Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 9417813, 9 pages
https://doi.org/10.1155/2022/9417813

Research Article

Minimum Aberration Split-Plot Designs When the Whole Plot and Subplot Factors Do Not Have the Same Importance

Shengli Zhao and Qianqian Zhao

School of Statistics and Data Science, Qufu Normal University, Qufu, China

Correspondence should be addressed to Qianqian Zhao; happy_carol@163.com

Received 26 March 2022; Accepted 5 July 2022; Published 27 September 2022

Academic Editor: He Chen

In practical factorial experiments, we sometimes find that complete randomization of the order of the runs is infeasible because it is more difficult to change the levels of some factors than the others, especially in some engineering experiments. Then, fractional factorial split-plot (FFSP) designs represent a practical option in such situations. The difficult-to-change factors are called whole plots (WP) factors, and the other factors are called subplot (SP) factors. The WP and SP factors do not have the same importance in many experiments. Then, the popular minimum aberration criterion is not suitable any more for choosing FFSP designs. This paper proposes two criteria for selecting FFSP designs. Algorithms for constructing the optimal FFSP designs under the two criteria are proposed. Some optimal designs under the two criteria are tabulated.

1. Introduction

Fractional factorial (FF) designs are widely used in various experiments, say agriculture, medicine, and engineering (see [1, 2]). Randomization is one of the fundamental principles in the design of an experiment. It provides protection against variables that are unknown to the experimenter but may impact the response. It reduces the unwanted influence of subjective judgment in treatment allocation [2]. However, complete randomization is sometimes inadvisable since the levels (settings) of some factors are very difficult or expensive to change. In such situations, fractional factorial split-plot (FFSP) designs, which involve a two-phase randomization, can be conveniently used to reduce costs and hence represent a practical design option.

In an FFSP design, the factors whose levels are difficult or expensive to change are called whole plot (WP) factors and the rest are called subplot (SP) factors. In the following, a “WP factor setting” means a combination of levels of the WP factors, and an “SP factor setting” means a combination of levels of the SP factors. The two-phase randomization process in an FFSP design is as follows (see [3]): (i) randomly choose any of the WP factor settings; (ii) run the experiment with the associated SP factor settings in a random order while holding the WP factor setting fixed; (iii) repeat steps (i) and (ii) till all the WP factor settings are covered. The two-step randomization results in two error sources in the analysis of variance (ANOVA), the WP and SP error terms. In general, the latter is smaller than the former. This means that the power to detect significant effects in data analysis is different for these two kinds of factors. So, the WP and SP factors cannot be treated equally.

Some practical examples also illustrate the difference of the WP and SP factors. Reference [4] cited an experiment from [1] (p. 629) as example in which the WP factors are more important than the SP factors. Reference [5] mentioned an example in which the SP factors (product design factors or control factors) are more important than the WP factors (environmental factors or noise factors). Reference [6] considered the following five scenarios for more flexible split-plot design choices: (i) basic screening; (ii) screening, with emphasis on SP effects; (iii) screening, with emphasis on WP effects; (iv) robust parameter design, with control factors at the SP level; (v) robust parameter design, with control factors at the WP level. Clearly, in scenarios (ii) and (iv), the SP factors are more important than the WP factors, and in (iii) and (v), the WP factors are more important than the SP factors.

One of the most important things for an experimenter is to choose the optimal experimental design according to some criterion. The minimum aberration (MA) criterion, which was first proposed by Fries and Hunter for regular two-level designs (see [7]), is popular and efficient for assessing designs. Up to now, many statisticians have been involved in extending MA criterion into wider applications such as FFSP designs. Reference [8] gave an algorithm to search MA FFSP designs. Reference [9] investigated some theoretical results and emphasized some of the differences between FFSP designs and FF designs. Reference [10] showed how the split-plot structure affects estimation, precision, and the use of resources. Reference [11] explored a minimum secondary aberration criterion called MSA-FFSP criterion. Reference [12] constructed MSA-FFSP designs in terms of consulting designs.

In the MA criterion, the WP and SP factors are treated with the same importance. From the above discussion, the WP and SP factors do not have the same importance in many experiments, thus new criterion needs to be developed. By adjusting the word lengths, reference [6] proposed different criteria for the five scenarios. These criteria are suitable for both regular and nonregular orthogonal designs. However, there are some defects in these criteria. For example, in scenario (ii), the words WW and SSS have the same length of 3, which contradicts the effect hierarchy principle somewhat, where WW and SSS denote the defining words involving 2 WP factors and 3 SP factors, respectively. Reference [4] proposed a WP-MA criterion for FFSP designs in which the WP factors are important and the experimenter is not interested in the SP effects. The WP-MA criterion, which first sequentially minimizes the WP wordlength pattern and then sequentially minimizes the SP wordlength pattern, gives full priority to the WP factors. Reference [13] constructed WP-MA FFSP designs via complementary designs. Conversely, in terms of the case that the SP factors are more important than the WP factors and the experimenter is not interested in the WP effects, Dang et al. [14] proposed the SP-MA criterion and studied the construction of SP-MA FFSP designs via complementary designs. Both the WP-MA and SP-MA criteria consider the different status of WP and SP factors, but these two criteria violate the effect hierarchy principle too when the SP/WP effects are also interested. Focusing on the WP factors, [15] proposed a WS-MA criterion for selecting FFSP designs and discussed the construction method of the optimal FFSP designs under the WS-MA criterion.

Considering scenarios (ii) and (iii) in [6], this paper proposes two wordlength patterns and two corresponding criteria, called W-MA criterion and S-MA criterion, respectively, for selecting FFSP designs. By intertwining the WP and SP wordlength patterns together in different forms, both the W-MA and S-MA criteria follow the effect hierarchy principle. As described above, there is a fundamental difference between the new proposed W-MA (S-MA) criterion and the WP-MA (SP-MA) criterion. In addition, the optimal designs in [13, 14] are confined to the case that the complementary sets are relatively small than the design sets themselves. To some extent, the new W-MA and S-MA criteria can help selecting optimal designs with parameters that are not within the scope of [13, 14]. The W-MA criterion proposed in this paper is consistent with the WS-MA criterion in [15] and the S-MA criterion is new. This paper proposes 3 algorithms for constructing W-MA and S-MA designs and these algorithms provide a supplement to the theoretical construction methods in [13–15].

The rest of the paper is organized as follows. Section 2 introduces some related definitions and notation and proposes the W-MA and S-MA criteria for selecting FFSP designs. Section 3 gives some theoretical results and algorithms that enable us to find the optimal FFSP designs under the W-MA and S-MA criteria. Section 4 tabulates some FFSP designs for practical use. Section 5 gives a conclusion.

2. Optimality Criteria for Regular Split-Plot Designs

Consider an experiment which involves n two-level factors with the two levels coded as 0 and 1, and suppose the experimenter can only afford 2^q runs. From 2^n runs of the full design, we select 2^q ones which satisfy $Bx = 0$ to constitute an FF design, where B is an $(n-q)\times n$ matrix over the Galois field GF (2) with full row rank and x is a row of the full design. Such an FF design is denoted as $2^{n-(n-q)}$ or 2^{n-k} $(k = n - q)$. The matrix B is called the generating matrix of the design. The vectors in the row space $\mathscr{R}(B)$ constitute the defining contrast subgroup, denoted as G. A nonzero vector $b \in \mathscr{R}(B)$ is called a defining vector. A fraction of the full design is called a regular design if it is determined by a defining contrast subgroup.

A vector $b' = (b_1, \ldots, b_n)$ with i nonzero entries represents an ith order effect of the factors corresponding to the i nonzero entries. It is said to be of the WP type if it involves only the WP factors, and of the SP type otherwise. The cosets of the defining contrast subgroup are called alias sets. An alias set is said to be a WP alias set if it contains at least a WP type effect, and an SP alias set if it contains only SP type effects. A nonzero vector $b' = (b_1, \ldots, b_n)$ can also be denoted by $\mathbf{1}^{b_1}\mathbf{2}^{b_2}\cdots\mathbf{n}^{b_n}$ with the convention that $\mathbf{i}^{b_i}$ is dropped for any $\mathbf{i}$ with $b_i = 0$. This system of notation is referred to as the compact notation, popularized by [16]. A defining vector b in such a compact notation is called a defining word and the number of nonzero entries of b is called its wordlength.

A regular two-level FFSP design with n_1 WP factors and n_2 SP factors is denoted as $2^{(n_1+n_2)-(k_1+k_2)}$, which is determined by k_1 independent WP type defining words and k_2 independent SP type defining words. A defining word is of WP type if it involves only WP factors and SP type if it involves at least one SP factor.

Take the construction of 2^{5-2} design d as an example. Let

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}. \quad (1)$$

Then, x's satisfied $Bx = 0$ over GF(2) constitute a 2^{5-2} design d. The nonzero vectors in $\mathcal{R}(B)$, $b_1' = (1,1,1,0,0)$, $b_2' = (1,0,0,1,1)$, and $b_3' = (0,1,1,1,1)$ are the defining vectors of design d, and corresponding $w_1 = \mathbf{123}$, $w_2 = \mathbf{145}$, and $w_3 = \mathbf{2345}$ in the compact notation, are the defining words of d. Then, the defining contrast subgroup of d is

$$G = \{I, \mathbf{123}, \mathbf{145}, \mathbf{2345}\}, \tag{2}$$

where I is the unity element, which corresponds to the zero vector in the row space $\mathcal{R}(B)$. If $\mathbf{1}, \mathbf{2}$, and $\mathbf{3}$ are WP factors, and $\mathbf{4}$ and $\mathbf{5}$ are SP factors, then d corresponds to an FFSP $2^{(3+2)-(1+1)}$ design with WP type defining word w_1 and SP type defining words $s_1 = w_2$ and $s_2 = w_3$, where w_1 and s_1 (or s_2) are two independent defining words.

Two $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs are isomorphic if the defining contrast subgroup of one design can be obtained from that of the other by permuting the WP factor labels and/or the SP factor labels. Isomorphic FFSP designs will be treated as the same in the following.

For a 2^{n-k} design d, let $A_i(d)$ be the number of defining words with wordlength i. The resolution of an FF design d is defined as the smallest i such that $A_i(d) \neq 0$. A 2^{n-k} design with resolution R is often denoted as 2_R^{n-k}. A 2^{n-k} design d is said to have minimum aberration (MA) if it sequentially minimizes

$$W(d) = (A_3(d), A_4(d), \ldots, A_n(d)). \tag{3}$$

Regarding an FFSP $2^{(n_1+n_2)-(k_1+k_2)}$ design d_0 as an FF 2^{n-k} design d, where $n = n_1 + n_2$ and $k = k_1 + k_2$. If the 2^{n-k} design d has MA, the $2^{(n_1+n_2)-(k_1+k_2)}$ design d_0 is said to have MA.

For a regular FFSP $2^{(n_1+n_2)-(k_1+k_2)}$ design d, let $A_{i,0}(d)$ be the number of WP type defining words with wordlength i, and $A_{i,1}(d)$ be the number of SP type defining words with wordlength i. The resolution of an FFSP design d is defined as the smallest i such that $A_{i,0}(d) \neq 0$ or $A_{i,1}(d) \neq 0$. A design with resolution one or two fails to ensure the estimability of all the main effects, even under the absence of all interactions. Since the main effects are almost invariably the objects of interest in a factorial experiment, only designs with $A_{1,0} = A_{1,1} = A_{2,0} = A_{2,1} = 0$ are considered. Define the following two $(n-2)$-dimensional sequences

$$W_1(d) = \left(A_{3,0}(d), A_{4,0}(d), \ldots, A_{n_1,0}(d), 0, \ldots, 0\right),$$
$$W_2(d) = \left(A_{3,1}(d), A_{4,1}(d), \ldots, A_{n_1,1}(d), \ldots, A_{n_1+n_2,1}(d)\right). \tag{4}$$

Among them, $W_1(d)$ and $W_2(d)$ are called the WP and SP wordlength patterns, respectively.

Considering the experiments, in which the WP factors are important and the experimenter is not interested in the SP effects, Wang et al. [4] proposed the following WP-MA criterion.

Definition 1. Let d_1 and d_2 be the two $2^{(n_1+n_2)-(k_1+k_2)}$ designs. Under the condition that WP factors are more likely to be significant than SP factors, d_1 is said to have less aberration of type WP than d_2 if either (i) $A_{i,0}(d_1) < A_{i,0}(d_2)$ for the smallest integer i such that $A_{i,0}(d_1) \neq A_{i,0}(d_2)$ or (ii) $A_{i,0}(d_1) = A_{i,0}(d_2)$ for any i but $A_{j,1}(d_1) < A_{j,1}(d_2)$ for the smallest integer j such that $A_{j,1}(d_1) \neq A_{j,1}(d_2)$. An FFSP design d is called a minimum aberration design of type WP (WP-MA) if no other design has less aberration of type WP than d.

The WP-MA criterion is suitable for selecting FFSP designs, in which the WP factors are more important than the SP factors and the experimenter is not interested in the SP effects. However, when the experimenter is interested in both the WP and SP effects, the WP-MA criterion violates the effect hierarchy principle, i.e., (i) lower order factorial effects are more likely to be important than higher order ones, and (ii) factorial effects of the same order are equally likely to be important.

Considering two of the five scenarios in [6], i.e., (ii) screening with emphasis on SP effects and (iii) screening with emphasis on WP effects, we propose the following two ranking wordlength patterns for FFSP designs:

$$W_w(d) = \left(A_{3,0}, A_{3,1}, \ldots, A_{n_1,0}, A_{n_1,1}, A_{n_1+1,1}, \ldots, A_{n_1+n_2,1}\right), \tag{5}$$

$$W_s(d) = \left(A_{3,1}, A_{3,0}, \ldots, A_{n_1,1}, A_{n_1,0}, A_{n_1+1,1}, \ldots, A_{n_1+n_2,1}\right). \tag{6}$$

Based on (5) and (6), we propose the following two criteria for selecting FFSP designs.

Definition 2. Let d_1 and d_2 be the two $2^{(n_1+n_2)-(k_1+k_2)}$ designs. Suppose i is the smallest integer such that $A_{i,j}(d_1) \neq A_{i,j}(d_2)$.

(1) Design d_1 is said to have less aberration of type W than design d_2 if either (i) $A_{i,0}(d_1) < A_{i,0}(d_2)$ or (ii) $A_{i,0}(d_1) = A_{i,0}(d_2)$ but $A_{i,1}(d_1) < A_{i,1}(d_2)$. An FFSP design d is said to have minimum aberration of type W (W-MA) if no other design has less aberration of type W than d.

(2) Design d_1 is said to have less aberration of type S than design d_2 if either (i) $A_{i,1}(d_1) < A_{i,1}(d_2)$ or (ii) $A_{i,1}(d_1) = A_{i,1}(d_2)$ but $A_{i,0}(d_1) < A_{i,0}(d_2)$. An FFSP design d is said to have minimum aberration of type S (S-MA) if no other design has less aberration of type S than d.

Clearly, a W-MA design d sequentially minimizes the components in $W_w(d)$ and an S-MA design d sequentially minimizes the components in $W_s(d)$. In fact, the MA FFSP designs are not always unique, the W-MA and S-MA designs are further discrimination of the MA FFSP designs.

3. Construction of the Optimal Split-Plot Designs

This section discusses the construction of the optimal FFSP designs under the W-MA and S-MA criteria. We first give a direct result for the case of $k_2 = 0$.

Theorem 1. *An FFSP $2^{(n_1+n_2)-(k_1+0)}$ design d has W-MA and S-MA if and only if its WP settings constitute an MA $2^{n_1-k_1}$ design.*

Proof. Note that when $k_2 = 0$, the SP factors are independent of the WP factors, which results in $A_{i,1} = 0$ and $i = 3, \ldots, n_1 + n_2$. Then, both (5) and (6) reduce to $(A_{3,0}, \ldots, A_{n_1,0})$, which is just the wordlength pattern of the $2^{n_1-k_1}$ design formed by the WP factors of the $2^{(n_1+n_2)-(k_1+0)}$ design d. Theorem 1 follows directly from Definition 2 and the definition of MA $2^{n_1-k_1}$ design. □

The construction of MA FF designs has been studied by many researchers in the past three decades. Reference [3] gave a good summary. Based on Theorem 1 and the MA designs given in Chapter 3 of [3], the W-MA and S-MA FFSP $2^{(n_1+n_2)-(k_1+0)}$ designs can be constructed by simply adding n_2 factors independent of the n_1 WP factors to represent the SP factors.

For $k_2 \neq 0$, we have the necessary condition for a $2^{(n_1+n_2)-(k_1+k_2)}$ design to have W-MA or S-MA as follows. □

Lemma 1. *If a $2^{(n_1+n_2)-(k_1+k_2)}$ design d has W-MA or S-MA, then each of the $n_1 + n_2$ factors is involved in some defining word of d.*

Proof. Suppose d is a $2^{(n_1+n_2)-(k_1+k_2)}$ design determined by $k_1 + k_2$ independent defining words $w_1, \ldots, w_{k_1}, s_1, \ldots, s_{k_2}$. Without loss of generality, suppose factor F is not involved in any defining word. By adding factor F to one of the SP type defining word, say s_1, we can obtain a $2^{(n_1+n_2)-(k_1+k_2)}$ design d'. According to whether or not factor F is involved, the defining words of d' can be classified into two groups. Let G_1 denote the group containing the defining words involving factor F and G_2 denote the other group. Each defining word in G_1 has one more factor (F) than the corresponding defining word of d. The defining words in the other group are the same as those of the design d. Thus, the design d' has less aberration of type W and S than d, and d cannot have W-MA or S-MA. □

By Lemma 1, we obtain the following result directly, which can help us construct some W-MA and S-MA designs with the special case of $k_2 = 1$. □

Corollary 1. *If a $2^{(n_1+n_2)-(k_1+1)}$ design d has W-MA or S-MA, the generating matrix B of d must have a submatrix $\widetilde{B}$ with n_2 columns, such that $\widetilde{B}$ has k_1 row vectors with all 0's and one row vector with all 1's.*

Recall that the two-step randomization in an FFSP design results in two sources of errors in ANOVA, the WP and SP error terms, and the former is larger than the latter in general. If a full design is considered (i.e., $k_1 = k_2 = 0$), this would entail lower estimation efficiency of WP effects compared to SP effects. In the same manner, in an FFSP design, estimation from a WP alias set has a lower efficiency than that from an SP alias set ([8, 10]). So, in a good FFSP design, the main effects of the SP factors should not appear in a WP alias set. From this point, we obtain the following two lemmas.

Lemma 2. *If a 2^{n-k} design d can be split into a $2^{(n_1+n_2)-(k_1+k_2)}$ design with $n_2 - k_2 = 1$, the generating matrix B of d must have a column with k_1 0's and k_2 1's. Especially, B must have a column with all 1's for $k_1 = 0$.*

Lemma 3. *If a 2^{n-k} design d can be split into a $2^{(n_1+n_2)-(k_1+k_2)}$ design, the generating matrix B of d must have the following form up to permutation of columns:*

$$B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}, \quad (7)$$

where B_{11} is a $k_1 \times n_1$ matrix with $\operatorname{rank}(B_{11}) = k_1$ and B_{22} is a $k_2 \times n_2$ matrix with $\operatorname{rank}(B_{22}) = k_2$, each row vector of B_{11} has at least three 1's and each row vector of B_{22} has at least two 1's.

Theorem 2. *Regarding an FFSPRegarding an FFSP$2^{(n_1+n_2)-(0+k_2)}$ design d_0 as an FF 2^{n-k_2} design d, where $n = n_1 + n_2$, if the 2^{n-k_2} design d has MA, then d_0 has W-MA and S-MA.*

Proof. Note that when $k_1 = 0$, we have $A_{i,0} = 0$ and $A_i = A_{i,1}$ for any $i = 3, \ldots, n_1 + n_2$. When regarding an FFSP $2^{(n_1+n_2)-(0+k_2)}$ design d_0 as an FF 2^{n-k_2} design d, both (5) and (6) of d_0 reduce to $(A_{3,1}, \ldots, A_{n_1+n_2,1})$, the wordlength pattern of d. Theorem 2 follows directly from Definition 2 and the definition of MA 2^{n-k_2} design. □

Theorem 2 implies that we can construct W-MA and S-MA $2^{(n_1+n_2)-(0+k_2)}$ designs via 2^{n-k_2} designs through decomposing n into suitable n_1 and n_2 with $n = n_1 + n_2$. Then, we propose the following Algorithm 1 to construct W-MA and S-MA $2^{(n_1+n_2)-(0+k_2)}$ designs via 2^{n-k_2} designs.

Reference [3] gave the 16-run MA $2^{(n_1+n_2)-(0+k_2)}$ designs. By Algorithm 1, we obtain all the 32-run MA $2^{(n_1+n_2)-(0+k_2)}$ designs with $n_1 + n_2 \leq 12$ in Table 1.

The WP settings of a $2^{(n_1+n_2)-(k_1+k_2)}$ design constitute a $2^{n_1-k_1}$ design. It is clear that the resolution of a $2^{(n_1+n_2)-(k_1+k_2)}$ design is no larger than that of the corresponding $2^{n_1-k_1}$ design. Note that the resolution of a $2^{n_1-k_1}$ design is at most n_1. Thus, if a 2^{n-k}_R design d can be split into a $2^{(n_1+n_2)-(k_1+k_2)}$ design with $k_1 > 0$, then $n_1 \geq R$. Therefore, for given n_1, n_2, k_1, and k_2, to construct the W-MA and S-MA $2^{(n_1+n_2)-(k_1+k_2)}$ designs, we narrow the search to 2^{n-k} designs with resolution $R \leq n_1$. In particular, if $k_1 = 1$, we select the defining word with length no more than n_1 as the first row of the generating matrix$B$.

From [3], we know that the maximum resolution of 2^{n-2} designs is $R_{\max} = [2n/3]$, here $[x]$ denotes the integer part of x.

Remark 1. Consider a $2^{n-2}_{R_{\max}}$ design d whose defining words have the same length, i.e., $A_{R_{\max}}(d) = 3$. Then, d has MA and $2n = 3R_{\max}$. For any $R_{\max} \leq n_1 \leq n - 2$, taking any defining word as the first row of generating matrix B of d, design d

Table 1: Catalogue of W-MA and S-MA $2^{(n_1+n_2)-(0+k_2)}$ designs with 32 runs.

n_1	n_2	k_2	s_i
1	5	1	**123456**
2	4	1	**123456**
3	3	1	**123456**
4	2	1	**123456**
1	6	2	**1236 12457**
2	5	2	**1236 12457**
3	4	2	**2356 12457**
4	3	2	**2356 12457**
1	7	3	**1236 1247 13458**
2	6	3	**1236 1247 13458**
3	5	3	**1246 2347 13458**
4	4	3	**2356 2457 13458**
1	8	4	**1236 1247 1258 13459**
2	7	4	**1236 1247 1258 13459**
3	6	4	**1256 2357 2458 13459**
4	5	4	**1256 2357 2458 13459**
1	9	5	**1236 1247 1258 13459 2345$\mathbf{t_0}$**
2	8	5	**1236 1247 1258 13459 2345$\mathbf{t_0}$**
3	7	5	**1246 2347 2458 12359 1345$\mathbf{t_0}$**
4	6	5	**1356 1457 2358 2459 12345$\mathbf{t_0}$**
1	10	6	**1236 1247 1348 1259 135$\mathbf{t_0}$ 145$\mathbf{t_1}$**
2	9	6	**1236 1247 1348 1259 135$\mathbf{t_0}$ 145$\mathbf{t_1}$**
3	8	6	**1246 1347 2348 1459 245$\mathbf{t_0}$ 345$\mathbf{t_1}$**
4	7	6	**1256 1357 1458 2359 245$\mathbf{t_0}$ 345$\mathbf{t_1}$**
1	11	7	**1236 1247 1348 2349 125$\mathbf{t_0}$ 135$\mathbf{t_1}$ 145$\mathbf{t_2}$**
2	10	7	**1236 1247 1348 2349 125$\mathbf{t_0}$ 135$\mathbf{t_1}$ 145$\mathbf{t_2}$**
3	9	7	**2346 1257 1358 1459 235$\mathbf{t_0}$ 245$\mathbf{t_1}$ 345$\mathbf{t_2}$**
4	8	7	**1256 1357 2358 1459 245$\mathbf{t_0}$ 345$\mathbf{t_0}$ 12345$\mathbf{t_2}$**

s_i denotes the independent SP-type defining words and $\mathbf{t_0}$, $\mathbf{t_1}$, $\mathbf{t_2}$, denote the factors **10**, **11**, **12**, respectively.

can be split into a $2^{(n_1+n_2)-(1+1)}$ design, which simultaneously has MA, W-MA, and S-MA.

Remark 2. Consider a 2_R^{n-2} design d whose defining words have different lengths. Suppose that taking each defining word as the first row of generating matrix B can generate a $2^{(n_1+n_2)-(1+1)}$ design. Then, $n_1 > R$. Denote the $2^{(n_1+n_2)-(1+1)}$ designs with the shortest word and longest word putting in the first row as d_1 and d_3, respectively, and the other design d_2. Then, under the W-MA criterion, d_3 is superior to d_2, and d_2 is superior to d_1, while under the S-MA criterion, d_1 is superior to d_2, and d_2 is superior to d_3.

The following example illustrates the use of Algorithm 2 and Remarks 1-2.

Example 1. Consider the construction of the W-MA and S-MA $2^{(n_1+n_2)-(1+1)}$ designs with $n_1+n_2=9$. The MA 2^{9-2} design d_0 has resolution 6 and the defining contrast subgroup is

$$G = \{I, \mathbf{123457}, \mathbf{123689}, \mathbf{456789}\}. \quad (8)$$

Since $A_6(d_0)=3$, pick any defining word as the first row of the generating matrix B, d_0 can be split into the $2^{(7+2)-(1+1)}$ and $2^{(6+3)-(1+1)}$ designs. Without loss of generality, take $w_1 = \mathbf{123457}$ and $s_1 = \mathbf{123689}$ to generate

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}. \quad (9)$$

Then, B satisfies the decomposition condition of Algorithm 2 for the $2^{(7+2)-(1+1)}$ (d_0') design and $2^{(6+3)-(1+1)}$ (d_0'', by permuting columns 6 and 7) design. By Remark 1, d_0' is the unique MA, W-MA, and S-MA $2^{(7+2)-(1+1)}$ design, with WP factors $\mathbf{1,2,3,4,5,6}$ and $\mathbf{7}(=\mathbf{12345})$ and SP factors $\mathbf{8,9}(=\mathbf{12368})$. The independent defining words of d_0' are w_1 (WP type) and s_1 (or $s_2 = \mathbf{456789}$, SP type). Similarly, the $2^{(6+3)-(1+1)}$ design d_0'' simultaneously has MA, W-MA, and S-MA. d_0'' has WP factors $\mathbf{1,2,3,4,5,6}(=\mathbf{12345})$ and SP factors $\mathbf{7,8,9}(=\mathbf{12378})$. The independent defining words of d_0'' are changed to $w_1 = \mathbf{123456}$ (WP type) and $s_1 = \mathbf{123789}$ (SP type) after the permutation of columns. Obviously, they have $W_w(d_0') = W_w(d_0'') = (0^6, 1, 2)$, where 0^s denotes s successive zeros. To construct $2^{(5+4)-(1+1)}$ design, consider the optimal 2_V^{9-2} design d_1 under the MA criterion, which has $W(d_1) = (0,0,1,1,1)$. The defining contrast subgroup of d_1 is

$$G = \{I, \mathbf{12345}, \mathbf{126789}, \mathbf{3456789}\}. \quad (10)$$

Denote $b_1' = (1,1,1,1,1,0,0,0,0)$, $b_2' = (1,1,0,0,0,1,1,1,1)$, and $b_3' = (0,0,1,1,1,1,1,1,1)$. Pick b_1', b_2', and b_3' as the first row of B, respectively. Without loss of generality, denote

$$B_1 = \begin{pmatrix} b_1' \\ b_2' \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_2' \\ b_1' \end{pmatrix}, \quad B_3 = \begin{pmatrix} b_3' \\ b_2' \end{pmatrix}. \quad (11)$$

Then, B_1 corresponds to $2^{(5+4)-(1+1)}$ ($\eth$), $2^{(6+3)-(1+1)}$, and $2^{(7+2)-(1+1)}$ (d_1') designs; B_2 corresponds to $2^{(6+3)-(1+1)}$ and $2^{(7+2)-(1+1)}$ (d_1'') designs; B_3 corresponds to a $2^{(7+2)-(1+1)}$ (d_1''') design. Among these FFSP designs split by d_1, $\eth$ is the unique MA and hence W-MA and S-MA $2^{(5+4)-(1+1)}$ design, the three $2^{(7+2)-(1+1)}$ designs are inferior to d_0', and the two $2^{(6+3)-(1+1)}$ designs are inferior to d_0''. W_w's for the three $2^{(7+2)-(1+1)}$ designs d_1', d_1'', d_1''' are, respectively, $W_w(d_1') = (0^4,1,0,0,1,0,1)$, $W_w(d_1'') = (0^4,0,1,1,0,0,1)$, and $W_w(d_1''') = (0^4,0,1,0,1,1,0)$. Obviously, under the W-MA criterion, d_1''' is superior to d_1'', and d_1'' is superior to d_1', while under the S-MA criterion, d_1' is superior to d_1'', and d_1'' is superior to d_1'''. Then, we go to the optimal 2_{IV}^{9-2} design d_2 under the MA criterion, which has $A_4(d_2) = A_6(d_2) = A_8(d_2) = 1$ and the other $A_i(d_2) = 0$. Pick the shortest word as the first row of B, we obtain the unique MA $2^{(4+5)-(1+1)}$ design, which also uniquely has W-MA and S-MA. At last we construct the unique MA $2^{(3+6)-(1+1)}$ design from the optimal 2_{III}^{9-2} design (see Table 1). Here, all

Step 1 For given n_1, n_2, and k_2, rank 2^{n-k_2} designs according to the MA criterion, say $d_1, d_2, \ldots$, where $n = n_1 + n_2$.
Step 2 Write down the generating matrix B of the design d_1 according to its defining contrast subgroup, where B is a $k_2 \times n$ matrix with full row rank.
Step 3 Try to decompose B into $(B_1 B_2)$ by permuting the columns such that B_1 and B_2 have n_1 and n_2 columns, respectively, and B_2 has full row rank. It also requires that B_2 has a column with all 1's if $n_2 - k_2 = 1$.
If d_1 can be split as above, then it is the W-MA and S-MA $2^{(n_1+n_2)-(0+k_2)}$ design. Otherwise, delete it from the list, go to d_2 and repeat Steps 2 and 3.

Algorithm 1: Construction algorithm to the W-MA and S-MA $2^{(n_1+n_2)-(0+k_2)}$ designs.

Step 1 For given n and k, rank 2^{n-k} designs according to the MA criterion.
Step 2 For given n_1 and n_2 satisfying $n = n_1 + n_2$, select the 2^{n-k} designs with resolution $R \le n_1$ according to the original order, say $d_1, d_2, \ldots$.
Step 3 For d_1, pick k defining words from its defining contrast subgroup and generate the generating matrix B.
Step 4 Try to decompose B into $\begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$ by permuting the columns, such that B_{11} is an n_1-dimensional non null row vector with at least three 1's, and B_{22} is a $(k-1) \times n_2$ matrix with $\text{rank}(B_{22}) = k-1$ and each row of B_{22} has at least two 1's.
(a) If B can be decomposed as above, then d_1 corresponds to the MA $2^{(n_1+n_2)-(1+(k-1))}$ design d_1'. If B_{11} corresponds to any of the shortest defining words, then d_1' is the unique S-MA $2^{(n_1+n_2)-(1+(k-1))}$ design. If B_{11} corresponds to any of the longest defining words, d_1' is the unique W-MA $2^{(n_1+n_2)-(1+(k-1))}$ design.
(b) Otherwise, delete it from the list, go to d_2 and repeat Steps 3 and 4.

Algorithm 2: Construction algorithm to the W-MA and S-MA $2^{(n1+n2)-(1+k2)}$.

128-run W-MA and S-MA $2^{(n_1+n_2)-(1+1)}$ designs are constructed according to Algorithm 2.

Now, we give three special cases of optimal $2^{(n_1+n_2)-(1+1)}$ designs.

Remark 3. For given $n_2 \ge 2$, $2^{(3+n_2)-(1+1)}$ design with WP factors $\mathbf{1, 2, 3 (= 12)}$ and SP factors $\mathbf{4, 5, \ldots, n_2 + 2, 145 \cdots (n_2 + 2)}$ is the unique MA $2^{(3+n_2)-(1+1)}$ design, which has $A_{3,0} = A_{n_2+1,1} = A_{n_2+2,1} = 1$.

Remark 4. For given $n_2 \ge 2$, $2^{(4+n_2)-(1+1)}$ design with WP factors $\mathbf{1, 2, 3, 4 (= 123)}$ and SP factors $\mathbf{5, \ldots, n_2 + 3, 125 \cdots (n_2 + 3)}$ is the unique MA $2^{(4+n_2)-(1+1)}$ design, which has $A_{4,0} = 1$ and $A_{n_2+2,1} = 2$.

Remark 5. For given $n_2 \ge 3$, $2^{(5+n_2)-(1+1)}$ design with WP factors $\mathbf{1, 2, 3, 4, 5 = (1234)}$ and SP factors $\mathbf{6, \ldots, n_2 + 4, 126 \cdots (n_2 + 4)}$ is the unique MA $2^{(5+n_2)-(1+1)}$ design, which has $A_{5,0} = 1$ and $A_{n_2+2,1} = A_{n_2+3,1} = 1$.

From Algorithm 2 and Remarks 1–5, we obtain all the W-MA and S-MA $2^{(n_1+n_2)-(1+1)}$ designs with 8, 16, 32, 64, and 128 runs, and all the W-MA, S-MA $2^{(n_1+n_2)-(1+2)}$, and $2^{(n_1+n_2)-(1+3)}$ designs with 8, 16, 32, and 64 runs, see Tables 2–4.

Now, we give a general algorithm to construct the W-MA and S-MA $2^{(n_1+n_2)-(k_1+k_2)}$ designs. By using this algorithm, we obtain some $2^{(n_1+n_2)-(2+1)}$ and $2^{(n_1+n_2)-(2+2)}$designs.

Table 2: Catalogue of W-MA and S-MA designs with $k_1 = k_2 = 1$.

n_1	n_2	w_i	s_i	$W_w(d)$	MAproperty
3	2	**123**	**145**	$(1, 1, 1, 0)$	*
4	2	**1234**	**1256**	$(0^2, 1, 2)$	*
3	3	**123**	**1456**	$(1, 0, 1, 1)$	*
4	3	**1234**	**12567**	$(0^2, 1, 0, 2)$	*
5	2	**12345**	**1267**	$(0^3, 1, 1, 1)$	W-MA
		1235	**12467**	$(0^2, 1, 0^2, 2)$	S-MA
3	4	**123**	**14567**	$(1, 0^2, 1, 1)$	*
5	3	**12345**	**12678**	$(0^4, 1, 1, 1)$	*
6	2	**123456**	**12378**	$(0^4, 0, 2, 1, 0)$	W-MA
		12346	**12578**	$(0^4, 1, 1, 0, 1)$	S-MA
4	4	**1234**	**125678**	$(0^2, 1, 0^2, 2)$	*
3	5	**123**	**145678**	$(1, 0^3, 1, 1)$	*
6	3	**123456**	**123789**	$(0^6, 1, 2)$	*
7	2	**123457**	**123689**	$(0^6, 1, 2)$	*
5	4	**12345**	**126789**	$(0^4, 1, 0, 1, 1)$	*
4	5	**1234**	**1256789**	$(0^2, 1, 0^3, 2)$	*
3	6	**123**	**1456789**	$(1, 0^4, 1, 1)$	*

w_i and s_i denote the independent WP- and SP-type defining words, respectively. *MA, W-MA, and S-MA design. 0^s denotes s successive zeros.

4. Tables of Optimal Designs

Some catalogues of W-MA and S-MA $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs are tabulated in Tables 1–6. In the tables, we use the compact notation to denote the factorial effects and defining words. For a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design, let $n_1 + n_2 = n$, $q_1 = n_1 - k_1$, $q_2 = n_2 - k_2$, and $q = q_1 + q_2$. Denote the WP factors as $\mathbf{1}, \ldots, \mathbf{q_1}, \mathbf{q_1 + 1}, \ldots, \mathbf{n_1}$, where the last k_1 factors are some interactions of $\mathbf{1}, \ldots, \mathbf{q_1}$, which generate the independent WP type defining words. The SP factors are

Table 3: Catalogue of W-MA and S-MA designs with $k_1 = 1$ and $k_2 = 2$.

n_1	n_2	w_i	s_i	$W_w(d)$	MA property
3	3	**123**	**145 246**	(1, 3, 3)	*
4	3	**1234**	**1256 1357**	$(0^2, 1, 6)$	*
3	4	**123**	**146 2457**	(1, 1, 3, 2)	1
4	4	**1234**	**1257 13568**	$(0^2, 1, 2, 4)$	*
5	3	**12345**	**1267 1368**	$(0^3, 3, 1, 3)$	W-MA
		1235	**2367 12468**	$(0^2, 1, 2, 0, 4)$	S-MA
3	5	**123**	**1457 2468**	(1, 0, 2, 3, 1)	*
4	5	**1234**	**12568 13579**	$(0^2, 1, 0, 4, 2)$	*
5	4	**12345**	**1268 14679**	$(0^3, 1, 1, 3, 2)$	W-MA
		1235	**12468 13479**	$(0^2, 1, 0^2, 4, 2)$	S-MA
6	3	**123456**	**12378 14579**	$(0^3, 1, 0, 4, 1, 1)$	W-MA
		1236	**12578 13479**	$(0^2, 1, 0^2, 4, 0, 2)$	S-MA
3	6	**123**	**14568 24579**	$(1, 0^2, 3, 3)$	*

w_i and s_i denote the independent WP- and SP-type defining words, respectively. *MA, W-MA, and S-MA design.

Table 4: Catalogue of W-MA and S-MA designs with $k_1 = 1$ and $k_2 = 3$.

n_1	n_2	w_i	s_i	$W_w(d)$	MA property
3	4	**123**	**145 246 1247**	$(1, 6, 7, 0^2, 1)$	*
4	4	**1234**	**1256 1357 2358**	$(0^2, 1, 13)$	*
3	5	**123**	**146 157 2458**	(1, 2, 7)	*
4	5	**1234**	**1257 1268 13569**	$(0^2, 1, 5, 8)$	*
5	4	**12345**	**1467 2468 3469**	$(0^3, 6, 1, 7)$	*
3	6	**123**	**1457 1468 2569**	(1, 0, 5, 6)	*
4	6	**1234**	**12568 12579 1367t** $_{\mathbf{0}}$	$(0^2, 1, 1, 8, 4)$	*
5	5	**12345**	**1268 12369 1467t** $_{\mathbf{0}}$	$(0^3, 2, 1, 7, 4)$	W-MA
		1235	**12468 12479 1367t** $_{\mathbf{0}}$	$(0^2, 1, 1, 0, 8, 4)$	S-MA
6	4	**123456**	**1278 13479 13567t** $_{\mathbf{0}}$	$(0^3, 2, 0, 8, 1, 3, 0, 1)$	W-MA
		12346	**1278 13579 1457t** $_{\mathbf{0}}$	$(0^3, 2, 1, 7, 0, 4)$	S-MA
3	7	**123**	**14568 24579 12467t** $_{\mathbf{0}}$	(1, 0, 1, 6, 6, 1)	*

w_i and s_i denote the independent WP- and SP-type defining words, respectively. *MA, W-MA, and S-MA design. $\mathbf{t_0}$ denotes the factor **10**.

Step 1 For given n_1 and n_2, rank 2^{n-k} designs according to the MA criterion, say $d_1, d_2, \ldots$, where $n = n_1 + n_2, k = k_1 + k_2$.
Step 2 Write down the generating matrix B of design d_1 according to its defining contrast subgroup, where B is a $k \times n$ matrix with rank(B) = k.
Step 3 Try to decompose B into $\begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$, such that up to permutation of columns, B_{11} is a $k_1 \times n_1$ matrix with rank$(B_{11}) = k_1$, and B_{22} is a $k_2 \times n_2$ matrix with rank$(B_{22}) = k_2$. Note that each row of B_{11} has at least three 1's and each row of B_{22} has at least two 1's.
(a) If B can be decomposed as above, then it is the MA $2^{(n_1+n_2)-(k_1+k_2)}$ design. If there is only one decomposition, it corresponds to the unique MA design, otherwise determine the W-MA and S-MA $2^{(n_1+n_2)-(k_1+k_2)}$ designs by comparing their wordlength patterns.
(b) Otherwise, delete it from the list and go to d_2 and Steps 2 and 3.

Algorithm 3: Construction algorithm to the W-MA and S-MA $2^{(n1+n2)-(k1+k2)}$.

denoted as $\mathbf{n_1} + \mathbf{1}, \ldots, \mathbf{n_1} + \mathbf{q}_2, \mathbf{n_1} + \mathbf{q}_2 + \mathbf{1}, \ldots, \mathbf{n}$, where the last k_2 factors correspond to the independent SP type defining words.

Table 1 lists all the 32-run W-MA and S-MA $2^{(n_1+n_2)-(0+k_2)}$ designs with $n_1 + n_2 \leq 12$. All the designs have MA and are split from the MA 2^{n-k_2} designs, where $n = n_1 + n_2$. All the W-MA and S-MA $2^{(n_1+n_2)-(1+1)}$ designs with 8, 16, 32, 64, and 128 runs are listed in Table 2, where $*$ in the last column denotes the design simultaneously has MA, W-MA, and S-MA. Tables 3 and 4, respectively, list all the W-MA and S-MA $2^{(n_1+n_2)-(1+2)}$ and $2^{(n_1+n_2)-(1+3)}$ designs with 8, 16, 32, and 64 runs. For given n_1, n_2, k_1, and k_2 in Tables 2–4, both the W-MA and S-MA designs are MA designs, though not isomorphic. Tables 5 and 6 give some $2^{(n_1+n_2)-(2+1)}$ and $2^{(n_1+n_2)-(2+2)}$ designs, which are obtained from Algorithm 3 and each which are obtained from Algorithm 3 and design simultaneously has MA, W-MA, and S-MA.

Use of the design tables is illustrated in the following example.

Table 5: Catalogue of W-MA and S-MA designs with $k_1 = 2$ and $k_2 = 1$.

n_1	n_2	w_i	s_i	$W_w(d)$
5	2	**124 135**	**2367**	(2, 0, 1, 2, 0, 2)
5	3	**124 135**	**23678**	$(2, 0, 1, 0^2, 2, 2)$
6	2	**1235 1246**	**13478**	$(0^2, 3, 0^2, 4)$
5	4	**124 135**	**236789**	$(2, 0, 1, 0^3, 2, 2)$
6	3	**1235 1246**	**134789**	$(0^2, 3, 0^4, 4)$
7	2	**1236 12457**	**13489**	$(0^2, 1, 0, 2, 2, 0, 2)$
5	5	**124 135**	**236789t$_0$**	$(2, 0, 1, 0^3, 2, 2)$
6	4	**1235 1246**	**134789t$_0$**	$(0^2, 3, 0^5, 4)$
7	3	**1236 12457**	**13489t$_0$**	$(0^2, 1, 0, 2, 0^2, 2, 0, 2)$

w_i and s_i denote the independent WP and SP type defining words, respectively. Each design is MA, W-MA, and S-MA design and $\mathbf{t_0}$ denotes the factor **10**.

Table 6: Catalogue of W-MA and S-MA designs with $k_1 = 2$ and $k_2 = 2$.

n_1	n_2	w_i	s_i	$W_w(d)$
5	3	**124 135**	**167 2368**	(2, 1, 1, 6, 0, 4)
5	4	**124 135**	**168 23679**	(2, 1, 1, 2, 0, 4, 4, 1)
6	3	**1235 1246**	**1278 13479**	$(0^2, 3, 3, 0, 8)$
5	5	**124 135**	**23679 12368t$_0$**	$(2, 0, 1, 0^2, 5, 6, 0^2, 1)$
6	4	**1235 1246**	**1379 1278t$_0$**	$(0^2, 3, 0^2, 7, 0, 4)$
7	3	**1236 12457**	**12489 1358t$_0$**	$(0^2, 1, 1, 2, 6, 0, 4)$

w_i and s_i denote the independent WP and SP type defining words, respectively. Each design is MA, W-MA, and S-MA design and $\mathbf{t_0}$ denotes the factor **10**.

Example 2. Consider the W-MA and S-MA $2^{(5+2)-(1+1)}$ designs in Table 2. Here, $n = 7$, $q_1 = 4$, and $q_2 = 1$. For the W-MA design d_1, we have the WP type defining word $w_1(d_1) = \mathbf{12345}$ and the SP type defining word $s_1(d_1) = 1267$, which means that $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ and $\mathbf{5}\,(= \mathbf{1234})$ are the WP factors and $\mathbf{6}, \mathbf{7} = (\mathbf{126})$ are the SP factors. The two words $w_1(d_1)$ and $s_1(d_1)$ generate another SP type defining word, say $s_2(d_1) = w_1(d_1)s_1(d_1) = \mathbf{34567}$. Thus, we obtain $A_{50}(d_1) = 1$, $A_{41}(d_1) = A_{51}(d_1) = 1$, and the wordlength pattern $W_w(d_1) = (0^3, 1, 1, 1)$, here we omit two 0's after the last 1. Similarly, the S-MA $2^{(5+2)-(1+1)}$ design d_2 has two independent defining words: $w_1(d_2) = \mathbf{1235}$ (WP type) and $s_1(d_2) = \mathbf{12467}$ (SP type). Thus, the WP factors for d_2 are $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ and $\mathbf{5}\,(= \mathbf{123})$, and SP factors are $\mathbf{6}, \mathbf{7} = (\mathbf{1246})$. The wordlength pattern is $W_w(d_2) = (0^2, 1, 0^2, 2)$. Both d_1 and d_2 are MA $2^{(5+2)-(1+1)}$ designs and they have wordlength pattern $W(d_1) = W(d_2) = (0, 1, 2)$. Obviously, d_1 and d_2 are nonisomorphic.

5. Conclusions

FFSP designs are useful in many scientific, agricultural, industrial, and engineering experiments. In an FFSP design, the factors whose levels are difficult to change are called whole plot (WP) factors, and the rest are called subplot (SP) factors. An FFSP design includes a two-phase randomization, which leads to two sources of errors in ANOVA. Therefore, the SP factors and the WP factors have different status.

MA criterion treats the WP and SP factors with the same importance, which is inconsistent with the physical truth. In addition, the MA FFSP designs are not always unique up to isomorphism. All the criteria for the five scenarios in [6], the WP-MA criterion proposed in [4], and the SP-MA criterion proposed in [14] violate the effect hierarchy principle somewhat.

According to the effect hierarchy principle and some actual circumstances and needs, this paper proposes the W-MA and S-MA criteria, which, respectively, correspond to the cases of WP factors and SP factors being more important. By decomposing the generating matrix of 2^{n-k} FF designs, we give some algorithms for splitting a 2^{n-k} FF design into a W-MA or S-MA $2^{(n_1+n_2)-(k_1+k_2)}$ design. By using these algorithms, we obtain some W-MA and S-MA $2^{(n_1+n_2)-(k_1+k_2)}$ designs for small k_1 and k_2.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant nos. 11801308 and 12171277).

References

[1] D. C. Montgomery, *Design and Analysis of Experiments*, Wiley, New York, NY, USA, Eight Edition, 2012.

[2] C. F. J. Wu and M. Hamada, *Experiments: Planning, Analysis, and Optimization*, Wiley, New York, NY, USA, Second Edition, 2009.

[3] R. Mukerjee and C. F. J. Wu, "A mordern theory of factorial designs," *Springer Series in Statistics*, Springer science+-Business Media Inc, New York, NY, USA, 2006.

[4] C. C. Wang, Q. Q. Zhao, and S. L. Zhao, "Optimal fractional factorial split-plot designs when the whole plot factors are important," *Journal of Statistical Planning and Inference*, vol. 199, pp. 1–13, 2019.

[5] G. E. P. Box and S. Jones, "Split-plot designs for robust product experimentation," *Journal of Applied Statistics*, vol. 19, no. 1, pp. 3–26, 1992.

[6] J. G. Tichon, W. Li, and R. G. Mcleod, "Generalized minimum aberration two-level split-plot designs," *Journal of Statistical Planning and Inference*, vol. 142, no. 6, pp. 1407–1414, 2012.

[7] A. Fries and W. G. Hunter, "Minimum aberration 2 $^{k-p}$ designs," *Technometrics*, vol. 22, no. 4, pp. 601–608, 1980.

[8] D. Bingham and R. R. Sitter, "Minimum-aberration two-level fractional factorial split-plot designs," *Technometrics*, vol. 41, no. 1, pp. 62–70, 1999a.

[9] D. R. BinghamBingham and R. R. Sitter, "Some theoretical results for fractional factorial split-plot designs," *Annals of Statistics*, vol. 27, no. 4, pp. 1240–1255, 1999b.

[10] D. R. BinghamBingham and R. R. Sitter, "Design issues in fractional factorial split-plot experiments," *Journal of Quality Technology*, vol. 33, no. 1, pp. 2–15, 2001.

[11] R. Mukerjee and K. T. Fang, “Fractional factorial split-plot designs with minimum aberration and maximum estimation capacity,” *Statistica Sinica*, vol. 12, pp. 885–903, 2002.

[12] M. Y. Ai and R. C. Zhang, “Minimum secondary aberration fractional factorial split-plot designs in terms of consulting designs,” *Science in China, Series A*, vol. 49, no. 4, pp. 494–512, 2006.

[13] Q. Q. Zhao and S. L. Zhao, “Constructing minimum aberration split-plot designs via complementary sets when the whole plot factors are important,” *Journal of Statistical Planning and Inference*, vol. 209, pp. 123–143, 2020.

[14] H. P. Dang, S. L. Zhao, and Q. Q. Zhao, “Constructing minimum aberration split-plot designs via complementary designs when the subplot factors are more important,” *Communications in Statistics - Theory and Methods*, pp. 1–19, 2022.

[15] M. Y. Hu and S. L. Zhao, “Minimum aberration split-plot designs focusing on the whole plot factors,” *Mathematics*, vol. 10, no. 5, p. 700, 2022, https://doi.org/10.3390/math10050700.

[16] G. E. P. Box and J. S. Hunter, “The 2^{k-p} fractional factorial designs Part I,” *Technometrics*, vol. 3, no. 3, pp. 311–352, 1961.