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Research Article

An Extension of Bonferroni Mean under Cubic Pythagorean Fuzzy Environment and Its Applications in Selection-Based Problems

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Cubic Pythagorean fuzzy (CPF) set (CPFS) is a hybrid set that can describe both interval-valued Pythagorean fuzzy (IVPF) sets (IVPFSs) and Pythagorean fuzzy (PF) sets (PFSs) simultaneously for addressing information ambiguities. Since an aggregation operator (AO) is a significant mathematical approach in decision-making (DM) problems, this article presents some novel Bonferroni mean (BM) and weighted Bonferroni mean averaging operators between CPF-numbers (CPFNs) for aggregating the different preferences of the decision-makers. Then, using the proposed AOs, we develop a DM approach under the CPF environment and demonstrated it with a numerical example. Furthermore, a comparative study between the proposed and existing methods has been performed to demonstrate the practicality and efficiency of the proposed DM approach.

1. Introduction

Decision-making is a significant process in order to pick the best-suited alternative from among those available. In it, a number of scholars provided a variety of theories to make the best judgments. In the past, decisions were made based on crisp numbered data sets, but this resulted in insufficient outcomes that were less applicable to real-life operational scenarios. However, as time passes and the complications of the system increase, it becomes more difficult for the decision-maker to handle the inconsistencies in the data, and thus the traditional technique is unable to find the optimum alternative. Therefore, the scholars used fuzzy set (FS) theory [1], interval-valued fuzzy sets (IVFSs) [2], intuitionistic fuzzy sets (IFSs) [3], interval-valued intuitionistic fuzzy sets (IVIFSs) [4], PFS [5, 6], and IVPFS [7] to describe the information. Scholars have paid increasing attention to these ideas in recent decades and have efficiently implemented them in a variety of scenarios in the DM process. An aggregation operator, which generally takes the form of a mathematical formalism to accumulate all of the individual input data into a single one, is an important part of the DM process. For example, Xu and Yager [8] introduced certain geometric AOs to integrate various preferences of the decision-makers into intuitionistic fuzzy numbers (IFNs). Later, Wang and Liu [9] utilized Einstein norm operations to generalize these operators. For aggregating different intuitionistic fuzzy information, Garg [10] introduced generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein norm operations. Garg has presented a series of interactive AOs for IFNs in [11]. Garg [12] implemented the IFS concept to PFS and proposed generalized averaging AOs. The symmetric Pythagorean fuzzy AOs were proposed by Ma and Xu [13]. To solve DM problems, Garg [14] presented some improved interactive aggregation operators, while Wang and Liu [15] presented some hybrid weighted aggregation operators using Einstein norm operators. Apart from that, several other authors have given other approaches to solve DM problems, such as ranking functions [16] and AOs (see [17]).

As the above, AOs have been considered by many researchers during the DM process in which they have highlighted the contribution of each factor or its ordered position but cannot represent the interrelationships of the individual information. In our real-life situation, a relationship of different criteria such as importance, support, and impact on each other constantly plays a significant role throughout the aggregation process. To deal with it, Yager [18] developed the power average (PA) AOs to address this issue and implement it into DM analysis. Xu and Yager [19] and Yu [20] proposed the prioritized averaging and geometric AOs in an IFS environment. Further, Yager [21] presented the idea of the BM aggregation operators, which has the potential to represent the interrelationship between the input arguments. To alleviate the limitation of BM, Beliakov and James [22] introduced the generalized BM. To aggregate the intuitionistic fuzzy information, Xu and Yager [23] developed an intuitionistic fuzzy BM. These BM operators were generalized to the interval-valued IFSs environment by Xu and Chen [24]. The generalized intuitionistic fuzzy BMs were presented by Xia et al. [25]. The partitioned BM operators were described by Liu et al. [26] in an IFSs environment. Shi and He [27] discussed how to optimize BMs by applying them to different DM processes. In an intuitionistic fuzzy soft set environment, Garg and Arora [28] proposed the BM aggregation operator. The Pythagorean fuzzy Bonferroni mean (PFBM) is developed by Liang et al. [29], and several specific properties and cases are described. Wang and Li proposed a Pythagorean fuzzy interaction PFBM and weighted PFBM operators [30]. Nie et al. [31] proposed a PF partitioned normalized weighted BM operator with Shapley fuzzy measure.

All of the existing research and their respective applications are mostly focused on the FS, interval-valued FS (IVFS), IFS, PFS, IVIFS, and IVPFS. Then, Jun et al. [32] developed several logic operations of the cubic sets and familiarized the theory of cubic set (CS) and their operational laws such as P-union, P-intersection, R-union, and R-intersection of CS and investigated several related properties. CS has been employed in a variety of real-world applications. They use highly interconnected distinctive to solve complex issues in engineering, economics, and the environment. Because of the many uncertainty models for such situations, it is not always simple to apply standard approaches to obtain good results. Therefore, Khalil and Hassan [33] introduced the class of cubic soft algebras and their basic characteristics. Shi and Ye [34] proposed Dombi Aggregation Operators of Neutrosophic Cubic Sets for Multiple Attribute Decision-Making. Ye et al. [35] presented Multi-fuzzy Cubic Sets and Their Correlation Coefficients for multi-criteria Group Decision-Making. Garg et al. [36] proposed Correlation Measures for Cubic m-Polar Fuzzy Sets with Applications. Khan et al. [37, 38] presented some cubic AOs under this set, while Mahmood et al. [39] presented the concepts of cubic hesitant fuzzy sets and their AOs. The above concepts only provide information in the form of membership intervals and ignore the nonmembership section of the data entities, which also perform an important role in evaluating the alternative in the DM process. In the real world, expressing the value of a membership degree by an exact value in a fuzzy set is typically challenging. In such situations, an interval value and an exact value, rather than unique interval/exact values, may be convenient to communicate uncertainty and ambiguity in the real world. Hence, the hybrid form of an interval and an exact number may be a very useful term for a person to express certainty and uncertainty as a result of his or her uncertain assessment in multifaceted DM problems. Kaur and Garg [40, 41] introduced the concept of the cubic intuitionistic fuzzy set CIFS, which is characterized by two portions at the same time, one of which reflects membership degrees by an IVIF value and the other by an intuitionistic fuzzy value. Each CIFS component is represented as $([\xi^-, \xi^+], [\nu^-, \nu^+], \xi, \nu)$ that satisfies the conditions $\xi^+ + \nu^+ \leq 1$ and $\xi + \upsilon \leq\ 1$. However, in some actual cases, the sum of membership and non-membership grades may be greater than 1, but their square sum is less than or equal to 1. Therefore, Abbas et al. [42] presented the concept of the CPFS, which is the generalization of CIFS. At the same time, CPFS has two phases, one of which reflects the degree of membership by an IVPFS and the other by a PFS that satisfies the condition $(\xi^+)^2 + (\upsilon^+)^2 \leq 1$ and $\xi^2 + \upsilon^2 \leq 1$. Therefore, a CPFS is a hybrid set that includes both an IVPFS and a PFS. Obviously, the CIFS has the advantage of being able to contain a lot more information in order to express both the IVPFN and the PFN at the same time. Hence, CPFS has a high level of efficiency and importance when used to evaluate alternatives during the DM process because the general DM process may use IVPFS or PFS data, which may loss of some important evaluation information. There is currently no research on AOs that reflects the interrelationship between the multiple criteria of a DM process having CPF information.

According to the current communication, inspired by the BM notion and by utilizing the CPFS advantages, we suggest some new AOs called the CPF Bonferroni mean (CPFBM) and weighted cubic Pythagorean fuzzy Bonferroni mean (WCPFBM) to aggregate the preferences of decision-makers. We have also investigated various desirable properties of these operators in detail. The main advantage of the proposed AOs is that the interrelationships between aggregated values are taken into account. We also examine the characteristics of the proposed work and design some specific scenarios. The proposed AOs have been deduced from several previous studies, demonstrating that the proposed AOs are more flexible than the others. Finally, a DM method for rating the various alternatives based on the proposed AOs has been presented. Finally, a DM method for rating the various alternatives based on the proposed AOs has been presented. The list of abbreviations used in this article is provided in Table 1.

The rest of the article is organized as follows. The basic concepts are briefly discussed in Section 2. Section 3 presents AOs called CPF Bonferroni mean (CPFBM) and weighted CPF Bonferroni mean (WCPFBM), as well as their applications. To address multi-criteria DM (MCDM) problems, a DM approach based on proposed operators has been developed in Section 4. In Section 5, a numerical model is provided to illustrate the proposed approach and demonstrate its practicality and applicability. Section 6 presents the conclusion including closing remarks.

Table 1: List of abbreviations.

Symbols	Abbreviations
CPF	Cubic Pythagorean fuzzy
CPFS	Cubic Pythagorean fuzzy set
PF	Pythagorean fuzzy
IVPFS	Interval-valued Pythagorean fuzzy set
PFS	Pythagorean fuzzy set
AOs	Aggregation operators
DM	Decision-making
CPFNs	Cubic Pythagorean fuzzy numbers
FS	Fuzzy set
IVFSs	Interval-valued fuzzy sets
IFSs	Intuitionistic fuzzy sets
IVIFSs	Interval-valued intuitionistic fuzzy sets
IFNs	Intuitionistic fuzzy numbers
PA	Power average
BM	Bonferroni mean
PFBM	Pythagorean fuzzy Bonferroni mean
CS	Cubic set
CIFS	Cubic intuitionistic fuzzy set
CPFBM	Cubic Pythagorean fuzzy Bonferroni mean
WCPFBM	Weighted cubic Pythagorean fuzzy Bonferroni mean

2. Preliminaries

Some main theories associated with PFSs, IVPFSs, CSs, CIFSs and CPFSs are briefly addressed in this section.

2.1. Pythagorean Fuzzy Set

Definition 1 (see [5, 6]). Let S be a universal set. A PFS p over S is defined as

$$P = \{x, \xi_P(x), \upsilon_P(x) | x \varepsilon S\}, \tag{1}$$

where $0 \leq \xi_P(x), \upsilon_P(x) \leq 1$ and $(\xi_P(x)^2 + (\upsilon_P(x)^2 \leq 1$. This couple is denoted as $P = (\xi_P, \upsilon_P)$ and is referred to as a PF number (PFN).

Definition 2 (see [43]). Let $\alpha_1 = (\xi_{P_1}, \upsilon_{P_1})$, $\alpha_2 = (\xi_{P_2}, \upsilon_{P_2})$ and $\alpha = (\xi_P, \upsilon_P)$ be three PFNs and $\varphi \geq 0$ then

(i) $\alpha_1 \oplus \alpha_2 = (\sqrt{\xi_{P_1}^2 + \xi_{P_2}^2 - \xi_{P_1}^2 \xi_{P_2}^2}, \upsilon_{P_1}\upsilon_{P_2})$;

(ii) $\alpha_1 \otimes \alpha_2 = (\xi_{P_1}\xi_{P_2}, \sqrt{\upsilon_{P_1}^2 + \upsilon_{P_2}^2 - \upsilon_{P_1}^2 \upsilon_{P_2}^2})$;

(iii) $\varphi\alpha = (\sqrt{1 - (1 - \xi^2)^{\varphi}}, \upsilon^{\varphi})$;

(iv) $\alpha^{\varphi} = (\xi^{\varphi}, \sqrt{1 - (1 - \upsilon^2)^{\varphi}})$.

2.2. Interval-Valued Pythagorean Fuzzy Set

Definition 3 (see [7]). Let S be a fixed set. An IVPFS $\mathcal{Q}$ over S is defined as

$$\mathcal{Q} = \{x, \xi_{\mathcal{Q}}(x), \upsilon_{\mathcal{Q}}(x) | x \varepsilon S\}, \tag{2}$$

where $\xi_Q(x) = [\xi_Q^-(x), \xi_Q^+(x)]$, $\upsilon_Q(x) = [\upsilon_Q^-(x), \upsilon_Q^+(x)]$ such that $(\xi_{\mathcal{Q}}^+)^2 + (\upsilon_{\mathcal{Q}}^+)^2 \leq 1$. For the sake of simplicity, we denote this pair as $\beta = ([\xi_Q^-, \xi_Q^+], [\upsilon_Q^-, \upsilon_Q^+])$ and called as IVPF number (IVPFN).

Definition 4 (see [7]). Let $\beta_1 = ([\xi_1^-, \xi_1^+], [\upsilon_1^-, \upsilon_1^+])$, $\beta_2 = ([\xi_2^-, \xi_2^+], [\upsilon_2^-, \upsilon_2^+])$ and $\beta = ([\xi^-, \xi^+], [\upsilon^-, \upsilon^+])$ be three IVPFNs, then

(i) $\beta_1 \oplus \beta_2 = \left(\begin{array}{c} [\sqrt{(\xi_1^-)^2 + (\xi_2^-)^2 - (\xi_1^-)^2(\xi_2^-)^2}, \sqrt{(\xi_1^+)^2 + (\xi_2^+)^2 - (\xi_1^+)^2(\xi_2^+)^2}], \\ [\upsilon_1^-\upsilon_2^-, \upsilon_1^+\upsilon_2^+] \end{array}\right)$;

(ii) $\beta_1 \otimes \beta_2 = \left(\begin{array}{c} [\xi_1^-\xi_2^-, \xi_1^+\xi_2^+], \\ [\sqrt{(\upsilon_1^-)^2 + (\upsilon_2^-)^2 - (\upsilon_1^-)^2(\upsilon_2^-)^2}, \sqrt{(\upsilon_1^+)^2 + (\upsilon_2^+)^2 - (\upsilon_1^+)^2(\upsilon_2^+)^2}] \end{array}\right)$;

(iii) $\varphi\beta = ([\sqrt{1 - (1 - (\xi^-)^2)^{\varphi}}, \sqrt{1 - (1 - (\upsilon^+)^2)^{\varphi}}], [(\upsilon^-)^{\varphi}, (\upsilon^+)^{\varphi}])$;

(iv) $\beta^{\varphi} = ([(\xi^-)^{\varphi}, (\xi^+)^{\varphi}], [\sqrt{1 - (1 - (\upsilon^-)^2)^{\varphi}}, \sqrt{1 - (1 - (\upsilon^+)^2)\ \varphi}])$.

2.3. Cubic Set

Definition 5 (see [33]). Let S be a universal set. A CS over S is defined as

$$\mathscr{R} = \{x, \mathscr{R}_A(x), \lambda_A | x \varepsilon S\} \tag{3}$$

where $\mathscr{R}_A(x) = [\mathscr{R}^-(x), \mathscr{R}^+(x)]$ is IVF set in X and ϑ is a FS. A CS $\mathscr{R}$ is said to be an internal cubic set if $\mathscr{R}^-(x) \preccurlyeq (x) \preccurlyeq \mathscr{R}^+(x)$ and a CS $\mathscr{R}$ an external cubic set if $\vartheta(x) \notin (\mathscr{R}^-(x), \mathscr{R}^+(x))$. A CS $\mathscr{R} = \{x, \mathscr{R}_A(x), \lambda_A(x) | x \in X\}$ is simply denoted by $\eta = \mathscr{R}, \lambda$.

Definition 6 (see [32]). Let $\eta_1 = \mathscr{R}_1, \lambda_1$ and $\eta_2 = \mathscr{R}_2, \lambda_2$ be CSs in X. Then,

(i) *Equality*. If $\mathscr{R}_1 = \mathscr{R}_2$ and $\lambda_1 = \lambda_2$ then $\eta_1 = \eta_2$;

(ii) *P-Order*. If $\mathscr{R}_1 \subseteq \mathscr{R}_2$ and $\lambda_1 \preccurlyeq \lambda_2$ then $\eta_1 \subseteq_P \eta_2$;

(iii) *R-Order*. If $\mathscr{R}_1 \subseteq \mathscr{R}_2$ and $\lambda_1 \succcurlyeq \lambda_2$ then $\eta_1 \subseteq_R \eta_2$.

Definition 7 (see [32]). Let $\mathscr{R}_i = \left\{x, \mathscr{R}_{A_i}(x), \lambda_{A_i}(x) | x \in S\right\}$ be a collection CSs where $i \in \Delta$, then

(i) *P-Union*.
$\Delta_{i \in \Delta}^{P} \mathscr{R}_i = \left\{x, \underset{i \in \Delta}{\Delta} \mathscr{R}_{A_i}(x), \underset{i \in \Delta}{\Delta} \lambda_{A_i}(x) | x \in S\right\}$;

(ii) *P-Intersection*.
$\Delta_{i \in \Delta}^{P} \mathscr{R}_i = \left\{x, \underset{i \in \Delta}{\Delta} \mathscr{R}_{A_i}(x), \underset{i \in \Delta}{\Delta} \lambda_{A_i}(x) | x \in S\right\}$;

(iii) *R-Union*.
$\Delta_{i \in \Delta}^{P} \mathscr{R}_i = \left\{x, \underset{i \in \Delta}{\Delta} \mathscr{R}_{A_i}(x), \underset{i \in \Delta}{\Delta} \lambda_{A_i}(x) | x \in S\right\}$;

(iv) *R-Intersection*.
$\Delta_{i \in \Delta}^{r} R_i = \left\{x, \underset{i \in \Delta}{\Delta} R_{A_i}(x), \underset{i \in \Delta}{\Delta} \lambda_{A_i}(x) | x \in S\right\}$.

2.4. Cubic Intuitionistic Fuzzy Set

Definition 8 (see [40, 41]). Let X be a non-empty set. A CIFS $\ddot{C}$ over $x \in S$ is defined as follows:

$$B = \{x, B(x), \lambda(x) | x \in S\}, \tag{4}$$

where $B(x) = \{x, [\xi_B^-(x), \xi_B^+(x)], [\nu_B^-(x), \nu_B^+(x)] | x \in S\}$ is IVIFS while $\lambda(x) = \{x, \xi_\lambda(x), \nu_\lambda(x) | x \in X\}$ represents IFS such that $0^\circ\xi_B^-(x)^\circ\xi_B^+(x)^\circ 1$, $0\preccurlyeq\nu_B^-(x)\preccurlyeq\nu_B^+(x)\preccurlyeq 1$ and $0\prec\nu_B^+(x)+\nu_B^+(x)\prec 1$. Also, $0\prec\xi_\lambda(x),\nu_\lambda(x)\prec 1$ and $0\preccurlyeq\xi_\lambda(x)+\nu_\lambda(x)\preccurlyeq 1$. In order to keep it simple, the pair $\mathscr{B}=B,\lambda$, where $[\xi_B^-,\xi_B^+]$, $[\nu_B^-,\nu_B^+]$ and ξ_λ,ν_λ and called as CIF number (CIFN).

Definition 9 (see [41]). For a family of CIFS $\{\mathscr{B}_i, i\in\Delta\}$, then

(i) *P-Union.* $U_{i\in\Delta}^P\mathscr{B}_i=\begin{pmatrix}[\max_{i\in\Delta}(\xi_i^-),\max_{i\in\Delta}(\xi_i^+)],[\min_{i\in\Delta}(\nu_i^-),\min_{i\in\Delta}(\nu_i^+)],\\ \max_{i\in\Delta}\xi_i,\min_{i\in\Delta}\nu_i\end{pmatrix}$.

(ii) *P-Intersection.* $U_{i\in\Delta}^R\mathscr{B}_i=\begin{pmatrix}[\min_{i\in\Delta}(\xi_i^-),\min_{i\in\Delta}(\xi_i^+)],[\max_{i\in\Delta}(\nu_i^-),\max_{i\in\Delta}(\nu_i^+)],\\ \min_{i\in\Delta}\xi_i,\max_{i\in\Delta}\nu_i\end{pmatrix}$.

(iii) *R-Union.* $U_{i\in\Delta}^R\mathscr{B}_i=\begin{pmatrix}[\max_{i\in\Delta}(\xi_i^-),\max_{i\in\Delta}(\xi_i^+)],[\min_{i\in\Delta}(\nu_i^-),\min_{i\in\Delta}(\nu_i^+)],\\ \min_{i\in\Delta}\xi_i,\max_{i\in\Delta}\nu_i\end{pmatrix}$.

(iv) *R-Intersection.* $U_{i\in\Delta}^P\mathscr{B}_i=\begin{pmatrix}[\max_{i\in\Delta}(\xi_i^-),\max_{i\in\Delta}(\xi_i^+)],[\min_{i\in\Delta}(\nu_i^-),\min_{i\in\Delta}(\nu_i^+)],\\ \min_{i\in\Delta}\xi_i,\max_{i\in\Delta}\nu_i\end{pmatrix}$.

Definition 10 (see [40]). Let $\mathscr{B}_1=([\xi_1^-,\xi_1^+],[\nu_1^-,\nu_1^+],\xi_1,\nu_1)$ and $\mathscr{B}_2=([\xi_2^-,\xi_2^+],[\nu_2^-,\nu_2^+],\xi_2,\nu_2)$ be two CIFSs in S. Then,

(i) *Equality.* $\mathscr{B}_1=\mathscr{B}_2$, if and only if $[\xi_1^-,\xi_1^+]=[\xi_2^-,\xi_2^+]$, $[\nu_1^-,\nu_1^+]=[\nu_2^-,\nu_2^+]$, $\xi_1=\xi_2$ and $\nu_1=\nu_2$.

(ii) *P-Order.* $B_1\subseteq_P B_2$ if $[\xi_1^-,\xi_1^+]\subseteq[\xi_2^-,\xi_2^+]$, $[\nu_1^-,\nu_1^+]\supseteq[\nu_2^-,\nu_2^+]$, $\xi_1\preccurlyeq\xi_2$ and $\nu_1\succcurlyeq\nu_2$.

(iii) *R-Order.* $\mathscr{B}_1\subseteq_R\mathscr{B}_2$ if $[\xi_1^-,\xi_1^+]\subseteq[\xi_2^-,\xi_2^+]$, $[\nu_1^-,\nu_1^+]\supseteq[\nu_2^-,\nu_2^+]$, $\xi_1\succcurlyeq\xi_2$ and $\nu_1\preccurlyeq\nu_2$.

Definition 11 (see [40]). Let $\mathscr{B}=([\xi^-,\xi^+],[\nu^-,\nu^+],\xi,\nu)$, $\mathscr{B}_1=([\xi_i^-,\xi_i^+],[\nu_i^-,\nu_i^+],\xi_i,\nu_i)i=1,2$ be the collections of CIFNs, and $\psi{>}0$ be a real number, then

$$\begin{aligned}
\mathscr{B}_1\oplus\mathscr{B}_2&=\begin{pmatrix}\left[1-\prod_{i=1}^{2}(1-\xi_i^-),1-\prod_{i=1}^{2}(1-\xi_i^+)\right],\left[\prod_{i=1}^{2}\nu_i^-,\prod_{i}^{2}\nu_i^+\right],\\ \prod_{i=1}^{2}\xi_i,1-\prod_{i=1}^{2}(1-v_i),\end{pmatrix},\\
\mathscr{B}_1\otimes\mathscr{B}_2&=\begin{pmatrix}\left[\prod_{i=1}^{2}\xi_i^-,\prod_{i}^{2}\xi_i^+\right],\left[1-\prod_{i=1}^{2}(1-\nu_i^-),1-\prod_{i=1}^{2}(1-\nu_i^+)\right],\\ 1-\prod_{i=1}^{2}(1-\xi_i),\prod_{i=1}^{2}v_i\end{pmatrix},\\
\varphi\mathscr{B}&=\begin{pmatrix}\left[1-(1-\xi^-)^\psi,1-(1-\xi^+)^\psi\right],\left[(\nu^-)^\psi,(\nu^+)^\psi\right],\\ \xi^\psi,1-(1-\nu)^\psi\end{pmatrix}\\
\mathscr{B}^\psi&=\begin{pmatrix}\left[(\xi^-)^\psi,(\xi^+)^\psi\right],\left[1-(1-\nu^-)^\psi,1-(1-\nu^+)^\psi\right],\\ 1-(1-\xi)^\psi,\nu^\psi\end{pmatrix},
\end{aligned}\tag{5}$$

2.5. Cubic Pythagorean Fuzzy Set

Definition 12. (see [39]). Let S be a non-empty set. A CPFS $\mathscr{C}$ over S is defined as follows:

$$\mathscr{C}=\{x,D(x),\rho(x)|x\in S\},\tag{6}$$

where $D(x)=\{x,[\xi_D^-(x),\xi_D^+(x)],[\nu_D^-(x),\nu_D^+(x)]\}$ represents IVPFS while $\rho(x)=\{x,\xi_\rho(x),\nu_\rho(x)\}$ represents PFS for all $x\in S$ such that $0\preccurlyeq\xi_D^-(x)\preccurlyeq\xi_D^+(x)\preccurlyeq 1$, $0\preccurlyeq\nu_D^-(x)\preccurlyeq\nu_D^+(x)\preccurlyeq 1$ and $0\preccurlyeq(\xi_D^+(x)^2+(\nu_D^+(x)^2\preccurlyeq 1$. Also, $0\preccurlyeq\xi_\rho(x),\nu_\rho(x)\preccurlyeq 1$ and $0\preccurlyeq(\xi_\rho(x)^2+(\nu_\rho(x)^2\preccurlyeq 1$. In order to keep it simple, the pair $\mathscr{C}=D,\rho$, where $[\xi_D^-,\xi_D^+]$, $[\nu_D^-,\nu_D^+]$ and μ_ρ,ξ_ρ and called as CPF number (CPFN).

Definition 13 (see [39]). Let $\vartheta=([\mu_\vartheta^-,\mu_\vartheta^+],[\nu_\vartheta^-,\nu_\vartheta^+],\mu_\vartheta,\nu_\vartheta)$ be a CPFN then score (sc) and accuracy function (ac) is defined as follows:

$$sc(\vartheta)=\frac{1}{2}\left[\frac{(\mu_\vartheta^-)^2+(\mu_\vartheta^+)^2-(\nu_\vartheta^-)^2-(\nu_\vartheta^+)^2}{2}-\left(\mu_\vartheta^2-\nu_\vartheta^2\right)\right],\tag{7}$$

where $-1\preccurlyeq sc(\beta)\preccurlyeq 1$.

$$ac(\beta)=\frac{1}{2}\left[\frac{(\mu_\vartheta^-)^2+(\mu_\vartheta^+)^2+(\nu_\vartheta^-)^2+(\nu_\vartheta^+)^2}{2}+\left(\mu_\vartheta^2+\nu_\vartheta^2\right)\right]\tag{8}$$

where $0\preccurlyeq sc(\vartheta)\preccurlyeq 1$.

Definition 14 (see [39]). Let $\vartheta = ([\xi^-, \xi^+], [\nu^-, \nu^+], \xi, \nu)$, $\vartheta_i = ([\xi_i^-, \xi_i^+], [\nu_i^-, \nu_i^+], \xi_i, \nu_i)\,(i = 1, 2))$ be the collections of CPFNs, and $\varphi > 0$ be a real number, then

$$
\begin{aligned}
\vartheta_1 \cup \vartheta_2 &= \begin{pmatrix} \left[\max\left(\xi_{\vartheta_1}^-, \xi_{\vartheta_2}^-\right), \max\left(\xi_{\vartheta_1}^+, \xi_{\vartheta_2}^+\right)\right], \left[\min\left(\nu_{\vartheta_1}^-, \nu_{\vartheta_2}^-\right), \min\left(\nu_{\vartheta_1}^+, \nu_{\vartheta_2}^+\right)\right]; \\ \max\left(\xi_{\vartheta_1}, \xi_{\vartheta_2}\right), \min\left(\nu_{\vartheta_1}, \nu_{\vartheta_2}\right) \end{pmatrix} \\
\vartheta_1 \cap \vartheta_2 &= \begin{pmatrix} \left[\min\left(\xi_{\vartheta_1}^-, \xi_{\vartheta_2}^-\right)\right], \left[\min\left(\xi_{\vartheta_1}^+, \xi_{\vartheta_2}^+\right)\right], \left[\max\left(\nu_{\vartheta_1}^-, \nu_{\vartheta_2}^-\right)\right], \left[\max\left(\nu_{\vartheta_1}^+, \nu_{\vartheta_2}^+\right)\right]; \\ \min\left(\xi_{\vartheta_1}, \xi_{\vartheta_2}\right), \max\left(\upsilon_{\vartheta_1}, \upsilon_{\vartheta_2}\right) \end{pmatrix} \\
\delta\vartheta &= \begin{pmatrix} \left[\sqrt{1 - \left(1 - (\xi_\vartheta^-)^2\right)^\varphi}, \sqrt{1 - \left(1 - (\xi_\vartheta^+)^2\right)^\varphi}\right], \left[(\nu_\vartheta^-)^\varphi, (\nu_\vartheta^+)^\varphi\right]; \\ \xi_\vartheta^\varphi, \sqrt{1 - \left(1 - \nu_\vartheta^2\right)^\varphi} \end{pmatrix} \\
\vartheta^\delta &= \begin{pmatrix} \left[(\xi_\vartheta^-)^\varphi, (\xi_\vartheta^+)^\varphi\right], \left[\sqrt{1 - \left(1 - (\nu_\vartheta^-)^2\right)^\varphi}, \sqrt{1 - \left(1 - (\nu_\vartheta^+)^2\right)^\varphi}\right]; \\ \sqrt{1 - \left(1 - \xi_\vartheta^2\right)^\varphi}, \nu_\vartheta^\varphi \end{pmatrix} \\
\vartheta_1 \oplus \vartheta_2 &= \begin{pmatrix} \begin{bmatrix} \sqrt{\left(\xi_{\vartheta_1}^-\right)^2 + \left(\xi_{\vartheta_2}^-\right)^2 - \left(\xi_{\vartheta_1}^-\right)^2\left(\xi_{\vartheta_2}^-\right)^2}, \\ \sqrt{\left(\xi_{\vartheta_1}^+\right)^2 + \left(\xi_{\vartheta_2}^+\right)^2 - \left(\xi_{\vartheta_1}^+\right)^2\left(\xi_{\vartheta_2}^+\right)^2} \end{bmatrix}, \left[\nu_{\vartheta_1}^- \nu_{\vartheta_2}^-, \nu_{\vartheta_1}^+ \nu_{\vartheta_2}^+\right]; \\ \xi_{\vartheta_1}\xi_{\vartheta_2}, \sqrt{\nu_{\vartheta_1}^2 + \nu_{\vartheta_2}^2 - \nu_{\vartheta_1}^2 \nu_{\vartheta_2}^2} \end{pmatrix} \\
\vartheta_1 \otimes \vartheta_2 &= \begin{pmatrix} \left[\xi_{\vartheta_1}^- \xi_{\vartheta_2}^-, \xi_{\vartheta_1}^+ \xi_{\vartheta_2}^+\right], \begin{bmatrix} \sqrt{\left(\nu_{\vartheta_1}^-\right)^2 + \left(\nu_{\vartheta_2}^-\right)^2 - \left(\nu_{\vartheta_1}^-\right)^2\left(\nu_{\vartheta_2}^-\right)^2}, \\ \sqrt{\left(\nu_{\vartheta_1}^+\right)^2 + \left(\nu_{\vartheta_2}^+\right)^2 - \left(\nu_{\vartheta_1}^+\right)^2\left(\nu_{\vartheta_2}^+\right)^2} \end{bmatrix}; \\ \sqrt{\xi_{\vartheta_1}^2 + \xi_{\vartheta_2}^2 - \xi_{\vartheta_1}^2 \xi_{\vartheta_2}^2}, \nu_{\vartheta_1}\nu_{\vartheta_2} \end{pmatrix}.
\end{aligned}
\tag{9}
$$

2.6. Bonferroni Mean Operators

Definition 15 (see [44]). Let $a, b \geq 0$ and $q_i\,(i = 1, 2, \ldots, n)$ be a collection of non-negative numbers, then BM is defined as

$$
BM^{a,b}(q_1, q_2, \ldots, q_n) = \left(\frac{1}{n(n+1)} \sum_{\substack{i, j=1 \\ i \neq j}}^{n} q_i^a q_j^b \right)^{1/a+b}. \tag{10}
$$

In [43], Liang et al. generalized this BM for PF environments, giving the following concepts.

Definition 16 (see [43]). Let $\alpha_i = (\xi_i, \upsilon_i)\,(i = 1, 2, \ldots, n)$ be the collection of PFNs. For any $a, b \geq 0$, the PF Bonferroni mean (PFBM) is defined as

$$
PFBM^{a,b}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n(n+1)} \left(\oplus_{\substack{i, j=1 \\ i \neq j}}^{n} \left(\alpha_i^a \oplus \alpha_j^b\right) \right) \right)^{1/a+b}. \tag{11}
$$

Definition 17 (see [43]). Let $\alpha_i = (\xi_i, \upsilon_i)\,(i = 1, 2, \ldots, n)$ be the collection of PFNs and $a, b \geq 0$. $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)^T$ is the weight vector of α_i such that $\Delta_i \epsilon [0, 1]$ and $\sum_{i=1}^{n} \Delta_i = 1$, then weighted PF Bonferroni mean (WPFBM) is defined as

$$
WPFBM^{a,b}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n(n+1)} \left(\oplus_{\substack{i, j=1 \\ i \neq j}}^{n} \left(\Delta_I \alpha_i^a\right) \Delta \left(\Delta_J \alpha_j^b\right) \right) \right)^{1/a+b}. \tag{12}
$$

3. CPF-Bonferroni Mean Aggregation Operators

In this section, a series of aggregation operators, namely, CPFBM and WCPFBM are presented.

3.1. CPF Bonferroni Mean Operators

Definition 18 A CPF Bonferroni mean (CPFBM) operator is a mapping CPFBM : $\Phi \longrightarrow \Phi$ defined on a set of CPFNs $\vartheta_i\,(i = 1, 2, \ldots, n)$ and is given by

$$CPFBM^{a,b}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \left(\frac{1}{n(n+1)}\left(\oplus^{n}_{\substack{i,\, j=1\\ i\neq j}}\left(\vartheta_i^a \oplus \vartheta_j^b\right)\right)\right)^{1/a+b}, \tag{13}$$

where $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are CPFNs, and a and b are positive real numbers.

Example 1. Let $\vartheta_1 = ([0.4, 0.5], [0.5, 0.7], 0.4, 0.7)$, and $\vartheta_2 = ([0.2, 0.4], [0.7, 0.8], 0.3, 0.6)$ be two CPFNs. Then, use CPFBM to aggregate these three CPFNs. The steps are outlined below. (Supposes $a = b = 2$ and $n = 3$).

By using (11), we get

$$\sqrt{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\xi^{-}_{\vartheta_i}\right)^{2a}\left(\xi^{-}_{\vartheta_j}\right)^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}} = \sqrt{\left(1-\left(\left(1-0.4^{2a}\times 0.2^{2b}\right)\times\left(1-0.2^{2a}\times 0.4^{2b}\right)\right)^{\frac{1}{6}}\right)^{1/a+b}} = 0.5231, \tag{14}$$

$$\sqrt{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\xi^{+}_{\vartheta_i}\right)^{2a}\left(\xi^{+}_{\vartheta_j}\right)^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}} = \sqrt{\left(1-\left(\left(1-0.5^{2a}\times 0.4^{2b}\right)\times\left(1-0.5^{2a}\times 0.4^{2b}\right)\right)^{1/6}\right)^{1/a+b}} = 0.6314, \tag{15}$$

$$\begin{aligned}&\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu^{-}_{\vartheta_i}\right)^{2}\right)^{a}\left(1-\left(\nu^{-}_{\vartheta_j}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}\\ &= \sqrt{1-\left(1-\left(1-\left(1-0.5^2\right)^a(1-0.7)^b\right)\times\left(1-\left(1-0.7^2\right)^a\left(1-0.5^2\right)^b\right)^{1/6}\right)^{1/a+b}} = 0.4789,\end{aligned} \tag{16}$$

$$\begin{aligned}&\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu^{+}_{\vartheta_i}\right)^{2}\right)^{a}\left(1-\left(\nu^{+}_{\vartheta_j}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}\\ &= \sqrt{1-\left(1-\left(1-\left(1-0.7^2\right)^a\left(1-0.8^2\right)^b\right)\times\left(1-\left(1-0.8^2\right)^a\left(1-0.7^2\right)^b\right)^{1/6}\right)^{1/a+b}} = 0.6476,\end{aligned} \tag{17}$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_j}^2\right)^b\right)^{1/n(n+1)}\right)^{1/a+b}}$$
$$=\sqrt{1-\left(1-\left(1-\left(1-0.5^2\right)^a\left(1-0.3^2\right)^b\right)\times\left(1-\left(1-0.3^2\right)^a\left(1-0.5^2\right)^b\right)^{1/6}\right)^{1/a+b}}=4150, \tag{18}$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_j}^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}}=\sqrt{\left(1-\left(\left(1-0.7^{2a}\times 0.6^{2b}\right)\times\left(1-0.6^{2a}\times 0.7^{2b}\right)\right)\frac{1}{6}\right)^{1/a+b}}=0.6246. \tag{19}$$

Thus,

$$\mathrm{CPFBM}^{a,b}\left(\vartheta_1,\vartheta_2,\vartheta_3\right)=\left(1/2(2+1)\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{2}\left(\vartheta_i^1\oplus\vartheta_j^1\right)\right)\right)^{1/2+2}=\left(\left\langle\begin{bmatrix}0.5231,\\ 0.6314\end{bmatrix},\begin{bmatrix}0.4789,\\ 0.6476\end{bmatrix}\right\rangle,\left\langle\begin{matrix}0.4150,\\ 0.6246\end{matrix}\right\rangle\right). \tag{20}$$

Proposition 1. *Let* $\vartheta_i=\left(\left\langle\left[\xi_i^-,\xi_i^+\right],\left[\nu_i^-,\nu_i^+\right]\right\rangle,\left\langle\xi_i,\nu_i\right\rangle\right)(i=1,2)$*) be the collections of CPFNs and* $a,b\geq 0$*. For any* i*,* j*, and* $i\neq j$*, we have*

$$\vartheta_i^a\otimes\vartheta_j^b=\left(\begin{matrix}\left\langle\begin{bmatrix}\left(\xi_{\vartheta_i}^-\right)^a\left(\xi_{\vartheta_j}^-\right)^b,\\ \left(\xi_{\vartheta_i}^+\right)^b\left(\xi_{\vartheta_j}^+\right)^b\end{bmatrix},\begin{bmatrix}\sqrt{1-\left(1-\left(\nu_{\vartheta_i}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_j}^-\right)^2\right)^b},\\ \sqrt{1-\left(1-\left(\nu_{\vartheta_i}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_j}^+\right)^2\right)^b}\end{bmatrix}\right\rangle;\\ \left\langle\sqrt{1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_j}^2\right)^b},\nu_{\vartheta_i}^a\nu_{\vartheta_j}^b\right\rangle\end{matrix}\right). \tag{21}$$

Proof. By Definition 14, we have

$$\vartheta_i^a=\left(\begin{matrix}\left\langle\left[\left(\xi_{\vartheta_i}^-\right)^a,\left(\xi_{\vartheta_i}^+\right)^a\right],\left[\sqrt{1-\left(1-\left(\nu_{\vartheta_i}^-\right)^2\right)^a},\sqrt{1-\left(1-\left(\nu_{\vartheta_i}^+\right)^2\right)^a}\right]\right\rangle;\\ \left\langle\sqrt{1-\left(1-\xi_{\vartheta_i}^2\right)^a},\nu_{\vartheta_i}^a\right\rangle\end{matrix}\right) \tag{22}$$

and

$$\vartheta_j^b = \left(\begin{array}{c} \langle \left[\left(\xi^-_{\vartheta_j}\right)^b, \left(\xi^+_{\vartheta_j}\right)^b \right], \left[\sqrt{1-\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b}, \sqrt{1-\left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^b} \right] \rangle; \\ \langle \sqrt{1-\left(1-\xi^2_{\vartheta_j}\right)^b}, \nu^b_{\vartheta_j} \rangle \end{array} \right) \tag{23}$$

Then,

$$\begin{aligned} & \left(\begin{array}{c} \langle \left[\left(\xi^-_{\vartheta_i}\right)^a, \left(\xi^+_{\vartheta_i}\right)^a \right], \left[\sqrt{1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a}, \sqrt{1-\left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^a} \right] \rangle; \\ \langle \sqrt{1-\left(1-\xi^2_{\vartheta_i}\right)^a}, \nu^a_{\vartheta_i} \rangle \end{array} \right) \\ & \otimes \left(\begin{array}{c} \langle \left[\left(\xi^-_{\vartheta_j}\right)^b, \left(\xi^+_{\vartheta_j}\right)^b \right], \left[\sqrt{1-\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b}, \sqrt{1-\left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^b} \right] \rangle; \\ \langle \sqrt{1-\left(1-\xi^2_{\vartheta_j}\right)^b}, \nu^b_{\vartheta_j} \rangle \end{array} \right) \\ & = \begin{array}{l} \langle \left[\begin{array}{c} \left(\xi^-_{\vartheta_i}\right)^a \left(\xi^-_{\vartheta_j}\right)^b, \\ \left(\xi^+_{\vartheta_i}\right)^b \left(\xi^+_{\vartheta_j}\right)^b \end{array} \right], \left[\begin{array}{c} \sqrt{1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b}, \\ \sqrt{1-\left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^b} \end{array} \right] \rangle; \\ \langle \sqrt{1-\left(1-\xi^2_{\vartheta_i}\right)^a \left(1-\xi^2_{\vartheta_j}\right)^b}, \nu^a_{\vartheta_i} \nu^b_{\vartheta_j} \rangle \end{array} \end{aligned} \tag{24}$$

□

Proposition 2. *Let $\vartheta_i = (\langle [\xi_i^-, \xi_i^+], [\nu_i^-, \nu_i^+] \rangle, \langle \xi_i, \nu_i \rangle) i = 1, 2)$ be the collections of CPFNs and $a, b \geq 0$. For any i, j and $i \neq j$, we have*

$$= \left(\begin{array}{c} \left[\begin{array}{c} \left(\vartheta_i^a \otimes \vartheta_j^b\right) \oplus \left(\vartheta_i^a \otimes \vartheta_j^b\right) \\ \sqrt{1-\left(1-\left(\xi^-_{\vartheta_i}\right)^{2a}\left(\xi^-_{\vartheta_i}\right)^{2b}\left(1-\left(\xi^-_{\vartheta_j}\right)^{2a}\left(\xi^-_{\vartheta_j}\right)^{2b}\right)\right)}, \\ \sqrt{1-\left(1-\left(\xi^+_{\vartheta_i}\right)^{2a}\left(\xi^+_{\vartheta_i}\right)^{2b}\left(1-\left(\xi^+_{\vartheta_j}\right)^{2a}\left(\xi^+_{\vartheta_j}\right)^{2b}\right)\right)}, \end{array} \right] \\ \langle \left[\sqrt{\left(1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b\right)\left(1-\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^a\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^b\right)}, \right] \rangle \\ \sqrt{\left(1-\left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^a\left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^b\right)\left(1-\left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^a\left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^b\right)}, \\ \langle \begin{array}{c} \sqrt{\left(1-\left(1-\xi^2_{\vartheta_i}\right)^a\left(1-\xi^2_{\vartheta_j}\right)^b\right)\left(1-\left(1-\xi^2_{\vartheta_j}\right)^a\left(1-\xi^2_{\vartheta_i}\right)^b\right)}, \\ \sqrt{1-\left(1-\nu^{2a}_{\vartheta_i}\nu^{2b}_{\vartheta_j}\right)\left(1-\nu^{2a}_{\vartheta_j}\nu^{2b}_{\vartheta_i}\right)} \end{array} \rangle \end{array} \right). \tag{25}$$

Proof. We can get the following result from Proposition 1:

$$\vartheta_i^a \otimes \vartheta_j^b = \left(\begin{array}{c} \left\langle \begin{bmatrix} (\xi^-_{\vartheta_i})^a (\xi^-_{\vartheta_j})^b, \\ (\xi^+_{\vartheta_i})^b (\xi^+_{\vartheta_j})^b \end{bmatrix}, \begin{bmatrix} \sqrt{1-(1-(\nu^-_{\vartheta_i})^2)^a (1-(\nu^-_{\vartheta_j})^2)^b}, \\ \sqrt{1-(1-(\nu^+_{\vartheta_i})^2)^a (1-(\nu^+_{\vartheta_j})^2)^b} \end{bmatrix} \right\rangle; \\ \left\langle \sqrt{1-(1-\xi^2_{\vartheta_i})^a (1-\xi^2_{\vartheta_j})^b}, \nu^a_{\vartheta_i} \nu^b_{\vartheta_j} \right\rangle \end{array} \right),$$

$$\vartheta_j^a \otimes \vartheta_i^b = \left(\begin{array}{c} \left\langle \begin{bmatrix} (\xi^-_{\vartheta_j})^a (\xi^-_{\vartheta_i})^b, \\ (\xi^+_{\vartheta_j})^b (\xi^+_{\vartheta_i})^b \end{bmatrix}, \begin{bmatrix} \sqrt{1-(1-(\nu^-_{\vartheta_j})^2)^a (1-(\nu^-_{\vartheta_i})^2)^b}, \\ \sqrt{1-(1-(\nu^+_{\vartheta_j})^2)^a (1-(\nu^+_{\vartheta_i})^2)^b} \end{bmatrix} \right\rangle; \\ \left\langle \sqrt{1-(1-\xi^2_{\vartheta_j})^a (1-\xi^2_{\vartheta_i})^b}, \nu^a_{\vartheta_j} \nu^b_{\vartheta_i} \right\rangle \end{array} \right)$$

By Definition 14, we have $(\vartheta_i^a \otimes \vartheta_j^b) \oplus (\vartheta_i^a \otimes \vartheta_j^b)$

$$\begin{aligned}
&= \left\langle \begin{bmatrix} \left(\xi^-_{\vartheta_i}\right)^a \left(\xi^-_{\vartheta_j}\right)^b, \\ \left(\xi^+_{\vartheta_i}\right)^b \left(\xi^+_{\vartheta_j}\right)^b \end{bmatrix}, \begin{bmatrix} \sqrt{1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b}, \\ \sqrt{1-\left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^b} \end{bmatrix} \right\rangle; \\
&\quad \left\langle \sqrt{1-\left(1-\xi^2_{\vartheta_i}\right)^a \left(1-\xi^2_{\vartheta_j}\right)^b}, \nu^a_{\vartheta_i} \nu^b_{\vartheta_j} \right\rangle \\
&\oplus \left\langle \begin{bmatrix} \left(\xi^-_{\vartheta_j}\right)^a \left(\xi^-_{\vartheta_i}\right)^b, \\ \left(\xi^+_{\vartheta_j}\right)^b \left(\xi^+_{\vartheta_i}\right)^b \end{bmatrix}, \begin{bmatrix} \sqrt{1-\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^b}, \\ \sqrt{1-\left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^a \left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^b} \end{bmatrix} \right\rangle; \\
&\quad \left\langle \sqrt{1-\left(1-\xi^2_{\vartheta_j}\right)^a \left(1-\xi^2_{\vartheta_i}\right)^b}, \nu^a_{\vartheta_j} \nu^b_{\vartheta_i} \right\rangle \\
&= \left(\begin{array}{c} \begin{bmatrix} \sqrt{\left(\left(\xi^-_{\vartheta_i}\right)^a \left(\xi^-_{\vartheta_j}\right)^b\right)^2 + \left(\left(\xi^-_{\vartheta_j}\right)^a \left(\xi^-_{\vartheta_i}\right)^b\right)^2 - \left(\left(\xi^-_{\vartheta_i}\right)^a \left(\xi^-_{\vartheta_j}\right)^b\right)^2 \left(\left(\xi^-_{\vartheta_j}\right)^a \left(\xi^-_{\vartheta_i}\right)^b\right)^2}, \\ \sqrt{\left(\left(\xi^+_{\vartheta_i}\right)^a \left(\xi^+_{\vartheta_j}\right)^b\right)^2 + \left(\left(\xi^+_{\vartheta_j}\right)^a \left(\xi^+_{\vartheta_i}\right)^b\right)^2 - \left(\left(\xi^+_{\vartheta_i}\right)^a \left(\xi^+_{\vartheta_j}\right)^b\right)^2 \left(\left(\xi^+_{\vartheta_j}\right)^a \left(\xi^+_{\vartheta_i}\right)^b\right)^2}, \end{bmatrix} \\ \left\langle \begin{bmatrix} \sqrt{\left(1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b\right)} \sqrt{\left(1-\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^b\right)} \\ \sqrt{\left(1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b\right)} \sqrt{\left(1-\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^b\right)} \end{bmatrix} \right\rangle \\ \left\langle \begin{array}{c} \sqrt{\left(1-\left(1-\xi^2_{\vartheta_i}\right)^a \left(1-\xi^2_{\vartheta_j}\right)^b\right)} \sqrt{\left(1-\left(1-\xi^2_{\vartheta_j}\right)^a \left(1-\xi^2_{\vartheta_i}\right)^b\right)} \\ \sqrt{\left(\nu^a_{\vartheta_i} \nu^b_{\vartheta_j}\right)^2 + \left(\nu^a_{\vartheta_j} \nu^b_{\vartheta_i}\right)^2 - \left(\nu^a_{\vartheta_i} \nu^b_{\vartheta_j}\right)^2 \left(\nu^a_{\vartheta_j} \nu^b_{\vartheta_i}\right)^2} \end{array} \right\rangle \end{array} \right) \\
&= \left(\begin{array}{c} \begin{bmatrix} \sqrt{1-\left(1-\left(\xi^-_{\vartheta_i}\right)^{2a} \left(\xi^-_{\vartheta_i}\right)^{2b} \left(1-\left(\xi^-_{\vartheta_j}\right)^{2a} \left(\xi^-_{\vartheta_j}\right)^{2b}\right)\right)}, \\ \sqrt{1-\left(1-\left(\xi^+_{\vartheta_i}\right)^{2a} \left(\xi^+_{\vartheta_i}\right)^{2b} \left(1-\left(\xi^+_{\vartheta_j}\right)^{2a} \left(\xi^+_{\vartheta_j}\right)^{2b}\right)\right)}, \end{bmatrix} \\ \left\langle \begin{bmatrix} \sqrt{\left(1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b\right) \left(1-\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^a \left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^b\right)}, \\ \sqrt{\left(1-\left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^a \left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^b\right) \left(1-\left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^a \left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^b\right)}, \end{bmatrix} \right\rangle \\ \left\langle \begin{array}{c} \sqrt{\left(1-\left(1-\xi^2_{\vartheta_i}\right)^a \left(1-\xi^2_{\vartheta_j}\right)^b\right) \left(1-\left(1-\xi^2_{\vartheta_j}\right)^a \left(1-\xi^2_{\vartheta_i}\right)^b\right)}, \\ \sqrt{1-\left(\left(1-\nu^{2a}_{\vartheta_i} \nu^{2b}_{\vartheta_j}\right) \left(1-\nu^{2a}_{\vartheta_j} \nu^{2b}_{\vartheta_i}\right)\right)} \end{array} \right\rangle \end{array} \right).
\end{aligned} \tag{26}$$

Thus, Proposition 2 holds. □

Proposition 3. *Let $\vartheta_i = (\langle[\xi_i^-, \xi_i^+], [\nu_i^-, \nu_i^+]\rangle, \langle \xi_i, \nu_i\rangle)\,(i = 1, 2)$ be the collection of CPFNs and $a, b \geq 0$. We can calculate the following by using the value of $k \oplus_{i=1}^{k} (\vartheta_i^a \otimes \vartheta_{k+1}^b)$.*

$$
= \left(\begin{array}{c} \left\langle \begin{array}{c} \left[\sqrt{1-\prod_{i=1}^{k}\left(\left(1-\left(\xi_{\vartheta_i}^-\right)^{2a}\left(\xi_{\vartheta_{k+1}}^-\right)^{2b}\right)\right)}, \sqrt{1-\prod_{i=1}^{k}\left(\left(1-\left(\xi_{\vartheta_i}^+\right)^{2a}\left(\xi_{\vartheta_{k+1}}^+\right)^{2b}\right)\right)}\right], \\ \left[\sqrt{\prod_{i=1}^{k}\left(1-\left(1-\left(\nu_{\vartheta_i}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k+1}}^-\right)^2\right)^b\right)}, \sqrt{\prod_{i=1}^{k}\left(1-\left(1-\left(\nu_{\vartheta_i}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k+1}}^+\right)^2\right)^b\right)}\right] \end{array} \right\rangle; \\ \left\langle \sqrt{\prod_{i=1}^{k}\left(1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_{k+1}}^2\right)^b\right)}, \sqrt{1-\prod_{i=1}^{k}\left(\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_{k+1}}^{2b}\right)\right)} \right\rangle \end{array} \right), \tag{27}
$$

where $1 \leq k \leq n$.

Proof. Mathematical induction on n. □

Step 1. By adopting Proposition 1, we obtain the following result when $k = 2$:

$$
\vartheta_i^a \otimes \vartheta_{2+1}^b = \left\langle \begin{array}{c} \left[\begin{array}{c} \left(\xi_{\vartheta_i}^-\right)^a\left(\xi_{\vartheta_{2+1}}^-\right)^b, \\ \left(\xi_{\vartheta_i}^+\right)^b\left(\xi_{\vartheta_{2+1}}^+\right)^b \end{array}\right], \left[\begin{array}{c} \sqrt{1-\left(1-\left(\nu_{\vartheta_i}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{2+1}}^-\right)^2\right)^b}, \\ \sqrt{1-\left(1-\left(\nu_{\vartheta_i}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{2+1}}^+\right)^2\right)^b} \end{array}\right] \end{array} \right\rangle; \\ \left\langle \sqrt{1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_{2+1}}^2\right)^b}, \nu_{\vartheta_i}^a\nu_{\vartheta_{2+1}}^b \right\rangle, \tag{28}
$$

Thus, we have

$$\oplus_{i=1}^{2}\left(\vartheta_i^a\otimes\vartheta_3^b\right)=\left(\vartheta_1^a\otimes\vartheta_3^b\right)\oplus\left(\vartheta_1^a\otimes\vartheta_3^b\right)$$
$$=\left(\begin{array}{c}\left\langle\begin{bmatrix}\left(\xi_{\vartheta_1}^-\right)^a\left(\xi_{\vartheta_3}^-\right)^b,\\ \left(\xi_{\vartheta_1}^+\right)^b\left(\xi_{\vartheta_3}^+\right)^b\end{bmatrix},\begin{bmatrix}\sqrt{1-\left(1-\left(\nu_{\vartheta_1}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^-\right)^2\right)^b},\\ \sqrt{1-\left(1-\left(\nu_{\vartheta_i}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^+\right)^2\right)^b}\end{bmatrix}\right\rangle;\\ \left\langle\sqrt{1-\left(1-\xi_{\vartheta_1}^2\right)^a\left(1-\xi_{\vartheta_3}^2\right)^b},\nu_{\vartheta_1}^a\nu_{\vartheta_3}^b\right\rangle\end{array}\right)$$
$$\otimes\left(\begin{array}{c}\left\langle\begin{bmatrix}\left(\xi_{\vartheta_1}^-\right)^a\left(\xi_{\vartheta_3}^-\right)^b,\\ \left(\xi_{\vartheta_1}^+\right)^b\left(\xi_{\vartheta_3}^+\right)^b\end{bmatrix},\begin{bmatrix}\sqrt{1-\left(1-\left(\nu_{\vartheta_1}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^-\right)^2\right)^b},\\ \sqrt{1-\left(1-\left(\nu_{\vartheta_i}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^+\right)^2\right)^b}\end{bmatrix}\right\rangle;\\ \left\langle\sqrt{1-\left(1-\xi_{\vartheta_1}^2\right)^a\left(1-\xi_{\vartheta_3}^2\right)^b},\nu_{\vartheta_1}^a\nu_{\vartheta_3}^b\right\rangle\end{array}\right)\tag{29}$$
$$=\left(\begin{array}{c}\begin{bmatrix}\sqrt{1-\left(1-\left(\xi_{\vartheta_1}^-\right)^{2a}\left(\xi_{\vartheta_3}^-\right)^{2b}\left(1-\left(\xi_{\vartheta_1}^-\right)^{2a}\left(\xi_{\vartheta_3}^-\right)^{2b}\right)\right)},\\ \sqrt{1-\left(1-\left(\xi_{\vartheta_1}^+\right)^{2a}\left(\xi_{\vartheta_3}^+\right)^{2b}\left(1-\left(\xi_{\vartheta_1}^+\right)^{2a}\left(\xi_{\vartheta_3}^+\right)^{2b}\right)\right)},\end{bmatrix}\\ \left\langle\begin{bmatrix}\sqrt{\left(1-\left(1-\left(\nu_{\vartheta_1}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^-\right)^2\right)^b\right)\left(1-\left(1-\left(\nu_{\vartheta_1}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^-\right)^2\right)^b\right)},\\ \sqrt{\left(1-\left(1-\left(\nu_{\vartheta_1}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^+\right)^2\right)^b\right)\left(1-\left(1-\left(\nu_{\vartheta_1}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_3}^+\right)^2\right)^b\right)},\end{bmatrix}\right\rangle\\ \left\langle\begin{array}{c}\sqrt{\left(1-\left(1-\xi_{\vartheta_1}^2\right)^a\left(1-\xi_{\vartheta_3}^2\right)^b\right)\left(1-\left(1-\xi_{\vartheta_1}^2\right)^a\left(1-\xi_{\vartheta_3}^2\right)^b\right)},\\ \sqrt{1-\left(\left(1-\nu_{\vartheta_1}^{2a}\nu_{\vartheta_3}^{2b}\right)\left(1-\nu_{\vartheta_1}^{2a}\nu_{\vartheta_3}^{2b}\right)\right)}\end{array}\right\rangle\end{array}\right),$$

Step 2. We can obtain the following result. Eq. (14) holds for $k=k_0,\oplus_{i=1}^{k_0}(\vartheta_i^a\otimes\vartheta_{k+1}^b)$

$$=\left(\left\langle\left[\sqrt{1-\prod_{i=1}^{k_0}\left(\left(1-\left(\xi_{\vartheta_i}^-\right)^{2a}\left(\xi_{\vartheta_{k_0+1}}^-\right)^{2b}\right)\right)},\sqrt{1-\prod_{i=1}^{k_0}\left(\left(1-\left(\xi_{\vartheta_i}^+\right)^{2a}\left(\xi_{\vartheta_{k_0+1}}^+\right)^{2b}\right)\right)}\right],\left[\sqrt{\prod_{i=1}^{k_0}\left(1-\left(1-\left(\nu_{\vartheta_i}^-\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+1}}^-\right)^2\right)^b\right)},\right.\right.\right.$$
$$\left.\left.\sqrt{\prod_{i=1}^{k_0}\left(1-\left(1-\left(\nu_{\vartheta_i}^+\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+1}}^+\right)^2\right)^b\right)}\right]\right\rangle;\left\langle\sqrt{\prod_{i=1}^{k_0}\left(1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_{k_0+1}}^2\right)^b\right)},\sqrt{1-\prod_{i=1}^{k_0}\left(\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_{k_0+1}}^{2b}\right)\right)}\right\rangle\right),\tag{30}$$

Step 3. When $k=k_0+1$, then we have ,, $\oplus_{i=1}^{k_0+1}(\vartheta_i^a\otimes\vartheta_{k_0+2}^b)=\oplus_{i=1}^{k_0}(\vartheta_i^a\otimes\vartheta_{k_0+2}^b)\oplus(\vartheta_i^a\otimes\vartheta_{k_0+2}^b)$,

$$
\begin{aligned}
&= \Bigg(\Big\langle \Bigg[\sqrt{1-\prod_{i=1}^{k_0}\left(\left(1-\left(\xi_{\vartheta_i}^{-}\right)^{2a}\left(\xi_{\vartheta_{k_0+1}}^{-}\right)^{2b}\right)\right)}, \sqrt{1-\prod_{i=1}^{k_0}\left(\left(1-\left(\xi_{\vartheta_i}^{+}\right)^{2a}\left(\xi_{\vartheta_{k_0+1}}^{+}\right)^{2b}\right)\right)} \Bigg], \\
&\Bigg[\sqrt{\prod_{i=1}^{k_0}\left(1-\left(1-\left(\nu_{\vartheta_i}^{-}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+1}}^{-}\right)^2\right)^b\right)}, \sqrt{\prod_{i=1}^{k_0}\left(1-\left(1-\left(\nu_{\vartheta_i}^{+}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+1}}^{+}\right)^2\right)^b\right)} \Bigg] \Big\rangle; \\
&\Big\langle \sqrt{\prod_{i=1}^{k_0}\left(1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_{k_0+1}}^2\right)^b\right)}, \sqrt{1-\prod_{i=1}^{k_0}\left(\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_{k_0+1}}^{2b}\right)\right)} \Big\rangle \Bigg) \\
&\oplus \left(\begin{array}{c} \Big\langle \left[\begin{array}{c} \left(\xi_{\vartheta_{k_0+1}}^{-}\right)^a\left(\xi_{\vartheta_{k_0+2}}^{-}\right)^b, \\ \left(\xi_{\vartheta_{k_0+1}}^{+}\right)^b\left(\xi_{\vartheta_{k_0+2}}^{+}\right)^b \end{array} \right], \left[\begin{array}{c} \sqrt{1-\left(1-\left(\nu_{\vartheta_{k_0+1}}^{-}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+2}}^{-}\right)^2\right)^b}, \\ \sqrt{1-\left(1-\left(\nu_{\vartheta_{k_0+1}}^{+}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+2}}^{+}\right)^2\right)^b} \end{array} \right] \Big\rangle; \\ \Big\langle \sqrt{1-\left(1-\xi_{\vartheta_{k_0+1}}^2\right)^a\left(1-\xi_{\vartheta_{k_0+2}}^2\right)^b}, \nu_{\vartheta_{k_0+1}}^a\nu_{\vartheta_{k_0+2}}^b \Big\rangle \end{array} \right) \\
&= \Bigg(\Big\langle \Bigg[\sqrt{1-\prod_{i=1}^{k_0+1}\left(\left(1-\left(\xi_{\vartheta_i}^{-}\right)^{2a}\left(\xi_{\vartheta_{k_0+2}}^{-}\right)^{2b}\right)\right)}, \sqrt{1-\prod_{i=1}^{k_0+1}\left(\left(1-\left(\xi_{\vartheta_i}^{+}\right)^{2a}\left(\xi_{\vartheta_{k_0+2}}^{+}\right)^{2b}\right)\right)} \Bigg], \Bigg[\sqrt{\prod_{i=1}^{k_0+1}\left(1-\left(1-\left(\nu_{\vartheta_i}^{-}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+2}}^{-}\right)^2\right)^b\right)}, \\
&\sqrt{\prod_{i=1}^{k_0+1}\left(1-\left(1-\left(\nu_{\vartheta_i}^{+}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_{k_0+2}}^{+}\right)^2\right)^b\right)} \Bigg] \Big\rangle; \Big\langle \sqrt{\prod_{i=1}^{k_0+1}\left(1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_{k_0+2}}^2\right)^b\right)}, \sqrt{1-\prod_{i=1}^{k_0+1}\left(\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_{k_0+2}}^{2b}\right)\right)} \Big\rangle \Bigg),
\end{aligned}
\tag{31}
$$

Hence, (31) is also true for $k = k_0 + 1$. As a result, Proposition 3 holds true.

Proposition 4. *Let* $\vartheta_i = (\langle[\xi_i^-, \xi_i^+], [\nu_i^-, \nu_i^+]\rangle, \langle\xi_i, \nu_i\rangle)$ $i = 1, 2$ *be the collections of CPFNs and* $a, b \geq 0$. *Given the value of k, we have* $\oplus_{j=1}^{k}(\vartheta_{k+1}^a \otimes \vartheta_i^b)$.

$$
= \left(\begin{array}{c} \Big\langle \begin{array}{c} \Bigg[\sqrt{1-\prod_{j=1}^{k}\left(\left(1-\left(\xi_{\vartheta_{k+1}}^{-}\right)^{2a}\left(\xi_{\vartheta_j}^{-}\right)^{2b}\right)\right)}, \sqrt{1-\prod_{j=1}^{k}\left(\left(1-\left(\xi_{\vartheta_{k+1}}^{+}\right)^{2a}\left(\xi_{\vartheta_j}^{+}\right)^{2b}\right)\right)} \Bigg], \\ \Bigg[\sqrt{\prod_{j=1}^{k}\left(1-\left(1-\left(\nu_{\vartheta_{k+1}}^{-}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_j}^{-}\right)^2\right)^b\right)}, \sqrt{\prod_{j=1}^{k}\left(1-\left(1-\left(\nu_{\vartheta_{k+1}}^{+}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_j}^{+}\right)^2\right)^b\right)} \Bigg] \end{array} \Big\rangle; \\ \Big\langle \sqrt{\prod_{j=1}^{k}\left(1-\left(1-\xi_{\vartheta_{k+1}}^2\right)^a\left(1-\xi_{\vartheta_j}^2\right)^b\right)}, \sqrt{1-\prod_{j=1}^{k}\left(\left(1-\nu_{\vartheta_{k+1}}^{2a}\nu_{\vartheta_j}^{2b}\right)\right)} \Big\rangle \end{array} \right). \tag{32}
$$

Proof. The proof is similar to that of proportion 3, so it is omitted. □

Proposition 5. *Let* $\vartheta_i = (\langle[\xi_i^-, \xi_i^+], [\nu_i^-, \nu_i^+]\rangle, \langle\xi_i, \nu_i\rangle)$ $(i = 1, 2)$) *be the collection of CPFNs and* $a, b \geq 0$. *For any* i, j *and* $i \neq j$, *we have* $\oplus_{\substack{i,j=1 \\ i\neq j}}^{n}(\vartheta_i^a \oplus \vartheta_j^b)$.

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \left[\sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\left(1-\left(\xi_{\vartheta_i}^{-}\right)^{2a}\left(\xi_{\vartheta_j}^{-}\right)^{2b}\right)\right)}, \sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\left(1-\left(\xi_{\vartheta_i}^{+}\right)^{2a}\left(\xi_{\vartheta_j}^{+}\right)^{2b}\right)\right)} \right], \\ \left[\sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{-}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{-}\right)^{2}\right)^{b}\right)}, \sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{+}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{+}\right)^{2}\right)^{b}\right)} \right] \end{array} \right\rangle; \\ \left\langle \sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\left(1-\xi_{\vartheta_i}^{2}\right)^{a}\left(1-\xi_{\vartheta_j}^{2}\right)^{b}\right)}, \sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_j}^{2b}\right)\right)} \right\rangle \end{array} \right), \tag{33}$$

Proof. The proof is similar to that of proportion 3, so it is omitted. □

Proposition 6. *Let* $\vartheta_i = (\langle[\xi_i^-, \xi_i^+], [\nu_i^-, \nu_i^+]\rangle, \langle\xi_i, \nu_i\rangle)(i = 1, 2))$ *be the collection of CPFNs and* $a, b \geq 0$*. For any* i*,* j *and* $i \neq j$*, we have* $1/n(n+1)\left(\oplus_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\vartheta_i^{a} \oplus \vartheta_j^{b}\right)\right)$.

$$= \left(\begin{array}{c} \left\langle \begin{array}{c} \left[\sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\left(1-\left(\xi_{\vartheta_i}^{-}\right)^{2a}\left(\xi_{\vartheta_j}^{-}\right)^{2b}\right)\right)^{1/n(n+1)}}, \sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\left(1-\left(\xi_{\vartheta_i}^{+}\right)^{2a}\left(\xi_{\vartheta_j}^{+}\right)^{2b}\right)\right)^{1/n(n+1)}} \right], \\ \left[\begin{array}{c} \sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{-}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{-}\right)^{2}\right)^{b}\right)^{1/n(n+1)}}, \\ \sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{+}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{+}\right)^{2}\right)^{b}\right)^{1/n(n+1)}} \end{array} \right] \end{array} \right\rangle; \\ \left\langle \sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-\left(1-\xi_{\vartheta_i}^{2}\right)^{a}\left(1-\xi_{\vartheta_j}^{2}\right)^{b}\right)^{1/n(n+1)}}, \sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_j}^{2b}\right)\right)^{1/n(n+1)}} \right\rangle \end{array} \right) \tag{34}$$

Proof. Based on Eq. (29), we have

$$\left(\oplus^{n}_{\substack{i,j=1\\ i\neq j}}\left(\vartheta_i^a\oplus\vartheta_j^b\right)\right) = \left(\begin{array}{c}\left\langle \begin{array}{c}\left[\sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1-\left(\xi^-_{\vartheta_i}\right)^{2a}\left(\xi^-_{\vartheta_j}\right)^{2b}\right)\right)}, \sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1-\left(\xi^+_{\vartheta_i}\right)^{2a}\left(\xi^+_{\vartheta_j}\right)^{2b}\right)\right)}\right],\\ \left[\sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu^-_{\vartheta_i}\right)^2\right)^a\left(1-\left(\nu^-_{\vartheta_j}\right)^2\right)^b\right)} \sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu^+_{\vartheta_i}\right)^2\right)^a\left(1-\left(\nu^+_{\vartheta_j}\right)^2\right)^b\right)}\right]\end{array}\right\rangle;\\ \left\langle \sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\xi^2_{\vartheta_i}\right)^a\left(1-\xi^2_{\vartheta_j}\right)^b\right)}, \sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1-\nu^{2a}_{\vartheta_i}\nu^{2b}_{\vartheta_j}\right)\right)}\right\rangle\end{array}\right), \tag{35}$$

By using Definition 14, we have

$$1/n(n+1)\left(\oplus^{n}_{\substack{i,j=1\\ i\neq j}}\left(\vartheta_i^a\oplus\vartheta_j^b\right)\right)$$

$$= 1/\mathbf{n}(\mathbf{n}+1)\left(\begin{array}{c}\left\langle \begin{array}{c}\left[\sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}((1-(\xi^-_{\vartheta_i})^{2a}(\xi^-_{\vartheta_j})^{2b}))}, \sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}((1-(\xi^+_{\vartheta_i})^{2a}(\xi^+_{\vartheta_j})^{2b}))}\right],\\ \left[\sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(\nu^-_{\vartheta_i})^2)^a(1-(\nu^-_{\vartheta_j})^2)^b)}, \sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(\nu^+_{\vartheta_i})^2)^a(1-(\nu^+_{\vartheta_j})^2)^b)}\right]\end{array}\right\rangle;\\ \left\langle \sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\xi^2_{\vartheta_i})^a(1-\xi^2_{\vartheta_j})^b)}, \sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}((1-\nu^{2a}_{\vartheta_i}\nu^{2b}_{\vartheta_j}))}\right\rangle\end{array}\right)$$

$$= \left(\begin{array}{c}\left\langle \begin{array}{c}\left[\sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}((1-(\xi^-_{\vartheta_i})^{2a}(\xi^-_{\vartheta_j})^{2b}))^{1/n(n+1)}}, \sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}((1-(\xi^+_{\vartheta_i})^{2a}(\xi^+_{\vartheta_j})^{2b}))^{1/n(n+1)}}\right],\\ \left[\begin{array}{c}\sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(\nu^-_{\vartheta_i})^2)^a(1-(\nu^-_{\vartheta_j})^2)^b)^{1/n(n+1)}},\\ \sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(\nu^+_{\vartheta_i})^2)^a(1-(\nu^+_{\vartheta_j})^2)^b)^{1/n(n+1)}}\end{array}\right]\end{array}\right\rangle;\\ \left\langle \sqrt{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\xi^2_{\vartheta_i})^a(1-\xi^2_{\vartheta_j})^b)^{1/n(n+1)}}, \sqrt{1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}((1-\nu^{2a}_{\vartheta_i}\nu^{2b}_{\vartheta_j}))^{1/n(n+1)}}\right\rangle\end{array}\right).$$

□

Theorem 1. *Let* $\vartheta_i = (\langle[\xi_i^-, \xi_i^+], [\nu_i^-, \nu_i^+]\rangle, \langle \xi_i, \nu_i\rangle) i = 1, 2)$ *be the collection of CPFNs and* $a, b \geq 0$*. By using the CPFBM operator, the aggregated value is also a CPFN, and*

$$
\begin{aligned}
CPFBM^{a,b}\left(\vartheta_1,\,\vartheta_2,\ldots,\vartheta_n\right) &= \left(\frac{1}{n(n+1)}\left(\oplus_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(\vartheta_i^{a}\oplus\vartheta_j^{b}\right)\right)\right)^{1/a+b}\\
&= \left(\begin{array}{c}
\left\langle\begin{array}{c}
\left[\begin{array}{c}
\sqrt{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1-\left(\xi_{\vartheta_i}^{-}\right)^{2a}\left(\xi_{\vartheta_j}^{-}\right)^{2b}\right)\right)^{1/n(n+1)}\right)^{1/a+b}}\,,\\
\sqrt{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\xi_{\vartheta_i}^{+}\right)^{2a}\left(\xi_{\vartheta_j}^{+}\right)^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}}
\end{array}\right],\\
\left[\begin{array}{c}
\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{-}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{-}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}\,,\\
\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{+}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{+}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}
\end{array}\right]
\end{array}\right\rangle;\\
\left\langle\begin{array}{c}
\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\xi_{\vartheta_i}^{2}\right)^{a}\left(1-\xi_{\vartheta_j}^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}\,,\\
\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_j}^{2b}\right)\right)^{1/n(n+1)}\right)^{1/a+b}}
\end{array}\right\rangle
\end{array}\right),
\end{aligned}
\tag{36}
$$

Proof. By Proposition 6 and Definition 14, we have

$$CPFBM^{a,b}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \left(\frac{1}{n(n+1)} \left(\oplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\vartheta_i^a \oplus \vartheta_j^b \right) \right) \right)^{1/a+b},$$

$$= \left(\begin{array}{c} \left[\sqrt{1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \left(\xi_{\vartheta_i}^{-} \right)^{2a} \left(\xi_{\vartheta_j}^{-} \right)^{2b} \right) \right)^{1/n(n+1)}}, \sqrt{1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \left(\xi_{\vartheta_i}^{+} \right)^{2a} \left(\xi_{\vartheta_j}^{+} \right)^{2b} \right) \right)^{1/n(n+1)}} \right], \\ \left\langle \left[\begin{array}{c} \sqrt{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \left(\nu_{\vartheta_i}^{-} \right)^2 \right)^a \left(1 - \left(\nu_{\vartheta_j}^{-} \right)^2 \right)^b \right)^{1/n(n+1)}}, \\ \sqrt{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \left(\nu_{\vartheta_i}^{+} \right)^2 \right)^a \left(1 - \left(\nu_{\vartheta_j}^{+} \right)^2 \right)^b \right)^{1/n(n+1)}} \end{array} \right] \right\rangle; \\ \left\langle \sqrt{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \xi_{\vartheta_i}^2 \right)^a \left(1 - \xi_{\vartheta_j}^2 \right)^b \right)^{1/n(n+1)}}, \sqrt{1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \nu_{\vartheta_i}^{2a} \nu_{\vartheta_j}^{2b} \right) \right)^{1/n(n+1)}} \right\rangle \end{array} \right)$$

$$
= \left(\begin{array}{c} \left\langle \begin{array}{c} \left[\begin{array}{c} \sqrt{\left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \left(\xi_{\vartheta_i}^{-}\right)^{2a} \left(\xi_{\vartheta_j}^{-}\right)^{2b}\right)\right)^{1/n(n+1)}\right)^{1/a+b}}, \\ \sqrt{\left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \left(\xi_{\vartheta_i}^{+}\right)^{2a} \left(\xi_{\vartheta_j}^{+}\right)^{2b}\right)\right)^{1/n(n+1)}\right)^{1/a+b}} \end{array} \right], \\ \left[\begin{array}{c} \sqrt{1 - \left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \left(\nu_{\vartheta_i}^{-}\right)^{2}\right)^{a} \left(1 - \left(\nu_{\vartheta_j}^{-}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}, \\ \sqrt{1 - \left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \left(\nu_{\vartheta_i}^{+}\right)^{2}\right)^{a} \left(1 - \left(\nu_{\vartheta_j}^{+}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}} \end{array} \right] \end{array} \right\rangle; \\ \left\langle \begin{array}{c} \sqrt{1 - \left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \xi_{\vartheta_i}^{2}\right)^{a} \left(1 - \xi_{\vartheta_j}^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}, \\ \sqrt{1 - \left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \nu_{\vartheta_i}^{2a} \nu_{\vartheta_j}^{2b}\right)\right)^{1/n(n+1)}\right)^{1/a+b}} \end{array} \right\rangle \end{array} \right), \tag{37}
$$

As a result, Theorem 1 holds true. □

Example 2. Let $\vartheta_1 = ([0.4, 0.5], [0.5, 0.7], 0.4, 0.7)$, $\vartheta_2 = ([0.2, 0.4], [0.7, 0.8], 0.3, 0.6)$ and $\vartheta_1 = ([0.5, 0.6], [0.4, 0.5], 0.3, 0.2)$ be three CPFNs. Then, use CPFBM to aggregate these three CPFNs. The steps are outlined below. (Supposes $a = b = 1$ and $n = 3$).

By using (21), we get

$$\sqrt{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\xi_{\vartheta_i}^{-}\right)^{2a}\left(\xi_{\vartheta_j}^{-}\right)^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{\left(1-\left(\begin{array}{c}\left(1-0.4^{2a}\times 0.2^{2b}\right)\times\left(1-0.2^{2a}\times 0.4^{2b}\right)\times\left(1-0.2^{2a}\times 0.5^{2b}\right)\\ \times\left(1-0.5^{2a}\times 0.2^{2b}\right)\times\left(1-0.4^{2a}\times 0.5^{2b}\right)\times\left(1-0.5^{2a}\times 0.4^{2b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=0.3709$$

$$\sqrt{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\xi_{\vartheta_i}^{+}\right)^{2a}\left(\xi_{\vartheta_j}^{+}\right)^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{\left(1-\left(\begin{array}{c}\left(1-0.5^{2a}\times 0.4^{2b}\right)\times\left(1-0.5^{2a}\times 0.4^{2b}\right)\times\left(1-0.4^{2a}\times 0.6^{2b}\right)\\ \times\left(1-0.6^{2a}\times 0.4^{2b}\right)\times\left(1-0.5^{2a}\times 0.6^{2b}\right)\times\left(1-0.6^{2a}\times 0.5^{2b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=0.5005$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{-}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{-}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{1-\left(1-\left(\begin{array}{c}\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.7\right)^{b}\right)\times\left(1-\left(1-0.7^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.7^{2}\right)^{a}\left(1-0.4^{2}\right)^{b}\right)\times\left(1-\left(1-0.4^{2}\right)^{a}\left(1-0.7^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.4^{2}\right)^{b}\right)\times\left(1-\left(1-0.4^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=0.5460$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{+}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{+}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{1-\left(1-\left(\begin{array}{c}\left(1-\left(1-0.7^{2}\right)^{a}\left(1-0.8^{2}\right)^{b}\right)\times\left(1-\left(1-0.8^{2}\right)^{a}\left(1-0.7^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.8^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\times\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.8^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.7^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\times\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.7^{2}\right)^{b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=0.6839$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\xi_{\vartheta_i}^{2}\right)^{a}\left(1-\xi_{\vartheta_j}^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{1-\left(1-\left(\begin{array}{c}\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.3^{2}\right)^{b}\right)\times\left(1-\left(1-0.3^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.3^{2}\right)^{a}\left(1-0.3^{2}\right)^{b}\right)\times\left(1-\left(1-0.3^{2}\right)^{a}\left(1-0.3^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.3^{2}\right)^{b}\right)\times\left(1-\left(1-0.3^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=3352$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_j}^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{\left(1-\left(\begin{array}{c}\left(1-0.7^{2a}\times0.6^{2b}\right)\times\left(1-0.6^{2a}\times0.7^{2b}\right)\times\left(1-0.6^{2a}\times0.2^{2b}\right)\\ \times\left(1-0.2^{2a}\times0.6^{2b}\right)\times\left(1-0.7^{2a}\times0.2^{2b}\right)\times\left(1-0.2^{2a}\times0.7^{2b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}}$$

$$=0.5203=\sqrt{\left(1-\left(\begin{array}{c}\left(1-0.5^{2a}\times0.4^{2b}\right)\times\left(1-0.5^{2a}\times0.4^{2b}\right)\times\left(1-0.4^{2a}\times0.6^{2b}\right)\\ \times\left(1-0.6^{2a}\times0.4^{2b}\right)\times\left(1-0.5^{2a}\times0.6^{2b}\right)\times\left(1-0.6^{2a}\times0.5^{2b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}} \tag{38}$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{-}\right)^{2}\right)^{a}\left(1-\left(\nu_{\vartheta_j}^{-}\right)^{2}\right)^{b}\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{1-\left(1-\left(\begin{array}{c}\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.7\right)^{b}\right)\times\left(1-\left(1-0.7^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.7^{2}\right)^{a}\left(1-0.4^{2}\right)^{b}\right)\times\left(1-\left(1-0.4^{2}\right)^{a}\left(1-0.7^{2}\right)^{b}\right)\\ \times\left(1-\left(1-0.5^{2}\right)^{a}\left(1-0.4^{2}\right)^{b}\right)\times\left(1-\left(1-0.4^{2}\right)^{a}\left(1-0.5^{2}\right)^{b}\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=0.5460, \tag{39}$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\nu_{\vartheta_i}^{+}\right)^2\right)^a\left(1-\left(\nu_{\vartheta_j}^{+}\right)^2\right)^b\right)^{1/12}\right)^{1/a+b}}$$

$$=\sqrt{1-\left(1-\left(\begin{array}{c}\left(1-\left(1-0.7^2\right)^a\left(1-0.8^2\right)^b\right)\times\left(1-\left(1-0.8^2\right)^a\left(1-0.7^2\right)^b\right)\\ \times\left(1-\left(1-0.8^2\right)^a\left(1-0.5^2\right)^b\right)\times\left(1-\left(1-0.5^2\right)^a\left(1-0.8^2\right)^b\right)\\ \times\left(1-\left(1-0.7^2\right)^a\left(1-0.5^2\right)^b\right)\times\left(1-\left(1-0.5^2\right)^a\left(1-0.7^2\right)^b\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=0.6839 \tag{40}$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\xi_{\vartheta_i}^2\right)^a\left(1-\xi_{\vartheta_j}^2\right)^b\right)^{1/n(n+1)}\right)^{1/a+b}}$$

$$=\sqrt{1-\left(1-\left(\begin{array}{c}\left(1-\left(1-0.5^2\right)^a\left(1-0.3^2\right)^b\right)\times\left(1-\left(1-0.3^2\right)^a\left(1-0.5^2\right)^b\right)\\ \times\left(1-\left(1-0.3^2\right)^a\left(1-0.3^2\right)^b\right)\times\left(1-\left(1-0.3^2\right)^a\left(1-0.3^2\right)^b\right)\\ \times\left(1-\left(1-0.5^2\right)^a\left(1-0.3^2\right)^b\right)\times\left(1-\left(1-0.3^2\right)^a\left(1-0.5^2\right)^b\right)\end{array}\right)^{1/12}\right)^{1/a+b}}=3352 \tag{41}$$

$$\sqrt{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\nu_{\vartheta_i}^{2a}\nu_{\vartheta_j}^{2b}\right)^{1/n(n+1)}\right)^{1/a+b}}, \tag{42}$$

Thus,

$$CPFBM^{a,b}\left(\vartheta_1,\vartheta_2,\vartheta_3\right)=\left(1/3\left(3+1\right)\left(\oplus_{\substack{i,\,j=1\\ i\neq j}}^{3}\left(\vartheta_i^1\oplus\vartheta_j^1\right)\right)\right)^{1/1+1}=\left(\left\langle\begin{bmatrix}0.3709,\\ 0.5005\end{bmatrix},\begin{bmatrix}0.5460,\\ 0.6839\end{bmatrix}\right\rangle,\left\langle\begin{matrix}0.3352,\\ 0.5203\end{matrix}\right\rangle\right). \tag{43}$$

Based on the results of (see [18]), we may conclude the following corollaries for Theorem 1.

Corollary 1 (idempotency). *If any PCFNs $\vartheta_i=\left(\left\langle\left[\xi_i^-,\xi_i^+\right],\left[\nu_i^-,\nu_i^+\right]\right\rangle,\left\langle\xi_i,\nu_i\right\rangle\right)(i=1,2,\ldots,n)$ are equal, i.e., $\vartheta_i=\vartheta=\left(\left\langle\left[\xi^-,\xi^+\right],\left[\nu^-,\nu^+\right]\right\rangle,\left\langle\xi,\nu\right\rangle\right)$ then $CPFBM^{a,b}$ $(\vartheta_1,\vartheta_2,\ldots,\vartheta_n)=\vartheta=\left(\left\langle\left[\xi^-,\xi^+\right],\left[\nu^-,\nu^+\right]\right\rangle,\left\langle\xi,\nu\right\rangle\right)$.*

Corollary 2 (commutativity). *If β_i' is any permutation of $\beta_i(i1,2,\ldots,n)$ then we have the following relationship $CPFBM^{a,b}(\beta_1',\beta_2',\ldots,\beta_n')=CPFBM^{a,b}(\beta_1,\beta_2,\ldots,\beta_n)$.*

Corollary 3 (monotonicity). *Let $\vartheta_i=\left(\left\langle\left[\xi_{\vartheta_i}^-,\xi_{\vartheta_i}^+\right],\left[\nu_{\vartheta_i}^-,\nu_{\vartheta_i}^+\right]\right\rangle,\left\langle\xi_{\vartheta_i},\nu_{\vartheta_i}\right\rangle\right)$ and $\widetilde{\vartheta}_i=\left(\left\langle\left[\xi_{\widetilde{\vartheta}_i}^-,\xi_{\widetilde{\vartheta}_i}^+\right],\left[\nu_{\widetilde{\vartheta}_i}^-,\nu_{\widetilde{\vartheta}_i}^+\right]\right\rangle,\left\langle\xi_{\widetilde{\vartheta}_i},\nu_{\widetilde{\vartheta}_i}\right\rangle\right)$ be the collections of CPFNs. Assisted by CPFBM operator, we assume*

$CPFBM^{a,b}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = (\langle [\xi_{\vartheta}^-, \xi_{\vartheta}^+], [\nu_{\vartheta}^-, \nu_{\vartheta}^+] \rangle, \langle \xi_{\vartheta}, \nu_{\vartheta} \rangle)$ and $CPFBM^{a,b}(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \ldots, \tilde{\vartheta}_n) = (\langle [\xi_{\tilde{\vartheta}}^-, \xi_{\tilde{\vartheta}}^+], [\nu_{\tilde{\vartheta}}^-, \nu_{\tilde{\vartheta}}^+] \rangle, \langle \xi_{\tilde{\vartheta}}, \nu_{\tilde{\vartheta}} \rangle)$. If $\xi_{\vartheta_i}^- \le \xi_{\tilde{\vartheta}_i}^-$, $\xi_{\vartheta_i}^+ \le \xi_{\tilde{\vartheta}_i}^+$, $\nu_{\vartheta_i}^- \ge \nu_{\tilde{\vartheta}_i}^-$, $\nu_{\vartheta_i}^+ \ge \nu_{\tilde{\vartheta}_i}^+$, $\xi_{\vartheta_i} \ge \xi_{\tilde{\vartheta}_i}$ and $\nu_{\vartheta_i} \le \nu_{\tilde{\vartheta}_i}$ holds, then $\xi_{\vartheta}^- \le \xi_{\tilde{\vartheta}}^-$, $\xi_{\vartheta}^+ \le \xi_{\tilde{\vartheta}}^+$, $\nu_{\vartheta}^- \ge \nu_{\tilde{\vartheta}}^-$, $\nu_{\vartheta}^+ \ge \nu_{\tilde{\vartheta}}^+$, $\xi_{\vartheta} \ge \xi_{\tilde{\vartheta}}$ and $\nu_{\vartheta} \le \nu_{\tilde{\vartheta}}$.

3.2. Weighted CFP Bonferroni Mean. The CPFBM described in Theorem 1 does not take the significance of the aggregated arguments into account. The weight vector of the criterion is a crucial component of the aggregate in various real-world MCDM problems. As a result, the WCPFBM operator is defined as follows.

Definition 19. Let $\vartheta_i = ([\xi_{\vartheta_i}^-, \xi_{\vartheta_i}^+], [\nu_{\vartheta_i}^-, \nu_{\vartheta_i}^+], \xi_{\vartheta_i}, \nu_{\vartheta_i})\,(i = 1, 2, \ldots, n)$ be the collection of CPFNs and a, b are any non-**negative** real numbers. $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)^T$ is the weight vector of ϑ_i such that $\Delta_i \in [0, 1]$ and $\sum_{i=1}^{n} \Delta_i = 1$. Then, weighted CPF Bonferroni mean (WCPFBM) operator is defined as

$$WCPFBM^{a,b}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \left(\frac{1}{n+1} \left(\oplus_{\substack{i, j = 1 \\ i \ne j}}^{n} \left((\Delta_i \vartheta_i)^a \otimes (\Delta_j \vartheta_j)^b \right) \right) \right)^{1/a+b}, \quad (44)$$

Theorem 2. *Let $\vartheta_i = ([\xi_{\vartheta_i}^-, \xi_{\vartheta_i}^+], [\nu_{\vartheta_i}^-, \nu_{\vartheta_i}^+], \xi_{\vartheta_i}, \nu_{\vartheta_i})\,(i = 1, 2, \ldots, n)$ be the collection of CPFNs and a, b are any non-negative real numbers. $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)^T$ is the weight vector of ϑ_i such that $\Delta_i \in [0, 1]$ and $\sum_{i=1}^{n} \Delta_i = 1$. Then, the aggregated values obtained by WCPFBM are also a CPFN, and*

$$WCPFBM^{a,b}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \left(\begin{array}{c} \left\langle \left[\begin{array}{c} \sqrt{\left(1 - \prod_{\substack{i,j=1 \\ i \ne j}}^{n} \left(1 - \left(1 - \left(1 - \left(\xi_{\vartheta_i}^-\right)^2 \right)^{\omega_i} \right)^a 1 - \left(1 - \left(1 - \left(\xi_{\vartheta_j}^-\right)^2 \right)^{\omega_j} \right)^b \right)^{1/n(n+1)} \right)^{1/a+b}}, \\ \sqrt{\left(1 - \prod_{\substack{i,j=1 \\ i \ne j}}^{n} \left(1 - \left(1 - \left(1 - \left(\xi_{\vartheta_i}^+\right)^2 \right)^{\omega_i} \right)^a 1 - \left(1 - \left(1 - \left(\xi_{\vartheta_j}^+\right)^2 \right)^{\omega_j} \right)^b \right)^{1/n(n+1)} \right)^{1/a+b}} \end{array} \right], \right\rangle \\ \left[\begin{array}{c} \sqrt{1 - \left(1 - \prod_{\substack{i,j=1 \\ i \ne j}}^{n} \left(1 - \left(1 - \left(\nu_{\vartheta_i}^-\right)^{2\omega_i} \right)^a \left(1 - \left(\nu_{\vartheta_j}^-\right)^{2\omega_i} \right)^b \right)^{\frac{1}{n(n+1)}} \right)^{1/a+b}}, \\ \sqrt{1 - \left(1 - \prod_{\substack{i,j=1 \\ i \ne j}}^{n} \left(1 - \left(1 - \left(\nu_{\vartheta_i}^+\right)^{2\omega_i} \right)^a \left(1 - \left(\nu_{\vartheta_j}^+\right)^{2\omega_i} \right)^b \right)^{1/n(n+1)} \right)^{1/a+b}} \end{array} \right]; \\ \left\langle \begin{array}{c} \sqrt{1 - \left(1 - \prod_{\substack{i,j=1 \\ i \ne j}}^{n} \left(1 - \left(1 - \xi_{\vartheta_i}^{2\omega_i} \right)^a \left(1 - \xi_{\vartheta_j}^2 \right)^{b\omega_i} \right)^{1/n(n+1)} \right)^{1/a+b}}, \\ \sqrt{1 - \prod_{\substack{i,j=1 \\ i \ne j}}^{n} \left(1 - \left(1 - \left(1 - \left(\nu_{\vartheta_i}^-\right)^2 \right)^{\omega_i} \right)^a 1 - \left(1 - \left(1 - \left(\nu_{\vartheta_j}^-\right)^2 \right)^{\omega_j} \right)^b \right)^{1/n(n+1)}} \end{array} \right\rangle \end{array} \right). \quad (45)$$

Proof. We omitted this proof since it is similar to Theorem 2.

The outcomes of the CPFBM and WCPFBM operators are CPFNs according to Theorems 1 and 2. WCPFBM increases the significance of the aggregated arguments in addition to CPFBM. We present a technique for applying the CPFBM and WCPFBM operator to MCDM with the help of the abovementioned conclusions. The new approach is designed as follows. □

4. DM Approach Based on Proposed CPF Bonferroni Mean Operators

A new technique of processing MCDM is developed based on the abovementioned discussion, which featured the notion of MB operators in a CPF environment, as well as its core algorithms and the CPF-matrix collection approach.

Step 4. In this step, the finite set of alternatives $\mathscr{A}_i\,(i = 1, 2, \ldots, m)$ is determined. We select the set of criteria $\mathbb{C}_j\,(j = 1, 2, \ldots, n)$ defending on the real DM problem and produce the weight vector Δ_i such that $\Delta_i \in [0, 1]$ and $\sum_{i=1}^{n} \Delta_i = 1$. The CPF decision matrix is then formed as follows:

$$\mathscr{D} = \left(\vartheta_{ij}\right)_{m\times n} = \left(\left[\xi_{ij}^-, \xi_{ij}^+\right], \left[\nu_{ij}^-, \nu_{ij}^+\right], \xi_{ij}, \nu_{ij}\right)_{m\times n} = \begin{pmatrix} \left(\begin{bmatrix} \xi_{11}^-, \\ \xi_{11}^+ \end{bmatrix}, \begin{bmatrix} \nu_{11}^-, \\ \nu_{11}^+ \end{bmatrix}, \begin{matrix} \xi_{11}, \\ \nu_{11} \end{matrix}\right) & \cdots & \left(\begin{bmatrix} \xi_{1n}^-, \\ \xi_{1n}^+ \end{bmatrix}, \begin{bmatrix} \nu_{1n}^-, \\ \nu_{1n}^+ \end{bmatrix}, \begin{matrix} \xi_{1n}, \\ \nu_{1n} \end{matrix}\right) \\ \vdots & \ddots & \vdots \\ \left(\begin{bmatrix} \xi_{m1}^-, \\ \xi_{m1}^+ \end{bmatrix}, \begin{bmatrix} \nu_{m1}^-, \\ \nu_{m1}^+ \end{bmatrix}, \begin{matrix} \xi_{m1}, \\ \nu_{m1} \end{matrix}\right) & \cdots & \left(\begin{bmatrix} \xi_{mn}^-, \\ \xi_{mn}^+ \end{bmatrix}, \begin{bmatrix} \nu_{mn}^-, \\ \nu_{mn}^+ \end{bmatrix}, \begin{matrix} \xi_{mn}, \\ \nu_{mn} \end{matrix}\right) \end{pmatrix}. \tag{46}$$

Step 5. Use the normalization procedure given in Eq. (16) to change the confidence and confidence and capability of cost type into benefit type, in each of these collective information decision-matrixes.

$$\Delta_{ij} = \begin{cases} \left(\begin{bmatrix} \xi_{ij}^-, \\ \xi_{11}^+ \end{bmatrix}, \begin{bmatrix} \nu_{ij}^-, \\ \nu_{ij}^+ \end{bmatrix}, \begin{matrix} \xi_{ij}, \\ \nu_{ij} \end{matrix}\right); \text{given the criteria of benefit type} \\ \left(\begin{bmatrix} \nu_{ij}^-, \\ \nu_{ij}^+ \end{bmatrix}, \begin{bmatrix} \xi_{ij}^-, \\ \xi_{ij}^+ \end{bmatrix}, \begin{matrix} \nu_{ij}, \\ \xi_{ij} \end{matrix}\right); \text{given the criteria of cost type} \end{cases}, \tag{47}$$

and hence integrate it into the decision-matrix $\mathscr{R}_{ij} = (\Delta_{ij})_{m\times n}$.

Step 6. Use the proposed AOs to aggregate the different preference values $\Delta_{ij}\,(j = 1, 2, \ldots, n)$ of the alternative $\mathscr{A}_i\,(i = 1, 2, \ldots m)$ as follows:

$$\begin{aligned} &a,bWCPFBM^{a,b}\left(\Delta_{i1},\Delta_{i2},\ldots,\Delta_{\mathrm{in}}\right) = \left(\frac{1}{n+1}\left(\oplus_{\substack{i=1\\ i\neq j}}^{n}\left(\left(\Delta_i\Delta_i\right)^a \otimes \left(\Delta_j\Delta_j\right)^b\right)\right)\right)^{1/a+b} \\ &CPFBM \\ &\left(\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{\mathrm{in}}\right) = \left(\frac{1}{n(n+1)}\left(\oplus_{\substack{i=1\\ i\neq j}}^{n}\left(\Delta_i^a \Delta\Delta_j^b\right)\right)\right)^{1/a+b}, \end{aligned} \tag{48}$$

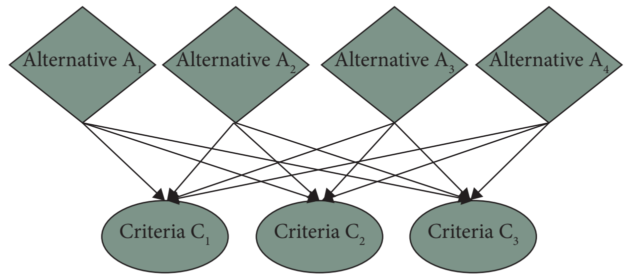

Figure 1: The multilevel layout of the problem.

Step 7. Use eq. (4) to compute the score values of each CPFN Δ_i as follows:

$$sc(\Delta_i) = \frac{1}{2}\left[\frac{(\xi_i^-)^2 + (\xi_i^+)^2 - (\nu_i^-)^2 - (\nu_i^+)^2}{2} - \left(\nu_i^2 - \xi_i^2\right)\right]. \quad (49)$$

Step 8. Arrange the alternatives $\mathscr{A}_i\,(i1, 2, \ldots, m)$ with respect to respective score values $sc(\Delta_i)$.

5. Case Study

These days, supply management is a major problem. From an industrial perspective, a company cannot obtain better performance levels unless its material is properly administered. Therefore, effective inventory management is the first step to achieving high levels of production. Any scarcity of raw material in inventory might cause the entire manufacturing cycle to be affected, resulting in a significant loss for the company. Consider the Case [40] of a food company that has to keep track of multiple inventory materials. The company primarily produces four different types of products $(\mathscr{A}_i)\,(i = 1, 2, 3, 4)$.

(i) $\mathscr{A}_1$: Beverages
(ii) $\mathscr{A}_2$: Edible Oils
(iii) $\mathscr{A}_3$: Pickles
(iv). $\mathscr{A}_4$: Bakery Items.

To prepare these food products, rearranging the stock the materials in the inventory will be decided on the basis of three criteria $(\mathbb{C}_j)\,(j = 1, 2, 3)$.

(i) $\mathbb{C}_1$: Cost Price
(ii) $\mathbb{C}_2$: Storage Facilities
(iii) $\mathbb{C}_3$: Staleness Level

And $\Delta = (0.2, 0.38, 0.32)^T$ is the weight vector of these features. The given alternatives are analyzed using these three criteria and their values are rated in terms of CPFNs. In each CPFN, the IVPFNs reflect the relative stock level in the inventory, and PFSs reflect the estimate of agreement and disagreement towards the present stock level for the following week. Since the company does not compromise on product quality, therefore, reducing the level of stagnation is the highest priority. The goal is then to identify the food products whose material supply must be re-ordered on a regular basis. The phases of the proposed approach have been carried out in the following order. The multilevel layout of the problem is shown in Figure 1.

Step 9. As indicated in Table 2, the favorites information for each alternative is signified in CPFNs, and the group ratings are given in the decision matrix.

Step 10. The CPFNs are normalized by using Eq. (20) and summarized in Table 3.

Step 11. We assume $a = b = 1$ for simplicity's convenience and then calculate the aggregated values of each alternative using Eq. (21) as follows: $\Delta_1 = \left(\left\langle \begin{bmatrix} 0.1022, \\ 0.1601 \end{bmatrix}, \begin{bmatrix} 0.7615, \\ 0.8317 \end{bmatrix} \right\rangle \left\langle \begin{matrix} 0.6688, \\ 0.1462 \end{matrix} \right\rangle\right)$, $\Delta_2 = \left(\begin{bmatrix} 0.1122, \\ 0.1735 \end{bmatrix}, \begin{bmatrix} 0.6300, \\ 0.7157 \end{bmatrix}, \begin{matrix} 0.7582, \\ 0.2125 \end{matrix}\right)$ $\Delta_3 = \left(\begin{bmatrix} 0.1288, \\ 0.1976 \end{bmatrix}, \begin{bmatrix} 0.7528, \\ 0.8288 \end{bmatrix}, \begin{matrix} 0.7462, \\ 0.1825 \end{matrix}\right)$ and $\Delta_4 = \left(\begin{bmatrix} 0.1430, \\ 0.2279 \end{bmatrix}, \begin{bmatrix} 0.6675, \\ 0.7427 \end{bmatrix}, \begin{matrix} 0.6927, \\ 0.2126 \end{matrix}\right)$.

Step 12. . The score values of each alternative are calculated using Eq. (22) as follows: $sc(\mathscr{A}_1) = -1.0437$, $sc(\mathscr{A}_2) = -0.9630$, $sc(\mathscr{A}_3) = -1.1224$ and $sc(\mathscr{A}_4) = -0.8970$

Step 13. According to the score values, the ranking order of the alternatives is concluded to be $\mathscr{A}_4 > \mathscr{A}_2 > \mathscr{A}_1 > \mathscr{A}_3$.

The suggested operators are symmetric with respect to the parameters a and b. Therefore, in order to examine the influence of parameter a and b on the final ranking of the alternatives, an investigation has been initiated by changing them simultaneously. The score values and raking order for different values of a and b are summarized in Table 3. According to Table 4, we can observe that the score values for different couples of the parameter a and b are different; however, the ranking orders of alternatives remain the same. This quality of the suggested operators is especially important in real-world DM problems. For example, it has been observed that as the parameters are increased, the score values of alternatives grow, giving us an optimistic impression of the decision makers. So, if the decision-makers are bullish, greater values for values for a and b can be allocated throughout the processing phase. If the decision-makers are bearish, lower values allocated can be provided to a and b. However, the ranking of alternatives cannot be

Table 2: Summarized rating values of each alternative in terms of CPFNs.

Alternatives	C_1		
A_1	$\left(\left\langle\begin{bmatrix}0.5,\\0.6\end{bmatrix},\begin{bmatrix}0.1,\\0.2\end{bmatrix}\right\rangle,\left\langle\begin{matrix}0.4,\\0.2\end{matrix}\right\rangle\right)$	$\left(\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{bmatrix}0.4,\\0.5\end{bmatrix},\begin{bmatrix}0.3,\\0.2\end{bmatrix}\right)$	$\left(\begin{bmatrix}0.4,\\0.6\end{bmatrix},\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{matrix}0.2,\\0.35\end{matrix}\right)$
A_2	$\left(\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{bmatrix}0.4,\\0.5\end{bmatrix},\begin{matrix}0.4,\\0.6\end{matrix}\right)$	$\left(\begin{bmatrix}0.15,\\0.25\end{bmatrix},\begin{bmatrix}0.3,\\0.35\end{bmatrix},\begin{matrix}0.2,\\0.3\end{matrix}\right)$	$\left(\begin{bmatrix}0.2,\\0.4\end{bmatrix},\begin{bmatrix}0.1,\\0.2\end{bmatrix},\begin{matrix}0.4,\\0.5\end{matrix}\right)$
A_3	$\left(\begin{bmatrix}0.5,\\0.6\end{bmatrix},\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{matrix}0.2,\\0.4\end{matrix}\right)$	$\left(\begin{bmatrix}0.4,\\0.6\end{bmatrix},\begin{bmatrix}0.25,\\0.35\end{bmatrix},\begin{matrix}0.3,\\0.4\end{matrix}\right)$	$\left(\begin{bmatrix}0.5,\\0.7\end{bmatrix},\begin{bmatrix}0.1,\\0.15\end{bmatrix},\begin{matrix}0.3,\\0.5\end{matrix}\right)$
A_4	$\left(\begin{bmatrix}0.3,\\0.5\end{bmatrix},\begin{bmatrix}0.1,\\0.3\end{bmatrix},\begin{matrix}0.1,\\0.3\end{matrix}\right)$	$\left(\begin{bmatrix}0.4,\\0.55\end{bmatrix},\begin{bmatrix}0.15,\\0.2\end{bmatrix},\begin{matrix}0.3,\\0.4\end{matrix}\right)$	$\left(\begin{bmatrix}0.4,\\0.5\end{bmatrix},\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{matrix}0.45,\\0.35\end{matrix}\right)$

Table 3: Decision matrix with normalization.

Alternatives	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$
$\mathscr{A}_1$	$\left(\begin{bmatrix}0.1,\\0.2\end{bmatrix},\begin{bmatrix}0.5,\\0.6\end{bmatrix},\begin{matrix}0.2,\\0.4\end{matrix}\right)$	$\left(\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{bmatrix}0.4,\\0.5\end{bmatrix},\begin{matrix}0.3,\\0.2\end{matrix}\right)$	$\left(\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{bmatrix}0.4,\\0.6\end{bmatrix},\begin{matrix}0.35,\\0.2\end{matrix}\right)$
$\mathscr{A}_2$	$\left(\begin{bmatrix}0.4,\\0.5\end{bmatrix},\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{matrix}0.6,\\0.4\end{matrix}\right)$	$\left(\begin{bmatrix}0.15,\\0.25\end{bmatrix},\begin{bmatrix}0.3,\\0.35\end{bmatrix},\begin{matrix}0.2,\\0.3\end{matrix}\right)$	$\left(\begin{bmatrix}0.1,\\0.2\end{bmatrix},\begin{bmatrix}0.2,\\0.4\end{bmatrix},\begin{matrix}0.5,\\0.4\end{matrix}\right)$
$\mathscr{A}_3$	$\left(\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{bmatrix}0.5,\\0.6\end{bmatrix},\begin{matrix}0.4,\\0.2\end{matrix}\right)$	$\left(\begin{bmatrix}0.4,\\0.6\end{bmatrix},\begin{bmatrix}0.25,\\0.35\end{bmatrix},\begin{matrix}0.3,\\0.4\end{matrix}\right)$	$\left(\begin{bmatrix}0.1,\\0.15\end{bmatrix},\begin{bmatrix}0.5,\\0.7\end{bmatrix},\begin{matrix}0.5,\\0.3\end{matrix}\right)$
$\mathscr{A}_4$	$\left(\begin{bmatrix}0.1,\\0.3\end{bmatrix},\begin{bmatrix}0.3,\\0.5\end{bmatrix},\begin{matrix}0.3,\\0.1\end{matrix}\right)$	$\left(\begin{bmatrix}0.4,\\0.55\end{bmatrix},\begin{bmatrix}0.15,\\0.2\end{bmatrix},\begin{matrix}0.3,\\0.4\end{matrix}\right)$	$\left(\begin{bmatrix}0.2,\\0.3\end{bmatrix},\begin{bmatrix}0.4,\\0.5\end{bmatrix},\begin{matrix}0.35,\\0.45\end{matrix}\right)$

Table 4: Influence on the score values and ranking with the variation of a and b.

a	b	$sc(\mathscr{A}_1)$	$sc(\mathscr{A}_2)$	$sc(\mathscr{A}_3)$	$sc(\mathscr{A}_4)$	Ranking
a = 1	b = 1	−1.0437	−0.9630	−1.1224	−0.8970	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 2	−1.0311	−0.9321	−1.0882	−0.8639	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 3	−1.0158	−0.8927	−1.0418	−0.8252	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 4	−1.0018	−0.8561	−0.9978	−0.7895	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_3 \succ \mathscr{A}_1$
	b = 5	−0.9897	−0.8243	−0.9595	−0.7582	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_3 \succ \mathscr{A}_1$
a = 2	b = 1	−1.0311	−0.9321	−1.0882	−0.8639	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 2	−1.0286	−0.9324	−1.0950	−0.8626	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 3	−1.0190	−0.9111	−1.0720	−0.8421	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 4	−1.0081	−0.8844	−1.0400	−0.8160	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 5	−0.9977	−0.8577	−1.0073	−0.7896	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
a = 3	b = 1	−1.0158	−0.8927	−1.0418	−0.8252	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 2	−1.0190	−0.9111	−1.0720	−0.8421	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 3	−1.0140	−0.9052	−1.0720	−0.8371	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 4	−1.0063	−0.8896	−1.0547	−0.8218	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 5	−0.9982	−0.8702	−1.0308	−0.8024	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
a = 4	b = 1	−1.0018	−0.8561	−1.0078	−0.7895	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 2	−1.0081	−0.8844	−1.0400	−0.8160	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 3	−1.0063	−0.8896	−1.0547	−0.8218	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 4	−1.0014	−0.8837	−1.0530	−0.8164	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 5	−1.0024	−0.8797	−1.0397	−0.8044	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
a = 5	b = 1	−0.9897	−0.8243	−0.9593	−0.7582	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_3 \succ \mathscr{A}_1$
	b = 2	−0.9977	−0.8577	−1.0073	−0.7896	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 3	−0.9982	−0.8702	−1.0308	−0.8024	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 4	−1.0024	−0.8797	−1.0397	−0.8044	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$
	b = 5	−1.0210	−0.8748	−1.0483	−0.7993	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_1 \succ \mathscr{A}_3$

changed, which indicates that the outcomes are objective and cannot be affected by decision-makers' negativity or positivity. As a consequence, the results obtained are legitimate.

By varying the parameter b, the variations of the score values of each alternative are summarized in Figures 2–4 which shows that the greatest score owing alternative remains $\mathscr{A}_4$ for all cases. However, in Figure 3, by holding $a = 1$ and varying b from 0 to 6, it is shown that when $a = 1$ and $b = 3.8580$ then, $sc(\mathscr{A}_1) = sc(\mathscr{A}_3) = -1.0038$, and thus, from (7), we get $ac(\mathscr{A}_1) = 1.0962$ and $ac(\mathscr{A}_3) = 1.1979$, as a result, the ranking order of the alternative for $a = 1$ and

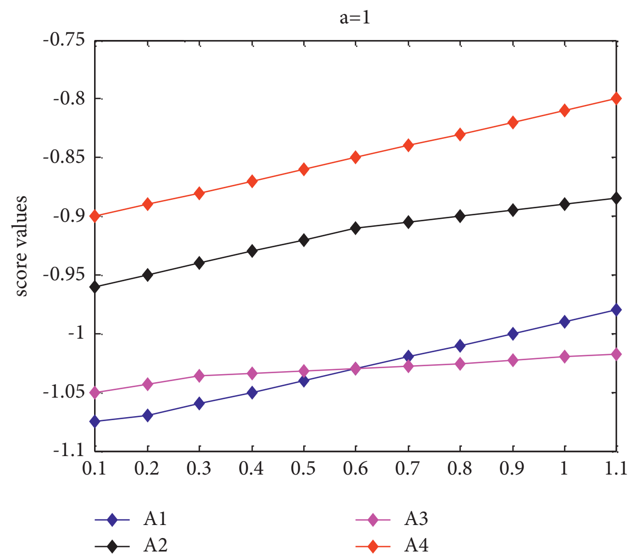

FIGURE 2: The influence of parameter b on the score values by fixing $a = 1$.

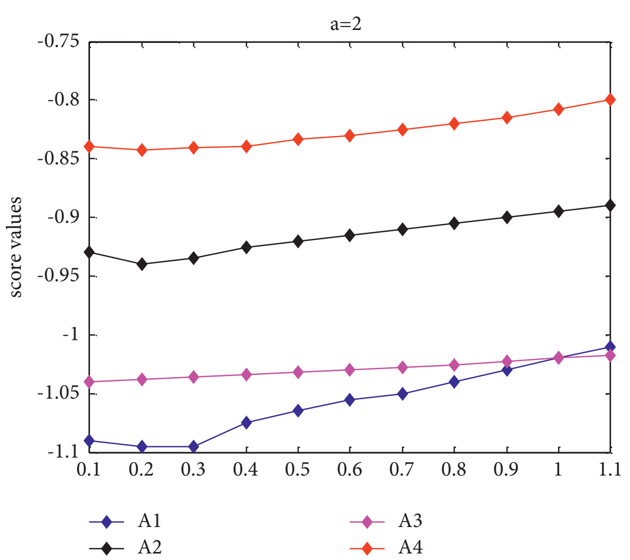

FIGURE 3: The influence of parameter b on the score values by fixing $a = 2$.

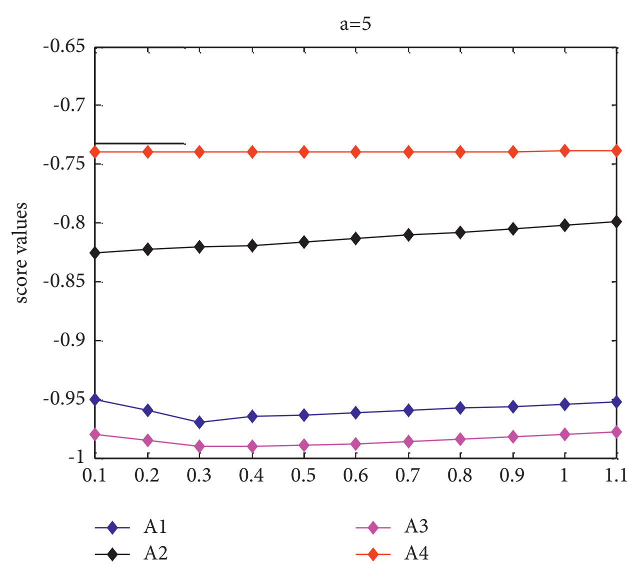

FIGURE 4: The influence of parameter b on the score values by fixing $a = 5$.

$b = 3.8580$ is $\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_3 \succ \mathscr{A}_1$. Similarly, in Figure 3, by holding $a = 2$ and $b = 5.4932$ then $sc(\mathscr{A}_1) = sc(\mathscr{A}_3) = -0.9924$; however, $ac(\mathscr{A}_1) =$ and $ac(\mathscr{A}_3) =$ and hence the ranking order of alternative for $a = 2$ and $b = 5.4932$ is also $\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_3 \succ \mathscr{A}_1$. The performance of score values by fixing $a = 5$ is shown in Figure 4.

5.1. Comparative Analysis. The dominance of the proposed aggregation operators with respect to the previous approaches such as interval-valued intuitionistic fuzzy aggregation operators [16, 45–50] and weighted cubic intuitionistic BM operator [40] is demonstrated. The weight vector is $\Delta = (0.2, 0.38, 0.42)^T$. Table 5 summarizes the optimal score values and rank order of the alternatives. According to this table, we concluded that the optimum alternative corresponds with the proposed approach results that authenticate the constancy of the approach with respect to state-of-the-art. The proposed DM approach under CPF environment covers much more estimation information on the alternatives by considering both PFSs and IVPFSs at the same time, whereas existing approaches include either PFS or IVPFS. Aa a result, the DM approaches under PFS or IVPFS may lose some valuable information, that may affect the DM results. In addition, the research concluded that the

Table 5: Alternatives with various methods are scored and ranked in order.

Approaches	Score values				Ranking
	$\mathscr{A}_1$	$\mathscr{A}_2$	$\mathscr{A}_3$	$\mathscr{A}_4$	
Garg [16]	0.1082	0.1101	0.1230	0.1649	$\mathscr{A}_4 \succ \mathscr{A}_3 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Wan et al. [45]	−0.2610	−0.0705	−0.1835	−0.0100	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_3 \succ \mathscr{A}_1$
Chen et al. [46]	−0.2608	−0.0613	−0.2154	−0.0315	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Sivaraman el al. [47]	−0.2120	−0.0792	−0.1910	0.0214	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Dugenci [48]	0.7106	0.7693	0.7316	0.7940	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Chen et al. [49]	−0.2633	−0.0630	−0.1425	0.0096	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Xu and chen [24]	−0.7152	−0.5847	0.6880	−0.5830	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Shi and hi [27]	−0.2680	−0.0685	−0.2244	−0.0301	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Wang and liu [50]	−0.2593	−0.0635	−0.1552	0.0194	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$
Kaur and garg [40] ($\mathbf{a} = \mathbf{b} = 1$)	−1.2942	−1.1989	−1.3185	−1.1474	$\mathscr{A}_4 \succ \mathscr{A}_2 \succ \mathscr{A}_2 \succ \mathscr{A}_1$

proposed computational process is different from the existing approaches, but the results of the proposed approach are more rational to reality in DM process. Finally, it is observed that the parameters a and b provide more ranges to the decision-makers to avail their required alternatives allocate the different score values of the alternatives for different values of a and b. Therefore, the approaches under CPS and IVPFS can be measured as a special case of the proposed approach.

6. Conclusion

The CPF set is the more flexible and appropriate tool for deal with fuzziness and uncertainties by interconnecting IVPFN and PFN simultaneously in DM problems. The main purpose of this paper is to present an aggregation operator BM whose significant characteristic is to capture the connections among the distinct influences. To accomplish this, we proposed two aggregation operators, i.e., CPFBM operator and WCPFBM operator, to aggregate the various preferences of the experts over the various attributes in the CPF context. Various desired properties of these aggregation operators are also investigated. Moreover, an approach for handling DM issues has been provided by changing the values of parameters a and b. It is observed that the parameter a and b makes the proposed operators more elastic and provides numerous ranges to the decision-maker for evaluating the decisions. A comparison with several existing operators reveals that the presented operators and their accompanying approaches offer the decision maker a more reliable, pragmatic, and optimistic character during the processing phase. As a result, we accomplish that the presented operators may be used to address the in real-world situations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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