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Research Article

Convergence and Stability of a Novel $\mathfrak{M}$-Iterative Algorithm with an Application

Manjinder Kaur[1,2] and Sumit Chandok[1]

[1]*School of Mathematics, Thapar Institute of Engineering & Technology, Patiala 147004, India*
[2]*Department of Mathematics, Lyallpur Khalsa College for Women, Guru Nanak Dev University, Jalandhar 144001, India*

Correspondence should be addressed to Sumit Chandok; sumit.chandok@thapar.edu

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In this manuscript, we propose a novel three - step iteration scheme called $\mathfrak{M}$ - iteration to approximate the invariant points for the class of weak contractions in the sense of Berinde and obtain that $\mathfrak{M}$ - iteration strongly converges to one and only one fixed point for Berinde mappings. Proving 'almost - stability' of $\mathfrak{M}$ - iteration, we compare the $\mathfrak{M}$ - iterative scheme with other chief iterative algorithms, namely. Picard, Mann, Ishikawa, Noor, S, normal-S, Abbas, Thakur, Varat, Ullah and F^* and claim that the framed iterative procedure converges to the invariant points of weak contractions at faster rate than other vital algorithms. Some numerical illustrations are adduced to strengthen our claim. We further ascertain data dependency through $\mathfrak{M}$ - iteration. Finally, we establish the solution of Caputo type fractional differential equation as an application and exhibit that $\mathfrak{M}$ - fixed point procedure converges to the solution of a fractional differential equation. The obtained results are not only new but, also, extend the scope of previous findings.

1. Introduction

Several nonlinear problems can be mathematically modelled via real valued self-mapping

$$M(x) = x. \tag{1}$$

Endowed with specific properties like continuity, contraction, nonexpansiveness etc. Acknowledging the work of Banach, mappings with contraction condition have proven to be fascinating in the theory of fixed points. But an obvious question arises:

What if the contraction condition on self - mapping is weaken?

Answering to this, Berinde in 2003 (see [1]) came up with a notion called weak contraction, sometimes termed as almost - contractions. He established that the class of these weak contractions is wider than contraction mappings, Zamfirescu mappings and developed the existence and uniqueness theorem for fixed points in the ambient domain.

Further, finding an analytical solution of (1) is not so straightforward that urges the researchers to numerical reckoning of fixed points. Owing to this limitation, researchers broached a number of approximation procedures to estimate fixed points of varying classes of nonlinear mappings, Picard [2], Mann [3], Ishikawa [4], S [5], K^* [6] are to name a few. Motivated by these works, we propose a new three - step approximation procedure called $\mathfrak{M}$ - iteration and prove that the suggested iterative scheme converges strongly to a unique fixed point for the class of Berinde's weak contractions. Also, we establish almost M-stability of our proposed algorithm in Subsection 3.1. In Section 4, we present that our algorithm has faster rate of convergence than certain other significant approximation procedures. Subsection 4.1, hooks our assertion of Section 4 numerically via few illustrations. In Section 5, we exhibit data dependency by $\mathfrak{M}$- iteration. From application perspective, we obtain a solution of Caputo type fractional differential equation in the last section.

2. Brief Preliminaries

In this section, we present few requisite definitions, notations and results.Berinde in 2003 [7] came up with a broader class of contraction mappings called weak contractions, sometimes known as 'almost contractions', which can be defined as:

Definition 1. Let $\breve{\mathscr{B}}$ be a Banach space. A self - mapping M on $\breve{\mathscr{B}}$ is called weak contraction if there exist a constant $\delta \in (0,1)$ and a nonnegative constant L, such that

$$\|Mx - My\| \leq \delta\|x - y\| + L\|y - Mx\|, \quad \text{for all } x,\ y \in \breve{\mathscr{B}}. \tag{2}$$

It is to be emphasized that weak contractions are different from contraction mappings (see Example 2.11 of [1]). He then manifests the subsequent theorem for the case of weak contraction mappings.

Theorem 1. *The self - mapping M on a Banach space $\breve{\mathscr{B}}$ agreeing (2) together with*

$$\|Mx - My\| \leq \delta\|x - y\| + L\|x - Mx\|, \quad \text{for all } x,\ y \in \breve{\mathscr{B}}, \tag{3}$$

ensues that the mapping M has a unique fixed point in $\breve{\mathscr{B}}$.

Our study incorporates a comparison of the various iterative procedures see Table 1 framed for the self - mapping $M: G \longrightarrow G$, where G is a nonempty subset of Banach space $\mathscr{B}$, and $\{l_n\}, \{m_n\}$ and $\{e_n\}$ are sequences in $(0,1)$.

The following two definitions were accorded by Berinde [14] to size up the rate of convergence of two iterative procedures.

Definition 2. Let $\{\imath_{\mathrm{n}}\}$ and $\{\mathrm{j}\}$ be two positive sequences in reals, converging to $\imath$ and respectively. Suppose

$$ɭ = \lim_{n\longrightarrow\infty} \frac{|\imath_{\mathrm{n}} - \imath|}{|j_n - |}. \tag{4}$$

(i) If ɭ=0, then $\{\imath_{\mathrm{n}}\}$ converges faster to 1 as compare to $\{j_n\}$ converging to.

(ii) If $0 <$ ɭ $< \infty$, then the rate of convergence of both $\{\imath_{\mathrm{n}}\}$ and $\{j_n\}$ are coequal.

Definition 3. Assume $\{k_n\}$ and $\{\rho_n\}$ are two iterative procedures such that each of them is converging to the common point k with error estimates:

$$\begin{aligned} |k_n - k| &\leq \imath_{\mathrm{n}}, \\ |\rho_n - k| &\leq . \end{aligned} \tag{5}$$

If $\lim_{n\longrightarrow\infty}(\imath_{\mathrm{n}}/j_n) = 0$, then $\{k_n\}$ converges faster than $\{\rho_n\}$.

An iterative procedure, in general, is prone to errors that may cause harm to its aspect of convergence. To address this problem, the concept of stability of a fixed point procedure was introduced in 1967 by Ostrowski [15]. However, it was Harder et al. [16], who, using the idea of Ostrowski, clearly defined the concept for the first time in 1988 and established the stability of Picard iteration with respect to α - contractions and Zamfirescu maps (see Theorems 1 and 2 of [16]) in the setting of metric space. He further showed that certain Mann iterations are stable with respect to Zamfirescu maps in the framework of normed linear space.

The following definition is based on the idea of Theorem 2 of Ostrowski [15].

Definition 4. Suppose, for a self - mapping M defined on Banach space $\breve{\mathscr{B}}$, we have:

(i) $Mg = g$,

(ii) $g_{n+1} = \theta(M, g_n)$, $n \in \mathbb{N} \cup \{0\}$, with $g_0 \in \breve{\mathscr{B}}$ where θ is certain function,

(iii) an approximate sequence $\{r_n\}$ of $\{g_n\}$ in $\breve{\mathscr{B}}$,

(iv) $\tilde{u}_n = \|r_{n+1} - \theta(M, r_n)\|$,

Then the iterative procedure g_{n+1} may be regarded as M - stable on condition that:

$$\lim_{n\longrightarrow\infty} \tilde{u}_n = 0 \text{ implies } \lim_{n\longrightarrow\infty} r_n = g. \tag{6}$$

In 1990, Rhoades [17] extended the results of Harder et al. [16] and showed the stability for some fixed point iterations (Picard, Mann). Osilike [18] introduced the concept of weak stability called 'almost-stability' for iterative algorithms as follows:

Definition 5. Suppose, for a self - mapping M defined on Banach space $\breve{\mathscr{B}}$, we have:

(i) $Mg = g$,

(ii) $g_{n+1} = \theta(M, g_n)$, $n \in \mathbb{N} \cup \{0\}$, with $g_0 \in \breve{\mathscr{B}}$ where θ is certain function,

(iii) an approximate sequence $\{r_n\}$ of $\{g_n\}$ in $\breve{\mathscr{B}}$,

(iv) $\tilde{u}_n = \|r_{n+1} - \theta(M, r_n)\|$,

Then the iterative procedure g_{n+1} may be regarded as almost M - stable if:

$$\sum_{n=0}^{\infty} \tilde{u}_n < \infty \text{ implies } \lim_{n\longrightarrow\infty} r_n = g. \tag{7}$$

Determination of fixed points is a challenging task, especially, in the case, when behaviour of an operator is not known. This propel the researchers to study about data dependency of fixed points. By data dependency for an iterative scheme, we mean, there exists an approximation by which one can find the fixed point of an unknown operator using the fixed point of a known operator. Markin [19] in 1973 was first to talk about continuous data dependency of fixed points. In 1998, Rus and Muresan [20] studied data dependency of weakly Picard operators. Then, again Rus et.al. in 2001 [21] and 2003 [22], discussed data dependence of fixed points of some multivalued weakly Picard operators.

Şoltuz [23] in 2004 established data dependence result for Mann- Ishikawa iteration scheme. He, together with Grosan [24] in 2008, demonstrated the result for Ishikawa iterative procedure for contractive - like mappings. In 2009, Olatinwo [25] documented couple of results on continuous data dependency of fixed points.

The following results will be needed in the sequel.

Definition 6 (see [1]). Suppose M and Q be two self - operators defined on a nonvoid set $G \subset \breve{\mathscr{B}}$ then the operator Q may be called an approximate operator of M if a positive constant $\varepsilon > 0$ exists such that

$$\|Mx - Qx\| \leq \varepsilon \text{ for all } x \in G. \tag{8}$$

Lemma 1. *[26] Let $\{\tilde{u}_n\}$ and $\{\tilde{v}_n\}$ be two nonnegative sequences in $\mathbb{R}$ and $0 \leq q < 1$, such that $\tilde{v}_{n+1} \leq q\tilde{v}_n + \tilde{u}_n$, for all $n \geq 0$. If $\lim_{n\longrightarrow\infty} \tilde{u}_n = 0$, then $\lim_{n\longrightarrow\infty} \tilde{v}_n = 0$.*

We need the following result to prove data dependency which is according to Şoltuz et al. [24], modified by Ali et al. [13].

Lemma 2. *Let there exists a positive integer m for a sequence $\left\{\grave{\mathfrak{a}}_n\right\}$ of positive reals, such that*

$$\grave{\mathfrak{a}}_{n+1} \leq \left(1 - \grave{\mathfrak{b}}_n\right)\grave{\mathfrak{a}}_n + \grave{\mathfrak{b}}_n\grave{\mathfrak{c}}_n \text{ for all } n \geq m, \tag{9}$$

where $\grave{\mathfrak{b}}_n \in (0, 1)$ for all $n \in \mathbb{N} \cup \{0\}$ such that series $\grave{\mathfrak{b}}_n$ is divergent and $\grave{\mathfrak{c}}_n$ is nonnegative bounded sequence. This gives:

$$0 \leq \lim_{n\longrightarrow\infty} \sup \grave{\mathfrak{a}}_n \leq \lim_{n\longrightarrow\infty} \sup \grave{\mathfrak{c}}_n. \tag{10}$$

Upcoming lemma is given by Şoltuz et al. [27] in 2007.

Lemma 3. *For a sequence $\{\tilde{\mathfrak{e}}_n\}$ of positive reals satisfying following inequality,*

$$\tilde{\mathfrak{e}}_{n+1} \leq \left(1 - \tilde{\mathfrak{d}}_n\right)\tilde{\mathfrak{e}}_n, \tag{11}$$

If $\tilde{\mathfrak{d}}_n \in (0, 1)$ and $\sum \tilde{\mathfrak{d}}_n = \infty$ then $\lim_{n\longrightarrow\infty}\tilde{\mathfrak{e}}_n = 0$.

3. Main Results

We begin with the introduction of a new three - step iteration called $\mathfrak{M}$ - iteration along these lines:

Let $M\colon G \longrightarrow G$ be a self - mapping where G is nonvoid, closed and convex subset of Banach space $\breve{\mathscr{B}}$. We define sequence $\{g_n\}$ in the following fashion:

$\mathfrak{M}$ - iteration

$$\begin{cases} g_0, & \in G, \\ g_{n+1}, & = M\left[(1 - e_n)Mh_n + e_n h_n\right], \\ h_n, & = M\left[(1 - l_n)f_n + l_n M f_n\right], \\ f_n, & = M\left[(1 - m_n)g_n + m_n M g_n\right],\ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{12}$$

where $\{l_n\}, \{m_n\}, \{e_n\}$ are sequences in $(0, 1)$.

We prove that $\mathfrak{M}$ - iteration defined by (12) converges strongly to fixed point for the class of weak contractions.

Theorem 2. *For a Banach space $\breve{\mathscr{B}}$ and nonvoid, closed and convex set $G \subseteq \breve{\mathscr{B}}$, let $M\colon G \longrightarrow G$ is Berinde weak contraction agreeing (3), then the sequence $\{g_n\}$ specified by $\mathfrak{M}$ - iteration (12) converges to unique fixed point of M subject to the sequences $\{l_n\}, \{m_n\}, \{e_n\}$ generated by (12) obeys one of the following:*

$$\begin{aligned} &(C_{\mathfrak{M}}1) \sum l_n = \infty, \\ &(C_{\mathfrak{M}}2) \sum m_n = \infty, \\ &(C_{\mathfrak{M}}3) \sum e_n = \infty. \end{aligned} \tag{13}$$

Proof. From Theorem 1, M possess a unique fixed point say g and using (3), we have

$$\begin{aligned} \|Mg_n - g\| &= \|Mg_n - Mg\| \leq \delta\|g_n - g\| + L\|g - Mg\| \\ &= \delta\|g_n - g\| \text{ for all } n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{14}$$

By $\mathfrak{M}$- iterative algorithm (12), we have,

$$\begin{aligned} \|f_n - g\| &= \|M((1 - m_n)g_n + m_n M g_n) - g\| \\ &\leq \delta\|(1 - m_n)g_n + m_n M g_n - g\| \\ &\leq \delta\left((1 - m_n)\|g_n - g\| + m_n\|Mg_n - g\|\right) \\ &\leq \delta\left((1 - m_n)\|g_n - g\| + \delta m_n\|g_n - g\|\right) \\ &= \delta((1 - m_n) + \delta m_n)\|g_n - g\| = \delta(1 - (1 - \delta)m_n)\|g_n - g\|, \end{aligned} \tag{15}$$

And

$$\begin{aligned} \|h_n - g\| &= \|M((1 - l_n)f_n + l_n M f_n) - g\| \\ &\leq \delta\|(1 - l_n)f_n + l_n M f_n - g\| \\ &\leq \delta\left((1 - l_n)\|f_n - g\| + l_n\|Mf_n - g\|\right) \\ &\leq \delta\left((1 - l_n)\|f_n - g\| + \delta l_n\|f_n - g\|\right) \\ &= \delta((1 - l_n) + \delta l_n)\|f_n - g\| \\ &= \delta(1 - (1 - \delta)l_n)\|f_n - g\|. \end{aligned} \tag{16}$$

Using (15) and (16), we get

$$\begin{aligned} \|g_{n+1} - g\| &= \|M((1 - e_n)Mh_n + e_n h_n) - Mg\| \\ &\leq \delta\|((1 - e_n)Mh_n + e_n h_n) - g\| \\ &\leq \delta\left((1 - e_n)\|Mh_n - Mg\| + e_n\|h_n - g\|\right) \\ &\leq \delta\left((1 - e_n)\delta\|h_n - g\| + e_n\|h_n - g\|\right) \\ &\leq \delta(1 - (1 - \delta)e_n)\|h_n - g\| \\ &\leq \delta^3(1 - (1 - \delta)l_n)(1 - (1 - \delta)m_n) \\ &\quad (1 - (1 - \delta)e_n)\|g_n - g\|. \end{aligned} \tag{17}$$

Inductively, we get

$$\|g_{n+1} - g\| \leq \delta^{3(n+1)} \prod_{t=1}^{t=n} \left(1 - (1-\delta) l_t\right)\left(1 - (1-\delta) m_t\right)\left(1 - (1-\delta) e_t\right)\|g_0 - g\|. \tag{18}$$

Implementing one of the condition ($C_{\mathfrak{M}}$ 1)-($C_{\mathfrak{M}}$ 3), we get

$$\left\|g_{n+1} - g\right\| \leq \delta^{3(n+1)}\left\|g_0 - g\right\|. \tag{19}$$

Because $0 < \delta < 1$, consequently $\{g_n\}$ strongly converges to g. □ □

3.1. Stability. This subsection embraces almost M - stability of $\mathfrak{M}$ - iteration (12) in the sense of Osilike.

Theorem 3. *For a Banach space $\breve{\mathscr{B}}$ and nonvoid, closed and convex set $G \subseteq \mathscr{B}$, let M: $G \longrightarrow G$ is Berinde weak contraction agreeing (3), then, iterative algorithm $\mathfrak{M}$ - iteration (12) with real sequences $\{l_n\}, \{m_n\}, \{e_n\}$ satisfying $l \leq l_n$, $m \leq m_n$, $e \leq e_n$ for some $l, m, e \in (0, 1)$ and for all $n \in \mathbb{N} \cup \{0\}$ is almost M - stable.*

Proof. Consider an arbitrary sequence $\{r_n\} \subseteq G$. The sequence defined by $\mathfrak{M}$ - iteration (12) is $g_{n+1} = \theta(M, g_n)$ and $\widetilde{u}_n = \|r_{n+1} - \theta(M, r_n)\|$, $n \in \mathbb{N} \cup \{0\}$.

Now, we have to prove that

$$\sum_{n=0}^{\infty} \widetilde{u}_n < \infty \text{ implies } \lim_{n \longrightarrow \infty} r_n = g. \tag{20}$$

Let $\sum_{n=0}^{\infty} \widetilde{u}_n < \infty$ then by (18) and $\mathfrak{M}$ - iteration (12), we get

$$\begin{aligned}
\left\|r_{n+1} - g\right\| &\leq \left\|r_{n+1} - \theta(M, r_n)\right\| + \left\|\theta(M, r_n) - g\right\| \\
&\leq \widetilde{u}_n + \delta^3\left(1 - (1-\delta) l_n\right)\left(1 - (1-\delta) m_n\right) \\
&\quad \left(1 - (1-\delta) e_n\right)\left\|r_n - g\right\| \\
&\leq \widetilde{u}_n + \delta^3 (1 - (1-\delta) l)(1 - (1-\delta) m) \\
&\quad (1 - (1-\delta) e)\left\|r_n - g\right\|.
\end{aligned} \tag{21}$$

Define $\widetilde{v}_n = \|r_n - g\|$ and $q = \delta^3 (1 - (1-\delta) l)(1 - (1-\delta) m)(1 - (1-\delta) e)$, then $0 \leq q < 1$. Therefore, we have

$$\widetilde{v}_{n+1} \leq q \widetilde{v}_n + \widetilde{u}_n. \tag{22}$$

Thus from Lemma 1, we get the result. □

4. Comparison with Various Iterative Algorithms

In this section, we prove that $\mathfrak{M}$ - iteration (12) converges faster than other leading algorithms for the case of weak contraction mappings.

Theorem 4. *For a Banach space $\breve{\mathscr{B}}$ and nonvoid, closed and convex 1set $G \subseteq \mathscr{B}$, let M: $G \longrightarrow G$ is Berinde weak contraction agreeing (3). Let all the iterative algorithms listed in Table 1 together with $\mathfrak{M}-$ iterative algorithm converges to common fixed point, g (say). Then, $\mathfrak{M}$ - iteration (12) converges to g faster than those mentioned in Table 1.*

Proof. For $n \in \mathbb{N} \cup \{0\}$, using Theorems 1 and 2 and inequality (18), we get

$$\begin{aligned}
\left\|g_{n+1} - g\right\| &\leq \delta^{3(n+1)} \prod_{t=1}^{t=n} \left(1 - (1-\delta) l_t\right)\left(1 - (1-\delta) m_t\right)\left(1 - (1-\delta) e_t\right)\left\|g_0 - g\right\| \\
&\leq \delta^{3(n+1)} (1 - (1-\delta) l)^n (1 - (1-\delta) m)^n (1 - (1-\delta) e)^n \left\|g_0 - g\right\| = \mu_n.
\end{aligned} \tag{23}$$

Symbolising Noor iteration by $g_{1,n}$, from [8],

$$
\begin{aligned}
\|f_{1,n}-g\| &= \|(1-m_n)g_{1,n}+m_nMg_{1,n}-g\| \\
&\le (1-m_n)\|g_{1,n}-g\|+\delta m_n\|g_{1,n}-g\|, \\
\|h_{1,n}-g\| &= \|(1-l_n)g_{1,n}+l_nMf_{1,n}-g\| \\
&\le \|(1-l_n)\|g_{1,n}-g\|+\delta l_n\|f_{1,n}-g\|, \\
\|g_{1,n+1}-g\| &= \|(1-e_n)g_{1,n}+e_nMh_{1,n}-g\| \\
&\ge (1-e_n)\|g_{1,n}-g\|-e_n\|Mh_{1,n}-g\| \\
&\ge (1-e_n)\|g_{1,n}-g\|-\delta e_n\|h_{1,n}-g\| \\
&\ge (1-e_n)\|g_{1,n}-g\|-\delta e_n\{(1-l_n)\|g_{1,n}-g\|+\delta l_n\|f_{1,n}-g\|\} \\
&\ge (1-e_n-\delta e_n(1-l_n))\|g_{1,n}-g\|-\delta^2 l_ne_n\{1-(1-\delta)m_n\}\|g_{1,n}-g\| \\
&\ge 1-e_n(1+\delta(1-l_n(1-\delta(1-m_n(1-\delta)))))\|g_{1,n}-g\|. \\
&\ge (1-(1+\delta)e_t)\|g_{1,n}-g\|\cdots \ge \prod_{t=0}^{t=n}(1-(1+\delta)e_t)\|g_{1,0}-g\|.
\end{aligned}
\tag{24}
$$

Therefore

$$
\frac{\|g_{n+1}-g\|}{\|g-g\|}\le \frac{\delta^{3(n+1)}(1-l(1-\delta))^n(1-m(1-\delta))^n(1-e(1-\delta))^n}{\prod_{t=0}^{t=n}(1-(1+\delta)e_t)}=\mathscr{W}_{1,n}\,(\text{say}). \tag{25}
$$

Also, we get

$$
\begin{aligned}
\frac{\mathscr{W}_{1,n+1}}{\mathscr{W}_{1,n}} &= \frac{\left(\delta^{3(n+1+1)}(1-l(1-\delta))^{n+1}(1-m(1-\delta))^{n+1}(1-e(1-\delta))^{n+1}\right)\left(\prod_{t=0}^{t=n}(1-(1+\delta)e_t)\right)}{\left(\prod_{t=0}^{t=n+1}(1-(1+\delta)e_t)\right)\left(\delta^{3(n+1)}(1-l(1-\delta))^n(1-m(1-\delta))^n(1-e(1-\delta))^n\right)} \\
&= \lim_{n\longrightarrow\infty}\frac{\delta^3(1-l(1-\delta))(1-m(1-\delta))(1-e(1-\delta))}{(1-e_{n+1}(1+\delta))} \\
&= \delta^3(1-l(1-\delta))(1-m(1-\delta))(1-e(1-\delta))<1.
\end{aligned}
\tag{26}
$$

In accordance with the Ratio test, (26) implies $\sum\mathscr{W}_{1,n}$ is convergent, which says $\lim\mathscr{W}_{1,n}=0$.

It validates that $\{g_n\}$ converges faster than $\{g_{1,n}\}$ to g.

If normal-S - iterative algorithm [9] be denoted by $g_{2,n}$, then

$$
\begin{aligned}
\|g_{2,n+1}-g\| &= \left\|M\left((1-l_n)g_{2,n}+l_nMg_{2,n}-g\right)\right\| \\
&\le \delta\left[(1-l_n)g_{2,n}+l_nMg_{2,n}-g\|\right] \\
&\le \delta\left[(1-l_n)\|g_{2,n}-g\|+\delta l_n\|g_{2,n}-g\|\right] \\
&= \delta(1-(1-\delta)l_n)\|g_{2,n}-g\| \\
&\le \delta(1-(1-\delta)l)\|g_{2,n}-g\|.
\end{aligned}
\tag{27}
$$

Inductively, we get

$$
\|g_{2,n+1}-g\|\le\delta^{n+1}(1-(1-\delta)l)^n\|g_{2,0}-g\|=\mu_{2,n}. \tag{28}
$$

Therefore

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}\frac{\mu_n}{\mu_{2,n}} &= \lim_{n\longrightarrow\infty}\frac{\delta^{3(n+1)}(1-(1-\delta)l)^n(1-(1-\delta)m)^n(1-(1-\delta)e)^n\|g_0-g\|}{\delta^{(n+1)}(1-(1-\delta)l)^n\|g_{2,0}-g\|} \\
&= \lim_{n\longrightarrow\infty}\delta^{2(n+1)}(1-(1-\delta)m)^n(1-(1-\delta)e)^n=0.
\end{aligned}
\tag{29}
$$

Hence, $\{g_n\}$ converges faster than $\{g_{2,n}\}$ to g.

If $g_{3,n}$ represents Abbas iterative scheme, then as shown in Theorem 3 of [10], we have

$$\|g_{3,n+1} - g\| \leq \delta^{n+1} (1 - (1 - \delta)lme)^n \|g_{3,0} - g\| = \mu_{3,n}. \quad (30)$$

Accordingly

$$\frac{\mu_n}{\mu_{4,n}} = \frac{\delta^{3(n+1)} (1 - (1 - \delta)l)^n (1 - (1 - \delta)m)^n (1 - (1 - \delta)e)^n \|g_0 - g\|}{\delta^{n+1} (1 - (1 - \delta)lme)^n \|g_{3,0} - g\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (31)$$

This gives $\{g_n\}$ converges faster than $\{g_{3,n}\}$ to g.

Considering Theorem 3 of [11] by Thakur et al. and using $g_{4,n}$ to mark the corresponding iterative scheme, we get

$$\begin{aligned}
&\|g_{4,n+1} - g\| \leq \delta^n (1 - (1 - \delta)m)^n \|g_{4,0} - g\| = \mu_{4,n},\\
&\frac{\mu_n}{\mu_{4,n}} = \frac{\delta^{3(n+1)} (1 - (1 - \delta)l)^n (1 - (1 - \delta)m)^n (1 - (1 - \delta)e)^n \|g_0 - g\|}{\delta^n (1 - (1 - \delta)m)^n \|g_{4,0} - g\|}\\
&= \delta^{2n+3} (1 - (1 - \delta)l)^n (1 - (1 - \delta)e)^n \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \quad (32)$$

Which establishes that $\{g_n\}$ converges faster than $\{g_{4,n}\}$ to g.

As proved by Sintunavarat et al. ([12], Theorem 2)

$$\|g_{5,n+1} - g\| \leq \delta^{(n+1)} (1 - (1 - \delta)le)^n \|g_{5,0} - g\| = \mu_{5,n}, \quad (33)$$

where $g_{5,n}$ denotes Varat's iterative algorithm.

Therefore,

$$\frac{\mu_n}{\mu_{5,n}} = \frac{\delta^{3(n+1)} (1 - (1 - \delta)l)^n (1 - (1 - \delta)m)^n (1 - (1 - \delta)e)^n \|g_0 - g\|}{\delta^{(n+1)} (1 - (1 - \delta)le)^n \|g_{5,0} - g\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (34)$$

Hence, $\{g_n\}$ converges faster than $\{g_{5,n}\}$ to g.

Theorem 4 of [6] by Ullah et al. elucidates that

$$\|g_{6,n} - g\| \leq \delta^{2(n+1)} \|g_{6,0} - g\| = \mu_{6,n} \quad (35)$$

where $g_{6,n}$ signifies AK iteration, which further gives

$$\begin{aligned}
\frac{\mu_n}{\mu_{6,n}} &= \frac{\delta^{3(n+1)} (1 - (1 - \delta)l)^n (1 - (1 - \delta)m)^n (1 - (1 - \delta)e)^n \|g_0 - g\|}{\delta^{2(n+1)} \|g_{6,0} - g\|}\\
&= \delta^{(n+1)} (1 - (1 - \delta)l)^n (1 - (1 - \delta)m)^n (1 - (1 - \delta)e)^n \frac{\|g_0 - g\|}{\|g_{6,0} - g\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \quad (36)$$

Consequently, $\{g_n\}$ converges faster than $\{g_{6,n}\}$ to g.

Let $g_{7,n}$ stands for F^* iteration. Theorem 1 of [13] conveys,

$$\|g_{7,n+1} - g\| \leq \delta^{2(n+1)} \|g_{7,0} - g\| = \mu_{7,n}. \quad (37)$$

Therefore

$$\begin{aligned}
\frac{\mu_n}{\mu_{6,n}} &= \frac{\delta^{3(n+1)} (1 - (1 - \delta)l)^n (1 - (1 - \delta)m)^n (1 - (1 - \delta)e)^n \|g_0 - g\|}{\delta^{2(n+1)} \|g_{7,0} - g\|}\\
&= \delta^{(n+1)} (1 - (1 - \delta)l)^n (1 - (1 - \delta)m)^n (1 - (1 - \delta)e)^n \frac{\|g_0 - g\|}{\|g_{7,0} - g\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \quad (38)$$

Hence, $\{g_n\}$ converges even faster than $\{g_{7,n}\}$ to g.

Also, in [12], it is shown that Varat iterative scheme has faster rate of convergence than Picard, Mann, Ishikawa and S - iteration. Henceforth Equations (26) - (38) confirms $\mathfrak{M}$ - iteration converges faster than all the approximation procedures enlisted in Table 1. □

4.1. Comparison Results. In this section, we give some examples to strengthen the above claim that $\mathfrak{M}$ - iteration converges faster than various interesting algorithms existing in the literature.

Example 1. Let us take Banach space $\breve{\mathscr{B}} = \mathbb{R}$ equipped with usual norm. We define a self map M on a subset G = [0, 100] of Banach space $\breve{\mathscr{B}}$ by:

$$Mx = x - \tan^{-1} x \text{ for all} x \in G. \tag{39}$$

It can be eventually establish that M is a weak contraction in sense of Berinde with unique fixed point $g = 0$. Choosing control sequences $l_n = 0.6021$, $m_n = 0.8654$, $e_n = 0.8453$, initial guess $g_0 = 45$, using Matlab R 2019(b) software, we prove our claim (see Figure 1 and Table 2, Table 3 for your reference).

Example 2. Let $\breve{\mathscr{B}}$ and G be same as in Example 1: we define M: G $\longrightarrow$ G by

$$Mx = \sqrt{x^2 - 9x + 45} \text{ for all } x \in G. \tag{40}$$

Here M is a weak contraction with unique fixed point $g = 5$. Through Matlab 2019(b)software, it is shown that $\mathfrak{M}$ - iteration converges faster than other iterative algorithms by selecting control sequences $l_n = 0.7021$, $m_n = 0.8484$, $e_n = 0.7558$ starting with guess value $g_n = 35$ (see Figure 2 and Table 4, Table 5 for your reference).

Remark 4.1. It can be easily infer from the above two examples that in the category of weak contractions, Picard iterative scheme has good speed of convergence when compared to Mann and Ishikawa iterative schemes (see Table 6).

5. Data Dependency

The present section marks the data dependency result for $\mathfrak{M}$ - iteration when applied to weak contractions. Here, instead of computing the fixed point of an operator, we approximate the operator with a given weak contraction operator for which it is possible to compute the fixed point and therefore, approximate the fixed point of an initial operator.

Theorem 5. *Let M be a weak contraction with Q an approximate operator of it and $\{g_n\}$ be a sequence governed by $\mathfrak{M}$ - iterative procedure (12) for M. Define sequence $\{q_n\}$ for Q as follows:*

$$\begin{cases} q_{n+1} = Q[(1-e_n)Qs_n + e_n s_n], \\ s_n = Q[(1-l_n)t_n + l_n Qt_n], \\ t_n = Q[(1-m_n)q_n + m_n Qq_n],\ n \in \mathbb{N}\cup\{0\}, \end{cases} \tag{41}$$

where $\{l_n\}, \{m_n\}, \{e_n\}$ are sequences in $(0,1)$ satisfying $(1/2) \le l_n$, $(1/2) \le m_n$, $(1/2) \le e_n$ for all $n \in \mathbb{N}\cup\{0\}$ and $\sum_{n=0}^{\infty} m_n = \infty$, $\sum_{n=0}^{\infty} l_n = \infty$, $\sum_{n=0}^{\infty} e_n = \infty$. If $Mg = g$ and $Qq = q$ such that $q_n \longrightarrow q$ as $n \longrightarrow \infty$, then

$$\|g - q\| \le \frac{14\varepsilon}{1-\delta}, \tag{42}$$

where $\varepsilon > 0$ is a fixed constant.

Proof. Consider

$$\begin{aligned} \|g_{n+1} - q_{n+1}\| &= \|M[(1-e_n)Mh_n + e_n h_n] - Q[(1-e_n)Qs_n + e_n s_n]\| \\ &\le \|M[(1-e_n)Mh_n + e_n h_n] - M[(1-e_n)Qs_n + e_n s_n]\| \\ &\quad + \|M[(1-e_n)Qs_n + e_n s_n] - Q[(1-e_n)Qs_n + e_n s_n]\| \\ &\le \delta\|(1-e_n)Mh_n + e_n h_n - ((1-e_n)Qs_n + e_n s_n)\| \\ &\quad + L\|(1-e_n)Mh_n + e_n h_n - M[(1-e_n)Mh_n + e_n h_n]\| + \varepsilon \\ &\le \delta(1-e_n)\|Mh_n - Qs_n\| + \delta e_n\|h_n - s_n\| \\ &\quad + L[\delta + (1-\delta)e_n]\|h_n - Mh_n\| + \varepsilon \\ &\le \delta(1-e_n)\left(\|Mh_n - Ms_n\| + \|Ms_n - Qs_n\|\right) + \delta e_n\|h_n - s_n\| \\ &\quad + L[\delta + (1-\delta)e_n]\|h_n - Mh_n\| + \varepsilon \\ &\le \delta(1-e_n)\left(\delta\|h_n - s_n\| + L\|h_n - Mh_n\| + \varepsilon\right) + \delta e_n\|h_n - s_n\| \\ &\quad + L[\delta + (1-\delta)e_n]\|h_n - Mh_n\| + \varepsilon \\ &\le \delta[1 - (1-\delta)(1-e_n)]\|h_n - s_n\| + L[1 - (1-2\delta)(1-e_n)]\|h_n - Mh_n\| \\ &\quad + \delta(1-e_n)\varepsilon + \varepsilon. \end{aligned} \tag{43}$$

Table 1: Table listing different iterative algorithms.

Picard [2]	$g_0 \in G$ $g_{n+1} = Mg_n,\ n \in \mathbb{N} \cup \{0\}$
Mann [3]	$g_0 \in G$ $g_{n+1} = (1-e_n)g_n + e_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
Ishikawa [4]	$g_0 \in G$ $g_{n+1} = (1-e_n)g_n + e_n M h_n$ $h_n = (1-l_n)g_n + l_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
Noor [8]	$g_0 \in G$ $g_{n+1} = (1-e_n)g_n + e_n M h_n$ $h_n = (1-l_n)g_n + l_n M f_n$ $f_n = (1-m_n)g_n + m_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
S (see [5])	$g_0 \in G$ $g_{n+1} = (1-e_n)Mg_n + e_n M h_n$ $h_n = (1-l_n)g_n + l_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
Normal - S [9]	$g_0 \in G$ $g_{n+1} = M[(1-e_n)g_n + e_n M g_n],\ n \in \mathbb{N} \cup \{0\}$
Abbas [10]	$g_0 \in G$ $g_{n+1} = (1-e_n)Mh_n + e_n M f_n$ $h_n = (1-l_n)Mg_n + l_n M f_n$ $f_n = (1-m_n)g_n + m_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
Thakur et al. [11]	$g_0 \in G$ $g_{n+1} = (1-e_n)Mf_n + e_n M h_n$ $h_n = (1-l_n)f_n + l_n M f_n$ $f_n = (1-m_n)g_n + m_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
Varat [12]	$g_0 \in G$ $g_{n+1} = (1-e_n)Mh_n + e_n M f_n$ $h_n = (1-l_n)g_n + l_n f_n$ $f_n = (1-m_n)g_n + m_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
K^* (Ullah et al.) [6]	$g_{n+1} = Mh_n$, $g_0 \in G$ $h_n = M((1-l_n)f_n + l_n M f_n)$ $f_n = (1-m_n)g_n + m_n M g_n,\ n \in \mathbb{N} \cup \{0\}$
F^* [13]	$g_0 \in G$ $g_{n+1} = Mh_n$ $h_n = M((1-l_n)g_n + l_n M g_n),\ n \in \mathbb{N} \cup \{0\}$

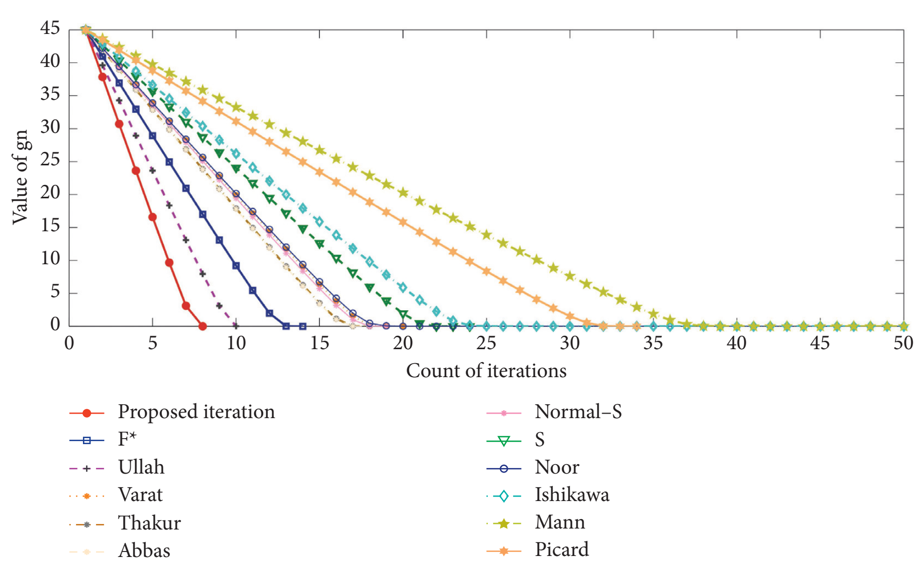

Figure 1: Convergence behaviour of approximation procedures corresponding to example 4.2.

Table 2: Table depicting comparison of iterative algorithms in context of Example 4.2.

Iter.No.	$\mathfrak{M}$	F^*	Ullah	Varat	Thakur	Abbas
1	37.8494	40.9722	39.6340	42.1944	41.9679	41.9550
2	30.7215	36.9505	34.2794	39.3916	38.9390	38.9134
3	23.6277	32.9361	28.9402	36.5919	35.9139	35.8755
4	16.5914	28.9311	23.6224	33.7959	32.8933	32.8422
5	9.6724	24.9380	18.3363	31.0042	29.8779	29.8141
6	3.1208	20.9608	13.1024	28.2175	26.8688	26.7925
7	$5.7901e^6$	17.0059	7.9694	25.4368	23.8675	23.7787
8	0	13.0844	3.1092	22.6636	20.8760	20.7749
9	0	9.2181	0.0219	19.8995	17.8972	17.7840
10	0	5.4594	0	17.1473	14.9354	14.8105
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
49	0	0	0	0	0	0
50	0	0	0	0	0	0

Let us take

$$\begin{aligned}\|h_n - s_n\| &= \|M[(1-l_n)f_n + l_n M f_n] - Q[(1-l_n)t_n + l_n Q t_n]\| \\ &\le \|M[(1-l_n)f_n + l_n M f_n] - M[(1-l_n)t_n + l_n Q t_n]\| \\ &\quad + \|M[(1-l_n)t_n + l_n Q t_n] - Q[(1-l_n)t_n + l_n Q t_n]\|, \\ \|h_n - s_n\| &\le \delta\|(1-l_n)f_n + l_n M f_n - [(1-l_n)t_n + l_n Q t_n]\| \\ &\quad + L\|(1-l_n)f_n + l_n M f_n - M[(1-l_n)f_n + l_n M f_n]\| + \varepsilon \\ &\le \delta(1-l_n)\|f_n - t_n\| + \delta l_n\|M f_n - Q t_n\| \\ &\quad + L[1-(1-\delta)l_n](1+\delta)\|f_n - M f_n\| + \varepsilon \\ &\le \delta(1-l_n)\|f_n - t_n\| + \delta l_n\|M f_n - M t_n\| + \delta l_n\|M f_n - Q t_n\| \\ &\quad + L[1-(1-\delta)l_n](1+\delta)\|f_n - M f_n\| + \varepsilon \\ &\le \delta(1-l_n)\|f_n - t_n\| + \delta l_n\left(\delta\|f_n - t_n\| + L\|f_n - M f_n\|\right) + \delta l_n \varepsilon \\ &\quad + L[1-(1-\delta)l_n](1+\delta)\|f_n - M f_n\| + \varepsilon \\ &\le \delta[1-(1-\delta)l_n]\|f_n - t_n\| + \delta l_n L\|f_n - M f_n\| + \delta l_n \varepsilon \\ &\quad + L[1-(1-\delta)l_n](1+\delta)\|f_n - M f_n\| + \varepsilon,\end{aligned} \tag{44}$$

$$\begin{aligned}\|f_n - t_n\| &= \|M[(1-m_n)g_n + m_n M g_n] - Q[(1-m_n)q_n + m_n Q q_n]\| \\ &\le \|M[(1-m_n)g_n + m_n M g_n] - M[(1-m_n)q_n + m_n Q q_n]\| + \varepsilon \\ &\le \delta\|(1-m_n)g_n + m_n M g_n - [(1-m_n)q_n + m_n Q q_n]\| \\ &\quad + L\|(1-m_n)g_n + m_n M g_n - M[(1-m_n)g_n + m_n M g_n]\| + \varepsilon, \\ \|f_n - t_n\| &\le \delta(1-m_n)\|g_n - q_n\| + \delta m_n\|M g_n - Q q_n\| \\ &\quad + L[1-(1-\delta)m_n](1+\delta)\|g_n - g\| + \varepsilon \\ &\le \delta[1-(1-\delta)m_n]\|g_n - q_n\| + \delta m_n L\|g_n - M g_n\| \\ &\quad + L[1-(1-\delta)m_n](1+\delta)\|g_n - g\| + \delta m_n \varepsilon + \varepsilon.\end{aligned} \tag{45}$$

It follows from (43), (44) and (45),

Table 3: Table depicting comparison of iterative algorithms in context of Example 4.2.

Iter.No.	Normal-S	S	Noor	Ishikawa	Mann	Picard
1	42.1431	42.6637	42.2219	42.9032	43.6910	43.4514
2	39.2890	40.3292	39.4465	40.8080	42.3825	41.9036
3	36.4383	37.9968	36.6743	38.7144	41.0747	40.3567
4	33.5913	35.6668	33.9058	36.6226	39.7675	38.8107
5	30.7488	33.3395	31.1415	34.5329	38.4609	37.2656
6	27.9117	31.0152	28.3824	32.4456	37.1551	35.7217
7	25.0809	28.6944	25.6292	30.3608	35.8501	34.1789
8	22.2580	26.3777	22.8836	28.2790	34.5458	32.6373
9	19.4450	24.0658	20.1472	26.2006	33.2425	31.0972
10	16.6449	21.7596	17.4227	24.1262	31.9401	29.5585
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
49	0	0	$5.5178e^{-27}$	$1.2394e^{-22}$	$1.0624e^{-11}$	0
50	0	0	$8.5360e^{-28}$	$1.9174e^{-23}$	$1.6436e^{-12}$	0

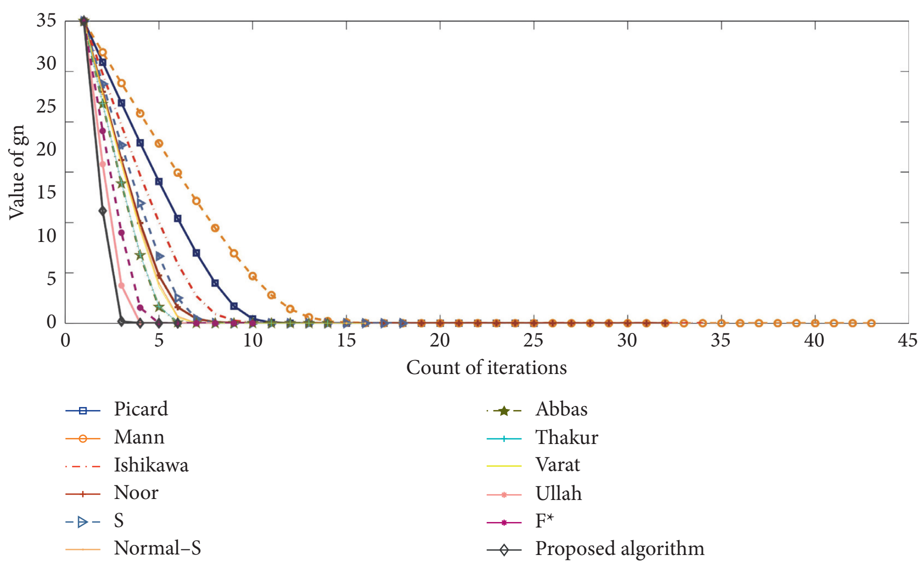

Figure 2: Convergence behaviour of approximation procedures corresponding to example 4.3.

$$
\begin{aligned}
\left\|g_{n+1}-q_{n+1}\right\| \leq\ & \delta^2\left[1-(1-\delta) l_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right]\left\|f_n-t_n\right\| \\
& +\delta^2 l_n L\left[1-(1-\delta)\left(1-e_n\right)\right]\left\|f_n-M f_n\right\|+\delta^2 l_n\left[1-(1-\delta)\left(1-e_n\right)\right] \varepsilon \\
& +\delta L\left[1-(1-\delta) l_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right](1+\delta)\left\|f_n-M f_n\right\| \\
& +\delta\left[1-(1-\delta)\left(1-e_n\right)\right] \varepsilon+L\left[1-(1-2 \delta)\left(1-e_n\right)\right]\left\|h_n-M h_n\right\|+\delta\left(1-e_n\right) \varepsilon+\varepsilon \\
\leq\ & \delta^3\left[1-(1-\delta) l_n\right]\left[1-(1-\delta) m_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right]\left\|g_n-q_n\right\| \\
& +\delta^3\left[1-(1-\delta) l_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right] m_n L\left\|g_n-M g_n\right\| \\
& +\delta^2 L\left[1-(1-\delta) l_n\right]\left[1-(1-\delta) m_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right](1+\delta)\left\|g_n-g\right\| \\
& +\delta^3\left[1-(1-\delta) l_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right] m_n \varepsilon \\
& +\delta^2\left[1-(1-\delta) l_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right] \varepsilon \\
& +\delta^2 l_n L\left[1-(1-\delta)\left(1-e_n\right)\right]\left\|f_n-M f_n\right\|+\delta^2 l_n\left[1-(1-\delta)\left(1-e_n\right)\right] \varepsilon \\
& +\delta L\left[1-(1-\delta) l_n\right]\left[1-(1-\delta)\left(1-e_n\right)\right](1+\delta)\left\|f_n-M f_n\right\| \\
& +\delta\left[1-(1-\delta)\left(1-e_n\right)\right] \varepsilon+L\left[1-(1-2 \delta)\left(1-e_n\right)\right]\left\|h_n-M h_n\right\|+\delta\left(1-e_n\right) \varepsilon+\varepsilon .
\end{aligned} \tag{46}
$$

Table 4: Table depicting comparison of iterative algorithms in context of Example 4.3.

Iter No.	$\mathfrak{M}$	F^*	Ullah	Varat	Thakur	Abbas
1	16.1635	24.0873	20.7939	27.7271	26.7974	26.7531
2	5.1889	13.9970	8.7498	20.7082	18.9507	18.8675
3	5.0000	6.5425	5.0203	14.1588	11.8512	11.7453
4	5.0000	5.0139	5.0000	8.6452	6.6766	6.6033
5	5.0000	5.0001	5.0000	5.5443	5.0828	5.0745
6	5	5.0000	5.0000	5.0217	5.0017	5.0015
7	5	5.0000	5.0000	5.0006	5.0000	5.0000
8	5	5.0000	5	5.0000	5.0000	5.0000
9	5	5.0000	5	5.0000	5.0000	5.0000
10	5	5	5	5.0000	5.0000	5.0000
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
49	5	5	5	5	5	5
50	5	5	5	5	5	5

Table 5: Table depicting comparison of iterative algorithms in context of Example 4.3.

Iter No.	Normal-S	S	Noor	Ishikawa	Mann	Picard
1	27.8515	28.7603	27.9749	29.7608	31.9035	30.9031
2	20.9458	22.6938	21.2166	24.6410	28.8410	26.8677
3	14.4823	16.9119	14.9509	19.7000	25.8202	22.9143
4	8.9719	11.6573	9.7321	15.0476	22.8520	19.0745
5	5.6829	7.5093	6.5898	10.9046	19.9515	15.4002
6	5.0315	5.4234	5.4501	7.7038	17.1408	11.9818
7	5.0010	5.0295	5.1223	5.9197	14.4530	8.9848
8	5.0000	5.0016	5.0329	5.2661	11.9393	6.6980
9	5.0000	5.0001	5.0088	5.0736	9.6796	5.4389
10	5.0000	5.0000	5.0024	5.0201	7.7917	5.0628
.	.	.	.	.	.	
.	.	.	.	.	.	
.	.	.	.	.	.	
49	5	5	5	5	5	5
50	5	5	5	5	5	5

Since $\delta < 1$, therefore, $\delta^2 < 1, \delta^3 < 1$. Also, $l_n, m_n, e_n < 1$, which gives, $[1 - (1 - \delta)l_n] < 1$, $[1 - (1 - \delta)m_n] < 1$ and $1 - (1 - \delta)(1 - e_n) < 1$. We get

$$\begin{aligned} \|g_{n+1} - q_{n+1}\| &\leq [1 - (1 - \delta)l_n]\|g_n - q_n\| + m_n L\|g_n - Mg_n\| + 2L\|g_n - g\| \\ &\quad + m_n\varepsilon + \varepsilon + l_n L\|f_n - Mf_n\| + l_n\varepsilon + 2L\|f_n - Mf_n\| + L\|h_n - Mh_n\| + 4\varepsilon \\ &\leq [1 - (1 - \delta)l_n]\|g_n - q_n\| + 2l_n L\|g_n - Mg_n\| + 4l_n L\|g_n - g\| \\ &\quad + 2l_n L\|f_n - Mf_n\| + 4l_n L\|f_n - Mf_n\| + 2l_n L\|h_n - Mh_n\| + 14 l_n\varepsilon. \end{aligned} \tag{47}$$

Define

$$\begin{aligned} \grave{\mathfrak{a}}_n &:= \|g_n - q_n\|;, \\ \grave{\mathfrak{b}}_n &:= l_n(1 - \delta);, \\ \grave{\mathfrak{c}}_n : &= \frac{2L\|g_n - Mg_n\| + 4L\|g_n - g\| + 6L\|f_n - Mf_n\| + 2L\|h_n - Mh_n\| + 14\varepsilon}{(1 - \delta)}. \end{aligned} \tag{48}$$

Thus, equation (46) can be rewritten as

$$\grave{\mathfrak{a}}_{n+1} \leq \left(1 - \grave{\mathfrak{b}}_n\right)\grave{\mathfrak{a}}_n + \grave{\mathfrak{b}}_n\grave{\mathfrak{c}}_n. \tag{49}$$

This calls for the application of lemma (2.3), which gives

$$\begin{aligned} 0 &\leq \limsup_{n\longrightarrow\infty}\|g_n - q_n\| \\ &\leq \limsup_{n\longrightarrow\infty}\frac{2L\|g_n - Mg_n\| + 4L\|g_n - g\| + 6L\|f_n - Mf_n\| + 2L\|h_n - Mh_n\| + 14\varepsilon}{(1 - \delta)} \\ &= \frac{14\varepsilon}{(1 - \delta)}. \end{aligned} \tag{50}$$

Table 6: Table enlisting count of iterations involved in Example 4.2 and Example 4.3.

Iterative Procedure	Iteration Count For example 4.2	Iteration Count For example 4.3
Picard	34	25
Mann	435	43
Ishikawa	422	34
Noor	416	32
S	24	18
Normal-S	19	16
Abbas	18	14
Thakur	18	14
Varat	20	16
Ullah	10	8
F^*	14	10
$\mathfrak{M}$	8	6

Using Theorem 2 and given hypotheses, we get

$$\|g - q\| \leq \frac{14\varepsilon}{1-\delta}. \tag{51}$$

The essence of Theorem 5 can be realised from the following example which has been inspired from [13]. □

Table 7: Approximated fixed point of operator Q by applying $\mathfrak{M}$ iterative procedure.

Iteration Count	Values obtained using algorithm (5.1)
1	0.9484
2	−0.0223
3	−0.0224

Example 3. Let us take Banach space $\breve{\mathscr{B}} = \mathbb{R}$ equipped with usual norm. We define a self map M on a subset $G = [-1, 1]$ of $\mathscr{B}$ by:

$$M(x) = \begin{cases} \left(\frac{13}{29}\right)\sin\left(\frac{13x}{29}\right), & \text{if } -1 \leq x < 0, \\ -\left(\frac{13}{29}\right)\sin\left(\frac{13x}{29}\right), & \text{if } 0 \leq x \leq 1. \end{cases} \tag{52}$$

It can be observed that M is a weak contraction in the sense of Berinde with $\delta \in ((169/841), 1]$ and a unique fixed point $g = 0 \in G$.

We further define another self map Q on G as follows:

$$Q(x) = \begin{cases} \frac{(x-0.09)}{5.01} + \frac{(x+0.2)^3}{97.06} - \frac{(x-0.6)^5}{7601.16} - \frac{(x+0.8)^7}{129996.03}, & \text{if } -1 \leq x < 0, \\ -\frac{x}{6.02} - \frac{(x-0.5)^3}{108.95} - \frac{(x+0.2)^5}{7598.27} + \frac{(x-0.6)^7}{230050.03}, & \text{if } 0 \leq x \leq 1. \end{cases} \tag{53}$$

Here $q = -0.0224$ is a unique fixed point lying in the domain of Q. Using MATLAB 2019(b) software, we get $\max_{x\in G}\|Mx - Qx\| = 0.0272$. Therefore, if we choose $\varepsilon = 0.0272$, then as per Berinde's definition of an approximate operator (Definition 6), Q is an approximate operator of G. Moreover, distance between fixed points of M and Q is $\|g - q\| = 0.0224$.

For $n \in \mathbb{N} \cup \{0\}$, choosing $l_n = m_n = e_n = (n + 1/n + 2)$ in (36) and using (48), we obtain approximated fixed point of the operator Q as visualized from Table 7.

However, using Theorem 5, we compute

$$\|g - q\| \leq \frac{14(0.0272)}{(1-(169/841))} = 0.4765. \tag{54}$$

From here, we can conclude that without knowing the fixed point of Q and in fact, without calculating it, we can approximate its fixed point by directly applying Theorem 5.

6. Application to Nonlinear Caputo Type Fractional Differential Equation

Fractional derivatives have been used through and through for mathematical modelling over recent years. For instance, numerous real world problems, signal processing, image restoration, determination of fluid level, biological algorithms, traffic flow, telecommunication etc., are being modelled in terms of nonlinear fractional order differential equations, majority of which do not possess exact solution and call for the approximate numerical solutions. Caputo et al. in [28] developed a novel fractional differential operator with a special feature of non singular kernel having exponential decay due to which it has attracted many researchers (see [29-31]).

$$\begin{aligned} &\mathscr{D}_{\mathfrak{t}}^{\flat}\grave{\mathfrak{u}}(\mathfrak{t}) = \varsigma(\mathfrak{t}, \grave{\mathfrak{u}}(\mathfrak{t}))0 < \flat < 1, \\ &\grave{\mathfrak{u}}(0) = \grave{\mathfrak{u}}_0. \end{aligned} \tag{55}$$

where $\mathscr{D}_{\mathfrak{t}}^{\flat}$ denotes Caputo type fractional differential equation, ς: $[0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $\mathfrak{u}$ is real valued with $\mathfrak{t} \in [0, 1]$.

The time dependent Caputo type fractional differential operator is defined by

$$\begin{aligned} \mathscr{D}_{\mathfrak{t}}^{\flat}\grave{\mathfrak{u}}(\mathfrak{t}) = \frac{(2-\flat)\mathscr{A}(\flat)}{2(1-\flat)}\int_0^{\mathfrak{t}} \exp\left[-\frac{\flat(\mathfrak{t}-\omega)}{1-\flat}\right] \\ \grave{\mathscr{U}}'(\zeta)d\zeta, \quad \mathfrak{t} \end{aligned} \tag{56}$$

where $\mathscr{A}(\flat)$ is a normalization function satisfying $\mathscr{A}(0) = \mathscr{A}(1) = 1$. The following lemma is due to [31].

Lemma 4. *The initial value problem (IVP)*

$$\begin{cases} \mathscr{D}_t^{\mathfrak{b}} \grave{\mathfrak{u}}(t) = \varsigma(t) 0 < \mathfrak{b} < 1, \\ \grave{\mathfrak{u}}(0) = \grave{\mathfrak{u}}_0 \in \mathbb{R}. \end{cases} \tag{57}$$

Possess following integeral solution

$$\grave{\mathfrak{u}}(t) = \grave{\mathfrak{u}}_0 + \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}(\varsigma(t) - \varsigma(0)) + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t \varsigma(\zeta) d\zeta. \tag{58}$$

Let $\alpha = [0,1]$ *and* $C(\alpha)$ *be the space of continuous functions defined on* $[0,1]$. *Let the Banach space* $\breve{\mathscr{B}} = \left\{\grave{\mathfrak{u}}(t) | \grave{\mathfrak{u}}(t) \in C(\alpha)\right\}$, *with norm* $\|\grave{\mathfrak{u}}\|_{\breve{\mathscr{B}}} = \max_{t\in\alpha}|\grave{\mathfrak{u}}(t)|$.

By Lemma 4, IVP (55) can be expressed as

$$\grave{\mathfrak{u}}(t) = \grave{\mathfrak{u}}_0 + \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}(\varsigma(t, \grave{\mathfrak{u}}(t)) - \varsigma(0, \grave{\mathfrak{u}}(0))) + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t \varsigma(\zeta, \grave{\mathfrak{u}}(\zeta)) d\zeta. \tag{59}$$

Let us define the operator $M: \breve{\mathscr{B}} \longrightarrow \breve{\mathscr{B}}$ *by*

$$M(\grave{\mathfrak{u}}(t)) = \grave{\mathfrak{u}}_0 + \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}(\varsigma(t, \grave{\mathfrak{u}}(t)) - \varsigma(0, \grave{\mathfrak{u}}(0))) + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t \varsigma(\zeta, \grave{\mathfrak{u}}(\zeta)) d\zeta. \tag{60}$$

Then the solution of IVP (55) is equivalent to the fixed point operator M.

We prove the following theorem.

Theorem 6. *For* $0 \le \alpha \le 1$, *let* $\varsigma: [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous function such that:*

(i $|\varsigma(t, \grave{\mathfrak{u}}(t)) - \varsigma(t, \grave{\mathfrak{v}}(t))| \le \delta|\grave{\mathfrak{u}}(t) - \grave{\mathfrak{v}}(t)| + L|\grave{\mathfrak{u}}(t) - M(\grave{\mathfrak{u}}(t))|$

(ii) $\max_{t\in\alpha}((2(1-\mathfrak{b})/(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})) + (2\mathfrak{b}/(2-\mathfrak{b})\mathscr{A}(\mathfrak{b}))t) < 1$.

Then the IVP (55) has unique solution which is fixed point of weak contraction M.

Proof. We denote fixed point operator M by

$$M(\grave{\mathfrak{u}}(t)) = \mathscr{K} + \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\varsigma(t, \grave{\mathfrak{u}}(t)) + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t \varsigma(\zeta, \grave{\mathfrak{u}}(\zeta)) d\zeta, \tag{61}$$

where $\mathscr{K} = \grave{\mathfrak{u}}_0 - (2(1-\mathfrak{b})/(2-\mathfrak{b})\mathscr{A}(\mathfrak{b}))\varsigma(0, \grave{\mathfrak{u}}_0)$.

For $\grave{\mathfrak{u}}(t), \grave{\mathfrak{v}}(t) \in \breve{\mathscr{B}}$, we have

$$\begin{aligned}
|M(\grave{\mathfrak{u}}(t)) - M(\grave{\mathfrak{v}}(t))| &= |\mathscr{K} + \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\varsigma(t, \grave{\mathfrak{u}}(t)) + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t \varsigma(\zeta, \grave{\mathfrak{u}}(\zeta)) d\zeta \\
&\quad - \mathscr{K} + \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\varsigma(t, \grave{\mathfrak{v}}(t)) + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t \varsigma(\zeta, \grave{\mathfrak{v}}(\zeta)) d\zeta| \\
&\le \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}|\varsigma(t, \grave{\mathfrak{u}}(t)) - \varsigma(t, \grave{\mathfrak{v}}(t))| \\
&\quad + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t |\varsigma(\zeta, \grave{\mathfrak{u}}(\zeta)) - \varsigma(\zeta, \grave{\mathfrak{v}}(\zeta))| d\zeta \\
&\le \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}[\delta|\grave{\mathfrak{u}}(t) - \grave{\mathfrak{v}}(t)| + L|\grave{\mathfrak{u}}(t) - M(\grave{\mathfrak{u}}(t))|] \\
&\quad + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^t \delta|\grave{\mathfrak{u}}(\zeta) - \grave{\mathfrak{v}}(\zeta)| + L|\grave{\mathfrak{u}}(\zeta) - M(\grave{\mathfrak{u}}(\zeta))| d\zeta \\
&\le \left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\delta + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\delta\int_0^t d\zeta\right)\|\grave{\mathfrak{u}}(t) - \grave{\mathfrak{v}}(t)\|_{\breve{\mathscr{B}}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}L|\grave{\mathfrak{u}}(\mathfrak{t})-M(\grave{\mathfrak{u}}(\mathfrak{t}))|\\
&+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}L\int_0^{\mathfrak{t}}|\grave{\mathfrak{u}}(\zeta)-M(\grave{\mathfrak{u}}(\zeta))|d\zeta,\\
|M(\grave{\mathfrak{u}}(\mathfrak{t}))-M(\grave{\mathfrak{v}}(\mathfrak{t}))| &\leq\left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\delta+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\delta\mathfrak{t}\right)\|\grave{\mathfrak{u}}(\mathfrak{t})-\grave{\mathfrak{v}}(\mathfrak{t}))\|_{\breve{\mathscr{B}}}\\
&+\left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}L+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}L\mathfrak{t}\right)\|\grave{\mathfrak{u}}(\mathfrak{t})-M(\grave{\mathfrak{u}}(\mathfrak{t}))\|_{\breve{\mathscr{B}}}\\
\max_{\mathfrak{t}\in\alpha}|M(\grave{\mathfrak{u}}(\mathfrak{t}))-M(\grave{\mathfrak{v}}(\mathfrak{t}))| &\leq\max_{\mathfrak{t}\in\alpha}\left[\left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\mathfrak{t}\right)\delta\|\grave{\mathfrak{u}}(\mathfrak{t})-\grave{\mathfrak{v}}(\mathfrak{t})\|_{\breve{\mathscr{B}}}\right.\\
&\left.+\left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\mathfrak{t}\right)L\|\grave{\mathfrak{u}}(\mathfrak{t})-M(\grave{\mathfrak{u}}(\mathfrak{t}))\|_{\breve{\mathscr{B}}}\right]\\
&\|M(\grave{\mathfrak{u}}(\mathfrak{t}))-M(\grave{\mathfrak{v}}(\mathfrak{t}))\|_{\breve{\mathscr{B}}}\\
&\leq\delta\|\grave{\mathfrak{u}}(\mathfrak{t})-\grave{\mathfrak{v}}(\mathfrak{t})\|_{\breve{\mathscr{B}}}+L\|\grave{\mathfrak{u}}(\mathfrak{t})-M(\grave{\mathfrak{u}}(\mathfrak{t}))\|_{\breve{\mathscr{B}}}.
\end{aligned}
\tag{62}
$$

Thus $M\colon \breve{\mathscr{B}}\longrightarrow\breve{\mathscr{B}}$ is a weak contraction. In view of Theorem 1, we get M has a unique fixed point which is the solution of IVP (55).

Losada et al. [31] proved the following theorem in context of nonlinear Caputo type fractional differential equation. □

Theorem 7. *For $0<\mathfrak{b}<1$, $T>0$, suppose the following conditions hold:*

(CM1) $\varsigma\colon [0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function such that there exists $\mathfrak{N}\in(0,1)$ satisfying,

$|\varsigma(\mathfrak{t},\tau_1)-\varsigma(\mathfrak{t},\tau_2)|\leq\mathfrak{N}|\tau_1-\tau_2|$ for all. $\tau_1,\tau_2\in\mathbb{R}$,

(CM2) $((2(1-\mathfrak{b})/(2-\mathfrak{b})\mathscr{A}(\mathfrak{b}))+(2\mathfrak{b}/(2-\mathfrak{b})\mathscr{A}(\mathfrak{b}))T)<1$.

Then the IVP (55) has unique solution (59) on \$ $C[0,T]$\$.

We now prove that - iteration converges to the solution of IVP (55)

Theorem 8. *If the conditions $(CM1)-(CM2)$ are satisfied, then for $\sum m_n=\infty$, the sequence $\{g_n\}$ generated by $\mathfrak{M}$ - iterative procedure converges to the solution say $\mathfrak{u}^{\#}$ of IVP (55) on $C[0,T]$, for some $g_0\in C[0,T]$.*

Proof. Let us define the operator M by:

$$
M(\grave{\mathfrak{u}}(\mathfrak{t}))=\grave{\mathfrak{u}}_0+\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}(\varsigma(\mathfrak{t},\grave{\mathfrak{u}}(\mathfrak{t}))-\varsigma(0,\grave{\mathfrak{u}}(0)))+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\varsigma(\zeta,\grave{\mathfrak{u}}(\zeta))d\zeta.
\tag{63}
$$

Which can be rewritten as:

$$
M(\grave{\mathfrak{u}}(\mathfrak{t}))=\mathscr{K}+\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}(\varsigma(\mathfrak{t},\grave{\mathfrak{u}}(\mathfrak{t})))+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\varsigma(\zeta,\grave{\mathfrak{u}}(\zeta))d\zeta,
\tag{64}
$$

where $\mathscr{K}=\grave{\mathfrak{u}}_0-(2(1-\mathfrak{b})/(2-\mathfrak{b})\mathscr{A}(\gamma))\varsigma(0,\grave{\mathfrak{u}}_0)$

In view of Theorem 7, the solution $\mathfrak{u}^{\#}$ of IVP (55) is fixed point of operator M.

Let $\{g_n\}$ be a sequence defined by $\mathfrak{M}$ - iterative procedure on fixed point operator M. We shall establish that $g_n\longrightarrow\mathfrak{u}^{\#}$ as $n\longrightarrow\infty$.

Define $\mathfrak{t}_n=(1-m_n)g_n+Mg_n$, $n\in\mathbb{N}\cup\{0\}$

Using condition $CM(1)$, we have:

$$
\begin{aligned}
\left|\mathfrak{t}_n-\mathfrak{u}^{\#}\right| &=\left|(1-m_n)g_n+m_nMg_n-\mathfrak{u}^{\#}\right|\leq(1-m_n)\left|g_n-\mathfrak{u}^{\#}\right|+m_n\left|Mg_n-M\mathfrak{u}^{\#}\right|\\
&=(1-m_n)\left|g_n-\mathfrak{u}^{\#}\right|+m_n|\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\left(\varsigma(\mathfrak{t},g_n(\mathfrak{t}))-\varsigma\left(\mathfrak{t},\mathfrak{u}^{\#}(\mathfrak{t})\right)\right)\\
&+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\varsigma(\mathfrak{t},g_n(\mathfrak{t}))
\end{aligned}
$$

$$\begin{aligned}\left|\mathfrak{t}_n - \mathbf{u}^{\#}\right| &\leq (1 - m_n)\left|g_n - \mathbf{u}^{\#}\right| + m_n\left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\left|\varsigma(\mathfrak{t}, g_n(\mathfrak{t})) - \varsigma\left(\mathfrak{t}, \mathbf{u}^{\#}(\mathfrak{t})\right)\right|\right.\\ &\quad\left.+\frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\left|\varsigma(\mathfrak{t}, g_n(\mathfrak{t})) - \varsigma\left(\mathfrak{t}, \mathbf{u}^{\#}(\mathfrak{t})\right)\right|d\mathfrak{t}\right)\\ &\leq (1 - m_n)\left|g_n - \mathbf{u}^{\#}\right| + m_n\left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\mathfrak{N}\left|g_n - \mathbf{u}^{\#}\right| + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\mathfrak{N}\left|g_n - \mathbf{u}^{\#}\right|d\mathfrak{t}\right)\\ \left\|\mathfrak{t}_n - \mathbf{u}^{\#}\right\| &\leq (1 - m_n)\left\|g_n - \mathbf{u}^{\#}\right\| + \left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})} + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\mathrm{T}\right)m_n\mathfrak{N}\left\|g_n - \mathbf{u}^{\#}\right\|\\ &\leq (1 - m_n(1-\mathfrak{N}))\left\|g_n - \mathbf{u}^{\#}\right\|.\end{aligned} \tag{65}$$

Using (65) and condition (CM1), we get:

$$\begin{aligned}\left|f_n - \mathbf{u}^{\#}\right| &= \left|M\mathfrak{t}_n - M\mathbf{u}^{\#}\right|\\ &= \left|\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\left(\varsigma(\mathfrak{t}, \mathfrak{t}_n(\mathfrak{t})) - \varsigma\left(\mathfrak{t}, \mathbf{u}^{\#}(\mathfrak{t})\right)\right) + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\varsigma(\mathfrak{t}, \mathfrak{t}_n(\mathfrak{t})) - \varsigma\left(\mathfrak{t}, \mathbf{u}^{\#}(\mathfrak{t})\right)d\mathfrak{t}\right|\\ &\leq \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\left|\varsigma(\mathfrak{t}, \mathfrak{t}_n(\mathfrak{t})) - \varsigma\left(\mathfrak{t}, \mathbf{u}^{\#}(\mathfrak{t})\right)\right|\\ &\quad + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\left|\varsigma(\mathfrak{t}, \mathfrak{t}_n(\mathfrak{t})) - \varsigma\left(\mathfrak{t}, \mathbf{u}^{\#}(\mathfrak{t})\right)\right|d\mathfrak{t}\\ &\leq \frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\mathfrak{N}\left|\mathfrak{t}_n - \mathbf{u}^{\#}\right| + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\int_0^{\mathfrak{t}}\mathfrak{N}\left|\mathfrak{t}_n - \mathbf{u}^{\#}\right|d\mathfrak{t}\\ \left\|f_n - \mathbf{u}^{\#}\right\| &\leq \left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})} + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\mathrm{T}\right)\mathfrak{N}\left\|\mathfrak{t}_n - \mathbf{u}^{\#}\right\|.\end{aligned} \tag{66}$$

Due to condition $(CM2)$, we obtain that:

$$\left\|f_n - \mathbf{u}^{\#}\right\| \leq \mathfrak{N}(1 - m_n(1-\mathfrak{N}))\left\|g_n - \mathbf{u}^{\#}\right\| \tag{67}$$

If we define $k_n = (1-l_n)f_n + l_n M f_n$, then in similar fashion as above, we can obtain:

$$\begin{aligned}\left\|k_n - \mathbf{u}^{\#}\right\| &\leq (1 - l_n(1-\mathfrak{N}))\left\|f_n - u^{\#}\right\|\\ &\leq (1 - l_n(1-\mathfrak{N}))\mathfrak{N}(1 - m_n(1-\mathfrak{N}))\left\|g_n - \mathbf{u}^{\#}\right\|\end{aligned} \tag{68}$$

Therefore using condition $(CM1)-(CM2)$,

$$\begin{aligned}\left\|h_n - \mathbf{u}^{\#}\right\| &= \left\|Mk_n - M\mathbf{u}^{\#}\right\|\\ &\leq \left(\frac{2(1-\mathfrak{b})}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})} + \frac{2\mathfrak{b}}{(2-\mathfrak{b})\mathscr{A}(\mathfrak{b})}\mathrm{T}\right)\mathfrak{N}\left\|k_n - \mathbf{u}^{\#}\right\|\\ &\leq \mathfrak{N}^2(1 - l_n(1-\mathfrak{N}))(1 - m_n(1-\mathfrak{N}))\left\|g_n - \mathbf{u}^{\#}\right\|.\end{aligned} \tag{69}$$

Likewise, we retrieve:

$$\begin{aligned}\left\|g_{n+1} - \mathbf{u}^{\#}\right\| &\leq \mathfrak{N}^3(1 - l_n(1-\mathfrak{N}))(1 - m_n(1-\mathfrak{N}))\\ &\quad(1 - e_n(1-\mathfrak{N}))\left\|g_n - \mathbf{u}^{\#}\right\|.\end{aligned} \tag{70}$$

Since $\mathfrak{N} < 1, l_n < 1, e_n < 1$, which gives $\mathfrak{N}^3 < 1$, $(1 - l_n(1-\mathfrak{N})) < 1$ and $(1 - e_n(1-\mathfrak{N})) < 1$.

Therefore (70) becomes:

$$\left\|g_{n+1} - \mathbf{u}^{\#}\right\| \leq (1 - m_n(1-\mathfrak{N}))\left\|g_n - \mathbf{u}^{\#}\right\|. \tag{71}$$

Define:

$$\begin{aligned}\widetilde{\mathfrak{d}}_n &= m_n(1-\mathfrak{N}),\\ \widetilde{\mathfrak{e}}_n &= \left\|g_n - \mathbf{u}^{\#}\right\|.\end{aligned} \tag{72}$$

where $\widetilde{\mathfrak{d}}_n \in (0,1)$ such that $\sum\widetilde{\mathfrak{d}}_n = \infty$.

Therefore, (71) can be viewed as:

$$\widetilde{\mathfrak{e}}_{n+1} = \left(1 - \widetilde{\mathfrak{d}}_n\right)\widetilde{\mathfrak{e}}_n. \tag{73}$$

Executing Lemma 3, we get $\mathfrak{M}$ - iteration converges to the solution $\mathbf{u}^{\#}$ of IVP (55). □

7. Conclusion

This article is endowed with introduction of a new three - step iterative technique called $\mathfrak{M}$ - iteration which converges strongly to the invariant point of almost - contraction mappings in Banach space. Followed by almost M - stability of $\mathfrak{M}$ approximation procedure, we presented that this novel technique has fast rate of convergence as compare to other vital procedures for the class of weak contractions which can be visualized from two numerical illustrations, namely. Example 1 and Example 2 presented here. We further established data dependency of our algorithm and demonstrated that Caputo type fractional differential equation possess a solution which, under certain circumstances, is a fixed point of Berinde contractions. Lastly, we showed that $\mathfrak{M}$ - iterative procedure converges to the solution of fractional differential equation.

Nevertheless, few obvious questions are engrossed in the field:

(1) Can one define an iterative technique which is even faster than $\mathfrak{M}$ iterative procedure for the class of weak contractions in the setting of Banach space?

(2) Does $\mathfrak{M}$ - iteration strongly converges to the fixed point of weak contractions in the framework of a space weaker than Banach space (quasi-Banach space or metric space, for instance)?

(3) Does $\mathfrak{M}$ - iterative algorithm converges for certain other class of mappings like quasi - nonexpansive or enriched contractions ?

Data Availability

No data were used to support this study.

Additional Points

This article is embraced with following key findings and implications: (i) Proposing a new fixed point approximation procedure namely $\mathfrak{M}$ -iteration which strongly converges to single fixed point of weak contractions. (ii) Stability discussion of $\mathfrak{M}$ - iteration followed by analytical and graphical contrast of $\mathfrak{M}$ - iteration with other leading algorithms and establishing that $\mathfrak{M}$ - iteration has faster rate of convergence than those taken into consideration via numerical illustrations executed through MATLAB R2019(b) software. (iii) Data Dependency result for $\mathfrak{M}$ - iteration. (iv) Finding solution of Caputo type fractional differential equation under certain conditions and manifesting that the new iterative procedure converges to the solution of fractional differential equation.

Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Authors' Contributions

All authors have made equal contribution towards the preparation of this article.

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