Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 9736427, 9 pages
https://doi.org/10.1155/2022/9736427

Research Article

Analytical and Numerical Study to a Forced Van der Pol Oscillator

Alvaro Salas,[1] **Lorenzo J. Martínez H**,[2] **and David L. Ocampo R**[2]

[1]*Universidad Nacional de Colombia, FIZMAKO Research Group, Manizales, Colombia*
[2]*Universidad Nacional de Colombia, Universidad de Caldas, Manizales, Colombia*

Correspondence should be addressed to Alvaro Salas; ahsalass@unal.edu.co

Received 12 November 2021; Revised 4 December 2021; Accepted 27 January 2022; Published 25 March 2022

Academic Editor: Sagheer Abbas

In this paper we show the way to apply some and analytical and numerical techniques in order to solve the forced Van der Pol oscillator. We illustrate the obtained results with examples. A comparison with Runge–Kutta numerical method is made in order to see the accuracy of the approximated analytical solution.

1. Introduction

We consider the following forced Van der Pol oscillator:

$$\begin{aligned} x-\varepsilon\left(1-x^{2}\right)\dot{x}+\alpha x &= F\cos(\Omega t),\\ x(0) &= x_{0},\\ x'(0) &= \dot{x}_{0}, \end{aligned} \tag{1}$$

In the case when $\alpha = 1$ and $\beta = F = 0$ we obtain the well known Van der Pol oscillator [1].

$$\begin{aligned} x-\varepsilon\left(1-x^{2}\right)\dot{x}+x &= 0,\\ x(0) &= x_{0},\\ x'(0) &= \dot{x}_{0}. \end{aligned} \tag{2}$$

Oscillator (2) may be solved using perturbation methods as described.

2. The Lindstedt–Poincaré Method

Let us consider the i.v.p.

$$\begin{aligned} x-\varepsilon\left(1-x^{2}\right)\dot{x}+\omega_0^2 x &= 0,\\ x(0) &= 2,\\ x'(0) &= 0. \end{aligned} \tag{3}$$

The Lindstedt–Poincaré method assumes the solution in the ansatz form

$$\begin{aligned} x(t) &= y_0(\omega t)+\varepsilon y_1(\omega t)+\varepsilon^2 y_2(\omega t)\\ &\quad+\varepsilon^3 y_3(\omega t)+\varepsilon^4 y_4(\omega t)+\cdots, \end{aligned} \tag{4}$$

where

$$\omega^2=\omega_0^2+\varepsilon\omega_1+\varepsilon^2\omega_2+\varepsilon^3\omega_3+\varepsilon^4\omega_4+\cdots. \tag{5}$$

Using this method gives the following solution up to ε^4:

$$x(t) = 2\cos(\omega t) + \left[\frac{3}{4\omega_0}\sin(\omega t) - \frac{1}{4\omega_0}\sin(3\omega t)\right]\varepsilon + \left[\frac{5}{96\omega_0^2}\cos(3\omega t) - \frac{5}{96\omega_0^2}\cos(5\omega t)\right]\varepsilon^2$$
$$+\left[-\frac{7}{256\omega_0^3}\sin(\omega t) + \frac{21}{256\omega_0^3}\sin(3\omega t) - \frac{35}{576\omega_0^3}\sin(5\omega t) + \frac{7}{576\omega_0^3}\sin(7\omega t)\right]\varepsilon^3 +$$
$$\left[-\frac{6301}{276480\omega_0^4}\cos(3\omega t) + \frac{1085}{27648\omega_0^4}\cos(5\omega t) - \frac{2149}{110592\omega_0^4}\cos(7\omega t) + \frac{61}{20480\omega_0^4}\cos(9\omega t)\right]\varepsilon^4, \quad (6)$$

where $\omega = \omega_0^2 - \frac{1}{8}\varepsilon^2 + \frac{23}{1536\omega_0^2}\varepsilon^4$.

This solution is periodic with period $T = 2\pi/\omega$.

3. He's Homotopy Perturbation Method

The homotopy perturbation method has been shown to solve a large class of nonlinear differential problems effectively, easily, and accurately; generally one iteration is enough for engineering applications with acceptable accuracy, making the method accessible to nonmathematicians. However, in this method secularity terms appears. We perform iterations until such terms appear. We construct the following homotopy [2].

$$\ddot{x} + \omega_0^2 x - p\varepsilon\left(1 - x^2\right)\dot{x} + \alpha p(1-p)x = 0, \quad (7)$$

where

$$x(t) = x_0(\omega t) + p x_1(\omega t) + p^2 x_2(\omega t) + p^3 x_3(\omega t) + p^4 x_4(\omega t) + \cdots,$$
$$\omega^2 = \omega_0^2 + p\omega_1 + p^2\omega_2 + p^3\omega_3 + p^4\omega_4 + \cdots. \quad (8)$$

After some algebra one finds that

$$x(t) = \sqrt{4 - c_0^2}\sin(\tau) + c_0\cos(\tau) + \frac{\varepsilon\sin(3\tau)\left(-3\sqrt{4-c_0^2}\,c_0^2\varepsilon + 3\sqrt{4-c_0^2}\,\varepsilon - 8c_0^3\omega_0 + 24c_0\omega_0\right)}{64\omega_0^2}$$
$$+\frac{\varepsilon\cos(3\tau)\left(-3c_0^3\varepsilon + 9c_0\varepsilon + 8\sqrt{4-c_0^2}\,c_0^2\omega_0 - 8\sqrt{4-c_0^2}\,\omega_0\right)}{64\omega_0^2} +$$
$$\frac{5\sqrt{4-c_0^2}\left(c_0^4 - 3c_0^2 + 1\right)\varepsilon^2\sin(5\tau)}{192\omega_0^2} - \frac{5c_0\left(c_0^4 - 5c_0^2 + 5\right)\varepsilon^2\cos(5\tau)}{192\omega_0^2}. \quad (9)$$

Here,

$$\tau = \left(c_1 + \sqrt{\omega_0^2 - \frac{\varepsilon^2}{8}}\right)t\alpha = 0,$$

$$\omega_1 = 0,$$

$$\omega_2 = -\frac{\varepsilon^2}{8}.$$

The constants c_0 and c_1 are determined from nthe initial conditions.

4. The Krylov–Bogoliubov–Mitropolsky Method (KBM)

The Krylov–Bogoliubov–Mitropolsky method (KBM) [2], [3] is a technique to give approximate analytical solution to the weakly nonlinear second-order equation

$$\frac{d^2u}{dt^2} + \omega_0^2 u = \varepsilon f\left(u, \frac{du}{dt}\right). \tag{10}$$

When $\varepsilon = 0$ the solution of (10) may be expressed as

$$u = a\cos(\omega_0 + \theta). \tag{11}$$

where a and θ are constants. For the case when $\varepsilon > 0$ is small, Krylov and Bogoliubov (1947) assumed that the solution is still given by (11) but with time-varying a and θ, and subject to the condition

$$\begin{aligned} \frac{du}{dt} &= -a\omega_0 \sin\phi, \\ \phi &= \omega_0 t + \theta. \end{aligned} \tag{12}$$

In the general case, the solution is assumed in the ansatz form

$$u = a\cos\psi + \sum_{n=1}^{N} \varepsilon^n u_n(a,\psi) + O\left(\varepsilon^{N+1}\right). \tag{13}$$

where each u_n is a periodic function of ψ with a period 2π, and a and ψ are assumed to vary with time according to

$$\begin{aligned} \frac{da}{dt} &= \sum_{n=1}^{N} \varepsilon^n A_n(a) + O\left(\varepsilon^{N+1}\right), \\ \frac{d\psi}{dt} &= \omega_0 + \sum_{n=1}^{N} \varepsilon^n \psi_n(a) + O\left(\varepsilon^{N+1}\right). \end{aligned} \tag{14}$$

In order to uniquely determine A_n and ψ_n, we require that no u_n contains $\cos\psi$. Let $N = 3$. Then,

$$\begin{aligned} \frac{du}{dt} = {} & -a\omega_0 \sin(\psi) + \left(\omega_0 u_{1,\psi} - a\psi_1 \sin(\psi) + A_1\cos(\psi)\right)\varepsilon + \left(A_1 u_{1,a} + \omega_0 u_{2,\psi} + \psi_1 u_{1,\psi} - a\psi_2\sin(\psi) + A_2\cos(\psi)\right)\varepsilon^2 \\ & + \left(A_2 u_{1,a} + A_1 u_{2,a} + \omega_0 u_{3,\psi} + \psi_2 u_{1,\psi} + \psi_1 u_{2,\psi} - a\psi_3\sin(\psi) + A_3\cos(\psi)\right)\varepsilon^3 + \cdots, \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{d^2u}{dt^2} = {} & -a\omega_0^2\cos(\psi) + \left(\omega_0^2 u_{1,\psi\psi} - 2a\psi_1\omega_0\cos(\psi) - 2A_1\omega_0\sin(\psi)\right)\varepsilon + \\ & \left(2A_1\omega_0 u_{1,a\psi} + 2\psi_1\omega_0 u_{1,\psi\psi} + \omega_0^2 u_{2,\psi\psi} + \sin(\psi)\left(-aA_1\dot{\psi}_1 - 2A_1\psi_1 - 2A_2\omega_0\right)t + ncosq(\psi)h\left(A_1\dot{A}_1 - a\left(2\psi_2\omega_0 + \psi_1^2\right)\right)\right)\varepsilon^2 \\ & + \begin{pmatrix} \dot{A}_1 A_1 u_{1,a} + A_1^2 u_{1,aa} + 2A_1\psi_1 u_{1,a\psi} + 2A_1\omega_0 u_{2,a\psi} + 2A_2\omega_0 u_{1,a\psi} + \\ A_1\dot{\psi}_1 u_{1,\psi} + 2\psi_2\omega_0 u_{1,\psi\psi} + 2\psi_1\omega_0 u_{2,\psi\psi} + \psi_1^2 u_{1,\psi\psi} + \omega_0^2 u_{3,\psi\psi} + \\ \sin(\psi)\left(-aA_2\dot{\psi}_1 - aA_1\dot{\psi}_2 - 2A_2\psi_1 - 2A_1\psi_2 - 2A_3\omega_0\right) + \\ \cos(\psi)\left(-2a\psi_3\omega_0 - 2a\psi_1\psi_2 + A_2\dot{A}_1 + A_1\dot{A}_2\right) \end{pmatrix}\varepsilon^3 + \cdots. \end{aligned} \tag{16}$$

Here,

$$\begin{aligned} \dot{\psi}_1 &= \psi_1'(t), \\ u_{2,\psi} &= \frac{\partial u_2}{\partial \psi}, \\ u_{2,\psi\psi} &= \frac{\partial^2 u_2}{\partial \psi^2}, \\ u_{1,a\psi} &= \frac{\partial^2 u_1}{\partial\psi\,\partial a}, \\ & \text{etc.} \end{aligned} \tag{17}$$

Following is the solution obtained using KBM to accuracy $O(\varepsilon^3)$ [2]:

$$\begin{aligned} x &= x(t) \\ &= a\cos(\psi) - \frac{1}{32}a^3\sin(3\psi)\varepsilon \\ &\quad - \left(\frac{\left(a^2+8\right)a^3}{1024}\cos(3\psi) + \frac{5a^5}{3072}\cos(5\psi)\right)\varepsilon^2, \end{aligned} \tag{18}$$

where $a = a(t)$ and $\psi = \psi(t)$ obey the odds

$$a'(t) = \frac{\varepsilon}{2}a(t) - \frac{\varepsilon}{8}a^3(t),$$
$$\psi'(t) = 1 - \frac{\varepsilon^2}{8} - \frac{\varepsilon^2 a(t)^4}{256} + \frac{\varepsilon^2 a(t)^2}{32}, \quad (19)$$

so that

$$a\left(t = \frac{2e^{\varepsilon t/2}}{\sqrt{e^{8c_1} + e^{\varepsilon t}}}\right),$$
$$\psi(t) = \frac{1}{16}\left(-\frac{e^{8c_1}\varepsilon}{e^{8c_1} + e^{\varepsilon t}} + \varepsilon \log\left(e^{8c_1} + e^{\varepsilon t}\right) - 2\left(\varepsilon^2 - 8\right)t\right) + c_2. \quad (20)$$

The constants c_1 and c_2 are determined from the initial conditions $x(0) = x_0$ and $x'(0) = \dot{x}_0$. For an illustration, see Figure 1 and Figure 2.

4.1. A More Accurate Analytical Solution. Let

$$u - \varepsilon\left(1 - u^2\right)\dot{u} + \omega_0^2 u = 0,$$
$$u(0) = u_0, \quad (21)$$
$$u'(0) = \dot{u}_0.$$

In view of the KBM, we assume the ansatz form

$$u = a \cos \psi + \sum_{n=1}^{3} \varepsilon^n u_n(a, \psi) + O\left(\varepsilon^4\right). \quad (22)$$

Hence,

$$\frac{da}{dt} = \sum_{n=1}^{3} \varepsilon^n A_n(a) + O\left(\varepsilon^4\right),$$
$$\frac{d\psi}{dt} = \omega_0 + \sum_{n=1}^{3} \varepsilon^n \psi_n(a) + O\left(\varepsilon^4\right). \quad (23)$$

Making use of (15) and (16), we will have

$$\ddot{u} - \varepsilon\left(1 - u^2\right)\dot{u} + \omega_0^2 u = H_1\varepsilon + H_2\varepsilon^2 + H_3\varepsilon^3 + O\left(\varepsilon^4\right), \quad (24)$$

where

$$H_1 = \frac{1}{4}\left(4\omega_0^2 u_{1,\psi\psi} + 4u_1\omega_0^2\right) + \frac{1}{4}\sin(\psi)\left(-\omega_0 a^3 + 4\omega_0 a - 8A_1\omega_0\right) - \frac{1}{4}a^3\omega_0 \sin(3\psi) - 2a\psi_1\omega_0\cos(\psi), \quad (25)$$

$$\begin{aligned} H_2 = {} & \frac{1}{4}\left(2a^2\omega_0 u_{1,\psi} + 8A_1\omega_0 u_{1,a\psi} + 8\psi_1\omega_0 u_{1,\psi\psi} - 4\omega_0 u_{1,\psi} + 4\omega_0^2 u_{2,\psi\psi} + 4u_2\omega_0^2\right) + \frac{1}{2}a^2\omega_0\cos(2\psi)u_{1,\psi} \\ & + \frac{1}{4}\sin(\psi)\left(a^3(-\psi_1) - 4aA_1\dot{\psi}_1 + 4a\psi_1 - 8A_1\psi_1 - 8A_2\omega_0\right) - \\ & \frac{1}{4}a^3\psi_1\sin(3\psi) + \frac{1}{4}\cos(\psi)\left(3a^2A_1 - 8a\psi_2\omega_0 - 4a\psi_1^2 - 4A_1 + 4A_1\dot{A}_1\right) + \frac{1}{4}a^2A_1\cos(3\psi) - a^2u_1\omega_0\sin(2\psi), \end{aligned} \quad (26)$$

$$\begin{aligned} H_3 = {} & \cos(\psi)\left(2au_1\omega_0 u_{1,\psi} + \frac{3a^2A_2}{4} - 2a\psi_3\omega_0 - 2a\psi_1\psi_2 - A_2 + A_2\dot{A}_1 + A_1\dot{A}_2\right) \\ & + \cos(2\psi)\left(\frac{1}{2}a^2A_1u_{1,a} + \frac{1}{2}a^2\omega_0u_{2,\psi} + \frac{1}{2}a^2\psi_1u_{1,\psi} + aA_1u_1\right) + \frac{1}{2}a^2A_1u_{1,a} + \frac{1}{2}a^2\omega_0u_{2,\psi} + \frac{1}{2}a^2\psi_1u_{1,\psi} - A_1u_{1,a} \\ & + A_1\dot{A}_1u_{1,a} + A_1^2u_{1,aa} + 2A_1\psi_1u_{1,a\psi} + 2A_2\omega_0u_{1,a\psi} + 2A_1\omega_0u_{2,a\psi} + A_1\dot{\psi}_1u_{1,\psi} + 2\psi_2\omega_0u_{1,\psi\psi} + \\ & 2\psi_1\omega_0u_{2,\psi\psi} + \psi_1^2u_{1,\psi\psi} - \omega_0u_{2,\psi} - \psi_1u_{1,\psi} + \omega_0^2u_{3,\psi\psi} + \sin(\psi) \\ & \left(-\frac{1}{4}a^3\psi_2 - aA_2\dot{\psi}_1 - aA_1\dot{\psi}_2 - au_1^2\omega_0 + a\psi_2 - 2A_2\psi_1 - 2A_1\psi_2 - 2A_3\omega_0\right) \\ & - \frac{1}{4}a^3\psi_2\sin(3\psi) + \frac{1}{4}a^2A_2\cos(3\psi) + \sin(2\psi)\left(a^2(-u_1)\psi_1 - a^2u_2\omega_0\right) + aA_1u_1 + u_3\omega_0^2. \end{aligned} \quad (27)$$

Equating to zero the coefficients of $\cos(\psi)$ and $\sin(\psi)$ in (25) gives

$$A_1 = \frac{1}{8}\left(4a - a^3\right),$$
$$\psi_1 = 0. \quad (28)$$

From the condition $H_1 = 0$, we get the second order linear ode

$$\frac{1}{4}\left(4\omega_0^2 u_{1,\psi\psi} + 4u_1\omega_0^2\right) - \frac{1}{4}a^3\omega_0\sin(3\psi) = 0. \quad (29)$$

The general solution to the ode (29) reads

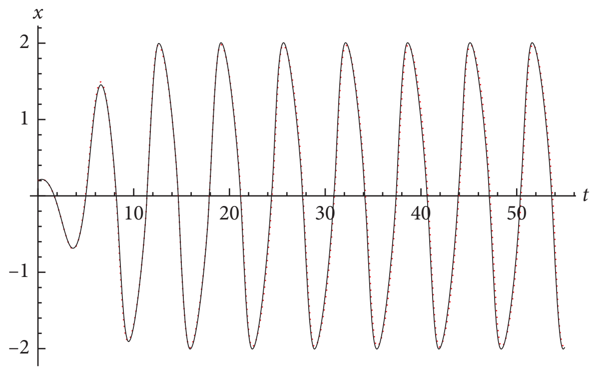

Figure 1: The Vander Pol oscillator for $\varepsilon = 0.75$.

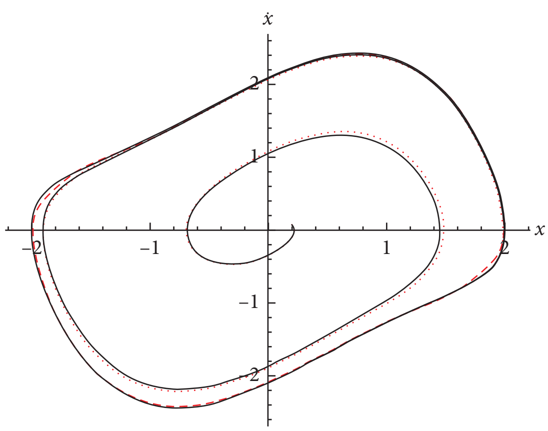

Figure 2: The Vander Pol oscillator for $\varepsilon = 0.75$.

$$u_1(a,\psi) = \frac{32c_1\omega_0^2}{32\omega_0^2}\cos(\psi) + \frac{\left(32c_2\omega_0^2 - 2a^3\omega_0\right)}{32\omega_0^2}\sin(\psi) - \frac{a^3}{32\omega_0}\sin(3\psi). \tag{30}$$

Define

$$c_1 = 0, \qquad c_2 = \frac{1}{16\omega_0}a^3. \tag{31}$$

Then,

$$u_1 = u_1(a,\psi) = -\frac{1}{32\omega_0}a^3\sin(3\psi). \tag{32}$$

Replacing the expressions (31) and (32) into (26), we obtain

$$H_2 = \frac{128\omega_0^4 u_2(a,\psi) + 128\omega_0^4 u_0^{(0,2)}(a,\psi)}{128\omega_0^2} - \frac{5a^5\omega_0^2}{128\omega_0^2}\cos(5\psi) - \frac{256A_2\omega_0^3}{128\omega_0^2}\sin(\psi) - \frac{1}{32}a^3\left(a^2+2\right)\cos(3\psi) + \left(\frac{3}{16}a^3 - \frac{1}{4}a - \frac{5}{128}a^5 - 2a\omega_0\psi_2\right)\cos(\psi), \tag{33}$$

Equating to zero the coefficients of $\sin(\psi)$ and $\sin(\psi)$ in (33), we get

$$A_2 = 0, \qquad \psi_2 = -\frac{1}{256\omega_0}\left(5a^4 - 24a^2 + 32\right). \tag{34}$$

From the condition $H_2 = 0$ we obtain the following linear second order ode

$$\omega_0^2\left(u_2(a,\psi) + u_2^{(0,2)}(a,\psi)\right) - \frac{5a^5\omega_0^2}{128\omega_0^2}\cos(5\psi) - \frac{1}{32}a^3\left(a^2+2\right)\cos(3\psi) = 0. \tag{35}$$

The general solution to the ode (35) is given by

$$u_2 = u_2(a,\psi) = -\frac{5a^5\omega_0^2}{3072\omega_0^4}\cos(5\psi) + c_4\sin(\psi) - \frac{1}{256}\frac{a^3}{\omega_0^2}\left(a^2+2\right)\cos(3\psi) - \frac{1}{128\omega_0^2}\left(a^5 + 2a^3 - 128c_3\omega_0^2\right)\cos(\psi). \tag{36}$$

Define

$$c_3 = \frac{1}{128}\frac{a^3}{\omega_0^2}\left(a^2+2\right), \qquad c_4 = 0. \tag{37}$$

Then,

$$u_2 = u_2(a,\psi) = -\frac{5a^5\omega_0^2}{3072\omega_0^4}\cos(5\psi) - \frac{1}{256}\frac{a^3}{\omega_0^2}\left(a^2+2\right)\cos(3\psi). \tag{38}$$

Finally, we must have $H_3 = 0$. Proceeding in a similar manner as before, we get

$$A_3 = -\frac{1}{8192\omega_0^2}\left(25a^7 - 184a^5 + 320a^3\right),$$

$$\psi_3 = 0. \tag{39}$$

$$u_3 = \frac{28a^7\sin(7\psi) - 6\left(71a^2+280\right)a^5\sin(\psi) + 5\left(3a^2+56\right)a^5\sin(5\psi) - 9\left(29a^4-168a^2+64\right)a^3\sin(3\psi)}{294912\omega_0^3}.$$

We thus have that

$$u = a\cos(\psi) - \frac{a^3\sin(3\psi)}{32\omega_0}\varepsilon - \left(\frac{5a^5\cos(5\psi)}{3072\omega_0^2} + \frac{\left(a^2+8\right)a^3\cos(3\psi)}{1024\omega_0^2}\right)\varepsilon^2 + \left(\frac{7a^7}{73728\omega_0^3}\sin(7\psi) + \frac{5\left(3a^2+56\right)a^5}{294912\omega_0^3}\sin(5\psi) - \frac{\left(29a^4-168a^2+64\right)a^3}{32768\omega_0^3}\sin(3\psi)\right)\varepsilon^3 + O\left(\varepsilon^4\right). \tag{40}$$

Also, from (23),

$$a\prime(t) = \frac{1}{2}\varepsilon a(t) - \left(\frac{5\varepsilon^3}{128\omega_0^2} + \frac{\varepsilon}{8}\right)a(t)^3 + \frac{23\varepsilon^3}{1024\omega_0^2}a(t)^5 - \frac{25\varepsilon^3}{8192\omega_0^2}a(t)^7, \tag{41}$$

Ode (41) is hard to solve in closed form. We use the approximation

$$\frac{a\varepsilon}{2} - \left(\frac{5\varepsilon^3}{128\omega_0^2} + \frac{\varepsilon}{8}\right)a^3 + \frac{23\varepsilon^3}{1024\omega_0^2}a^5 - \frac{25\varepsilon^3}{8192\omega_0^2}a^7 \approx ra + sa^3 \text{for} \quad |a| \le M,$$

where,

$$r = \frac{\varepsilon}{2} + \frac{\varepsilon^3\left(25M^2-184\right)M^4}{65536\omega_0^2}, \tag{42}$$

$$s = -\frac{\varepsilon}{8} - \frac{\varepsilon^3\left(175M^4 - 1472M^2 + 2560\right)}{65536\omega_0^2}.$$

Solving the ode

$$\frac{da}{dt} = ra + sa^3, \tag{43}$$

one gets

$$a = a(t) = \pm\frac{a_0\sqrt{r}e^{rt}}{\sqrt{r + a_0^2 s\sqrt{1 - a_0^2 se^{2rt}/a_0^2 s + r}}}. \tag{44}$$

The expression for $\psi = \psi(t)$ is obtained from the ode $\psi\prime(t) = \omega_0 - \varepsilon^2/8\omega_0 + \varepsilon^2/32\omega_0 a(t)^2 - \varepsilon^2/256\omega_0 a(t)^4$:

$$\psi(t)=\frac{1}{512s^2\omega_0}\left(\left(r-64s^2t-(r+8s)\log\left(1-\frac{a_0^2s}{r}\left(e^{2rt}-1\right)\right)\right)\varepsilon^2-\frac{\varepsilon^2r\left(a_0^2s+r\right)}{r-a_0^2s\left(e^{2rt}-1\right)}+s\left(a_0^2\varepsilon^2+512s\omega_0\left(\omega_0t+\psi_0\right)\right)\right). \quad (45)$$

Example 1. Let

$$\begin{aligned} u_0 &= u(0)\\ u\prime(0) &= \dot{u}_0\\ \varepsilon &= 0.75,\\ \omega_0 &= 1. \end{aligned} \quad (46)$$

The i.v.p to be solved reads

$$\begin{aligned} u-0.75\left(1-u^2\right)\dot{u}+u &= 0,\\ u(0) &= -0.104,\\ u\prime(0) &= -0.25. \end{aligned} \quad (47)$$

From (43),

$$\begin{aligned} r &= 0.366348,\\ s &= -0.0903511. \end{aligned} \quad (48)$$

We have

$$\begin{aligned} a(t) &= \frac{0.251949e^{0.366348t}}{\sqrt{1+0.0156555e^{0.732697t}}},\\ \psi(t) &= 0.239256\Bigg(-0.0903511\,(0.0351563-46.2598\,(t+2))\\ &\quad -\frac{0.07433}{0.00564694\left(e^{0.732697t}-1\right)+0.366348}+-0.522453t\\ &\quad -0.5625\left(0.356461\left(-\log\left(0.00564694\left(e^{0.732697t}-1\right)\right.\right.\right.\\ &\quad \left.\left.\left.+0.366348\right)-1.00417\right)+0.366348\right)\Bigg). \end{aligned} \quad (49)$$

See Figure 3.

5. Analytical Solution to a Forced Van der Pol Oscillator

Let us consider the i v p.

$$\ddot{x}-\varepsilon\left(1-x^2\right)\dot{x}+\omega_0^2x=F\cos(\Omega t). \quad (50)$$

given that

$$\begin{aligned} x(0) &= x_0,\\ x\prime(0) &= \dot{x}_0. \end{aligned} \quad (51)$$

We will assume the ansatz

$$x(t)=u(t)+C\cos(\Omega t)+D\sin(\Omega t). \quad (52)$$

Here, the function $u=u(t)$ is the solution to the i v p

$$\begin{aligned} u-\varepsilon\left(1-u^2\right)\dot{u}+\omega_0^2u &= 0,\\ u(0)=u_0 &= x_0-C,\\ u\prime(0)=\dot{u}_0 &= \dot{x}_0-D\Omega\,. \end{aligned} \quad (53)$$

We now will determine suitable values for the constants C and D. We have

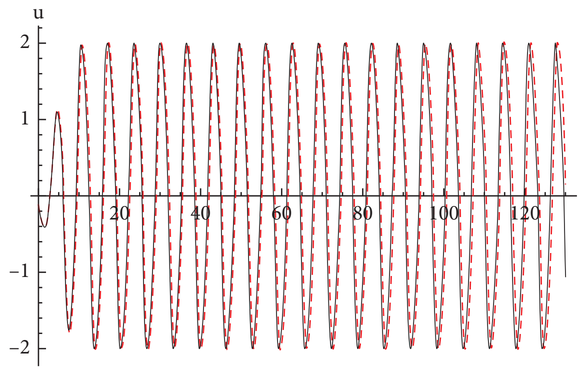

Figure 3: Comparison between analytical and numerical solution.

$$\begin{aligned}x-\varepsilon\left(1-x^{2}\right)\dot{x}+\omega_0^2x-F\cos(\Omega t)={}&\omega_0^2u(t)-\varepsilon\sin(2\Omega t)\left(C^2\Omega u(t)-C\,D\dot{u}_0-D^2\Omega u(t)\right)+\\&\frac{1}{2}\varepsilon\cos(2\Omega t)\left(C^2\dot{u}_0+4C\,D\Omega\,u(t)-D^2\dot{u}_0\right)+\frac{1}{2}\varepsilon\dot{u}_0\left(C^2+D^2\right)-\frac{1}{4}\cos(\Omega t)\\&\cdot\left(-4\varepsilon u(t)^2(D\Omega)-8C\varepsilon\dot{u}_0u(t)-C^2D\varepsilon\,\Omega-4C\omega_0^2+4C\Omega^2-D^3\varepsilon\Omega+4\,D\varepsilon\,\Omega+4F\right)-\\&\frac{1}{4}\varepsilon\sin(3\Omega t)\left(C^3\Omega-3CD^2\Omega\right)+\frac{1}{4}\varepsilon\cos(3\Omega t)\left(3C^2D\Omega-3\beta CD^2-D^3\Omega\right)-\frac{1}{4}\sin(\Omega t)\\&\cdot\left(4C\Omega\varepsilon u(t)^2-8\,D\varepsilon\,\dot{u}_0u(t)+C^3\varepsilon\Omega-3\beta C^2D\varepsilon+CD^2\varepsilon\Omega-4\,D\omega_0^2+4\,D\Omega^2-4C\varepsilon\Omega\right).\end{aligned}\tag{54}$$

We choose the values of C and D so that

$$\begin{aligned}-C^2D\varepsilon\,\Omega-4C\omega_0^2+4C\Omega^2-D^3\varepsilon\Omega+4\,D\varepsilon\,\Omega+4F&=0,\\C^3\varepsilon\Omega-3\beta C^2D\varepsilon+CD^2\varepsilon\Omega-4\,D\omega_0^2+4\,D\Omega^2-4C\varepsilon\Omega&=0.\end{aligned}\tag{55}$$

From (55), it follows that

$$\varepsilon^2F^2\Omega^2C^3+8\varepsilon^2F\Omega^2\left(\Omega^2-\omega_0^2\right)C^2+16\left(\Omega^2-\omega_0^2\right)^2\left(\varepsilon^2\Omega^2+\Omega^4-2\Omega^2\omega_0^2+\omega_0^4\right)C+16F\left(\Omega^2-\omega_0^2\right)^3=0.\tag{56}$$

We choose the least in magnitude real root C to (56) and the least in magnitude real root D to (57).

Example 2. Let $0\le t\le T=130$, $\varepsilon=0.1$, $\omega_0=1$, $F=1,\Omega=5.2$, $x_0=0$ and $\dot{x}_0=-0.183$. Consider the i.v.p.

$$\ddot{x}(t)-0.1\left(1-x(t)^2\right)\dot{x}(t)+x(t)=\cos(5.2t)\wedge x(0)=0\wedge x'(0)=-0.183.\tag{57}$$

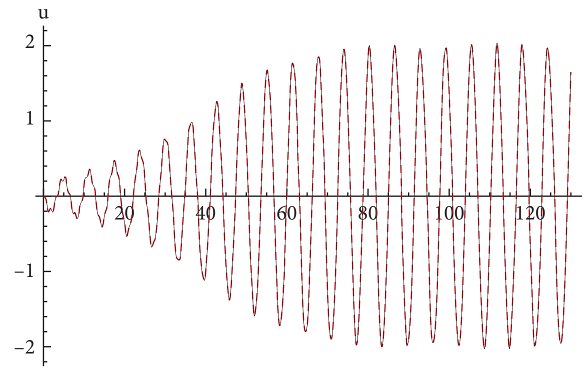

Figure 4: Comparison between the approximate analytical solution (dashed curve) and the Runge–Kutta numerical.

Our calculations give

$$\begin{aligned} a_0 &= 0.185483, \\ \psi_0 &= 4.50402, \\ C &= -0.0383872.\ D = -0.000766281. \\ r &= 0.049979, \\ s &= -0.0124918. \end{aligned} \tag{58}$$

$$a(t) = -\frac{0.186053e^{0.049979t}}{\sqrt{1+0.00865188e^{0.099958t}}}.$$

$$\psi(t) = 0.00625262\ \log\left(0.0495503 + 0.000428704e^{0.099958t}\right) + 0.99875t +$$

$$= \frac{1}{-1.38308e^{0.099958t} - 159.859} + 4.52869.$$

The [4] expression for $u = u(t)$ is [5] obtained from (40). In Figure 4, we compare[6] the approximate [7] analytical solution (dashed curve) with the Runge–Kutta numerical solution. The error equals 0.0756086.

Data Availability

No data were used to support this paper

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] B. V. D. Pol, "On a type of osciuation hysteresis in a simple triode generator," *Philosophical Magazine*, vol. 43, pp. 177–193, 1922.
[2] H. A. Nayfeh, *Perturbation Methods*, Wiley, NJ, USA, 1973.
[3] N. Krylow and N. N. Bogoliubov, *Introduction to Nonlinear Mechanics*, Princeton University Press, NJ, USA, 1947.
[4] F. Verlhultst, *Nonlinear Differential Equations and Dynamical Systems*, Springer-Verlag, Heidelberg, Germany, 1996.
[5] A. R. E. Oliveira, "History of krylov-bogoliubov-mitropolsky methods of nonlinear oscillations," *Advances in Historical Studies*, vol. 6, no. 1, pp. 40–55, 2017.
[6] J. H. He, "Homotopy Perturbation Method with an Auxiliary Term," *Abstract and Applied Analysis*, vol. 20127 pages, 2012, https://doi.org/10.1155/2012/857612, Article ID 857612.
[7] H. Alvaro and S. Salas, "Analytical Approximant to a Quadratically Damped Duffing Oscillator," *The Scientific World Journal*, vol. 2022, Article ID 3131253, 10 pages, 2022.