Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 9514053, 7 pages
https://doi.org/10.1155/2022/9514053

Research Article

On Reduce Differential Transformation Method for Solving Damped Kawahara Equation

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Received 12 May 2022; Accepted 11 July 2022; Published 3 August 2022

Academic Editor: A. K. Alomari

The damped Kawahara equation (KE) is nonintegrable equation and does not have analytical integration. In this work, the powerful numerical method, which is the reduce differential transformation method (RDTM), is devoted to solve the damped KE. The accuracy of the method is proved. The results are compared with the different numerical methods. The numerical solution is axi-symmetric wave and shows the effect of damping term successfully. We confirmed that the RDTM is useful for solving nonintegrable equations.

1. Introduction

The partial deferential equations (PDEs) describe several important applications in many branches of science such as physics, engineering, medicine, and fluid dynamic [1–4]. Mathematicians put forth high efforts to develop methods that are able to find solutions of these PDEs [5–8]. Usually, as the PDEs describe a problem very well with taking all issues in account, there are some terms appear and make the PDEs are not solvable. Therefore, the mathematicians improved the computational methods to find different types of solutions such as exact, approximate, equivalent, numerical, and analytical.

One of the well-known PDEs is Korteweg-de Vries (KdV) equation and its family. The fifth order of KDV is also known as Kawahara equation (KE). T. Ono and K. Ono [9] were the first to discover this type of equation during the study of magneto-acoustic waves in a cool collision-free plasma. Kawahara numerically investigated this type of equation and discovered that it has both oscillatory and monotone solitary wave solutions [10]. In a fluid medium like shallow water, the equation describes the propagation of soliton waves. The KE is governed by the following equation [11]:

$$\partial_t a + \alpha a \partial_x a + \beta \partial_{3x} a - \gamma \partial_{5x} a = 0, \tag{1}$$

where α, β, and γ are constants. The KE has been solved analytically and numerically in many researches [12–14]. The obtained solutions are N-soliton solutions [15], various solitons solutions [16], soliton and breathers [17], and different types of N-soliton and lump solutions [18]. The numerical solutions are obtained by using modified variational iteration algorithm-I and II [19, 20], differential quadrature [21], hybridizable discontinuous Galerkin (HDG) [22], and others. However, if a collisional effect is taken into account in applications of KE equation, we obtain the damping term, and KE becomes damped KE with the following form:

$$\partial_t a + \alpha a \partial_x a + \beta \partial_{3x} a - \gamma \partial_{5x} a + Ca = 0, \tag{2}$$

where $C = m/2$ and m is the frequency of the ion-neutral collision. The damping term makes the (2) nonintegrable equation. In order to obtain the solutions, we aim to use a new improved technique.

The differential transformation method (DTM) is based on Taylor series expansion but differs from the typical high-order Taylor series method, which takes a long time to calculate [23]. The DTM is one of the most powerful numerical methods. Pukhov was the first who used the DTM to tackle linear and nonlinear initial value problems in electric circuit analysis [24]. Chen and Ho developed the DTM for solving PDEs and found closed form series solutions for a variety of linear and nonlinear initial value problems [25]. Abdel-Halim Hassan demonstrated that the DTM can be used on a wide range of PDES and easily obtain closed form solutions [26–28].

If the series of the solution has a closed form, then the numerical solution can be convergent to the exact solution, but this is not usually the case, especially in most realistic cases. Thus, the obtained solution is in series form. Since it is based on Taylor series, which is the local convergent [29], the DTM finds the solutions in small domain and about the initial point. It has been improved recently to reduce differential transformation method (RDTM) [30]. Keskin was the first who proposed the RDTM for finding exact solutions to PDEs [31, 32]. Keskin and Oturanc created RDTM in recent years, in which the differential transformation is applied solely to one domain (time domain) [31]. The RDTM is a very effective and powerful tool for solving exact or approximate mathematical modeling solutions for a wide range of problems in technology, economics, engineering disciplines, and natural sciences such as biology, physics, chemistry, and earth science. It can solve both linear and nonlinear problems and provides results in the form of quick convergent successive approximations. The solutions by RDTM can also be classified as semiapproximate solution since the method applies the iteration only for the time domain. This technique is powerful compared to DTM and other methods.

The novelty of this paper is proving that the RDTM is able to solve the class of nonintegrable equations, which does not have exact solutions. Such equations appear usually in physics applications when viscosity and ion-collisions are taken into account. We chose damped KE as an example of nonintegrable equations and devoted the RDTM to investigate the solution in long domain.

The following is how the article is structured: Section 2 describes the used methods briefly, Section 3 presents the numerical solutions for KE and damped KE by RDTM, and Section 4 includes the conclusion of the work.

2. The Methodology

The DTM and its improved version (RDTM) are based on the following list of definitions.

Definition 1 (differential transformation in two dimensions). The basic concept of the two-dimensional differential transform is as follows: let $y(x,t)$ be analytic and continuously differentiable with respect to t and x,

$$Y(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} y(x,t)\right]_{x=x_0,t=t_0}. \tag{3}$$

The converted function is $Y(k,h)$, where $Y(k,h)$ is the spectrum function [33]. The original function (lower case) $y(x,t)$ is represented in this paper, whereas the converted function (upper case) $Y(k,h)$ is represented. Using the two-dimensional differential transformation (3), we present the differential transformation for several operators in Table 1.

Definition 2 (inverse differential transformation in two dimensions). The inverse differential transform of $Y(k,h)$ is defined as follows [33]:

$$y(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} Y(k,h)(x-x_0)^k (t-t_0)^h. \tag{4}$$

Taking (3) and (4) together and assuming $x_0 = t_0 = 0$ yields to

$$y(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} y(x,t)\right]_{x=0,t=0} x^k t^h = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} Y(k,h) x^k t^h. \tag{5}$$

Definition 3 (reduce differential transformation and its inverse in two dimensions). If $a(x,t)$ is analytical function in the domain of interest, then the spectrum function is used

$$A_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} a(x,t)\right]_{t=t_0}, \tag{6}$$

where $a(x,t)$ is reduced transformed function. Lowercase $a(x,t)$ refers to the original function, whereas uppercase $A_k(x)$ refers to the reduced transformed function. The differential inverse transformation of $A_k(x)$ is defined as [30]

$$a(x,t) = \sum_{k=0}^{\infty} A_k(x)(t-t_0)^k. \tag{7}$$

Combining (6) and (7) gives

$$a(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} a(x,t)\right]_{t=t_0} (t-t_0)^k. \tag{8}$$

Table 2 shows the list of reduce differential transformation for several operators.

3. Numerical Simulation

3.1. Kawahara Equation. The first application is applying the DTM and RDTM into KE (1) in order to prove the accuracy of RDTM. In addition, we aim to prove the power of RDTM comparing to other methods in literature. Let's consider KE (1) with $\alpha = \beta = \gamma = 1$ and subjects to the initial condition [35].

$$a(x,0) = \frac{-72}{169} + \frac{105}{169}\operatorname{sech}^4(gx). \tag{9}$$

TABLE 1: The fundamental operations by the two-dimensional differential transform method [33].

Original function	Transformed function
$y(x,t) = a(x,t) \pm b(x,t)$	$Y(k,h) = A(k,h) \pm B(k,h)$
$y(x,t) = ca(x,t)$	$Y(k,h) = cA(k,h)$
$y(x,t) = \partial/\partial x a(x,t)$	$Y(k,h) = (k+1)A(k+1,h)$
$y(x,t) = \partial/\partial x a(x,t)$	$Y(k,h) = (h+1)A(k,h+1)$
$y(x,t) = \partial r + s/\partial x^r \partial t^s a(x,t)$	$Y(k,h) = (k+r)!(h+s)!/k!h!A(k+r,h+s)$
$y(x,t) = a(x,t)b(x,t)$	$Y(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} A(r,h-s)B(k-r,s)$
$y(x,t) = x^m t^n$	$Y(k,h) = \delta(k-m,h-n) = \begin{cases} 1 & k=m, h=n \\ 0 & \text{Otherwise} \end{cases}$
$y(x,t) = \partial/\partial x a(x,t)\partial/\partial t b(x,t)$	$Y(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} (k-r+1)(h-s+1)A(k-r+1,s)B(r,h-s+1)$

TABLE 2: The fundamental operations of the two-dimensional RDTM [30, 34].

Original function	Reduced transformed function
$a(x,t) = a(x,t) \pm b(x,t)$	$A_k(x) = A_k(x) \pm B_k(x)$
$a(x,t) = ca(x,t)$	$A_k(x) = cA_k(x)$
$a(x,t) = \partial/\partial x a(x,t)$	$A_k(x) = \partial/\partial x A_k(x)$
$a(x,t) = \partial/\partial t a(x,t)$	$A_k(x) = (k+1)A_{k+1}(x)$
$a(x,t) = {}^{r+s}\partial/\partial x^r \partial t^s a(x,t)$	$A(k,h) = (k+s)!/k!\partial^r/\partial x^r A_{k+s}(x)$
$a(x,t) = a(x,t)b(x,t)$	$A_k(x) = \sum_{r=0}^{k} A_r(x)B_{k-r}(x)$
$a(x,t) = x^m t^n$	$A_k(x) = x^m\delta(k-n) = \begin{cases} x^m & k=n \\ 0 & \text{Otherwise} \end{cases}$

The exact solution of this equation is given by

$$a(x,t) = \frac{-72}{169} + \frac{105}{169}\operatorname{sech}^4(g(x+ft)). \quad (10)$$

where $g = 1/2\sqrt{13}$ and $f = 36/169$.

We get the following scheme by using DTM in Definition 1 for $k, h = 0, 1, 2, \ldots, N$, where N is the number of iterations:

$$\begin{aligned} A(k,h+1) = \frac{1}{h+1}\Bigg[&(k+1)(k+2)(k+3)(k+4)(k+5)A(k+5,h) - (k+1)(k+2) \\ &\cdot(k+3)A(k+3,h) - \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)A(r,h-s)A(k-r+1,s)\Bigg]. \end{aligned} \quad (11)$$

The initial condition is transformed into the following:

$$A(k,0) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}a(x,0)\right]_{x=x_0,t=t_0}. \quad (12)$$

The recursive equations deduced from (11) for different values of k, h are obtained as [36]

$$\begin{aligned} &k=0, h=0: \ A(0,1) = 120A(5,0) - 6A(3,0) - A(0,0)A(1,0), \\ &k=1, h=0: \ A(1,1) = 720A(6,0) - 24A(4,0) - 2A(0,0)A(2,0) - A^2(1,0), \\ &k=2, h=0: \ A(2,1) = 2520A(7,0) - 60A(5,0) - 3A(0,0)A(3,0) - 3A(1,0)A(2,0), \\ &\vdots \end{aligned} \quad (13)$$

We have noticed in Figure 1 that the numerical solution converges to exact solution in small interval about $(-4, 4)$ and diverges after that. Because of this disadvantage of DTM, the scheme is improved to RDTM as follows:

$$A_{k+1} = \frac{1}{k+1}\left[-\sum_{r=0}^{k} A_r \frac{\partial}{\partial x}A_{(k-r)} - \frac{\partial^3}{\partial x^3}A_k + \frac{\partial^5}{\partial x^5}A_k\right]. \quad (14)$$

The errors between the solution by RDTM and exact solution in defferent time are shown in Table 3. The solutions by DTM and RDTM are compared with the numerical solutions by optimal homotopy asymptotic method (OHAM) [35], homotopy perturbation and variational iteration method (VHPM) [37], homotopy perturbation method (HPM) [38], and Laplace homotopy perturbations method (LHPM) [39] in Table 4. The comparison reveals the accuracy of these methods. From

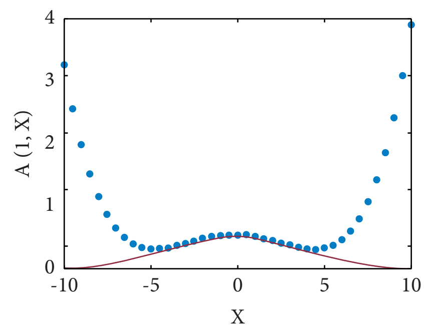


Figure 1: The comparisons of the solution by differential transform method with exact solution.

Table 3: Absolute error of the RDTM at time $t = 2, 4, 6, 8, 10$ and $1 \le x \le 10$.

x/t	2	4	6	8	10
1	8.1981×10^{-8}	5.0281×10^{-6}	5.7678×10^{-5}	3.3236×10^{-4}	1.3115×10^{-3}
2	7.1506×10^{-8}	4.4635×10^{-6}	5.2118×10^{-5}	3.0536×10^{-4}	1.2212×10^{-3}
3	7.7606×10^{-8}	4.9584×10^{-6}	5.8989×10^{-5}	3.5029×10^{-4}	1.4145×10^{-3}
4	8.6109×10^{-8}	5.3515×10^{-6}	6.1119×10^{-5}	3.4313×10^{-4}	1.2843×10^{-3}
5	7.6421×10^{-8}	4.5042×10^{-6}	4.8623×10^{-5}	2.5813×10^{-4}	9.2089×10^{-4}
6	5.9589×10^{-8}	3.4236×10^{-6}	3.6229×10^{-5}	1.8994×10^{-4}	6.7507×10^{-4}
7	4.6679×10^{-8}	2.6715×10^{-6}	2.8250×10^{-5}	1.4850×10^{-4}	5.3085×10^{-4}
8	3.8055×10^{-8}	2.1839×10^{-6}	2.3184×10^{-5}	1.2249×10^{-4}	4.4052×10^{-4}
9	3.2162×10^{-8}	1.8529×10^{-6}	1.9753×10^{-5}	1.0484×10^{-4}	3.7888×10^{-4}
10	2.7982×10^{-8}	1.6178×10^{-6}	1.7310×10^{-5}	9.2209×10^{-5}	3.3450×10^{-4}

Table 4: When the proposed method's finding are compared to the results in at time $t = 0.1$ and $0.1 \le x \le 0.5$.

x	OHAM	VHPM	HPM	LHPM	DTM	RDTM
0.1	1.58×10^{-6}	2.18×10^{-9}	5.00×10^{-5}	3.05×10^{-16}	1.08×10^{-5}	4.44×10^{-16}
0.2	2.10×10^{-6}	4.24×10^{-9}	1.89×10^{-4}	4.16×10^{-16}	1.06×10^{-5}	4.44×10^{-16}
0.3	2.62×10^{-6}	6.28×10^{-9}	2.18×10^{-4}	1.17×10^{-15}	1.04×10^{-5}	1.22×10^{-15}
0.4	3.13×10^{-6}	8.28×10^{-9}	9.01×10^{-5}	6.11×10^{-16}	1.01×10^{-5}	7.77×10^{-16}
0.5	3.63×10^{-6}	1.02×10^{-6}	1.31×10^{-3}	8.88×10^{-16}	9.75×10^{-6}	8.88×10^{-16}

Table 4, we realized that the accuracy of RDTM and LHAM is better than that of the other methods, but RDTM is faster than LHPM. The speed of RDTM is 5.65 seconds, while for LHPM is 15.97 seconds for 6 iterations. Therefore, RDTM is the optimal iteration method. Figure 2 shows the plot of the numerical solution of KE with IC [9].

The second example is, KE (1), where $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.4$ and subjects to the IC [13].

$$a(x,0) = Q\left(\operatorname{sech}^4\left(\left(\frac{x}{w}\right)\right)\right), \tag{15}$$

where $Q = 105\beta^2/169\alpha\gamma$ and $W = \sqrt{52\gamma/\beta}$, and the exact solution of this equation as follows [13]:

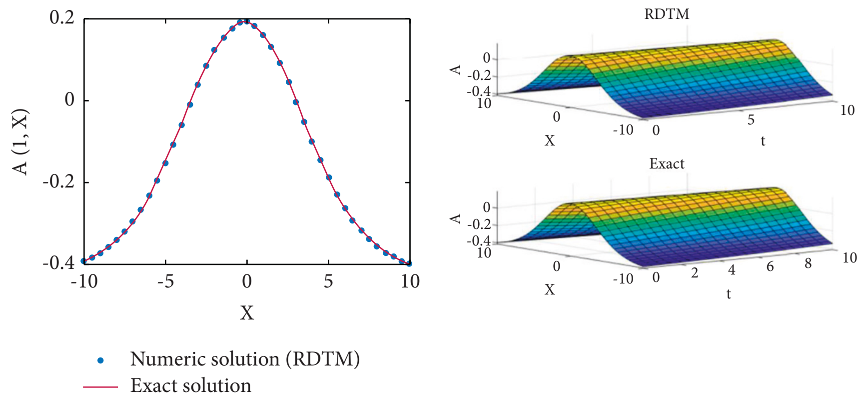
$$a(x,t) = Q\operatorname{sech}^4\left[\frac{1}{W}\left(x - \frac{36\beta^2}{169\gamma}t\right)\right]. \tag{16}$$

The numerical result is obtained by RDTM and proposed in Figure 3.

3.2. Damped Kawahara Equation. Because there is damping term in the Kawahara equation, the energy of the soliton is not conserved and decays with increasing both c and t, (2) is nonintegrable Hamiltonian system. We consider damped Kawahara (2) with $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.4$ and subject to the IC [13]. Since we do not have exact solution, we can use the initial condition of Kawahara equation as initial condition of the damped Kawahara [13]. The scheme of the damped KE by RDTM is as follows:

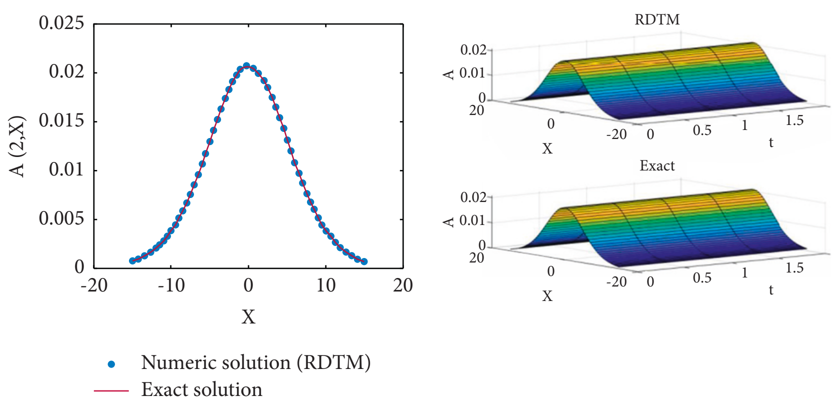
$$A_{k+1} = \frac{1}{k+1}\left[-\alpha\sum_{r=0}^{k} A_r \frac{\partial}{\partial x}A_{(k-r)} - \beta\frac{\partial^3}{\partial x^3}A_k + \gamma\frac{\partial^5}{\partial x^5}A_k - CA_k\right]. \tag{17}$$

The numerical solution is shown in Figure 4. The amplitude of the wave decrease as the damping parameter increases.



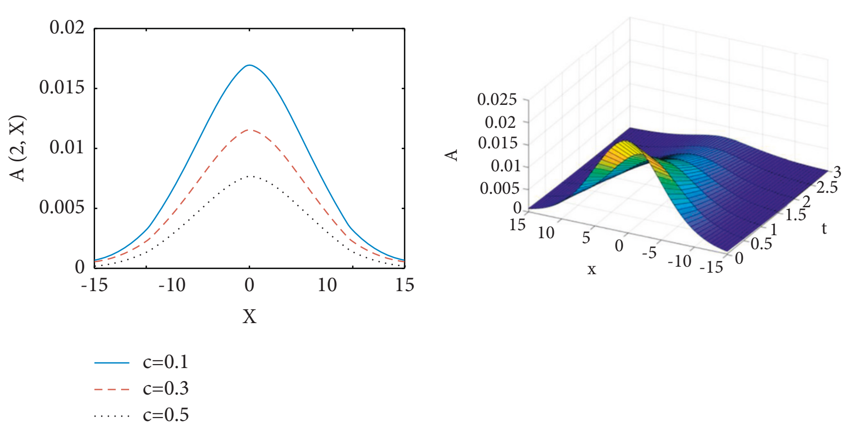
(a) (b)

Figure 2: Compression between the numerical solutions by RDTM and exact solution of Kawahara equation, where $\alpha = \beta = \gamma = 1$. (a) $t = 1$; $-10 \leq x \leq 10$. (b) $-10 \leq x \leq 10$ and $0 \leq t \leq 10$.



(a) (b)

Figure 3: Compression between the numerical solutions by RDTM and exact solution of Kawahara equation where $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.4$. (a) $-15 \leq x \leq 15$ and $t = 2$. (b) $-15 \leq x \leq 15$ and $0 \leq t \leq 2$.



(a) (b)

Figure 4: The plot of numerical solution of damped Kawahara equation via RDTM. (a) For $t = 2$ and $-15 \leq x \leq 15$. (b) For $c = 0.7$, $-15 \leq x \leq 15$ and $0 \leq t \leq 3$.

4. Discussion and Conclusion

This paper studies the KdV-fifth order (Kawahara equation) within two cases: integrable KE and nonintegrable KE. The integrable KE has been solved in literature via different methods such as OHAM, VHPM, HPM, and LHAM. In this article, it is solved by DTM and RDTM to prove that RDTM converges to the solution faster than other methods with high accuracy. The new contribution in this work is solving nonintegrable KE, which includes damping term by RDTM. The two-dimensional DTM obtains the solutions in series form, but it is different from the traditional high-order Taylors series method, because it does not need symbolic computation of derivative for each term. Also, it does not require linearization, discretization, or other complected computation process. Therefore, the DTM is faster than the Taylors series method. The DTM has been developed for solving ordinary and partial differential either linear or nonlinear equations. The improved version of the DTM is theRDTM, which is powerful to find numerical solutions for integrable equations as well as nonintegrable equations in several branches of science. MATLAB has been used for computations in this article. In future work, the RDTM can be applied to solve different new systems in physics and engineering that generate nonintegrable equations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

The Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU), Jeddah, Saudi Arabia has founded this project, under grant no. (KEP-MSc: 35-665-1443). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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