Research Article

Spectral Solutions for Fractional Black–Scholes Equations

M. A. Abdelkawy and António M. Lopes

1Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia
2Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
3LAETA/INEGI, Faculty of Engineering, University of Porto, Porto, Portugal

Correspondence should be addressed to António M. Lopes; aml@fe.up.pt

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1. Introduction

Fractional calculus [1–3] is the branch of calculus that generalizes the derivative and integral operators to non-integer orders. During the last few decades, the fractional calculus gained increasing importance in several fields, such as mechanics [4, 5], biology [6, 7], physics [8], chemistry [9, 10], viscoelasticity [11], engineering [12], and finance/economics [13, 14]. Fractional differential equations can be used to represent accurately a variety of phenomena. However, many equations cannot be solved analytically, and, thus, numerical approaches are required to find approximate solutions. The finite difference [15, 16] and finite element [17–20] methods have been widely used, as well as other numerical schemes, such as the Galerkin [21], Petrov-Galerkin [22, 23], pseudospectral [24–26], and fast predictor-corrector [27] methods.

The Black–Scholes model (BSM) is a mathematical description of pricing evolution [28, 29]. Specifically, the model estimates financial instruments via time variation. These instruments (such as stocks or futures) track a log-normal distribution of prices. Based on this hypothesis and taking into account other variables, the BSM derives the price of call options. Despite there being several models that describe pricing evolution, the BSM has been one of the most significant and prevalent in the last decades, and its generalization to fractional order was proposed [30, 31].

In [32], the Laplace homotopy analysis approach was employed to solve the fractional BSM (FBSM) in the sense of the Caputo–Fabrizio derivative. In [33], the authors employed the Sumudu and Laplace transforms to address various economic models with different fractional operators. In [34, 35], the Adomian and fractional Adomian decomposition were adopted to solve the FBSM. In [36], the time-fractional BSM was solved using the generalized differential transform technique. Moreover, in [37], the existence and uniqueness solution of the European-type option pricing model was discussed, while in [38], the multivariate Padé approximation was utilized to solve the Caputo fractional European vanilla call option pricing problem.
In most circumstances, it is impossible to provide explicit analytical solutions to space and/or time-fractional differential equations. Hence, developing effective numerical techniques is a critical necessity. Many high-accuracy numerical approaches have been devised to tackle diverse issues in various applications. Because of their high-order precision, fractional differential and integral equations [39] have seen tremendous development in recent years. But, contrasting with the effort put into evaluating finite difference and element schemes, little research has been dedicated to designing and assessing global spectral schemes [40]. In cases with periodic boundary conditions, spectral techniques based on the Fourier expansion have been used. When compared to other schemes, spectral techniques take the main place due to their robustness and exponential rates of convergence [41–44]. Despite some drawbacks, such as the inability to represent physical processes in spectral space and the difficulty in parallelizing on distributed memory computers, they reveal excellent accuracy, high speed of convergence, and simplicity in solving many types of differential equations [45].

The spectral collocation method is particularly valuable because it can estimate the solution of a wide range of equations. Moreover, its exponential rate of convergence is extremely useful in delivering very exact solutions. The collocation approach has grown in prominence in recent decades for dealing with specific difficulties posed by fractional derivatives.

In this paper, a new spectral collocation approach is proposed to solve the FBSPM [31, 46, 47], given by

\[
\frac{\partial^{\mu} Y(x, t)}{\partial t^{\mu}} + \frac{1}{2} a^2 x^2 \frac{\partial^2}{\partial x^2} (Y(x, t)) + r \frac{\partial}{\partial x} (Y(x, t)) + \mathbb{H}(x, t) = 0, \quad (x, t) \in \overline{\Omega} \times \overline{\Omega},
\]

where \( \mathbb{H}(x, t) \) denotes the right Riemann–Liouville fractional derivative:

\[
\frac{\partial^{\mu} Y(x, t)}{\partial t^{\mu}} = \frac{1}{\Gamma(1 - \mu)} \frac{\partial}{\partial \tau} \int_0^\tau \frac{Y(x, s) - Y(x, \tau)}{(s - \tau)^\mu} ds,
\]

with \( 0 < \mu < 1 \).

Herein, the shifted fractional Jacobi–Gauss–Radau collocation (SFJ-GR-C) and shifted fractional Jacobi–Gauss–Lobatto collocation (SFJ-GL-C) techniques are utilized to solve the FBSPM. We interpolate the independent variables using the shifted fractional-order Jacobi nodes, and we approximate the solution of the model by a sequence of shifted fractional-order Jacobi orthogonal functions. After that, the residuals at the shifted fractional-order Jacobi quadrature locations are estimated. This yields an algebraic system of equations that can be solved using a suitable approach. The accuracy of the new method is demonstrated using numerical examples.

The paper is organized into four sections: Section 2 introduces the new spectral collocation technique. Section 3 applies the method to solve numerical examples. Section 4 summarizes the main conclusions.

2. Fully Spectral Collocation Technique

We start by mapping the variable \( \tau \) as \( \tau = t - G \). Additionally, the Riemann–Liouville is transformed to the Caputo fractional derivative \( \frac{\partial^{\mu} Y(x, t)}{\partial t^{\mu}} \) [31, 46]:

\[
\frac{\partial^{\mu} (Y(x, t))}{\partial t^{\mu}} + \frac{1}{2} a^2 x^2 \frac{\partial^2}{\partial x^2} (Y(x, t)) - r x \frac{\partial}{\partial x} (Y(x, t)) + \mathbb{H}(x, t) = 0, \quad (x, t) \in \overline{\Omega} \times \overline{\Omega},
\]

where \( \overline{\Omega} \equiv [0, \overline{\mathcal{L}}] \) and \( \overline{\mathcal{L}} \equiv [0, \overline{T}] \). Conditions are

\[
\begin{align*}
Y(x, 0) &= \Theta_1(x), \quad x \in \overline{\Omega}, \\
Y(0, t) &= \Theta_2(t), \\
Y(G, t) &= \Theta_3(t), \quad t \in \overline{\mathcal{L}},
\end{align*}
\]

and \( \frac{\partial^{\mu} Y(x, t)}{\partial t^{\mu}} \) is a Caputo fractional derivative.

We express the truncated solution as

\[
\begin{align*}
Y_{N, M}(x, t) &= \sum_{r_1=0}^{N} \sum_{r_2=0}^{M} c_{r_1 r_2} G_{x, r_1}^{\alpha_{1, 2}, \beta_{1, 2}}(x) G_{x, r_2, \lambda_1}(t),
\end{align*}
\]

where \( G_{x, r_1}^{\alpha_{1, 2}, \beta_{1, 2}}(x) \) denotes the shifted fractional Jacobi functions on \( [0, \overline{\mathcal{L}}] \) [see [48, 49] for extra information].

\[
\frac{\partial}{\partial x} \left( \frac{\partial^{\mu} Y_{N, M}(x, t)}{\partial t^{\mu}} \right) = \sum_{r_1=0}^{N} \sum_{r_2=0}^{M} c_{r_1 r_2} G_{x, r_1}^{\alpha_{1, 2}, \beta_{1, 2}}(x) G_{x, r_2, \lambda_1}(t).
\]

Similarly, we have [48, 49]

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial^{\mu} Y_{N, M}(x, t)}{\partial t^{\mu}} \right) = \sum_{r_1=0}^{N} \sum_{r_2=0}^{M} c_{r_1 r_2} G_{x, r_1}^{\alpha_{1, 2}, \beta_{1, 2}}(x) G_{x, r_2, \lambda_1}(t).
\]

The time Caputo fractional derivative, on the contrary, is obtained as [48, 49]

\[
\frac{\partial^{\mu}}{\partial t^{\mu}} \left( \frac{\partial^{\mu} Y_{N, M}(x, t)}{\partial t^{\mu}} \right) = \sum_{r_1=0}^{N} \sum_{r_2=0}^{M} c_{r_1 r_2} G_{x, r_1}^{\alpha_{1, 2}, \beta_{1, 2}}(x) G_{x, r_2, \lambda_1}(t).
\]

Previous computations are indicated at certain nodes as
\[
\left( \frac{\partial Y}{\partial t} (X, t) \right)_{t=\nu, \alpha, \beta, \gamma} = \sum_{r_{1}=0, \ldots, \nu} \gamma_{r_{1}} \sum_{r_{2}=0, \ldots, \nu} \gamma_{r_{2}} \left( X_{r_{1}, \alpha, \beta, \gamma} (t) \right) \left( X_{r_{2}, \alpha, \beta, \gamma} (t) \right)
\]

Otherwise, the initial boundary can be obtained by
\[
\sum_{r_{1}=0, \ldots, \nu} \gamma_{r_{1}} \sum_{r_{2}=0, \ldots, \nu} \gamma_{r_{2}} \left( X_{r_{1}, \alpha, \beta, \gamma} (0) \right) \left( X_{r_{2}, \alpha, \beta, \gamma} (0) \right) = \Theta_{1} (x),
\]

Eq. (3) is constrained to be zero at \((N - 1) \times (M)\) points:
\[
\Delta \left( X_{r_{1}, \alpha, \beta, \gamma} (t) \right) = \Theta_{2} (x, t, \alpha, \beta, \gamma),
\]

where
\[
\Delta \left( X_{r_{1}, \alpha, \beta, \gamma} (t) \right) = \sum_{r_{1}=0, \ldots, \nu} \gamma_{r_{1}} \sum_{r_{2}=0, \ldots, \nu} \gamma_{r_{2}} \left( X_{r_{1}, \alpha, \beta, \gamma} (t) \right) \left( X_{r_{2}, \alpha, \beta, \gamma} (t) \right)
\]

\[
- \frac{1}{2} \Delta ^{2} (x, t, \alpha, \beta, \gamma) \sum_{r_{1}=0, \ldots, \nu} \gamma_{r_{1}} \sum_{r_{2}=0, \ldots, \nu} \gamma_{r_{2}} \left( X_{r_{1}, \alpha, \beta, \gamma} (t) \right) \left( X_{r_{2}, \alpha, \beta, \gamma} (t) \right)
\]

\[
- r X_{r_{1}, \alpha, \beta, \gamma} \sum_{r_{2}=0, \ldots, \nu} \gamma_{r_{2}} \left( X_{r_{1}, \alpha, \beta, \gamma} (t) \right) \left( X_{r_{2}, \alpha, \beta, \gamma} (t) \right)
\]

where \(n = 1, \ldots, N - 1\), and \(m = 1, \ldots, M\), and
Table 1: The $L_{\infty}$ errors between the exact and approximate solutions achieved by our method and those reported in [47] for Example 1, taking $\alpha = 0.25, r_1 = 0.01875, r_2 = 0.05, \text{and } \mu = 0.6.$

<table>
<thead>
<tr>
<th>$(\mathcal{N}', \mathcal{M})$</th>
<th>$\alpha_1 = \beta_1 = 0$</th>
<th>$\alpha_1 = 1/2, \beta_1 = 0$</th>
<th>$\alpha_1 = 1/2, \beta_1 = 1/2$</th>
<th>$\alpha_2 = \beta_2 = 1/2$</th>
<th>$\alpha_2 = \beta_2 = -1/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4, 4)</td>
<td>$2.55732 \times 10^{-2}$</td>
<td>$1.09933 \times 10^{-2}$</td>
<td>$1.26953 \times 10^{-2}$</td>
<td>(7, 20)</td>
<td>$8.6814 \times 10^{-5}$</td>
</tr>
<tr>
<td>(8, 8)</td>
<td>$1.60814 \times 10^{-6}$</td>
<td>$1.8991 \times 10^{-6}$</td>
<td>$1.4466 \times 10^{-6}$</td>
<td>(9, 20)</td>
<td>$4.3312 \times 10^{-9}$</td>
</tr>
<tr>
<td>(12, 12)</td>
<td>$3.43343 \times 10^{-11}$</td>
<td>$3.95182 \times 10^{-11}$</td>
<td>$2.89728 \times 10^{-11}$</td>
<td>(11, 20)</td>
<td>$1.8476 \times 10^{-11}$</td>
</tr>
<tr>
<td>(16, 16)</td>
<td>$4.21885 \times 10^{-15}$</td>
<td>$5.32907 \times 10^{-15}$</td>
<td>$3.58257 \times 10^{-15}$</td>
<td>(13, 20)</td>
<td>$5.7662 \times 10^{-14}$</td>
</tr>
<tr>
<td>(20, 20)</td>
<td>$4.08849 \times 10^{-15}$</td>
<td>$6.66294 \times 10^{-16}$</td>
<td>$6.66134 \times 10^{-16}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Finally, a solvable linear algebraic system is provided.

It should be mentioned that the convergence spectral rate of Jacobi polynomials has been thoroughly studied in the literature [50, 51], while the convergence spectral rate of fractional Jacobi functions has been addressed in [48, 49, 52], to cite a few.

3. Numerical Results

In this section, we demonstrate the resilience and accuracy of our strategy to solve two problems. The algorithm was implemented in MATHEMATICA version 12.2, running in an Intel(R) Core(TM) i7-10510U CPU @ 2.30 GHz on a PC with 8.00 GB of RAM. In what follows, we take $\mathcal{O} \equiv [0, 1]$ and $\mathcal{Q} \equiv [0, 1].$

Example 1. We solve the FBSM [47].

$$\frac{\partial^\mu}{\partial t^\mu} (\mathcal{Y}(x, t)) - \frac{1}{2} \alpha^2 \frac{\partial^2}{\partial x^2} (\mathcal{Y}(x, t)) - r_1 \frac{\partial}{\partial x} (\mathcal{Y}(x, t))$$

$$+ r_2 \mathcal{Y}(x, t) = \mathbb{H}(x, t),$$

with

$$\mathcal{Y}(x, 0) = \cos(\pi x), \quad x \in [0, 1],$$

$$\mathcal{Y}(0, t) = 1 + t^\mu, \quad \mathcal{Y}(1, t) = -t^\mu - 1, \quad t \in [0, 1],$$

choosing $\mathbb{H}(x, t)$ such that

$$\mathcal{Y}(x, t) = (1 + \sin(\pi x)) \left( t^\mu + \frac{x}{2} \right).$$

Table 1 summarizes the $L_{\infty}$ errors between the exact and approximate solutions achieved by our method and those reported in [47], taking $\alpha = 0.25, r_1 = 0.01875, r_2 = 0.05,$ and $\mu = 0.6.$ We verify that the new strategy is superior and that good estimates are obtained with a small number of collocation points. Figures 1 and 2 present the 3D charts of the numerical solution and the absolute error, respectively, when $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0, \lambda_1 = 1, \lambda_2 = 1,$ and $\mathcal{N}' = \mathcal{M} = 20.$ Figure 3 depicts the corresponding exact and numerical solutions along $x.$ Figures 4 and 5 represent the absolute errors along $x$ and $t,$ respectively, while Figure 6 shows the maximum absolute error ($M_E$) convergence for the following cases:

Case I: $\alpha_1 = 1/2, \alpha_1 = 1/3, \alpha_2 = 2/3, \beta_2 = 1/2, \lambda_1 = 0.1, \lambda_2 = 1.$

Case II: $\alpha_1 = 1/3 = \alpha_2 = \beta_2 = 1/2, \lambda_1 = 0.5, \lambda_2 = 1.$

Case III: $\alpha_1 = 1/3 - \alpha_2 = -\beta_2 = 1/2, \lambda_1 = 0.9, \lambda_2 = 1, \lambda_3 = 1.$

Example 2. We solve the FBSM [46].

$$\frac{\partial^\mu}{\partial t^\mu} (\mathcal{Y}(x, t)) - \frac{1}{2} \alpha^2 \frac{\partial^2}{\partial x^2} (\mathcal{Y}(x, t)) - r x \frac{\partial}{\partial x} (\mathcal{Y}(x, t))$$

$$+ r (\mathcal{Y}(x, t)) = \mathbb{H}(x, t),$$

(18)

with

$$\mathcal{Y}(x, 0) = e^x + x + 1, \quad x \in [0, 1],$$

$$\mathcal{Y}(0, t) = t^\mu + 1,$$

$$\mathcal{Y}(1, t) = t^\mu + e + 2, \quad t \in [0, 1],$$

choosing $\mathbb{H}(x, t)$ such that $\mathcal{Y}(x, t) = t^\mu + e^x + x + 1.$

Table 2 summarizes the $L_{\infty}$ errors between the exact and approximate solutions achieved by our method and those reported in [46], taking $\alpha = 0.1, r = 0.06,$ and $0 < \mu < 1.$ We verify that the new strategy is superior and that good estimates are obtained with a small number of collocation points. Figures 7 and 8 present the 3D charts of the numerical solution and the absolute error, respectively, when $\alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1,$ and $\mathcal{N}' = \mathcal{M} = 14.$ Figure 9 depicts the corresponding exact and numerical solutions along $x.$ Figures 10 and 11 represent the absolute errors along $x$ and $t,$ respectively, while Figure 12 shows the $M_E$ convergence for the following cases:

Case I: $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0, \mu = 0.6, \lambda_1 = 1, \lambda_2 = 1.$
Case II: \[ \alpha_1 = 0.5, \beta_1 = 0, \alpha_2 = -1/2, \beta_2 = 0, \mu = 0.6, \lambda_1 = 1, \lambda_2 = 1. \]

Case III: \[ \alpha_1 = \beta_1 = -\alpha_2 = -\beta_2 = 1/2, \mu = 0.6, \lambda_1 = 1, \lambda_2 = 1. \]

Figures 8, 10, and 11 show that the results are fairly precise, as the absolute error approaches zero. We can also see in Figures 7 and 9 that the approximate solution matches precisely the exact solution. In addition, Figure 12 clearly reveals that the method yields exponential convergence of the error.

To sum up, we emphasize that the majority of numerical techniques for the problem at hand are based on orthogonal polynomials. The use of classical polynomials in problems with nonsmooth solutions leads to low
accuracy or even failure to converge. Using the fractional, rather than the classical, Jacobi functions mitigates the problem. With the proposed method, all numerical computations could be completed with good precision and a low number of degrees of freedom. Moreover, our technique outperformed other existing approaches. Finally, we can conclude that the fully spectral collocation approach is a useful, efficient, and acceptable strategy for dealing with problems that have singular solutions.
Figure 7: Numerical solution of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1, \) and \( N = M = 14. \)

Figure 8: Absolute error of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1, \) and \( N = M = 14. \)

Figure 9: Exact and numerical solutions along \( x \) of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1, \) and \( N = M = 14. \)

Figure 10: Absolute error along \( t \) of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1, \) and \( N = M = 14. \)

Figure 11: Absolute error along \( x \) of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1, \) and \( N = M = 14. \)

Figure 12: The \( M_E \) convergence of Example 2 for the three cases.
4. Conclusion

The FBSM was treated using a fully spectral collocation technique for the two independent variables $x$ and $t$. It was verified that the new technique is superior in terms of accuracy and efficiency to other methods, for both smooth and nonsmooth solutions. To deal with the FBSM, we devised an approach that yields an algebraic system from which an approximated solution can be computed. The simulation results revealed that the proposed approach is effective for the goal at hand. Furthermore, because of its ease of use, our technique is relevant to a wide range of fractional problems. In the future, we can concentrate on the usage of the spectral Galerkin and the tau approaches for solving more complicated pricing models, such as the tempered FBSM. Finally, we should mention that the maximum absolute error for a given boundary value problem with a smooth solution is exponentially convergent. For nonsmoothness in time (or in space), the method’s order of convergence degrades. This, however, can be mitigated by employing the fractional-order Jacobi functions described herein.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


