive integers with $\gamma > 0$ and $n_{0(i)}'$ is the ith largest term of n_i, i.e., $n_{o(i)}' \le n_{o(i-1)}'$, where $n_i' = n\dot{\omega}_i n_i$. Further, the weighted vector is denoted and defined by $\dot{\omega} = (\dot{\omega}_1, \dot{\omega}_2, \ldots, \dot{\omega}_n)^{\text{Ŧ}}, \dot{\omega}_i \in [0,1]$, where $\sum_{i=1}^n \dot{\omega}_i = 1$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\text{Ŧ}}$, $\omega_i \in [0,1]$, $\sum_{i=1}^n \omega_i = 1$.

3. Proposed CSVNHFS

In this study, we explored two sets named as CSVNSs and CSVNHFSs and their algebraic laws.

3.1. Complex Single-Valued Neutrosophic Fuzzy Set (CSVNFS)

Definition 11. The CSVNFS is initiated by

$$N = \left\{\frac{(u, \text{Ŧ}(u), \text{Ą}(u), F(u))}{u \in X}\right\}, \quad (10)$$

where $\text{Ŧ}(u) = t' e^{i\theta_{t'}}$, $\text{Ą}(u) = a' e^{i\theta_{a'}}$, and $F(u) = f' e^{i\theta_{f'}}$ with the conditions $0 \le t' + a' + f' \le 3$ and $0 \le \theta_{t'} + \theta_{a'} + \theta_{f'} \le 6$, where $0 \le t', a', f' \le 1$ and $0 \le \theta_{t'}, \theta_{a'}, \theta_{f'} \le 2$. Further, $u = \{\text{Ŧ}(u), \text{Ą}(u), F(u)\}$ represents the complex single-valued neutrosophic fuzzy number (CSVNFN). Symbolically, $n = (\text{Ŧ}, \text{Ą}, F) = (t' e^{i\theta_{t'}}, a' e^{i\theta_{a'}}, f' e^{i\theta_{f'}})$.

Definition 12. Let $n_1 = (\text{Ŧ}_1, \text{Ą}_1, F_1) = (t_1' e^{i\theta_{t_1'}}, a_1' e^{i\theta_{a_1'}}, f_1' e^{i\theta_{f_1'}})$ and $n_2 = (\text{Ŧ}_2, \text{Ą}_2, F_2) = (t_2' e^{i\theta_{t_2'}}, a_2' e^{i\theta_{a_2'}}, f_2' e^{i\theta_{f_2'}})$ be two CSVNFNs with $\gamma > 0$. Then,

(1) $n_1 \oplus n_2 = \left((t_1' + t_2' - t_1' t_2') e^{i(\theta_{t'_1} + \theta_{t'_2} - \theta_{t_1'}\theta_{t_2'}/2\pi)}, (a_1' a_2') e^{i(\theta_{a_1'}\theta_{a_2'}/2\pi)} (f_1' f_2') e^{i(\theta_{f_1'}\theta_{f_2'}/2\pi)} \right)$.
(2) $n_1 \otimes n_2 = \left((t_1' t_2') e^{i(\theta_{t_1'}\theta_{t_2'}/2\pi)}, (a_1' + a_2' - a_1' a_2') e^{i(\theta_{a_1'} + \theta_{a_2'} - \theta_{a_1'}\theta_{a_2'}/2\pi)} (f_1' + f_2' - f_1' f_2') e^{i(\theta_{f_1'} + \theta_{f_2'} - \theta_{f_1'}\theta_{f_2'}/2\pi)} \right)$.
(3) $\gamma n_1 = ((1 - (1 - t_1')^\gamma) e^{i2\pi(1 - (1 - \theta_{t_1'}/2\pi)^\gamma)}, a_1^{'\gamma} e^{i\theta_{a_1'}^\gamma}, f_1^{'\gamma} e^{i\theta_{f_1'}^\gamma})$.
(4) $n_1^\gamma = (t_1^{'\gamma} e^{i\theta_{t'_1}^\gamma}, (1 - (1 - a_1')^\gamma) e^{i2\pi(1 - (1 - \theta_{a_1'}/2\pi)^\gamma)}, (1 - (1 - f_1')^\gamma) e^{i2\pi(1 - (1 - \theta_{f_1'}/2\pi)^\gamma)})$.

Theorem 1. *Let* $n_1 = (\text{Ŧ}_1, \text{Ą}_1, F_1) = (t_1' e^{i\theta_{t_1'}}, a_1' e^{i\theta_{a_1'}}, f_1' e^{i\theta_{f_1'}})$ *and* $n_2 = (\text{Ŧ}_2, \text{Ą}_2, F_2) = (t_2' e^{i\theta_{t_2'}}, a_2' e^{i\theta_{a_2'}}, f_2' e^{i\theta_{f_2'}})$ *be two CSVNFNs with* $\gamma, \gamma_1, \gamma_2 > 0$. *Then,*

(1) $n_1 \oplus n_2 = n_2 \oplus n_1$.
(2) $n_1 \otimes n_2 = n_2 \otimes n_1$.
(3) $\gamma(n_1 \oplus n_2) = \gamma n_2 \oplus \gamma n_1$.
(4) $\gamma_1 n_1 \oplus \gamma_2 n_1 = (\gamma_1 + \gamma_2) n_1$.
(5) $n_1^\gamma \otimes n_2^\gamma = (n_1 \otimes n_2)^\gamma$.
(6) $n_1^{\gamma_1} \otimes n_1^{\gamma_2} = n_1^{\gamma_1 + \gamma_2}$.

Proof. It can be easily derived, so we omit it here. □

Definition 13. Let $n = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{a_i}}, f_i' e^{i\theta_{f_i}})$ be CSVNFN. Then,

$$\text{Ş}(n) = \frac{1}{6}\left\{(t_i' + (1 - a_i') + (1 - f_i')) + \frac{1}{2}\left(\theta_{t_i} + (1 - \theta_{a_i}) + (1 - \theta_{f_i})\right)\right\} \quad (11)$$

is called the score function (SF), and the accuracy function (AF) is defined as

$$H(n) = \frac{1}{6}\left\{((1 - t_i') + a_i' + f_i') + \frac{1}{2}\left((1 - \theta_{t_i}) + \theta_{a_i} + \theta_{f_i}\right)\right\}. \quad (12)$$

If we considered the two CSVNFNs $n_1 = (\text{Ŧ}_1, \text{Ą}_1, F_1) = (t_1' e^{i\theta_{t_1'}}, a_1' e^{i\theta_{a_1'}}, f_1' e^{i\theta_{f_1'}})$ and $n_2 = (\text{Ŧ}_2, \text{Ą}_2, F_2) = (t_2' e^{i\theta_{t'_2}}, a_2' e^{i\theta_{a'_2}}, f_2' e^{i\theta_{f'_2}})$, then

(1) If $\text{Ş}(n_1) > \text{Ş}(n_2)$, then $n_1 > n_2$.

(2) If $\S(n_1) < \S(n_2)$, then $n_1 < n_2$.
(3) If $\S(n_1) = \S(n_2)$, then $n_1 = n_2$.
(1) If $H(n_1) > H(n_2)$, then $n_1 > n_2$.
(2) If $H(n_1) < H(n_2)$, then $n_1 < n_2$.
(3) If $H(n_1) = H(n_2)$, then $n_1 = n_2$.

3.2. Complex Single-Valued Neutrosophic Hesitant Fuzzy Set (CSVNHFS)

Definition 14. The CSVNHFS is denoted and defined by

$$N = \left\{ \frac{(u, Ŧ(u), Ą(u), F(u))}{u \in X} \right\}, \tag{13}$$

where $Ŧ(u) = \left\{ t = t' e^{i\theta_{t'}} / t \in Ŧ(u) \right\}$, $Ą(u) = \left\{ a = a' e^{i\theta_{a'}} / a \in Ą(u) \right\}$, and $F(u) = \left\{ f = f' e^{i\theta_{f'}} / f \in F(u) \right\}$ with the conditions $0 \le \max(t') + \max(a') + \max(f') \le 3$ and $0 \le \max(\theta_{t'}) + \max(\theta_{a'}) + \max(\theta_{f'}) \le 6$, where $0 \le t', a', f' \le 1$ and $0 \le \theta_{t'}, \theta_{a'}, \theta_{f'} \le 2$. Further, $n = \{Ŧ(u), Ą(u), F(u)\}$ represents the CSVNHFN; simply, we write $n = (Ŧ, Ą, F) = (t' e^{i\theta_{t'}}, a' e^{i\theta_{a'}}, f' e^{i\theta_{f'}})$.

Definition 15. Let $n_1 = (Ŧ_1, Ą_1, F_1) = (t_1' e^{i\theta_{t'1}}, a_1' e^{i\theta_{a'1}}, f_1' e^{i\theta_{f'1}})$ and $n_2 = (Ŧ_2, Ą_2, F_2) = (t_2' e^{i\theta_{t_2'}}, a_2' e^{i\theta_{a_2'}}, f_2' e^{i\theta_{f_2'}})$ be two CSVNHFNs. Then,

(1) $n_1 \cup n_2 = (Ŧ_1 \cup Ŧ_2, Ą_1 \cap Ą_2, F_1 \cap F_2) = \begin{pmatrix} (t_1' \vee t_2') e^{i(\theta_{t_1'} \vee \theta_{t_2'})}, \ (a_1' \wedge a_2') e^{i(\theta_{a_1'} \wedge \theta_{a_2'})} \\ (f_1' \wedge f_2') e^{i(\theta_{f_1'} \wedge \theta_{f_2'})} \end{pmatrix}$.

(2) $n_1 \cap n_2 = (Ŧ_1 \cap Ŧ_2, Ą_1 \cup Ą_2, F_1 \cup F_2) = \begin{pmatrix} (t_1' \wedge t_2') e^{i(\theta_{t_1'} \wedge \theta_{t_2'})} \\ (a_1' \vee a_2') e^{i(\theta_{a_1'} \vee \theta_{a_2'})} \\ (f_1' \vee f_2') e^{i(\theta_{f_1'} \vee \theta_{f_2'})} \end{pmatrix}$.

Definition 16. Let $n_1 = (Ŧ_1, Ą_1, F_1) = (t_1' e^{i\theta_{t_1'}}, a_1' e^{i\theta_{a_1'}}, f_1' e^{i\theta_{f_1'}})$ and $n_2 = (Ŧ_2, Ą_2, F_2) = (t_2' e^{i\theta_{t_2'}}, a_2' e^{i\theta_{a_2'}}, f_2' e^{i\theta_{f_2'}})$ be two CSVNHFNs with $\gamma > 0$. Then,

(1) $n_1 \oplus n_2 = (Ŧ_1 \oplus Ŧ_2, Ą_1 \oplus Ą_2, F_1 \oplus F_2) = \coprod_{\substack{t_1' \in Ŧ_1, a_1' \in Ą_1, f_1' \in F_1, \\ t_2' \in Ŧ_2, a_2' \in Ą_2, f_2' \in n_2}} \begin{pmatrix} \begin{pmatrix} t_1' + t_2' - \\ t_1' t_2' \end{pmatrix} e^{i \begin{pmatrix} \theta_{t_1'} + \theta_{t_2'} - \\ \theta_{t_1'} \theta_{t_2'} / 2\pi \end{pmatrix}}, (a_1' a_2') e^{i(\theta_{a_1'} \theta_{a_2'} / 2\pi)} \\ (f_1' f_2') e^{i(\theta_{f_1'} \theta_{f_2'} / 2\pi)} \end{pmatrix}$.

(2) $n_1 \otimes n_2 = (Ŧ_1 \otimes Ŧ_2, Ą_1 \otimes Ą_2, F_1 \otimes F_2) = \coprod_{\substack{t_1' \in Ŧ_1, a_1' \in Ą_1, f_1' \in F_1 \\ t_2' \in Ŧ_2, a_2' \in Ą_2, f_2' \in F_2}} \begin{pmatrix} (t_1' t_2') e^{i(\theta_{t_1'} \theta_{t_2'} / 2\pi)}, \begin{pmatrix} a_1' + a_2' - \\ a_1' a_2' \end{pmatrix} e^{i \begin{pmatrix} \theta_{a_1'} + \theta_{a_2'} - \\ \theta_{a_1'} \theta_{a_2'} / 2\pi \end{pmatrix}}, \\ \begin{pmatrix} f_1' + f_2' - \\ f_1' f_2' \end{pmatrix} e^{i \begin{pmatrix} \theta_{f_1'} + \theta_{f_2'} - \\ \theta_{f_1'} \theta_{f_2'} / 2\pi \end{pmatrix}} \end{pmatrix}$.

(3) $\gamma n_1 = \coprod_{t_1' \in Ŧ_1, a_1' \in Ą_1, f_1' \in F_1} \left((1 - (1 - t_1')^{\gamma}) e^{i2\pi(1 - (1 - \theta_{t_1'}/2\pi)^{\gamma})}, \ a_1'^{\gamma} e^{i\theta_{a_1'}^{\gamma}}, f_1'^{\gamma} e^{i\theta_{f_1'}^{\gamma}} \right)$.

(4) $n_1^{\gamma} = \coprod_{t_1' \in Ŧ_1, a_1' \in Ą_1, f_1' \in F_1} \left(t_1'^{\gamma} e^{i\theta_{t_1'}^{\gamma}}, (1 - (1 - a_1')^{\gamma}) e^{i2\pi(1 - (1 - \theta_{a_1'}/2\pi)^{\gamma})} (1 - (1 - f_1')^{\gamma}) e^{i2\pi(1 - (1 - \theta_{f_1'}/2\pi)^{\gamma})} \right)$.

Theorem 2. *Let $n_1 = (Ŧ_1, Ą_1, F_1) = (t_1' e^{i\theta_{t_1'}}, a_1' e^{i\theta_{a_1'}}, f_1' e^{i\theta_{f_1'}})$ and $n_2 = (Ŧ_2, Ą_2, F_2) = (t_2' e^{i\theta_{t_2'}}, a_2' e^{i\theta_{a_2'}}, f_2' e^{i\theta_{f_2'}})$ be two CSVNHFNs with a positive real number $\gamma, \gamma_1, \gamma_2 > 0$. Then,*

(1) $n_1 \oplus n_2 = n_2 \oplus n_1$.
(2) $n_1 \otimes n_2 = n_2 \otimes n_1$.
(3) $\gamma(n_1 \oplus n_2) = \gamma n_2 \oplus \gamma n_1 1/2$.
(4) $\gamma_1 n_1 \oplus n_2 \gamma_1 = (\gamma_1 + \gamma_2) n_1$.
(5) $n_1^{\gamma} \otimes n_2^{\gamma} = (n_1 \otimes n_2)^{\gamma}$.
(6) $n_1^{\gamma_1} \otimes n_1^{\gamma_2} = n_1^{\gamma_1 + \gamma_2}$.

Definition 17. Let $n = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{a_i}}, f_i' e^{i\theta_{f_i}})$ be CSVNHFN. Then,

$$\S(n) = \frac{1}{6} \left\{ \left(\frac{1}{\alpha} \sum_{i=1}^{\alpha} t_i' + \frac{1}{\beta} \sum_{i=1}^{\beta} (1 - a_i') + \frac{1}{\gamma} \sum_{i=1}^{\gamma} (1 - f_i') \right) + \frac{1}{2} \left(\frac{1}{\alpha} \sum_{i=1}^{\alpha} \theta_{t_i'} + \frac{1}{\beta} \sum_{i=1}^{\beta} (2 - \theta_{a_i'}) + \frac{1}{\gamma} \sum_{i=1}^{\gamma} (2 - \theta_{f_i'}) \right) \right\} \tag{14}$$

Is called the SF, and the AF is denoted and defined by

$$H(n) = \frac{1}{6} \left\{ \left(\frac{1}{\alpha} \sum_{i=1}^{\alpha} (1 - t_i') + \frac{1}{\beta} \sum_{i=1}^{\beta} a_i' + \frac{1}{\gamma} \sum_{i=1}^{\gamma} f_i' \right) + \frac{1}{2} \left(\frac{1}{\alpha} \sum_{i=1}^{\alpha} (1 - \theta_{t_i'}) + \frac{1}{\beta} \sum_{i=1}^{\beta} \theta_{a_i'} + \frac{1}{\gamma} \sum_{i=1}^{\gamma} \theta_{f_i'} \right) \right\}. \tag{15}$$

If we considered the two CSVNHFNs $n_1 = (Ŧ_1, Ą_1, F_1) = (t_1' e^{i\theta_{t_1'}}, a_1' e^{i\theta_{a_1'}}, f_1' e^{i\theta_{f_1'}})$ and $n_2 = (Ŧ_2, Ą_2, F_2) = (t_2' e^{i\theta_{t'2}}, a_2' e^{i\theta_{a'2}}, f_2' e^{i\theta_{f'2}})$, then

(1) If $Ş(n_1) > Ş(n_2)$, then $n_1 > n_2$.
(2) If $Ş(n_1) < Ş(n_2)$, then $n_1 < n_2$.
(3) If $Ş(n_1) = Ş(n_2)$, then $n_1 = n_2$.
 (1) If $H(n_1) > H(n_2)$, then $n_1 > n_2$.
 (2) If $H(n_1) < H(n_2)$, then $n_1 < n_2$.
 (3) If $H(n_1) = H(n_2)$, then $n_1 = n_2$.

4. Some Aggregation Operators Based on CSVNHFSs

In this section, we propose new aggregation operators called CSVNHFGWA operator, CSVNHFGOWA operator, and CSVNHFGHWA operator to aggregate the CSVNHFNs effectively. Throughout the paper, X represents the fixed set and the weighted vector is denoted and defined by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{Ŧ}, \omega_i \in [0,1]$, where $\sum_{i=1}^{n} \omega_i = 1$.

Definition 18. The CSVNHFGWA operator is given by CSVNHFGWA: $\Omega^n \longrightarrow \Omega$:

$$\text{CSVNHFGWA}(n_1, n_2, \ldots, n_n) = \left(\sum_{i=1}^{n} \omega_i n_i^{\gamma}\right)^{1/\gamma}, \tag{16}$$

where Ω represents the family of all CSVNHFNs with $\gamma > 0$. The CSVNHFN is of the form $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{a_i}}, f_i' e^{i\theta_{f_i}})\,(i = 1, 2, \ldots, n)$.

Theorem 3. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{a_i'}}, f_i' e^{i\theta_{a_i}})\,(i = 1, 2, \ldots, n)$ be the family of CSVNHFNs with $\gamma > 0$. Then, consider the concept of CSVNHFGWA operator, and we get CSVNHFGWA$(n_1, n_2, \ldots, n_n)$.*

$$\begin{pmatrix} \left(1 - \prod_{i=1}^{n} \coprod_{t_i'} (1 - (t_i')\gamma)^{\omega_i}\right) e^{i2\pi \left(1 - \prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(1 - \left(\theta_{t_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}} \\ \left(1 - \left(1 - \prod_{i=1}^{n} \coprod_{a_i' \in A_i} (1 - (1 - a_i')\gamma)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi \left(1 - \left(1 - \prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(1 - \left(1 - \theta_{a_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right)} \\ \left(1 - \left(1 - \prod_{i=1}^{n} \coprod_{f_i' \in F_i} (1 - (1 - f_i')\gamma)^{\omega_i}\right)^{1/\gamma}\right) \end{pmatrix}. \tag{17}$$

Proof. (1) First, we have proven that

$$\sum_{i=1}^{n} \omega_i n_i^{\gamma} = \begin{pmatrix} \left(1 - \prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(1 - (t_i')^{\gamma}\right)^{\omega_i}\right) e^{i2\pi \left(1 - \prod_{i=1}^{n} \coprod_{t_i'} \left(1 - (\theta f_i'/2\pi)^{\gamma}\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(1 - (1 - a_i')^{\gamma}\right)^{\omega_i}\right) e^{i2\pi \left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(1 - (1 - \theta a_i'/2\pi)^{\gamma}\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{n} \coprod_{f_i' \in f_i} \left(1 - (1 - f_i')^{\gamma}\right)^{\omega_i}\right) e^{i2\pi \left(\prod_{i=1}^{n} \coprod_{f_i' \in f_i} \left(1 - (1 - \theta f_i'/2\pi)^{\gamma}\right)^{\omega_i}\right)} \end{pmatrix}. \tag{18}$$

We utilize the mathematical induction on n to proof equation (18). □

Case 1. If we considered $n = 1$,

$$n_1^{\gamma} = \coprod_{t_1' \in T_1, a_1' \in A_1, f_1' \in f_1} \begin{pmatrix} (t_1')^{\gamma} e^{i\left(\theta_{t_1'}\right)^{\gamma}}, \left(1 - (1 - a_1')^{\gamma}\right) e^{i2\pi\left(1 - \left(1 - \theta_{a_1'}/2\pi\right)\right)} \\ \left(1 - (1 - f_1')^{\gamma}\right) ei2\pi\left(1 - \left(1 - \theta_{f_1'}/2\pi\right)\right) \end{pmatrix},$$

$$\omega_1 n_1^{\gamma} = \coprod_{t_1' \in T_1, a_1' \in A_1, f_1' \in f_1} \begin{pmatrix} \left(1 - \left(1 - (t_1')^{\gamma}\right)^{\omega^1}\right) e^{i2\pi\left(1 - \left(1 - \left(\theta_{t_1'}\right)^{\gamma}\right)^{\omega_1}\right)} \\ \left(1 - (1 - (a_1'))^{\gamma}\right)^{\omega^1} e^{i2\pi\left(1 - \left(1 - \theta_{a_1'}/2\pi\right)^{\gamma}\right)^{\omega_1}} \\ \left(1 - (1 - (f_1'))^{\gamma}\right)^{\omega^1} e^{i2\pi\left(1 - \left(1 - \theta_{f_1'}/2\pi\right)^{\gamma}\right)^{\omega_1}} \end{pmatrix}. \tag{19}$$

It is true for $n = 1$.

Case 2. If $n = k$ is right, then

$$\sum_{i=1}^{k} \omega_{\mathrm{in}_i^{\gamma}} = \begin{pmatrix} \left(1 - \prod_{i=1}^{k} \coprod_{t_i' \in T_i} \left(1 - (t_i')^{\gamma}\right)^{\omega_i}\right) e^{i2\pi\left(1 - \prod_{i=1}^{k} \coprod_{t_i' \in T_i} \left(1 - \left(\theta_{t_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{k} \coprod_{a_i' \in A_i} \left(1 - (1 - a_i')^{\gamma}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{k} \coprod_{a_i' \in A_i} \left(1 - \left(1 - \theta_{a_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{k} \coprod_{f_i' \in f_i} \left(1 - (1 - f_i')^{\gamma}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{k} \coprod_{f_i' \in f_i} \left(1 - \left(1 - \theta_{f_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)} \end{pmatrix}. \tag{20}$$

Then, we checked for $n = k + 1$, and we get

$$\omega_{k+1} n_{k+1}^{\gamma} = \coprod_{t_{k+1'} \in T_{k+1'}, a_{k+1'} \in A_{k+1'}, f_{k+1'} \in F_{k+1'}} \begin{pmatrix} \left(1 - \left(1 - (t_{k+1'})^{\gamma}\right)^{\omega_{k+1}}\right) e^{i2\pi\left(1 - \left(1 - \left(\theta_{t_{k+1}'}/2\pi\right)^{\gamma}\right)^{\omega_{k+1}}\right)} \\ \left(1 - (1 - a_{k+1'})^{\gamma}\right)^{\omega_{k+1}} e^{i2\pi\left(1 - \left(1 - \theta_{a_{k+1}'}/2\pi\right)^{\gamma}\right)^{\omega_{k+1}}} \\ \left(1 - (1 - f_{k+1'})^{\gamma}\right)^{\omega_{k+1}} e^{i2\pi\left(1 - \left(1 - \theta_{f_{k+1}'}/2\pi\right)^{\gamma}\right)^{\omega_{k+1}}} \end{pmatrix}, \tag{21}$$

and

$$\sum_{i=1}^{k+1}\omega_i n_i{}^{\gamma}=\left(\begin{array}{c}\left(\begin{array}{c}\left(\prod_{i=1}^{k}\coprod_{a_i'\in A_i}\left(1-\left(1-a_i'\right)^{\gamma}\right)^{\omega_i}\right)e^{i2\pi\left(\prod_{i=1}^{k}\coprod_{a_i'\in A_i}\left(1-\left(1-{}^{\theta}a_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)},\\ \left(\prod_{i=1}^{k}\coprod_{f_i'\in F_i}\left(1-\left(1-f_i'\right)^{\gamma}\right)^{\omega_i}\right)e^{i2\pi\left(\prod_{i=1}^{k}\coprod_{f\in F_i}\left(1-\left(1-{}^{\theta}f_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)},\\ \left(1-\prod_{i=1}^{k}\coprod_{t_i'\in T_i}\left(1-\left(t_i'\right)^{\gamma}\right)^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{k}\coprod_{t_i'\in T_i}\left(1-\left({}^{\theta}t_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)}\end{array}\right)\\ \coprod_{t_{k+1}'\in T_{k+1},a_{k+1}'\in A_{k+1},f_{k+1}'\in F_{k+1}}\left(\begin{array}{c}\left(1-\left(1-\left(a_{k+1}'\right)^{\gamma}\right)^{\omega_{k+1}}\right)e^{i\left(1-\left(1-\left({}^{\theta}a'k+1/2\pi\right)^{\gamma}\right)^{\omega_{k+1}}\right)},\\ \left(1-\left(1-\left(f_{k+1}'\right)^{\gamma}\right)^{\omega_{k+1}}\right)e^{i\left(1-\left(1-\left({}^{\theta}f'k+1/2\pi\right)^{\gamma}\right)^{\omega_{k+1}}\right)},\\ \left(1-\left(1-\left(t_{k+1}'\right)^{\gamma}\right)^{\omega_{k+1}}\right)e^{i\left(1-\left(1-\left({}^{\theta}t'k+1/2\pi\right)^{\gamma}\right)^{\omega_{k+1}}\right)},\end{array}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(1-\prod_{i=1}^{k+1}\coprod_{a_i'\in T_i}\left(1-\left(1-a_i'\right)^{\gamma}\right)^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{k+1}\coprod_{a_i'\in A_i}\left(1-\left({}^{\theta}a_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)}\\ \left(1-\prod_{i=1}^{k+1}\coprod_{t_i'\in T_i}\left(1-\left(f_i'\right)^{\gamma}\right)^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{k+1}\coprod_{f_i\in\notin F_i}\left(1-\left(1-{}^{\theta}f_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)}\\ \left(1-\prod_{i=1}^{k+1}\coprod_{t_i'\in T_i}\left(1-\left(t_i'\right)^{\gamma}\right)^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{k+1}\coprod_{t_i'\in T_i}\left(1-\left({}^{\theta}t_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)}\end{array}\right). \tag{22}$$

It is true also for $n=k+1$, so it is true for all n.

Now, we have

$$\mathrm{CSVNHFGWA}\left(n_1,n_2,\ldots,n_n\right)=\left(\sum_{i=0}^{n}\omega_i n_i\right)=\left(\begin{array}{c}\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_i}\left(1-t_i'\right)^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_i}\left(1-\left({}^{\theta}t_i'/2\pi\right)\right)^{\omega_i}\right)},\\ \left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin A_i}\left(a_i'\right)^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin A_i}\left({}^{\theta}a_i'/2\pi\right)^{\omega_i}\right)},\\ \left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F}\left(f_i'\right)^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left({}^{\theta}f_i'/2\pi\right)^{\omega_i}\right)},\end{array}\right). \tag{23}$$

Hence, the result is completed.

Next, we state some properties for CSVNHFGWA operator.

Theorem 4. *Let $n=n_i$ $(i=1,2,\ldots,n)$ be the family of CSVNHFNs with $\gamma>0$. Then, $\mathrm{CSVNHFGWA}(n_1,n_2,\ldots,n_n)=n$.*

Proof. If $n = n_i$, then

$$\begin{aligned}\text{CSVNHFGWA}\left(n_1, n_2, \ldots, n_n\right) &= \left(\sum_{i=1}^{n} \omega_i n_i^{\gamma}\right)^{1/\gamma}\\ &= \left(\sum_{i=1}^{n} \omega_i n^{\gamma}\right)^{1/\gamma}\\ &= \left(n^{\gamma} \sum_{i=1}^{n} \gamma\omega_i\right)^{1/\gamma}\\ &= n.\end{aligned} \tag{24}$$

Hence, the result is completed. □

Theorem 5. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{a_i'}}, f_i' e^{i\theta_{f_i}})\,(i = 1, 2, \ldots, n)$ be the family of CSVNHFNs with $\gamma > 0$. If $n_i \geq n_{j'}$, then*

$$\begin{aligned}&\text{CSVNHFGWA}\left(n_1, n_2, \ldots, n_n\right)\\ &\quad\geq \text{CSVNHFGWA}\left(n_1', n_2', \ldots, n_n'\right).\end{aligned} \tag{25}$$

Proof. We considered $n_i \geq n_{i'}$, that is, $t_i' \geq t_i''$, $a_i' \leq a_i''$, $f_i' \leq f_i''$ and $\theta_{t_i'} \geq \theta_{t_i'}$, $\theta_{a_i'} \leq \theta_{a_i'}$, $\theta_{f_i'} \leq \theta_{f_i'}$ for all i; then, firstly we prove for membership grades such that

$$\begin{aligned}&\left(1-(t_i')^{\gamma}\right)^{\omega_i} e^{i2\pi\left(1-\left(\theta_{t_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}}\\ &\quad\leq \left(1-(t_i'')^{\gamma}\right)^{\omega_i} e^{i2\pi\left(1-\left(\theta_{t_i'}'/2\pi\right)^{\gamma}\right)^{\omega_i}} \prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i} \left(1-(t_i')^{\gamma}\right)^{\omega_i} e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i}\left(1-\left(\theta_{t_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)}\\ &\quad\leq \prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i} \left(1-(t_i'')^{\gamma}\right)^{\omega_i} e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i}\left(1-\left(\theta_{t_i'}'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)} \left(1-\prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i} \left(1-(t_i')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma} e^{i2\pi\left(1-\prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i}\left(1-\left(\theta_{t_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}\\ &\quad\geq \left(1-\prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i} \left(1-(t_i'')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma} e^{i2\pi\left(1-\prod_{i=1}^{n} \coprod_{t_i' \in \text{Ŧ}_i}\left(1-\left(\theta_{t_i'}'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}.\end{aligned} \tag{26}$$

Similarly, for falsity and non-membership grades, we get

$$\begin{aligned}&\left(1-\left(1-\prod_{i=1}^{n} \coprod_{a_i' \in \text{Ą}_i} \left(1-(1-a_i')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n} \coprod_{a_i' \in \text{Ą}_i}\left(1-\left(1-\theta_{a_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right)}\\ &\quad\leq \left(1-\left(1-\prod_{i=1}^{n} \coprod_{a_i' \in \text{Ą}_i} \left(1-(1-a_i'')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n} \coprod_{a_i' \in \text{Ą}_i}\left(1-\left(1-\theta_{a_i'}'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right)},\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\left(1-\left(1-\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(1-(1-f_i')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n} \coprod_{f_i' \in F_i}\left(1-\left(1-\theta_{f_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right)}\\ &\quad\leq \left(1-\left(1-\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(1-(1-f_i'')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n} \coprod_{f_i' \in F_i}\left(1-\left(1-\theta_{f_i'}'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right)}.\end{aligned} \tag{28}$$

Hence, we combine the above equations such that

$$\begin{pmatrix} \left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_i}\left(1-(t_i')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma} e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_i}\left(1-\left({}^{\theta}t_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin A_i}\left(1-(a_i')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin A_i}\left(1-\left({}^{\theta}a_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left(1-(f_i')^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left(1-\left({}^{\theta}f_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \end{pmatrix}$$
$$\geq \begin{pmatrix} \left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_i}\left(1-\left(t_i'\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma} e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_i}\left(1-\left({}^{\theta'}t_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin A_i}\left(1-\left(a_i'\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin A_i}\left(1-\left({}^{\theta'}a_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left(1-\left(f_i'\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left(1-\left({}^{\theta'}f_i'/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \end{pmatrix} \tag{29}$$

So,

$$\begin{aligned} &\text{CSVNHFGWA}\left(n_1, n_2, \ldots, n_n\right) \\ &\quad \geq \text{CSVNHFGWA}\left(n_1', n_2', \ldots, n_n'\right). \end{aligned} \tag{30}$$

Hence, the result is completed. □

Theorem 6. *Let* $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\,(i = 1, 2, \ldots, n)$ *be the family of CSVNHFNs which lies between max and min operators with* $\gamma > 0$. *Then,*

$$\begin{aligned} \min\left(n_1, n_2, \ldots, n_n\right) &\leq \text{CSVNHFGWA}\left(n_1, n_2, \ldots, n_n\right) \\ &\leq \max\left(n_1, n_2, \ldots, n_n\right). \end{aligned} \tag{31}$$

Proof. We know that $\alpha = \min(n_1, n_2, \ldots, n_n)$ and $\beta = \max(n_1, n_2, \ldots, n_n)$; then,

$$\alpha \leq \text{CSVNHFGWA}\left(n_1, n_2, \ldots, n_n\right) \leq \beta. \tag{32}$$

Then, we get

$$\left(\sum_{i=1}^{n}\omega_i \alpha^{\gamma}\right)^{1/\gamma} \leq \left(\sum_{i=1}^{n}\omega_i n_i^{\gamma}\right)^{1/\gamma} \leq \left(\sum_{i=1}^{n}\omega_i \beta^{\gamma}\right)^{1/\gamma}. \tag{33}$$

This implies that

$$\alpha \leq \left(\sum_{i=1}^{n}\omega_i n_i^{\gamma}\right)^{1/\gamma} \leq \beta, \tag{34}$$

i.e.,

$$\begin{aligned} \min\left(n_1, n_2, \ldots, n_n\right) &\leq \text{CSVNHFGWA}\left(n_1, n_2, \ldots, n_n\right) \\ &\leq \max\left(n_1, n_2, \ldots, n_n\right). \end{aligned} \tag{35}$$

Hence, the result is completed. □

Remark 1. The aim of this study is to discover the particular cases of the presented approach.

(1) If $\gamma \longrightarrow 0$, our proposed work will be reduced to CSVNHF weighted geometric operator:

$$\mathrm{CSVNHFWG}(n_1,n_2,\ldots,n_n)=\prod_{i=1}^{n}(n_i)^{\omega_i}=\left(\begin{array}{c}\left(\prod_{i=1}^{n}\coprod_{t_i'\in Ŧ_i}(t_i')^{\omega_i}\right)e^{i2\pi\left(\prod_{i=1}^{n}\coprod_{t_i'\in Ŧ_i}(\theta_{t_i'}/2\pi)^{\omega_i}\right)}\\ \left(1-\prod_{i=1}^{n}\coprod_{a_i'\in Ȃ_i}(1-a_i')^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{a_i'\in Ȃ_i}(1-\theta_{a_i'}/2\pi)^{\omega_i}\right)}\\ \left(1-\prod_{i=1}^{n}\coprod_{f_i'\in F_i}(1-f_i')^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{f_i'\in F_i}(1-\theta f_i'/2\pi)^{\omega_i}\right)}\end{array}\right). \tag{36}$$

(2) If $\gamma=1$, our proposed work will be reduced to CSVNHF weighted averaging operator:

$$\mathrm{CSVNHFGWA}(n_1,n_2,\ldots,n_n)=\left(\sum_{i=0}^{n}\omega_i n_i\right)=\left(\begin{array}{c}\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_{\bar{i}}}(1-t_i')^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_{\bar{i}}}\left(1-\left({}^{\theta}t_i'/2\pi\right)\right)^{\omega_i}\right)},\\ \left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin Ȃ_i}(a_i')^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin Ȃ_i}\left({}^{\theta}a_i'/2\pi\right)^{\omega_i}\right)},\\ \left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}(f_i')^{\omega_i}\right)e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left({}^{\theta}f_i'/2\pi\right)^{\omega_i}\right)},\end{array}\right). \tag{37}$$

If $\gamma=2$, our proposed work will be reduced to CSVNHF weighted quadratic averaging operator:

$$\begin{aligned}&\mathrm{CSVNHFGWQA}(n_1,n_2,\ldots,n_n)\\&=\left(\sum_{i=0}^{n}\omega_i n_i\right)^{1/2}=\left(\begin{array}{c}\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_{\bar{i}}}\left(1-(t_i')^2\right)\right)^{1/2}e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{t_i'\notin T_{\bar{i}}}\left(1-\left({}^{\theta}t_i'/2\pi\right)^2\right)^{\omega_i}\right)^{1/2}},\\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin Ȃ_i}\left(1-(1-a_i')^2\right)^{\omega_i}\right)^{1/2}\right)e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\coprod_{a_i'\notin Ȃ_i}\left(1-\left({}^{\theta}a_i'/2\pi\right)^2\right)^{\omega_i}\right)^{1/2}\right)},\\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left(1-(1-f_i')^2\right)^{\omega_i}\right)^{1/2}\right)e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\coprod_{f_i'\notin F_i}\left(1-\left({}^{\theta}f_i'/2\pi\right)^2\right)^{\omega_i}\right)^{1/2}\right)},\end{array}\right).\end{aligned} \tag{38}$$

Definition 19. The CSVNHFGOWA operator is given by CSVNHFGOWA: $\Omega^n\longrightarrow\Omega$:

$$\mathrm{CSVNHFGOWA}(n_1,n_2,\ldots,n_n)=\left(\sum_{i=1}^{n}\omega_i n_{o(i)}^{\gamma}\right)^{1/\gamma}, \tag{39}$$

where Ω represents the family of all CSVNHFNs with $\gamma > 0$ and $n_{o(i)}$ is the ordered CSVNHFNs which is an ascending ordered (AO) or descending ordered (DO) i.e., $n_{o(i)} \leq n_{o(i-1)}$. The CSVNHFN is of the form $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\,(i = 1, 2, \ldots, n)$.

Theorem 7. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\,(i = 1, 2, .., n)$ be the family of CNHFNs with $\gamma > 0$. Then, considering the concept of CSVNHFGOWA, we get*

$$\text{CSVNHFGOWA}\left(n_1, n_2, \ldots, n_n\right),$$

$$\text{CSVNHF/GOWA}\left(n_1, n_2, \ldots, n_n\right) = \begin{pmatrix} \left(1 - \prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(1 - \left(t_i'\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma} e^{i2\pi\left(1 - \prod_{i=1}^{n} \coprod_{t_i' \in T_i}\left(1 - \left(\theta_{t_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \\ \left(1 - \prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(1 - \left(a_i'\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma} e^{i2\pi\left(1 - \prod_{i=1}^{n} \coprod_{a_i' \in A_i}\left(1 - \left(\theta_{a_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \\ \left(1 - \prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(1 - \left(f_i'\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma} e^{i2\pi\left(1 - \prod_{i=1}^{n} \coprod_{f_i' \in F_i}\left(1 - \left(\theta_{f_i'}/2\pi\right)^{\gamma}\right)^{\omega_i}\right)^{1/\gamma}}, \end{pmatrix}. \tag{40}$$

Proof. Straightforward. □

Theorem 8. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\,(i = 1, 2, \ldots, n)$ be the family of CSVNHFNs with $\gamma > 0$. Then,*

$$\text{CSVNHFGOWA}\left(n_1, n_2, \ldots, n_n\right) = n. \tag{41}$$

Proof. Straightforward. □

Theorem 9. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\,(i = 1, 2, .., n)$ be the family of CSVNHFNs with $\gamma > 0$. If $n_i \geq n_{j'}$, then*

$$\text{CSVNHFGOWA}\left(n_1, n_2, \ldots, n_n\right) \geq \text{CSVNHFGOWA}\left(n_1', n_2', \ldots, n_n'\right). \tag{42}$$

Proof. Straightforward. □

Theorem 10. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\,(i = 1, 2, \ldots, n)$ be the family of CSVNHFNs which lies between max and min operators with $\gamma > 0$. Then,*

$$\begin{aligned} \min\left(n_1, n_2, .., n_n\right) &\leq \text{CSVNHFGOWA}\left(n_1, n_2, \ldots, n_n\right) \\ &\leq \max\left(n_1, n_2, \ldots, n_n\right). \end{aligned} \tag{43}$$

Proof. We know that $\alpha = \min\left(n_1, n_2, \ldots, n_n\right)$ and $\beta = \max\left(n_1, n_2, \ldots, n_n\right)$; then,

$$\alpha \leq \text{CSVNHFGOWA}\left(n_1, n_2, \ldots, n_n\right) \leq \beta. \tag{44}$$

Then, we get

$$\left(\sum_{i=1}^{n} \omega_i \alpha^{\gamma}\right)^{1/\gamma} \leq \left(\sum_{i=1}^{n} \omega_i n_{o(i)}^{\gamma}\right)^{1/\gamma} \leq \left(\sum_{i=1}^{n} \omega_i \beta^{\gamma}\right)^{1/\gamma}. \tag{45}$$

This implies that

$$\alpha \leq \left(\sum_{i=1}^{n} \omega_i n_{o(i)}^{\gamma}\right)^{1/\gamma} \leq \beta. \tag{46}$$

That is,

$$\begin{aligned} \min\left(n_1, n_2, .., n_n\right) &\leq \text{CSVNHFGOWA}\left(n_1, n_2, \ldots, n_n\right) \\ &\leq \max\left(n_1, n_2, \ldots, n_n\right). \end{aligned} \tag{47}$$

Hence, the result is completed. □

Remark 2. The aim of this study is to discover the particular cases of the presented approach.

(1) If $\gamma \longrightarrow 0$, our proposed work will be reduced to CSVNHF ordered weighted geometric operator:

$$\mathrm{CSVNHFGOWA}(n_1, n_2, \ldots, n_n) = \prod_{i=1}^{n} \left(n_{0(i)}\right)^{\omega_i} = \begin{pmatrix} \left(\prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(t'_{o(i)}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(\theta t_{o(i)}/2\pi\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(a'_{o(i)}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(\theta a_{o(i)}/2\pi\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(f'_{o(i)}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(\theta f_{o(i)}/2\pi\right)^{\omega_i}\right)} \end{pmatrix}. \quad (48)$$

(2) If $\gamma = 1$, our proposed work will be reduced to CSVNHF ordered weighted averaging operator:

$$\mathrm{CSVNHFGOWA}(n_1, n_2, \ldots, n_n) = \prod_{i=1}^{n} \left(n_{0(i)}\right)^{\omega_i} = \begin{pmatrix} \left(\prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(t_{o(i)'}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(\theta t_{o(i)}/2\pi\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(a_{o(i)'}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(\theta a_{o(i)}/2\pi\right)^{\omega_i}\right)} \\ \left(\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(f_{o(i)'}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(\theta f_{o(i)}/2\pi\right)^{\omega_i}\right)} \end{pmatrix}. \quad (49)$$

(3) If $\gamma = 2$, the proposed work will be reduced to CSVNHF ordered weighted quadratic averaging operator:

$$\mathrm{CSVNHFGOWA}(n_1, n_2, \ldots, n_n) = \left(\sum_{i=1}^{n} \omega_i n_{o(i)}\right) = \begin{pmatrix} 1 - \left(\prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(t_{o(i)'}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(\theta t_{o(i)}/2\pi\right)^{\omega_i}\right)}, \\ 1 - \left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(a_{o(i)'}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(\theta a_{o(i)}/2\pi\right)^{\omega_i}\right)}, \\ 1 - \left(\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(f_{o(i)'}\right)^{\omega_i}\right) e^{i2\pi\left(\prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(\theta f_{o(i)}/2\pi\right)^{\omega_i}\right)}, \end{pmatrix}. \quad (50)$$

Definition 20. The CSVNHFGHWA operator is given by CSVNHFGHWA: $\Omega^n \longrightarrow \Omega$:

$$\mathrm{CSVNHFGHWA}(n_1, n_2, \ldots, n_n) = \left(\sum_{i=1}^{n} \omega_i \dot{n}_{o(i)}^{\gamma}\right)^{1/\gamma}, \quad (51)$$

where Ω represents the family of all CSVNHFNs with $\gamma > 0$ and $\dot{n}_{o(i)} = n\dot{\omega}_i n_i$ i.e. $n_{o(n)} \leq n_{o(i-1)}$. The CSVNHFN is of the form $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\ (i = 1, 2, \ldots, n)$. Where $\sum_{i=1}^{n} \dot{\omega}_i = 1$.

Theorem 11. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\ (i = 1, 2, \ldots, n)$ be the family of CSVNHFNs with $\gamma > 0$. Then, considering the concept of CSVNHFGHWA, we get*

$$\mathrm{CSVNHFGHWA}(n_1, n_2, \ldots, n_n) = \begin{pmatrix} \left(1 - \prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(1 - \left(t'_{o(i)}\right)^{\gamma}\right)^{w_i}\right)^{1/\gamma} e^{i2\pi\left(1 - \prod_{i=1}^{n} \coprod_{t_i' \in T_i} \left(1 - \left(\theta_{t' o(i)}/2\pi\right)^{\gamma}\right)^{w_i}\right)^{1/\gamma}}, \\ \left(1 - \left(1 - \prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(1 - \left(1 - \dot{a}'_{o(i)}\right)^{\gamma}\right)^{w_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1 - \left(1 - \prod_{i=1}^{n} \coprod_{a_i' \in A_i} \left(1 - \left(1 - \theta_{\dot{a}'_{o(i)}}/2\pi\right)^{\gamma}\right)^{w_i}\right)^{1/\gamma}\right)}, \\ \left(1 - \left(1 - \prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(1 - \left(1 - \dot{f}'_{o(i)}\right)^{\gamma}\right)^{w_i}\right)^{1/\gamma}\right) e^{i2\pi\left(1 - \left(1 - \prod_{i=1}^{n} \coprod_{f_i' \in F_i} \left(1 - \left(1 - \theta_{\dot{f}'_{o(i)}}/2\pi\right)^{\gamma}\right)^{w_i}\right)^{1/\gamma}\right)} \end{pmatrix}. \tag{52}$$

Proof. Straightforward. □

Theorem 12. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\ (i = 1, 2, \ldots, n)$ be the family of CSVNHFNs with $\gamma > 0$. If $n_i \geq n_{j'}$, then*

$$\begin{aligned} &\mathrm{CSVNHFGHWA}(n_1, n_2, \ldots, n_n) \\ &\quad \geq \mathrm{CSVNHFGHWA}(n_1', n_2', \ldots, n_n'). \end{aligned} \tag{53}$$

Proof. Straightforward. □

Theorem 13. *Let $n_i = (t_i' e^{i\theta_{t_i}}, a_i' e^{i\theta_{t_i}}, f_i' e^{i\theta_{t_i}})\ (i = 1, 2, \ldots, n)$ be the family of CSVNHFNs which lies between max and min operators with $\gamma > 0$. Then,*

$$\begin{aligned} \min(n_1, n_2, \ldots, n_n) &\leq \mathrm{CSVNHFGHWA}(n_1, n_2, \ldots, n_n) \\ &\leq \max(n_1, n_2, \ldots, n_n). \end{aligned} \tag{54}$$

Proof. We know that $\alpha = \min(n_1, n_2, \ldots, n_n)$ and $\beta = \max(n_1, n_2, \ldots, n_n)$; then,

$$\alpha \leq \mathrm{CSVNHFGHWA}(n_1, n_2, \ldots, n_n) \leq \beta. \tag{55}$$

Then, we get

$$\left(\sum_{i=1}^{n} \omega_i \alpha^{\gamma}\right)^{1/\gamma} \leq \left(\sum_{i=1}^{n} \omega_i \dot{n}_{o(i)}^{\gamma}\right)^{1/\gamma} \leq \left(\sum_{i=1}^{n} \omega_i \beta^{\gamma}\right)^{1/\gamma}. \tag{56}$$

This implies that

$$\alpha \leq \left(\sum_{i=1}^{n} \omega_i \dot{n}_{o(i)}^{\gamma}\right)^{1/\gamma} \leq \beta. \tag{57}$$

That is,

$$\begin{aligned} \min(n_1, n_2, .., n_n) &\leq \mathrm{CSVNHFGHWA}(n_1, n_2, \ldots, n_n) \\ &\leq \max(n_1, n_2, \ldots, n_n). \end{aligned} \tag{58}$$

Hence, the result is completed. □

Remark 3. The aim of this study is to discover the cases of the presented approach.

(1) If $\gamma \longrightarrow 0$, our proposed work will be reduced to CSVNHF hybrid weighted geometric (CSVNHFHWG) operator:

$$\mathrm{CSVNHFGHWA}\left(n_1, n_2, \ldots, n_n\right) = \prod_{i=1}^{n}\left(\dot{n}_{o(i)}\right)^{w_i} = \begin{pmatrix} \left(\prod_{i=1}^{n}\coprod_{t_i' \in T_i}\left(\dot{t}'_{o(i)}\right)^{w_i}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{t_i' \in T_i}\left(\theta_{t' o(i)}/2\pi\right)^{w_i}\right)}, \\ \left(1-\prod_{i=1}^{n}\coprod_{\dot{a}_i' \in A_i}\left(1-\dot{a}'_{o(i)}\right)^{w_i}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{a_i' \in A_i}\left(1-\theta_{\dot{a}'_{o(i)}}/2\pi\right)^{w_i}\right)}, \\ \left(1-\prod_{i=1}^{n}\coprod_{f_i' \in F_i}\left(1-\dot{f}'_{o(i)}\right)^{w_i}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{f_i' \in F_i}\left(1-\theta_{\dot{f}'_{o(i)}}/2\pi\right)^{w_i}\right)} \end{pmatrix}. \tag{59}$$

(2) If $\gamma = 1$, our proposed work will be reduced to CSVNHF hybrid weighted averaging (CSVNHFHWA) operator:

$$\mathrm{CSVNHFGHWA}\left(n_1, n_2, \ldots, n_n\right) = \left(\sum_{i=1}^{n} W_i \dot{n}_{o(i)}\right) = \begin{pmatrix} \left(1-\prod_{i=1}^{n}\coprod_{\dot{t}_i' \in T_i}\left(1-\dot{t}'_{o(i)}\right)^{w_i}\right) e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{\dot{t}_i' \in T_i}\left(1-\theta_{t' o(i)}/2\pi\right)^{w_i}\right)}, \\ \left(\prod_{i=1}^{n}\coprod_{\dot{a}_i' \in A_i}\left(1-\dot{a}'_{o(i)}\right)^{w_i}\right) e^{i2\pi\left(\prod_{i=1}^{n}\coprod_{\dot{a}_i' \in A_i}\left(\theta_{\dot{a}'_{o(i)}}/2\pi\right)^{w_i}\right)}, \\ \left(\prod_{i=1}^{n}\coprod_{\dot{f}_i' \in F}\left(\dot{f}'_{o(i)}\right)^{w_i}\right) e^{i2\pi\left(\prod_{i=1}^{n}\coprod_{\dot{f}_i' \in F_i}\left(\theta_{\dot{f}'_{o(i)}}/2\pi\right)^{w_i}\right)} \end{pmatrix}. \tag{60}$$

(3) If $\gamma = 2$, our proposed work will be reduced to CSVNHF hybrid weighted quadratic averaging (CSVNHFHWQA) operator:

$$\begin{aligned} &\mathrm{CSVNHFGHWQA}\left(n_1, n_2, \ldots, n_n\right) \\ &= \left(\sum_{i=1}^{n} W_i \dot{n}^2_{o(i)}\right)^{1/2} = \begin{pmatrix} \left(1-\prod_{i=1}^{n}\coprod_{\dot{t}_i' \in T_i}\left(1-\left(t'_{o(i)}\right)^2\right)^{w_i}\right)^{1/2} e^{i2\pi\left(1-\prod_{i=1}^{n}\coprod_{\dot{t}_i' \in T_i}\left(1-\left(\theta_{t' o(i)}/2\pi\right)^2\right)^{w_i}\right)^{1/2}} \\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{\dot{a}_i' \in A_i}\left(1-\left(1-\dot{a}'_{o(i)}\right)^2\right)^{w_i}\right)^{1/2}\right) e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\coprod_{\dot{a}_i' \in A_i}\left(1-\left(1-\theta_{\dot{a}'_{o(i)}}/2\pi\right)^2\right)^{w_i}\right)^{1/2}\right)} \\ \left(1-\left(1-\prod_{i=1}^{n}\coprod_{\dot{f}_i' \in F}\left(1-\left(1-\dot{f}'_{o(i)}\right)^2\right)^{w_i}\right)^{1/2}\right) e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\coprod_{\dot{f}_i' \in F_i}\left(1-\left(1-\theta_{\dot{f}'_{o(i)}}/2\pi\right)^2\right)^{w_i}\right)^{1/2}\right)} \end{pmatrix}. \end{aligned} \tag{61}$$

5. Proposed MADM Method

The decision-making strategy aims to grow the chance of the benefits and reduce the chance of the cost during the decision-making procedure for simplifying genuine life dilemmas. Several people have worked on decision-making strategies under the presence of a crisp set. In several situations, it is very complicated to study the decision-making strategy considering fuzzy sets because they have covered a lot of possibilities. Inspired by the MADM technique, we will employ it here in the presence of the proposed works.

5.1. Decision-Making Algorithm. During the decision-making process, ambiguity and complexity always occur in our day-to-day life. This study aims to employ the decision-making strategy in the presence of proposed works. For this, consider a finite number of alternatives $X = \{u_1, u_2, \ldots, u_m\}$ and attributes $C = \{c_1, c_2, \ldots, c_n\}$. To evaluate each alternative under each attribute, we assign a weight vector for attribute as $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{Ŧ}}, \sum_{i=1}^{n} \omega_i = 1$. Experts are invited to evaluate each alternative $Ą_i$ and give their preferences in terms of the CSVNHF information $n_{ij} = (t_{ij}, a_{ij}, f_{ij})(i, j = 1, 2, \ldots, n)$ for criteria C_j where $\left\{t_{ij} = t'_{ij}e^{i\theta_{t'_{ij}}}/t_{ij} \in Ŧ(u)\right\}$, $\left\{a_{ij} = a'_{ij}e^{i\theta_{a'_{ij}}}/a_{ij} \in Ą(u)\right\}$ and $\left\{f_{ij} = f'_{ij}e^{i\theta_{f'_{ij}}}/f_{ij} \in F(u)\right\}$, gives the TG, abstinence grade, and FG . Then, the following are the steps summarized to find the best alternative(s).

Step 1: collect the data in the shape of the CSVNHF setting and summarize them in the form of the matrix called as decision matrix.

Step 2: normalize the collective information, if needed by using the following equation.

$$D(r_{ij}) = \begin{cases} (t_{ij}, a_{ij}, f_{ij}), & \text{for benefit criteria}, \\ (f_{ij}, a_{ij}, t_{ij}), & \text{for cost criteria}. \end{cases} \tag{62}$$

Step 3: aggregate the collective information by using stated operators. For instance, utilize equation (7) to aggregate the information r_{ij} into $r_i = (t'_i e^{i\theta_{t_i}}, a'_i e^{i\theta_{a_i}}, f'_i e^{i\theta_{f_i}})$.

Step 4: compute the crisp value for each number $r_i = (t'_i e^{i\theta_{t_i}}, a'_i e^{i\theta_{a_i}}, f'_i e^{i\theta_{f_i}})$ by using the following equation:

$$Ş(r_i) = \frac{1}{6}\left\{\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha} t'_i + \frac{1}{\beta}\sum_{i=1}^{\beta}(1 - a'_i) + \frac{1}{\gamma}\sum_{i=1}^{\gamma}(1 - f'_i)\right) + \frac{1}{2}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\theta_{t'_i} + \frac{1}{\beta}\sum_{i=1}^{\beta}\left(2 - \theta_{a'_i}\right) + \frac{1}{\gamma}\sum_{i=1}^{\gamma}\left(2 - \theta_{f'_i}\right)\right)\right\}. \tag{63}$$

If for any two indices, $Ş(r_i)$ values are equal, then compute accuracy degree as

$$H(r_i) = \frac{1}{6}\left\{\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}(1 - t'_i) + \frac{1}{\beta}\sum_{i=1}^{\beta} a'_i + \frac{1}{\gamma}\sum_{i=1}^{\gamma} f'_i\right) + \frac{1}{2}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\left(1 - \theta_{t'_i}\right) + \frac{1}{\beta}\sum_{i=1}^{\beta}\theta_{a'_i} + \frac{1}{\gamma}\sum_{i=1}^{\gamma}\theta_{f'_i}\right)\right\}. \tag{64}$$

Step 5: rank the alternatives based on the crisp values and get the beneficial optimal.

5.2. Illustrated Example. To illustrate the working of the above stated algorithm to the decision-making process, we consider a numerical example which can be read as follows.

Consider a decision-making process related to the selection of the best enterprise for the investment. Since the market always shows an up and down phases at a regular interval of time, the decision maker will always look for the desired option for the investment. After deep analysis of the market scenario, an investor selects the following four potential options, considered as an alternative, which are characterized as

(i) $Ą_1$: invest in the local market.
(ii) $Ą_2$: invest in the southern Asian market.
(iii) $Ą_3$: invest in the northeastern market.
(iv) $Ą_4$: invest in the European market.

To access all these alternatives deeply, we have taken the following three attributes with attribute weights taken as $\omega = (0.5, 0.4, 0.1)^{\mathrm{Ŧ}}$:

(i) C_1C_1: economic growth.
(ii) C_2: profit in the long term.
(iii) C_3: social political impact analysis.

Then, the following steps of the stated algorithms are implemented to find the best alternative as follows.

Step 1: a senior expert from the market sector is hired for accessing the given alternatives under each attribute. Their corresponding rating is summarized in Table 1

Step 2: as all the criteria are of benefit types, there is no need for normalization.

Step 3: with attribute weights taken as $\omega = (0.5, 0.4, 0.1)^{\mathrm{Ŧ}}$, utilize CSVNHFGHA operator as stated in equation. (52) to aggregate the information for $\gamma = 1$.

Step 4: the score values of all the aggregated numbers (as obtained from Step 3) are calculated, and their results are summarized in Table 2 along with the ranking order.

Table 1: CSVNHF generalized aggregation operator decision matrix.

Data representations	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$
$A1$	$\left(\begin{array}{c}\begin{pmatrix}(0.7)e^{i.2(0.9)}\\(0.5)e^{i.2(0.4)}\\(0.7)e^{i.2(0.7)}\end{pmatrix}\\\begin{pmatrix}(0.6)e^{i.2(0.87)}\\(0.56)e^{i.2(0.45)}\\(0.76)e^{i.2(0.66)}\end{pmatrix}\\\begin{pmatrix}(0.67)e^{i.2(0.98)}\\(0.65)e^{i.2(0.67)}\\(0.14)e^{i.2(0.56)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.77)e^{i.2(0.8)}\\(0.78)e^{i.2(0.81)}\\(0.79)e^{i.2(0.82)}\end{pmatrix}\\\begin{pmatrix}(0.83)e^{i.2(0.67)}\\(0.84)e^{i.2(0.45)}\\(0.85)e^{i.2(0.67)}\end{pmatrix}\\\begin{pmatrix}(0.56)e^{i.2(0.78)}\\(0.56)e^{i.2(0.79)}\\(0.67)e^{i.2(0.69)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.13)e^{i.2(0.46)}\\(0.23)e^{i.2(0.64)}\\(0.45)e^{i.2(0.66)}\end{pmatrix}\\\begin{pmatrix}(0.45)e^{i.2(0.56)}\\(0.54)e^{i.2(0.65)}\\(0.55)e^{i.2(0.87)}\end{pmatrix}\\\begin{pmatrix}(0.89)e^{i.2(0.89)}\\(0.67)e^{i.2(0.66)}\\(0.47)e^{i.2(0.88)}\end{pmatrix}\end{array}\right)$
A_2	$\left(\begin{array}{c}\begin{pmatrix}(0.45)e^{i.2(0.93)}\\(0.56)e^{i.2(0.45)}\\(0.78)e^{i.2(0.76)}\end{pmatrix},\\\begin{pmatrix}(0.67)e^{i.2(0.88)}\\(0.6)e^{i.2(0.5)}\\(0.7)e^{i.2(0.7)}\end{pmatrix},\\\begin{pmatrix}(0.8)e^{i.2(0.3)}\\(0.6)e^{i.2(0.67)}\\(0.6)e^{i.2(0.56)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.77)e^{i.2(0.56)}\\(0.34)e^{i.2(0.89)}\\(0.56)e^{i.2(0.78)}\end{pmatrix},\\\begin{pmatrix}(0.67)e^{i.2(0.77)}\\(0.92)e^{i.2(0.66)}\\(0.57)e^{i.2(0.45)}\end{pmatrix},\\\begin{pmatrix}(0.67)e^{i.2(0.45)}\\(0.56)e^{i.2(0.79)}\\(0.45)e^{i.2(0.57)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.58)e^{i.2(0.69)}\\(0.67)e^{i.2(0.49)}\\(0.68)e^{i.2(0.59)}\end{pmatrix},\\\begin{pmatrix}(0.59)e^{i.2(0.58)}\\(0.39)e^{i.2(0.67)}\\(0.49)e^{i.2(0.34)}\end{pmatrix},\\\begin{pmatrix}(0.33)e^{i.2(0.55)}\\(0.44)e^{i.2(0.65)}\\(0.45)e^{i.2(0.77)}\end{pmatrix}\end{array}\right)$
A_3	$\left(\begin{array}{c}\begin{pmatrix}(0.7)e^{i.2(0.45)}\\(0.45)e^{i.2(0.45)}\\(0.45)e^{i.2(0.45)}\end{pmatrix}\\\begin{pmatrix}(0.76)e^{i.2(0.23)}\\(0.5)e^{i.2(0.43)}\\(0.7)e^{i.2(0.45)}\end{pmatrix}\\\begin{pmatrix}(0.54)e^{i.2(0.64)}\\(0.65)e^{i.2(0.57)}\\(0.46)e^{i.2(0.74)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.68)e^{i.2(0.85)}\\(0.86)e^{i.2(0.48)}\\(0.58)e^{i.2(0.84)}\end{pmatrix}\\\begin{pmatrix}(0.66)e^{i.2(0.45)}\\(0.9)e^{i.2(0.67)}\\(0.57)e^{i.2(0.36)}\end{pmatrix}\\\begin{pmatrix}(0.67)e^{i.2(0.56)}\\(0.45)e^{i.2(0.67)}\\(0.33)e^{i.2(0.34)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.45)e^{i.2(0.68)}\\(0.46)e^{i.2(0.48)}\\(0.67)e^{i.2(0.46)}\end{pmatrix}\\\begin{pmatrix}(0.5)e^{i.2(0.45)}\\(0.39)e^{i.2(0.67)}\\(0.49)e^{i.2(0.67)}\end{pmatrix}\\\begin{pmatrix}(0.56)e^{i.2(0.57)}\\(0.45)e^{i.2(0.45)}\\(0.67)e^{i.2(0.34)}\end{pmatrix}\end{array}\right)$
A_4	$\left(\begin{array}{c}\begin{pmatrix}(0.4)e^{i.2(0.45)}\\(0.56)e^{i.2(0.78)}\\(0.46)e^{i.2(0.46)}\end{pmatrix}\\\begin{pmatrix}(0.6)e^{i.2(0.9)}\\(0.7)e^{i.2(0.2)}\\(0.8)e^{i.2(0.3)}\end{pmatrix}\\\begin{pmatrix}(0.4)e^{i.2(0.47)}\\(0.5)e^{i.2(0.49)}\\(0.45)e^{i.2(0.59)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.8)e^{i.2(0.67)}\\(0.8)e^{i.2(0.54)}\\(0.9)e^{i.2(0.63)}\end{pmatrix}\\\begin{pmatrix}(0.66)e^{i.2(0.9)}\\(0.78)e^{i.2(0.88)}\\(0.89)e^{i.2(0.45)}\end{pmatrix}\\\begin{pmatrix}(0.56)e^{i.2(0.69)}\\(0.57)e^{i.2(0.67)}\\(0.68)e^{i.2(0.7)}\end{pmatrix}\end{array}\right)$	$\left(\begin{array}{c}\begin{pmatrix}(0.35)e^{i.2(0.48)}\\(0.67)e^{i.2(0.49)}\\(0.38)e^{i.2(0.51)}\end{pmatrix}\\\begin{pmatrix}(0.52)e^{i.2(0.65)}\\(0.53)e^{i.2(0.75)}\\(0.54)e^{i.2(0.85)}\end{pmatrix}\\\begin{pmatrix}(0.86)e^{i.2(0.75)}\\(0.87)e^{i.2(0.73)}\\(0.74)e^{i.2(0.71)}\end{pmatrix}\end{array}\right)$

Table 2: Ranking results of the CSVNHFSs.

Alternatives	Score function	Rank
A_1	$S(A_1) = 0.467$	4
A_2	$S(A_2) = 0.515$	2
A_3	$S(A_3) = 0.524$	1
A_4	$S(A_4) = 0.478$	3

Step 5: based on the score values, we rank the given alternatives as

$$A_3 \geq A_2 \geq A_4 \geq A_1, \tag{65}$$

and found that A_3 is the best one for this suitable job.

5.3. Comparative Analysis. To compare the study with the several existing studies, we compare the performance of the existing algorithms in [20, 21, 26, 31–34] under different environments. The result corresponding to each method is listed in Table 3. It is clear that from Table 3 that A_3 is the best alternative identified by all existing methods and the proposed method. Although the ranking result is same by all the methods, the proposed MADM algorithm has several advantages over such existing studies.

To highlight such things, we summarize the characteristics of the stated method over the existing ones in Table 4. It is seen from this table that proposed method in this paper is more generalized than the existing methods. Also, it is clear that the proposed concept is more general and reliable than IFS [3], CIFS [20], CNS [26], and NS [7], and all are

Table 3: Comparative study with the existing methods.

Method	Score values	Ranking results	Best alternatives
Ali and Smarandache [26]	Cannot be calculated	No	No
Alkouri and Salleh [20]	Cannot be calculated	No	No
Garg and Rani [31, 32]	Cannot be calculated	No	No
Rani and Garg [21, 33]	Cannot be calculated	No	No
Beg and Rashid [35]	$\text{Ş}(\underset{\sim}{A}_1) = 0.66, \text{Ş}(\underset{\sim}{A}_2) = 0.71, \text{Ş}(3) = 0.72, \text{Ş}(\underset{\sim}{A}_4) = 0.67$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$	$\underset{\sim}{A}_3$
Torra [27]	$\text{Ş}(\underset{\sim}{A}_1) = 0.573, \text{Ş}(\underset{\sim}{A}_2) = 0.601, \text{Ş}(3) = 0.603, \text{Ş}(\underset{\sim}{A}_4) = 0.579$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$	$\underset{\sim}{A}_3$
Beg and Rashid [35]	$\text{Ş}(\underset{\sim}{A}_1) = 0.47, \text{Ş}(\underset{\sim}{A}_2) = 0.52, \text{Ş}(3) = 0.53, \text{Ş}(\underset{\sim}{A}_4) = 0.49$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$	$\underset{\sim}{A}_3$
Proposed method	$\text{Ş}(\underset{\sim}{A}_1) = 0.467, \text{Ş}(\underset{\sim}{A}_2) = 0.515, \text{Ş}(3) = 0.524, \text{Ş}(\underset{\sim}{A}_4) = 0.478$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$	$\underset{\sim}{A}_3$

Table 4: Characteristic comparison between proposed work and existing methods.

Methods	Ability to integrate information	Ability to capture information using complex numbers	Ability to handle two-dimensional information	Flexible according to decision makers' preferences	Superior characteristic of the ideas	Dealing hesitant kind of information
Atanassov [3]	Yes	No	No	No	No	No
Alkouri and Salleh [20]	Yes	Yes	Yes	No	No	No
Garg and Rani [31, 32]	Yes	Yes	Yes	No	No	No
Rani and Garg [21, 33]	Yes	Yes	Yes	No	No	No
Smarandache [7]	Yes	No	No	Yes	Yes	No
Ali and Smarandache [26]	Yes	Yes	Yes	Yes	Yes	No
Proposed work	Yes	Yes	Yes	Yes	Yes	Yes

Table 5: Impact of the parameter γ on the ratings.

Parameter γ	Score values	Ranking
$\gamma = 1$	$\text{Ş}(\underset{\sim}{A}_1) = 0.467, \text{Ş}(\underset{\sim}{A}_2) = 0.515, \text{Ş}(\underset{\sim}{A}_3) = 0.524, \text{Ş}(\underset{\sim}{A}_4) = 0.478$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$
$\gamma = 4$	$\text{Ş}(\underset{\sim}{A}_1) = 0.496, \text{Ş}(\underset{\sim}{A}_2) = 0.539, \text{Ş}(\underset{\sim}{A}_3) = 0.545, \text{Ş}(\underset{\sim}{A}_4) = 0.518$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$
$\gamma = 10$	$\text{Ş}(\underset{\sim}{A}_1) = 0.522, \text{Ş}(\underset{\sim}{A}_2) = 0.566, \text{Ş}(\underset{\sim}{A}_3) = 0.571, \text{Ş}(\underset{\sim}{A}_4) = 0.553$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$
$\gamma = 15$	$\text{Ş}(\underset{\sim}{A}_1) = 0.531, \text{Ş}(\underset{\sim}{A}_2) = 0.57, \text{Ş}(\underset{\sim}{A}_3) = 0.58, \text{Ş}(\underset{\sim}{A}_4) = 0.56$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$
$\gamma = 20$	$\text{Ş}(\underset{\sim}{A}_1) = 0.536, \text{Ş}(\underset{\sim}{A}_2) = 0.584, \text{Ş}(\underset{\sim}{A}_3) = 0.588, \text{Ş}(\underset{\sim}{A}_4) = 0.57$	$\underset{\sim}{A}_3 \geq \underset{\sim}{A}_2 \geq \underset{\sim}{A}_4 \geq \underset{\sim}{A}_1$

considered as special cases of CSVNHFS. For instance, by taking $\gamma = 1$ in the proposed work, the stated method reduces to CSVNHFWA operators, and if $\gamma = 2$, then our proposed work is reduced to CSVNHFWG.

5.4. Sensitivity Analysis of the Parameter γ. In this section, we examined the influence of parameter γ on to the decision-making process. For this, we vary the value of the parameter γ from 1 to 20 (by taking arbitrary values) and implement the steps of the proposed algorithm on it. The final score values and the ranking order for each value of γ are recorded and listed in Table 5. Also, we have listed some special cases of the proposed operators with γ in Remarks 1–3. From this investigation, we noted that as we increase the value of γ, the score value corresponding to each alternative increases which shows the optimistic nature. Decision makers can select the suitable value as per their choice by seeing the overall score value of each alternative. From the above analysis, we conclude that the proposed strategies are more unrivaled and more broad than existing work.

6. Conclusion

The main contribution of this paper is discussed below.

(1) A novel concept of CSVNHFS is defined in the paper by incorporating the features of SVN, hesitant, and complex sets. The idea behind this set is to address the ambiguity in the data when it is arranged in the form of "yes," "abstinence," and "no" under the complex domain. In the presented set, each element is characterized with three independent hesitant degrees, namely, TG ($t' e^{i\theta_{t'}}$), abstinence ($a' e^{i\theta_{a'}}$), and FG ($f' e^{i\theta_{f'}}$), over the unit disc of complex plane with the conditions $0 \leq t' + a' + f' \leq 3$ and $0 \leq \theta_{t'} + \theta_{a'} + \theta_{f'} \leq 6$ where $0 \leq t', a', f' \leq 1$ and $0 \leq \theta_{t'}, \theta_{a'}, \theta_{f'} \leq 2$.

(2) The fundamental properties and the operations of the stated set are investigated. Also, it is analyzed that the proposed CSVNHFS is the generalization of the existing sets such as IFS [3], CIFS [20], CNS [26], and NS [7].

(3) By taking the features of the CSVNHFS, we defined generalized weighted average (CSVNHFGWA), ordered weighted (CSVNHFGOWA), and hybrid weighted averaging (CSVNHFGHWA) operators to aggregate different pairs of the given information. Some of their basic properties are also discussed.

(4) A MADM algorithm based on the stated operators is presented by using the features of CSVNHFS and illustrated with the numerical examples.

(5) To determine the supremacy and efficiency of investigated operators, we utilized the advantages, sensitive analysis, and geometrical expressions of the proposed work to discover the dominance of the elaborated approaches.

In the future, we will try to implement the application of the stated algorithm and extend it to different fuzzy environments [35–42].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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