Hindawi
Mathematical Problems in Engineering
Volume 2023, Article ID 1821347, 14 pages
https://doi.org/10.1155/2023/1821347

Research Article

On Statistical Inference of Generalized Pareto Distribution with Jointly Progressive Censored Samples with Binomial Removal

Hanan Haj Ahmad [1] **and Ehab M. Almetwally** [2,3]

[1] *Department of Basic Science, Preparatory Year Deanship, King Faisal University, Hofuf, Al-Ahsa 31982, Saudi Arabia*
[2] *Faculty of Business Administration, Delta University for Science and Technology, Gamasa 11152, Egypt*
[3] *The Scientific Association for Studies and Applied Research (SASAR), Al Manzalah, Egypt*

Correspondence should be addressed to Hanan Haj Ahmad; hananahm1@yahoo.com

Received 1 June 2022; Revised 7 July 2022; Accepted 13 July 2022; Published 21 April 2023

Academic Editor: Naeem Jan

A jointly censored sample is a very useful sampling technique in conducting comparative life tests of the products, its efficiency appears in permitting the selection of two samples from two manufacturing lines at the same time and conducting a life-testing experiment. This article presents estimation information of the joint generalized Pareto distributions parameters using Type-II progressive censoring scheme, which is carried out with binomial removal. The generalized Pareto distribution has many applications in different fields. We outline the problem of parameter estimation using the frequentest maximum likelihood and the Bayesian estimation methods. Furthermore, different interval estimation methods for estimating the four parameters were used: the asymptotic property of the maximum likelihood estimator, the credible confidence intervals, and the Bootstrap confidence intervals. The detailed numerical simulations have been considered to compare the performance of the proposed estimates. In addition, the applicability of the joint generalized Pareto censored model has been performed by applying a real data example.

1. Introduction

In many fields of manufacturing, the products may come from more than one production line under the same processing environment. Hence, selecting two samples from two production lines and conducting a test on a life-testing experiment is essential, therefore a jointly censored sample is quite useful in conducting comparative life tests of the products. Assume the two test samples are independent with sizes m and n, and they are selected from two production lines, then they are located simultaneously on a life-testing experiment. Furthermore, to optimize the cost and the experimental time of the economic life test procedure, suppose that we implement a joint progressive Type-II censoring (JPC) scheme and end the life-testing experiment once r failures occur. In this sequence, we are concerned with developing both point and interval estimation of the unknown parameters of the lifetime density function, and hence estimate the mean lifetimes of units manufactured by the two lines. In this article, we used the JPC scheme which was described by Rasouli and Balakrishnan [1]. The units under consideration are following two parameters generalized Pareto (GP) lifetime distribution. Although much work has been performed on different types of the progressive censoring schemes for one sample, few articles discussed the idea of two sample problems. Inference on the joint Type-II censoring scheme have been discussed earlier in the literature. See for example, Basu [2]; Johnson and Mehrotra [3]; Bhattacharyya and Mehrotra [4]; and Bhattacharyya [5], who have reviewed all issues related to this model. Recently, Balakrishnan and Rasouli [6] developed exact inferential methods based on maximum likelihood estimates (MLEs) and compared their performance with those based on approximate Bayesian and bootstrap methods. In 2010, Rasouli and Balakrishnan generalized the model to be a joint progressive Type-II censoring scheme with two exponential lifetime distributions. Ashour and Eraki [7] used the joint Type-II censoring idea for estimating the parameters of Weibull populations, see also Parsi et al. [8]; Doostparast et al. [9]; Balakrishnan et al. [10]; Mondal and Kundu [11]; Algarni et al. [12]; Shrahili et al. [13]; Alotaibi et al. [14].

GP distribution has many applications and it can model many real life distributions, recently many authors studied GP distribution, for example, one may refer to Martín et al. [15], who discussed baseline methods for the parameter estimation. Huang et al. [16] obtained statistical inference of dynamic conditional GP distribution with weather and air quality factors. Shui et al. [17] discussed outlier-robust truncated maximum likelihood parameter estimators of GP distribution. He et al. [18] introduced risk analysis by GP distribution. Mahgoun et al. [19] discussed GP distribution exploited for ship detection as a model for sea clutter in a Pol-SAR application.

In the present article, we aimed to work on a joint progressive censored data under GP lifetime units. Since not much work had been performed regarding the joint progressive censored samples under GP distribution with binomial removal of the censored units, we will focus our work on making statistical inference for the unknown parameters of GP distribution. Therefore, both frequentest and Bayesian point and interval estimation methods are investigated. Numerical techniques were used to compare the performance of estimation methods to select the most efficient one. Simulation analysis is implemented to obtain the point and interval estimation for the unknown parameters of the GP distribution. Monte Carlo simulation and Gibbs sampling techniques were used to generate samples from the joint GP distribution under the performed scheme, hence the simulation experiments can be obtained easily. Finally real data analysis is achieved to illustrate the purpose of this study.

The rest of the article is organized as follows. In Section 2, model description is given. Point estimation methods are studied in Section 3, while confidence intervals are obtained in Section 4. Data analysis and simulation are used in Section 5 to facilitate comparison between different types of point and interval estimation of GP parameters and a real data set is performed to check the advantage of the new scheme over the old one. Finally, an optimal censoring scheme is suggested in Section 6.

2. Model Description

Suppose $X_1, X_2, \ldots, X_m$ denote the ordered lifetimes of m units of population 1, and it is assumed that they are independent and identically distributed (i.i.d) from generalized Pareto distribution (GP) with shape and scale parameters θ_1 and λ_1, respectively. Similarly, it is assumed that $Y_1, Y_2, \ldots, Y_n$ denote the ordered lifetimes of n units of population 2, and it is assumed that they are independent and identically distributed (i.i.d) from generalized Pareto distribution with shape and scale parameters θ_2 and λ_2, respectively.

The generalized Pareto probability density function (pdf) is given by the following equation:

$$f(x;\theta,\lambda) = \begin{cases} \dfrac{1}{\lambda}\left(1+\dfrac{\theta x}{\lambda}\right)^{-1/\theta-1}, & \theta \neq 0, \\ \\ \dfrac{1}{\lambda} e^{-x/\lambda}, & \theta = 0, \end{cases} \tag{1}$$

and its cumulative density function (CDF) is as follows:

$$F(x;\theta,\lambda) = \begin{cases} 1-\left(1+\dfrac{\theta x}{\lambda}\right)^{-1/\theta}, & \theta \neq 0, \\ \\ 1-e^{-x/\lambda}, & \theta = 0, \end{cases} \tag{2}$$

where $\lambda > 0$ and for $\theta > 0$, $x > 0$, and for $\theta < 0$, $0 < x < -\lambda/$. For $\theta > 0$, the GP distribution is one of the several forms of the usual Pareto family of distribution often called the Pareto distribution. For <0, the support of the distribution is $0 < x < -(\lambda/\theta)$, and the GP distribution has bounded support. For $\theta \longrightarrow 0$, the GP distribution reduces to the exponential distribution. The special case where $\theta = -1$ corresponds to the uniform distribution $U(0,\lambda)$. In this work, we considered the case when $\theta > 0$, with support $x > 0$.

For the joint progressive censored sampling scheme described by Rasouli and Balakrishnan [1], assume $W_1 \leq \ldots \leq W_N$ are the ordered statistics of the $N = m + n$ random variables $\{X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n\}$. A JPC scheme between the two samples is described as follows. At the time of the first failure (can be from either X or Y), R_1 units are randomly withdrawn from the remaining $N-1$ surviving units. Next, at the time of the second failure (can be from either X or Y), R_2 units are randomly withdrawn from the remaining $N - R_1 - 2$ surviving units, and so on. Finally, at the time of the rth failure (can be again from either X or Y), all remaining $R_r = N - r - R_1 - \ldots - R_{r-1}$ surviving units are withdrawn from the life-testing experiment. Let $Ri = S_i + T_i$ and $i = 1, \ldots, r$, where S_i and T_i are the number of units withdrawn at the time of the ith failure that belong to X and Y sample, respectively, and they are unknown random variables. The data observed in this form will consist of (Z, W, S), where $W = (W_1, \ldots, W_r)$ with $r < N$ is a fixed integer, $Z = Z_1, \ldots, Z_r$, where $Z_j = 1$ if the j^{th} failure takes place from population 1 and $Z_j = 0$, and $S = (S_1, \ldots, S_r)$. The progressive Type-II censoring scheme $R = (R_1, \ldots R_r)$ has the decomposition $S + T = (S_1, \ldots, S_r) + (T_1, \ldots T_r)$.

The likelihood function of the joint progressive censored sample under generalized Pareto lifetime (JGP) can be written as follows:

$$L\left(\theta_1,\theta_2,\lambda_1,\lambda_2 \mid \mathbf{w},\mathbf{z},\mathbf{s}\right) = C\prod_{i=1}^{r}\frac{1}{\lambda_1^{z_i}}\frac{1}{\lambda_2^{1-z_i}}\left(1+\frac{\theta_1 w_i}{\lambda_1}\right)^{-z_i\left(1/\theta_1+1\right)-s_i/\theta_1}\left(1+\frac{\theta_2 w_i}{\lambda_2}\right)^{-\left(z_i-1\right)\left(1/\theta_2+1\right)-\left(t_i/\theta_2\right)}, 0\le w_1\le \ldots \le w_r, \tag{3}$$

where

$$C = \prod_{i=1}^{r}\left[\left(m-\sum_{j=1}^{i-1}\left(z_j-s_j\right)\right)z_i+\left(n-\sum_{j=1}^{i-1}\left(\left(R_j-s_j\right)-\left(1-z_j\right)\right)\right)\left(1-z_i\right)\right] \times\prod_{i=1}^{r-1}\frac{\binom{m-\sum_{j=1}^{i-1}\left(z_j-s_j\right)}{s_i}\binom{n-\sum_{j=1}^{i-1}\left(\left(R_j-s_j\right)-\left(1-z_j\right)\right)}{t_i}}{\binom{m+n-i-\sum_{j=1}^{i-1}R_j}{R_i}}, \tag{4}$$

and $m_r = \sum_{i=1}^{r} z_i$ and $n_r = \sum_{i=1}^{r}\left(1-z_i\right) = r-m_r$.

In the following section, we provided the point inference for two GP populations under the joint progressive censoring scheme. We obtained the maximum likelihood estimators (MLEs) and Bayes estimators of the unknown parameters; numerical methods will be used to obtain the estimated parameters.

3. Point Estimations

In this section, we will use likelihood inference together with the nonclassical Bayesian estimation method. Numerical methods was used to solve some nonlinear equations since it is impossible to write it in explicit forms. These methods will be used in Section 4 to obtain exact and approximate confidence intervals for the unknown parameters.

3.1. Maximum Likelihood Estimators (MLEs). The maximum likelihood estimation (MLE) is commonly used inferential statistics, MLE has many nice properties, such as invariance, consistency, and normal approximation properties. It depends basically on maximizing the likelihood function of distribution under consideration. Assume the log-likelihood function of the unknown parameters $\theta_1,\theta_2,\lambda_1$, and λ_2 is denoted by $\ell\left(\gamma;\ \mathbf{w},\mathbf{z},\mathbf{s}\right)$, where $\gamma = \left(\theta_1,\theta_2,\lambda_1,\lambda_2\right)$ is a vector of parameters. Now, taking partial derivatives of $\ell\left(\gamma;\ \mathbf{w},\mathbf{z}\right)$ with respect to the unknown parameters, we obtained the following equation:

$$\begin{aligned}\frac{\partial\ell\left(\gamma;\mathbf{w},\mathbf{z},\mathbf{s}\right)}{\partial\theta_1} &= -\sum_{i=1}^{r}\left[\left(\left(\frac{1}{\theta_1}+1\right)z_i+\frac{s_i}{\theta_1}\right)\frac{1}{\left(\lambda_1/w_i\right)+\theta_1}-\left(\frac{z_i+s_i}{\theta_1^2}\right)\ln\left(1+\theta_1\frac{w_i}{\lambda_1}\right)\right],\\ \frac{\partial\ell\left(\gamma;\mathbf{w},\mathbf{z},\mathbf{s}\right)}{\partial\theta_2} &= -\sum_{i=1}^{r}\left[\left(\left(\frac{1}{\theta_2}+1\right)\left(1-z_i\right)+\frac{t_i}{\theta_2}\right)\frac{1}{\left(\lambda_2/w_i\right)+\theta_2}-\left(\frac{\left(1-z_i\right)+t_i}{\theta_2^2}\right)\ln\left(1+\theta_2\frac{w_i}{\lambda_2}\right)\right].\end{aligned} \tag{5}$$

Solving this equation by equating it to zero will give $\widehat{\theta}_1$ and $\widehat{\theta}_2$. Numerical methods such as Newton-Raphson was suggested to be used to solve the above system of nonlinear equations. Now, for estimating λ_1 and λ_2, take the partial derivative with respect to λ_1 and λ_2 as follows:

$$\begin{aligned}\frac{\partial\ell\left(\gamma;\mathbf{w},\mathbf{z},\mathbf{s}\right)}{\partial\lambda_1} &= -\frac{m_r}{\lambda_1}+\sum_{i=1}^{r}\left(\left(\frac{1}{\theta_1}+1\right)z_i+\frac{s_i}{\theta_1}\right)\frac{\theta_1 w_i}{\lambda_1\left(\lambda_1+\theta_1 w_i\right)},\\ \frac{\partial\ell\left(\gamma;\mathbf{w},\mathbf{z},\mathbf{s}\right)}{\partial\lambda_2} &= -\frac{n_r}{\lambda_2}+\sum_{i=1}^{r}\left(\left(\frac{1}{\theta_2}+1\right)\left(1-z_i\right)+\frac{t_i}{\theta_2}\right)\frac{\theta_2 w_i}{\lambda_2\left(\lambda_2+\theta_2 w_i\right)}.\end{aligned} \tag{6}$$

Equating the partial derivatives to zero yields, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ is given by the following explicit forms:

$$\hat{\lambda}_1 = \hat{\theta}_1 w_i\left(\frac{\sum_{i=1}^{r}(1/\theta_1 + 1)z_i + s_i/\theta_1}{m_r} - 1\right),$$
$$\hat{\lambda}_2 = \hat{\theta}_2 w_i\left(\frac{\sum_{i=1}^{r}(1/\theta_2 + 1)(1 - z_i) + (t_i/\theta_2)}{n_r} - 1\right). \tag{7}$$

3.2. Bayes Estimators. In this section, Bayes estimates for the unknown parameters $\theta_1, \theta_2, \lambda_1$, and λ_2 are observed. In Bayesian method all parameters are considered as random variables with certain distribution called prior distribution. If prior information is not available which is usually the case, we need to select a prior distribution. Since the selection of prior distribution plays an important role in estimation of the parameters, our choice for the prior of $\theta_1, \theta_2, \lambda_1$, and λ_2 are the independent gamma distributions i.e., $G(a_1, b_1), G(a_2, b_2), G(a_3, b_3), G(a_4, b_4)$, respectively. The reason for choosing this prior density is that Gamma prior has flexible nature as a noninformative prior, specially when the values of hyperparameters are assumed to be zero. Thus, the suggested prior for $\theta_1, \theta_2, \lambda_1$, and λ_2 are

$$\begin{aligned} f_1(\theta_1) &\propto \theta_1^{a_1-1} e^{-b_1\theta_1},\\ f_2(\theta_2) &\propto \theta_2^{a_2-1} e^{-b_2\theta_2},\\ f_3(\lambda_1) &\propto \lambda_1^{a_3-1} e^{-b_3\lambda_1},\\ f_4(\lambda_2) &\propto \lambda_2^{a_4-1} e^{-b_4\lambda_2}, \end{aligned} \tag{8}$$

respectively, and $a_1, a_2, a_3, a_4, b_1, b_2, b_3,$ and b_4 are the hyperparameters of prior distributions. In Bayesian estimation method, we need to also determine the loss function. In this article, we considered quadratic loss function because this loss function is mostly used as symmetrical loss function and is defined as $L(\hat{\gamma}, \gamma) = (\hat{\gamma} - \gamma)^2$, where $\hat{\gamma}$ is the point estimate of the vector parameter γ. Under quadratic loss function, Bayes estimators are the posterior mean of the distribution.

The joint prior of $\theta_1, \theta_2, \lambda_1$, and λ_2 is as follows:

$$g(\gamma) \propto \theta_1^{a_1-1}\theta_2^{a_2-1}\lambda_1^{a_3-1}\lambda_2^{a4-1} e^{-b_1\theta_1 - b_2\theta_2 - b_2\lambda_1 - b_3\lambda_1 - b_4\lambda_2}, \tag{9}$$

where $\theta_1, \theta_2, \lambda_1, \lambda_2, a_1, a_2, a_3, b_1, b_2,$ and $b_3 > 0$, while the joint posterior of $\theta_1, \theta_2, \lambda_1$, and λ_2 is given by the following equation:

$$p\left(\frac{\gamma}{\mathbf{w}, \mathbf{z}, \mathbf{s}}\right) = \frac{L(\gamma \mid \mathbf{w}, \mathbf{z}, \mathbf{s}) g(\gamma)}{\int_{\theta_1}\int_{\theta_2}\int_{\lambda_1}\int_{\lambda_2} L(\gamma \mid \mathbf{w}, \mathbf{z}, \mathbf{s}) g(\gamma) d\theta_1 d\theta_2 d\lambda_1 d\lambda_2}, \tag{10}$$

where $L(\gamma \mid \mathbf{w}, \mathbf{z}, \mathbf{s})$ is the likelihood function of the GP distribution under PC scheme. Substituting $L(\gamma \mid \mathbf{w}, \mathbf{z}, \mathbf{s})$ and $g(\gamma)$ as defined in equation (1) and (#prior), respectively, the joint posterior is as follows:

$$\begin{aligned} p\left(\frac{\gamma}{\mathbf{w}, \mathbf{z}, \mathbf{s}}\right) &\propto \theta_1^{a_1-1}\theta_2^{a_2-1}\lambda_1^{a_3-z_i-1}\lambda_2^{a_4-z_i} e^{-b_1\theta_1 - b_2\theta_2 - b_3\lambda_1 - b_4\lambda_2} u_1(\theta_1, \lambda_1) u_2(\theta_2, \lambda_2)\\ &\propto G_{\theta_1}(a_1, b_1) G_{\theta_2}(a_2, b_2) G_{\lambda_1}(a_3 - z_i, b_3) G_{\lambda_2}(a_4 - z_i + 1, b_4) u_1(\theta_1, \lambda_1) u_2(\theta_2, \lambda_2), \end{aligned} \tag{11}$$

where $u_1(\theta_1, \lambda_1) = (1 + \theta_1 w_i/\lambda_1)^{-z_i(1/\theta_1+1)-(s_i/\theta_1)}$ and $u_2(\theta_2, \lambda_2) = (1 + \theta_2 w_i/\lambda_2)^{-z_i(1/\theta_2+1)-(t_i/\theta_2)}$.

The Bayes estimate of any function of $\theta_1, \theta_2, \lambda_1$, and λ_2, say $h(\theta_1, \theta_2, \lambda_1, \lambda_2)$, under the quadratic loss function is $\hat{h}_B(\theta_1, \theta_2, \lambda_1, \lambda_2) = E_{\theta_1,\theta_2,\lambda_1,\lambda_2/\text{data}}(h(\theta_1, \theta_2, \lambda_1, \lambda_2))$. Since it is difficult to compute this expected value analytically, we will use the Markov Chain Monte Carlo (MCMC) technique, see Lindley [20] and Karandikar [21].

We will use Gibbs sampling method to generate a sample from the posterior density function $p(\gamma/\mathbf{w}, \mathbf{z}, \mathbf{s})$ and compute Bayes estimates. Gibbs Sampling is a Markov chain Monte Carlo (MCMC) algorithm which is used to obtain a sequence of observations that are approximately following a certain probability distribution. In another words, it is a specific case of the Metropolis–Hastings algorithm. Gibbs Sampling is applicable when the joint distribution is not known explicitly or is difficult to sample from directly, but the conditional distribution of each variable is known and is easier to sample from. For more details, one may refer to Andrew et al. [22]; Muhammed and Almetwally [23]; and El-Sherpieny et al. [24]. For the purpose of generating a sample from the posterior distribution, we assumed that the pdf of prior densities are as described in equation (2). The full conditional posterior densities of $\theta_1, \theta_2, \lambda_1$, and λ_2 and the data is given by the following equation:

$$\begin{aligned}
&\pi\left(\frac{\theta_1}{\theta_2,\lambda_1,\lambda_2,\mathbf{w},\mathbf{z},\mathbf{s}}\right)\propto G_{\theta_1}(a_1,b_1)u_1(\theta_1,\lambda_1),\\
&\pi\left(\frac{\theta_2}{\theta_1,\lambda_1,\lambda_2,\mathbf{w},\mathbf{z},\mathbf{s}}\right)\propto G_{\theta_2}(a_2,b_2)u_2(\theta_2,\lambda_2),\\
&\pi\left(\frac{\lambda_1}{\theta_1,\theta_2,\lambda_2,\mathbf{w},\mathbf{z},\mathbf{s}}\right)\propto G_{\lambda_1}(a_3-z_i,b_3)u_1(\theta_1,\lambda_1),\\
&\pi\left(\frac{\lambda_2}{\theta_1,\theta_2,\lambda_1,\mathbf{w},\mathbf{z},\mathbf{s}}\right)\propto G_{\lambda_2}(a_4-z_i+1,b_4)u_2(\theta_2,\lambda_2).
\end{aligned}\tag{12}$$

To apply Gibbs technique, we need the following algorithm:

(1) Start with initial values $(\theta_1^0,\theta_2^0,\lambda_1^0,\lambda_2^0)$

(2) Use M-H algorithm to generate posterior sample for θ_1,θ_2, λ_1, and λ_2 from equation (8).

(3) Repeat Step 2 M times and obtain $(\theta_{11},\theta_{21},\lambda_{11},\lambda_{21})$, $(\theta_{12},\theta_{22},\lambda_{12},\lambda_{22}),\ldots,(\theta_{1M},\theta_{2M},\lambda_{1M},\lambda_{2M})$,

(4) After obtaining the posterior sample, the Bayes estimates of $\theta_1,\theta_2,\lambda_1$, and λ_2 with respect to quadratic loss function are as follows:

$$\begin{aligned}
\widehat{\theta}_1^{MC}&=\left[E_\pi\left(\frac{\theta_1}{\mathbf{w},\mathbf{z},\mathbf{s}}\right)\right]\approx\left(\frac{1}{M-M_0}\sum_{i=1}^{M-M_0}\theta_{1i}\right),\\
\widehat{\theta}_2^{MC}&=\left[E_\pi\left(\frac{\theta_2}{\mathbf{w},\mathbf{z},\mathbf{s}}\right)\right]\approx\left(\frac{1}{M-M_0}\sum_{i=1}^{M-M_0}\theta_{2i}\right),\\
\widehat{\lambda}_1^{MC}&=\left[E_\pi\left(\frac{\lambda_1}{\mathbf{w},\mathbf{z},\mathbf{s}}\right)\right]\approx\left(\frac{1}{M-M_0}\sum_{i=1}^{M-M_0}\lambda_{1i}\right),\\
\widehat{\lambda}_2^{MC}&=\left[E_\pi\left(\frac{\lambda_2}{\mathbf{w},\mathbf{z},\mathbf{s}}\right)\right]\approx\left(\frac{1}{M-M_0}\sum_{i=1}^{M-M_0}\lambda_{2i}\right),
\end{aligned}\tag{13}$$

where M_0 is the burn-in-period of Markov Chain.

4. Interval Estimation

Normal approximation method for constructing confidence intervals is an efficient method, it has an advantage and performs well when the sample size is large enough, otherwise this method will not be useful. If this is the case, bootstrap methods may provide more accurate approximate confidence intervals. In this section, four different approximate confidence interval methods are proposed. Our goal is to select the best interval with respect to the interval lengths, i.e., the interval with the shortest length is the best.

4.1. Asymptotic Confidence Interval. When the sample size is large enough, the normal approximation of the MLE can be used to construct asymptotic confidence intervals for the parameters θ_1,θ_2, λ_1, and λ_2. The asymptotic normality of MLE can be stated as $(\widehat{\gamma}-\gamma)\longrightarrow_d N_4(0,I^{-1}(\gamma))$, where $\gamma=(\theta_1,\theta_2\lambda_1,\lambda_2)$, is a vector of parameters, $\longrightarrow_d$ denotes convergence in distribution, and $I(\gamma)$ is the Fisher information matrix, i.e.,

$$I(\gamma)=-\begin{bmatrix}E(\ell_{\theta_1\theta_1}) & E(\ell_{\theta_1\theta_2}) & E(\ell_{\theta_1\lambda_1}) & E(\ell_{\theta_1\lambda_2})\\ E(\ell_{\theta_2\theta_1}) & E(\ell_{\theta_2\theta_2}) & E(\ell_{\theta_2\lambda_1}) & E(\ell_{\theta_2\lambda_2})\\ E(\ell_{\lambda_1\theta_1}) & E(\ell_{\lambda_1\theta_2}) & E(\ell_{\lambda_1\lambda_1}) & E(\ell_{\lambda_1\lambda_2})\\ E(\ell_{\lambda_2\theta_1}) & E(\ell_{\lambda_2\theta_2}) & E(\ell_{\lambda_2\lambda_1}) & E(\ell_{\lambda_2\lambda_2})\end{bmatrix}.\tag{14}$$

The expected values of the second order partial derivatives can be evaluated using integration techniques. Therefore, the $100(1-\xi)\%$ approximate CIs for θ_1,θ_2, λ_1, and λ_2 are

$$\begin{aligned}
&\widehat{\theta}_1\pm z_{\xi/2}\sqrt{v_{11}},\\
&\widehat{\theta}_2\pm z_{\xi/2}\sqrt{v_{22}},\\
&\widehat{\lambda}_1\pm z_{\xi/2}\sqrt{v_{33}},\\
&\widehat{\lambda}_2\pm z_{\xi/2}\sqrt{v_{44}},
\end{aligned}\tag{15}$$

respectively, where v_{11},v_{22},v_{33}, and v_{44} are the entries in the main diagonal of fisher matrix $I^{-1}(\gamma)$ and $z_{\xi/2}$ is the $(\xi/2)100\%$ lower percentile of standard normal distribution.

4.2. Bootstrap Confidence Interval. Since the asymptotic confidence intervals do not perform very well for small sample size, an alternative approach to the traditional one is used, namely, the bootstrap method. Parametric and nonparametric bootstrap methods are presented in Davison and Hinkley [25] and Efron and Tibshirani [26]. In this section, we used two parametric bootstrap methods: (a) percentile bootstrap and (b) t-bootstrap (see Hall [27] and Efron [28, 29]).

4.2.1. Percentile Bootstrap Confidence Interval. The confidence intervals based on percentile bootstrap are performed by using the following algorithm:

(1) Step (1): Given the original data set $(w,z,s)=\{(w_i,z_i,s_i),i=1,\ldots,k,1\le k<\max\{n,m\}\}$, and $z_i=0$ or 1 depending on whether the

failure is from population one or two. Estimate θ_1, θ_2, λ_1, and λ_2 using the maximum likelihood estimation (say $\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\lambda}_1$, and $\widehat{\lambda}_2$).

(2) Step (2): Generate a bootstrap sample (w^*, z^*, s^*) from joint Weibull distribution with parameters $\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\lambda}_1, \widehat{\lambda}_2$ obtained in Step (1).

(3) Step (3): With respect to (w^*, z^*, s^*) the bootstrap sample estimation is $\widehat{\theta}_1^{\,*}, \widehat{\theta}_2^{\,*}, \widehat{\lambda}_1^*, \widehat{\lambda}_2^*$.

(4) Step (4): Repeat Step 2 and 3 M-times to obtain different bootstrap samples.

(5) Step (5): Arrange the different bootstrap samples in an ascending order as $(\widehat{\psi}_j^{*[1]}, \widehat{\psi}_j^{*[2]}, \ldots, \widehat{\psi}_j^{*[M]})$, where $j = 1, 2, 3, 4$ and $(\widehat{\psi}_1^* = \widehat{\theta}_1^{\,*}, \widehat{\psi}_2^* = \widehat{\theta}_2^{\,*}, \widehat{\psi}_3^* = \widehat{\lambda}_1^*, \widehat{\psi}_4^* = \widehat{\lambda}_2^*)$.

A two-sided 100(1-ξ)% percentile bootstrap confidence intervals for the unknown parameters θ_1, θ_2, λ_1, and λ_2 are given by the following equation:

$$\left(\widehat{\psi}_j^{*\,[M_{\xi/2}]}, \widehat{\psi}_j^{*\,[M_{1-\xi/2}]}\right), j = 1, 2, 3, 4. \tag{16}$$

4.2.2. Bootstrap-t Confidence Intervals. For this method, use the following algorithm:

(1) Given the original data set $(w, z, s) = \{(w_i, z_i, s_i), i = 1, \ldots, k, 1 \le k < \max\{n, m\}\}$, and $z_i = 0$ or 1 depending on whether the failure is from population one or two. Estimate θ_1, θ_2, λ_1, and λ_2 using the maximum likelihood estimation (say $\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\lambda}_1$, and $\widehat{\lambda}_2$).

(2) Generate a bootstrap sample (w^*, z^*, s^*) from joint Weibull distribution with parameters $\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\lambda}_1$, and $\widehat{\lambda}_2$

(3) The bootstrap sample estimation is $\widehat{\theta}_1^{\,*}, \widehat{\theta}_2^{\,*}, \widehat{\lambda}_1^*$, and $\widehat{\lambda}_2^*$

(4) Compute the t-statistics $T_1 = \widehat{\theta}_1^{\,*} - \widehat{\theta}_1/\sqrt{\mathrm{Var}(\widehat{\theta}_1^{\,*})}$, $T_2 = \widehat{\theta}_2^{\,*} - \widehat{\theta}_2/\sqrt{\mathrm{Var}(\widehat{\theta}_2^{\,*})}$, $T_3 = \widehat{\lambda}_1^* - \widehat{\lambda}_1/\sqrt{\mathrm{Var}(\widehat{\lambda}_1^*)}$, and $T_4 = \widehat{\lambda}_2^* - \widehat{\lambda}_2/\sqrt{\mathrm{Var}(\widehat{\lambda}_2^*)}$, where $\mathrm{Var}(\widehat{\theta}_1^{\,*})$, $\mathrm{Var}(\widehat{\theta}_2^{\,*})$, $\mathrm{Var}(\widehat{\lambda}_1^*)$, and $\mathrm{Var}(\widehat{\lambda}_2^*)$ are the asymptotic variance of θ_1, θ_2, λ_1, and λ_2, respectively

(5) Repeat Steps 2 to 4 M times $T_j^{(1)}, T_j^{(2)}, \ldots, T_j^{(M)}, j = 1, 2, 3, 4$.

(6) Arrange the T values obtained in Step 5 in ascending order $T_j^{[1]}, T_j^{[2]}, \ldots, T_j^{[M]}, j = 1, 2, 3, 4$.

Two-sided $100(1-\xi)$% t-bootstrap confidence intervals for the unknown parameters θ_1, θ_2, λ_1, and λ_2 are given by the following equation:

$$\begin{aligned}
&\left(\widehat{\theta}_1 + T_1^{[M_{\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\theta}_1)}, \widehat{\theta}_1 + T_1^{[M_{1-\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\theta}_1)}\right),\\
&\left(\widehat{\theta}_2 + T_2^{[M_{\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\theta}_2)}, \widehat{\theta}_2 + T_2^{[M_{1-\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\theta}_2)}\right),\\
&\left(\widehat{\lambda}_1 + T_3^{[M_{\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\lambda}_1)}, \widehat{\lambda}_1 + T_3^{[M_{1-\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\lambda}_1)}\right),\\
&\left(\widehat{\lambda}_2 + T_4^{[M_{\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\lambda}_2)}, \widehat{\lambda}_2 + T_4^{[M_{1-\xi/2}]}\sqrt{\mathrm{Var}(\widehat{\lambda}_2)}\right).
\end{aligned} \tag{17}$$

4.3. Credible Intervals. Using MCMC techniques in Section 3.2, Bayes credible intervals of the parameters θ_1, θ_2, λ_1, and λ_2 can be obtained as follows:

(1) Arrange $\theta_{1i}, \theta_{2i}, \lambda_{1i}$, and λ_{2i}, ascending order as follow $\theta_{1[1]}, \theta_{1[2]}, \ldots, \theta_{1[M]}$, $\theta_{2[1]}, \theta_{2[2]}, \ldots, \theta_{2[M]}, \lambda_{1[1]}, \lambda_{1[2]}, \ldots, \lambda_{1[M]}$, and $\lambda_{2[1]}, \lambda_{2[2]}, \ldots, \lambda_{2[M]}$.

(2) Two-sided $100(1-\xi)$% credible intervals for the unknown parameters θ_1, θ_2, λ_1, and λ_2 are given by the following equation:

$$\begin{aligned}
&\left(\theta_{1[M_{\xi/2}]}, \theta_{1[M_{1-\xi/2}]}\right),\\
&\left(\theta_{2[M_{\xi/2}]}, \theta_{2[M_{1-\xi/2}]}\right),\\
&\left(\lambda_{1[M_{\xi/2}]}, \lambda_{1[M_{1-\xi/2}]}\right),\\
&\left(\lambda_{2[M_{\xi/2}]}, \lambda_{2[M_{1-\xi/2}]}\right).
\end{aligned} \tag{18}$$

5. Data Analysis and Simulations

In this section, comparisons are proposed regarding the different methods of point and interval estimation that were used in the previous sections. These comparisons need numerical analysis methods and simulation, Monte Carlo simulation was carried out. We analyzed a real data set for illustrative purposes; also, a simulation study was carried out to compare the performances of the different estimators, using different parameter values and different sampling schemes.

5.1. Simulation Study. In this subsection, the Monte Carlo simulation study was utilized to analyze the point and confidence interval estimation for the parameters of GP distribution based on the JPC scheme with binomial removal. The simulation results are summarized in Tables 1–3 and some concluding remarks of simulation results are pointed out. To evaluate the performance of the estimation procedures described in this article, we performed an

Table 1: MLE and Bayesian estimation methods for GP based on JPC scheme with binomial removal: Case 1.

				$\theta_1 = 1.2, \lambda_1 = 1.5, \theta_2 = 0.8, \lambda_2 = 1.3$							
P	m, n	r		MLE					Bayesian		
				Bias	MSE	L.CI	BP	BT	Bias	MSE	L.CI
0.15	30, 25	38	θ_1	−0.1070	0.3141	2.1575	0.3716	0.3734	0.1136	0.1125	1.1888
			λ_1	0.2343	0.6998	3.1495	0.4183	0.4171	0.1747	0.1459	1.2873
			θ_2	−0.1096	0.2406	1.8753	0.3039	0.3024	0.0901	0.0577	0.7950
			λ_2	0.2358	0.5393	2.7276	0.3379	0.3385	0.1382	0.0864	0.9927
		47	θ_1	−0.0765	0.2248	1.8353	0.2328	0.2342	0.1111	0.0986	1.1212
			λ_1	0.2113	0.5604	2.8165	0.4470	0.4429	0.1702	0.1445	1.2413
			θ_2	−0.0950	0.1736	1.5910	0.2195	0.2198	0.0809	0.0535	0.8268
			λ_2	0.2377	0.4801	2.5528	0.4226	0.4244	0.1374	0.0796	1.0029
0.5	30, 25	38	θ_1	−0.1258	0.2939	2.0682	0.3326	0.3299	0.1097	0.1174	1.1922
			λ_1	0.2718	0.8972	3.5587	0.3862	0.3869	0.1820	0.1577	1.3092
			θ_2	−0.1435	0.2237	1.7673	0.2677	0.2668	0.0750	0.0524	0.8106
			λ_2	0.2630	0.6031	2.8658	0.5155	0.5111	0.1297	0.0764	0.9194
		47	θ_1	−0.1223	0.2228	1.7879	0.2514	0.2510	0.0783	0.0981	1.1519
			λ_1	0.2481	0.6264	2.9476	0.4655	0.4635	0.1773	0.1480	1.2727
			θ_2	−0.1252	0.1738	1.5598	0.2193	0.2180	0.0675	0.0475	0.7839
			λ_2	0.2250	0.4979	2.6228	0.4034	0.4010	0.1136	0.0793	0.9689
0.15	50, 55	73	θ_1	−0.0498	0.1602	1.5574	0.1954	0.1963	0.0826	0.0793	1.0336
			λ_1	0.0862	0.3192	2.1900	0.2285	0.2283	0.1414	0.1293	1.2784
			θ_2	−0.0642	0.0900	1.1495	0.1465	0.1476	0.0528	0.0380	0.6929
			λ_2	0.1014	0.1951	1.6860	0.2062	0.2064	0.1034	0.0657	0.9164
		97	θ_1	−0.0344	0.1185	1.3436	0.1447	0.1426	0.0802	0.0696	0.9834
			λ_1	0.1305	0.2983	2.0802	0.1949	0.1942	0.1514	0.1277	1.2348
			θ_2	−0.0332	0.0683	1.0168	0.0948	0.0948	0.0481	0.0337	0.6791
			λ_2	0.0907	0.1614	1.5347	0.1677	0.1674	0.1020	0.0629	0.8676
0.5	50, 55	73	θ_1	−0.0738	0.1537	1.5101	0.1731	0.1745	0.0780	0.0794	1.0173
			λ_1	0.1170	0.3552	2.2919	0.2728	0.2726	0.1431	0.1267	1.2095
			θ_2	−0.0649	0.0961	1.1889	0.1330	0.1322	0.0547	0.0425	0.7491
			λ_2	0.1131	0.1998	1.6960	0.1809	0.1813	0.1055	0.0685	0.9047
		97	θ_1	−0.0243	0.1216	1.3644	0.1321	0.1311	0.0779	0.0730	0.9805
			λ_1	0.0812	0.2653	1.9947	0.1892	0.1907	0.1291	0.1147	1.1928
			θ_2	−0.0487	0.0707	1.0252	0.0921	0.0924	0.0401	0.0343	0.6885
			λ_2	0.0960	0.1664	1.5549	0.1453	0.1461	0.0999	0.0651	0.9256
0.15	110, 125	170	θ_1	−0.0220	0.0680	1.0192	0.0790	0.0796	0.0529	0.0507	0.8254
			λ_1	0.0474	0.1316	1.4108	0.1120	0.1118	0.1046	0.0891	1.0500
			θ_2	−0.0315	0.0371	0.7451	0.0552	0.0549	0.0236	0.0228	0.5809
			λ_2	0.0474	0.0706	1.0255	0.0741	0.0738	0.0747	0.0413	0.6996
		215	θ_1	−0.0223	0.0545	0.9111	0.0579	0.0582	0.0390	0.0416	0.7669
			λ_1	0.0568	0.1214	1.3485	0.0927	0.0927	0.0939	0.0803	0.9922
			θ_2	−0.0183	0.0301	0.6767	0.0463	0.0463	0.0238	0.0217	0.5584
			λ_2	0.0236	0.0545	0.9109	0.0641	0.0642	0.0532	0.0353	0.7036
0.5	110, 125	170	θ_1	−0.0193	0.0691	1.0283	0.0744	0.0739	0.0563	0.0517	0.8571
			λ_1	0.0524	0.1305	1.4020	0.0973	0.0969	0.0983	0.0837	1.0777
			θ_2	−0.0274	0.0358	0.7348	0.0538	0.0536	0.0265	0.0236	0.5889
			λ_2	0.0394	0.0747	1.0606	0.0752	0.0752	0.0690	0.0434	0.7391
		215	θ_1	−0.0295	0.0487	0.8575	0.0596	0.0589	0.0328	0.0352	0.7124
			λ_1	0.0852	0.1313	1.3815	0.0885	0.0881	0.1223	0.0917	1.0405
			θ_2	−0.0163	0.0281	0.6544	0.0438	0.0442	0.0277	0.0203	0.5575
			λ_2	0.0405	0.0577	0.9290	0.0615	0.0620	0.0667	0.0382	0.7144

Table 2: MLE and Bayesian estimation methods for GP based on JPC scheme with binomial removal: Case 2.

P	m, n	m_1, m_2		MLE Bias	MLE MSE	MLE L.CI	BP	BT	Bayesian Bias	Bayesian MSE	Bayesian L.CI
				$\theta_1 = 0.4, \lambda_1 = 0.65, \theta_2 = 0.7, \lambda_2 = 1.8$							
0.15	30, 25	38	θ_1	−0.0780	0.1139	1.2879	0.1900	0.1874	0.0783	0.0816	0.9430
			λ_1	0.1098	0.0792	1.0165	0.1622	0.1613	0.1690	0.0867	0.8643
			θ_2	−0.1446	0.2904	2.0358	0.2776	0.2749	0.0688	0.0549	0.7931
			λ_2	0.2431	1.2161	4.2186	0.4798	0.4786	0.0722	0.0522	0.8544
		47	θ_1	−0.0779	0.0859	1.1084	0.1465	0.1479	0.0636	0.0688	0.8687
			λ_1	0.0740	0.0613	0.9263	0.1070	0.1074	0.1202	0.0612	0.8002
			θ_2	−0.1332	0.2020	1.6836	0.2490	0.2465	0.0622	0.0552	0.8273
			λ_2	0.3044	0.9990	3.7337	0.4639	0.4627	0.0961	0.0592	0.8724
0.5	30, 25	38	θ_1	−0.1060	0.0964	1.1447	0.2016	0.2009	0.0747	0.0888	0.9539
			λ_1	0.0765	0.0713	1.0030	0.1335	0.1331	0.1230	0.0694	0.8319
			θ_2	−0.1336	0.2390	1.8442	0.2571	0.2577	0.0790	0.0537	0.8066
			λ_2	0.3110	1.1623	4.0485	0.7156	0.7204	0.0894	0.0563	0.8486
		47	θ_1	−0.0778	0.0819	1.0799	0.1827	0.1815	0.0524	0.0630	0.8295
			λ_1	0.0579	0.0606	0.9382	0.1462	0.1476	0.1133	0.0601	0.7943
			θ_2	−0.1297	0.1793	1.5808	0.2437	0.2426	0.0654	0.0499	0.7465
			λ_2	0.2946	0.9882	3.7236	0.6212	0.6204	0.0881	0.0577	0.8527
0.15	50, 55	73	θ_1	−0.0695	0.0574	0.8995	0.1200	0.1206	0.0480	0.0509	0.7484
			λ_1	0.0621	0.0399	0.7444	0.1024	0.1008	0.1066	0.0456	0.6818
			θ_2	−0.0659	0.0960	1.1874	0.1525	0.1513	0.0564	0.0403	0.7161
			λ_2	0.1542	0.4190	2.4657	0.2325	0.2315	0.0910	0.0572	0.8513
		97	θ_1	−0.0398	0.0449	0.8162	0.0751	0.0746	0.0441	0.0429	0.7176
			λ_1	0.0331	0.0285	0.6487	0.0600	0.0608	0.0698	0.0294	0.5969
			θ_2	−0.0434	0.0615	0.9579	0.1056	0.1063	0.0387	0.0301	0.6311
			λ_2	0.0879	0.2666	1.9955	0.1961	0.1963	0.0805	0.0595	0.9262
0.5	50, 55	73	θ_1	−0.0659	0.0565	0.8954	0.1122	0.1111	0.0452	0.0517	0.7435
			λ_1	0.0482	0.0420	0.7814	0.0829	0.0840	0.0917	0.0423	0.7060
			θ_2	−0.0693	0.0782	1.0624	0.1135	0.1149	0.0411	0.0330	0.6727
			λ_2	0.1510	0.3952	2.3933	0.2390	0.2389	0.1023	0.0636	0.8960
		97	θ_1	−0.0395	0.0451	0.8183	0.0804	0.0804	0.0442	0.0419	0.7157
			λ_1	0.0349	0.0297	0.6615	0.0679	0.0685	0.0741	0.0321	0.6107
			θ_2	−0.0350	0.0630	0.9745	0.1064	0.1071	0.0512	0.0314	0.6640
			λ_2	0.1246	0.2864	2.0413	0.1868	0.1855	0.0949	0.0613	0.8704
0.15	110, 125	170	θ_1	−0.0367	0.0286	0.6478	0.0486	0.0490	0.0155	0.0250	0.5653
			λ_1	0.0277	0.0176	0.5084	0.0416	0.0416	0.0536	0.0187	0.4729
			θ_2	−0.0250	0.0337	0.7136	0.0566	0.0564	0.0311	0.0218	0.5438
			λ_2	0.0504	0.1343	1.4238	0.1047	0.1060	0.0677	0.0533	0.8351
		215	θ_1	−0.0317	0.0207	0.5501	0.0386	0.0385	0.0063	0.0181	0.4915
			λ_1	0.0234	0.0130	0.4381	0.0285	0.0285	0.0474	0.0145	0.4177
			θ_2	−0.0223	0.0268	0.6364	0.0424	0.0427	0.0197	0.0184	0.5040
			λ_2	0.0615	0.1098	1.2770	0.0868	0.0872	0.0752	0.0506	0.8087
0.5	110, 125	170	θ_1	−0.0293	0.0259	0.6202	0.0520	0.0518	0.0275	0.0270	0.5755
			λ_1	0.0227	0.0164	0.4941	0.0343	0.0345	0.0475	0.0177	0.4701
			θ_2	−0.0265	0.0323	0.6974	0.0533	0.0530	0.0261	0.0208	0.5409
			λ_2	0.0585	0.1276	1.3821	0.1093	0.1087	0.0755	0.0506	0.8174
		215	θ_1	−0.0209	0.0214	0.5680	0.0388	0.0387	0.0175	0.0181	0.5027
			λ_1	0.0187	0.0136	0.4522	0.0302	0.0301	0.0412	0.0143	0.4153
			θ_2	−0.0227	0.0261	0.6273	0.0456	0.0450	0.0190	0.0177	0.5158
			λ_2	0.0606	0.1103	1.2806	0.0884	0.0879	0.0771	0.0512	0.8293

Table 3: MLE and Bayesian estimation methods for GP based on JPC scheme with binomial removal: Case 3.

			$\theta_1=3, \lambda_1=0.5, \theta_2=2.5, \lambda_2=0.8$								
P	m, n	m_1, m_2		MLE					Bayesian		
				Bias	MSE	L.CI	BP	BT	Bias	MSE	L.CI
0.15	30, 25	38	θ_1	−0.1225	0.8748	3.6367	0.6448	0.6399	0.0730	0.0684	0.9731
			λ_1	0.1229	0.1610	1.4979	0.2191	0.2171	0.1599	0.1098	1.0032
			θ_2	−0.1780	0.8485	3.5446	0.5199	0.5170	0.0417	0.0224	0.5386
			λ_2	0.2296	0.5346	2.7225	0.3544	0.3521	0.1445	0.1110	1.0498
		47	θ_1	−0.0844	0.6192	3.0683	0.3737	0.3722	0.0761	0.0723	0.9554
			λ_1	0.1238	0.1513	1.4461	0.2005	0.2008	0.1616	0.1098	1.0553
			θ_2	−0.1270	0.6350	3.0855	0.4516	0.4542	0.0407	0.0250	0.5879
			λ_2	0.1780	0.4050	2.3963	0.3548	0.3512	0.1225	0.0914	1.0154
0.5	30, 25	38	θ_1	−0.0495	0.8648	3.6421	0.7053	0.6970	0.0988	0.0769	1.0033
			λ_1	0.1364	0.2120	1.7250	0.2059	0.2046	0.1572	0.1196	1.0154
			θ_2	−0.1077	0.8035	3.4901	0.6011	0.6000	0.0522	0.0251	0.5755
			λ_2	0.2513	0.6866	3.0969	0.3506	0.3510	0.1621	0.1243	1.0919
		47	θ_1	−0.1095	0.6208	3.0601	0.4453	0.4476	0.0731	0.0729	0.9974
			λ_1	0.1183	0.1561	1.4787	0.1868	0.1868	0.1411	0.1097	1.0063
			θ_2	−0.1147	0.5906	2.9801	0.5159	0.5097	0.0406	0.0245	0.5773
			λ_2	0.2050	0.4206	2.4130	0.3703	0.3696	0.1411	0.0956	1.0342
0.15	50, 55	73	θ_1	−0.0846	0.4923	2.7318	0.3140	0.3109	0.0631	0.0780	1.0545
			λ_1	0.0946	0.0966	1.1610	0.1500	0.1498	0.1551	0.0998	0.9455
			θ_2	−0.0401	0.3687	2.3761	0.2793	0.2825	0.0431	0.0334	0.6636
			λ_2	0.0966	0.1640	1.5424	0.1806	0.1803	0.1292	0.0821	0.9686
		97	θ_1	−0.0720	0.3557	2.3221	0.2009	0.2007	0.0552	0.0797	1.0926
			λ_1	0.0745	0.0635	0.9440	0.0944	0.0942	0.1155	0.0639	0.8116
			θ_2	−0.0337	0.2424	1.9263	0.1773	0.1773	0.0383	0.0326	0.6818
			λ_2	0.0518	0.1103	1.2865	0.1188	0.1180	0.0996	0.0659	0.8705
0.5	50, 55	73	θ_1	−0.0257	0.4791	2.7127	0.3319	0.3348	0.0861	0.0829	1.0674
			λ_1	0.0706	0.0849	1.1090	0.1480	0.1488	0.1259	0.0783	0.8786
			θ_2	−0.0796	0.3202	2.1971	0.2434	0.2464	0.0344	0.0295	0.6497
			λ_2	0.1039	0.1732	1.5803	0.1623	0.1629	0.1264	0.0818	0.9442
		97	θ_1	−0.0451	0.3238	2.2247	0.2232	0.2228	0.0661	0.0743	1.0298
			λ_1	0.0553	0.0634	0.9631	0.0947	0.0937	0.1086	0.0652	0.7943
			θ_2	−0.0551	0.2395	1.9073	0.1972	0.1977	0.0333	0.0313	0.6756
			λ_2	0.0757	0.1240	1.3489	0.1472	0.1461	0.1006	0.0704	0.8533
0.15	110, 125	170	θ_1	−0.0409	0.2023	1.7567	0.1246	0.1256	0.0461	0.0729	1.0124
			λ_1	0.0308	0.0309	0.6784	0.0507	0.0503	0.0857	0.0437	0.6772
			θ_2	−0.0197	0.1467	1.5004	0.1134	0.1143	0.0341	0.0374	0.7304
			λ_2	0.0374	0.0508	0.8713	0.0614	0.0613	0.0834	0.0417	0.7118
		215	θ_1	−0.0325	0.1695	1.6096	0.1113	0.1114	0.0398	0.0735	1.0183
			λ_1	0.0276	0.0253	0.6138	0.0423	0.0419	0.0710	0.0308	0.5853
			θ_2	−0.0282	0.1052	1.2670	0.0873	0.0862	0.0218	0.0325	0.6899
			λ_2	0.0384	0.0424	0.7937	0.0516	0.0515	0.0813	0.0370	0.6699
0.5	110, 125	170	θ_1	−0.0442	0.1991	1.7415	0.1457	0.1465	0.0419	0.0745	1.1018
			λ_1	0.0365	0.0331	0.6992	0.0611	0.0621	0.0854	0.0384	0.6426
			θ_2	−0.0176	0.1338	1.4331	0.1051	0.1055	0.0351	0.0339	0.6972
			λ_2	0.0469	0.0617	0.9569	0.0712	0.0714	0.0908	0.0514	0.7523
		215	θ_1	−0.0125	0.1634	1.5846	0.1037	0.1032	0.0523	0.0728	1.0384
			λ_1	0.0366	0.0259	0.6143	0.0443	0.0443	0.0784	0.0307	0.5853
			θ_2	−0.0220	0.1122	1.3111	0.0867	0.0869	0.0286	0.0354	0.7265
			λ_2	0.0312	0.0471	0.8422	0.0527	0.0531	0.0704	0.0371	0.6477

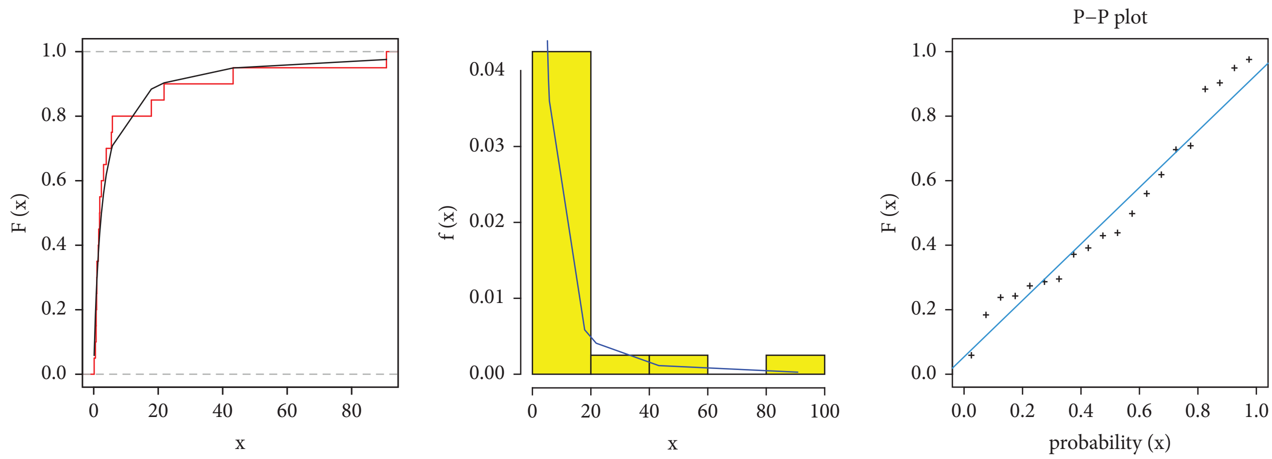

Figure 1: Estimated CDF, PDF, and PP plot of GP for data X.

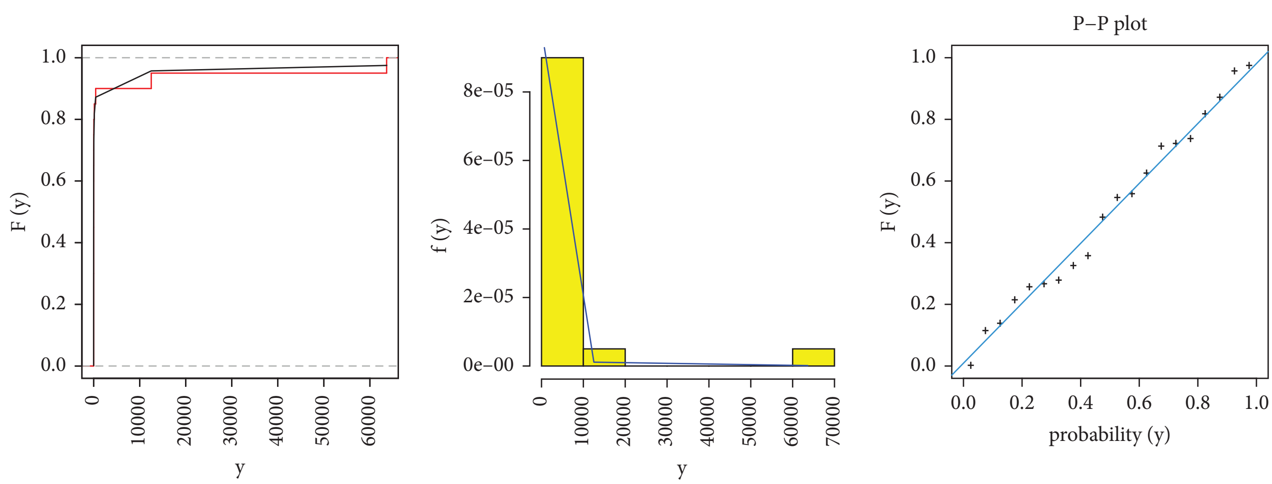

Figure 2: Estimated CDF, PDF, and PP plot of GP for data Y.

extensive simulation study. Since the JPC samples scheme with binomial removal of the GP distribution is given in terms of the cdf and pdf of GP distribution, it is easy to generate a random sample from this model.

$$x_i = \begin{cases} \dfrac{\lambda_1}{\theta_1}\left[(1-u_i)^{-\theta_1}-1\right], & \theta_1 \neq 0, \\ -\lambda_1 \ln(1-u_i), & \theta_1 = 0, \end{cases}$$
$$y_i = \begin{cases} \dfrac{\lambda_2}{\theta_2}\left[(1-v_i)^{-\theta_2}-1\right], & \theta_2 \neq 0, \\ -\lambda_2 \ln(1-v_i), & \theta_2 = 0, \end{cases} \tag{19}$$

where $0 < u < 1$ and $0 < v < 1$. Therefore, we need to determine some values of actual parameters for this model as follows:

Case 1. $\theta_1 = 1.2, \lambda_1 = 1.5, \theta_2 = 0.8, \lambda_2 = 1.3$.

Case 2. $\theta_1 = 0.4, \lambda_1 = 0.65, \theta_2 = 0.7, \lambda_2 = 1.8$.

Case 3. $\theta_1 = 3, \lambda_1 = 0.5, \theta_2 = 2.5, \lambda_2 = 0.8$.

Also, we need to suggest different samples sizes, hence when the samples size are $m = 30$ and $n = 25$, we selected different failure sizes for this sample as $r = 38$ and $r = 47$. When the sample sizes are $m = 50$ and $n = 55$, we selected different failure sizes as $r = 73$ and $r = 97$. Also, when the sample sizes are $m = 110$ and $n = 125$, we selected different failure sizes as $r = 170$ and $r = 215$. The probability of binomial removal for JPC is supposed to have two values as $P = 0.15$ and 0.5.

After generating the data for X and Y, we combined these variables to obtain the $W = \{X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n\}$ from two different samples with the same probabilities. Furthermore, $W_1 \leq W_2 \leq \ldots \leq W_N$ denote the order statistics, where $N = n + m$. Then, generate R-removal of censored form binomial with P probability. Generate progressive censored sample. For more information about generating progressive censored samples, see Balakrishnan [30] and Balakrishnan and Cramer [31].

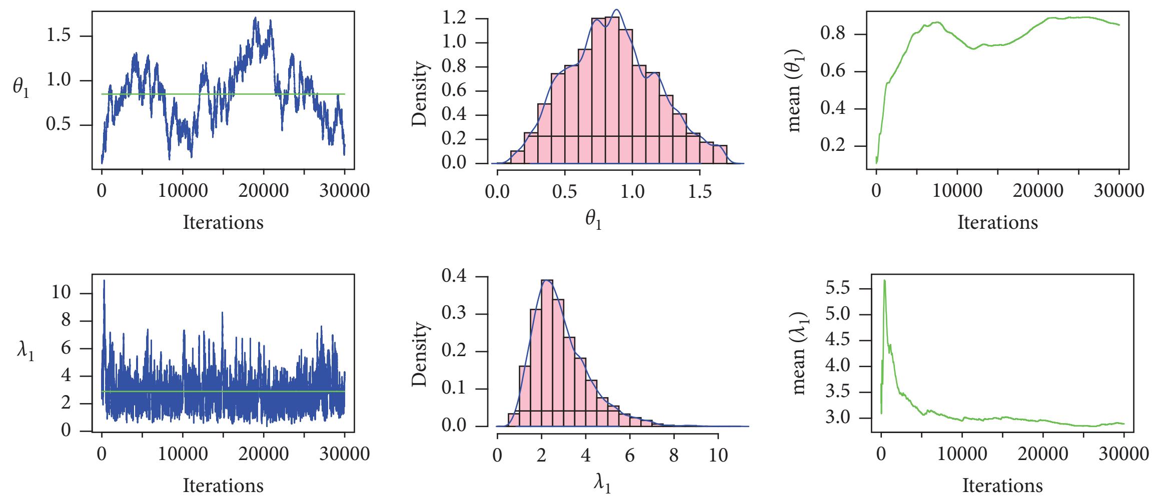

FIGURE 3: ACF, histogram, and trace plot of λ_1 and θ_1.

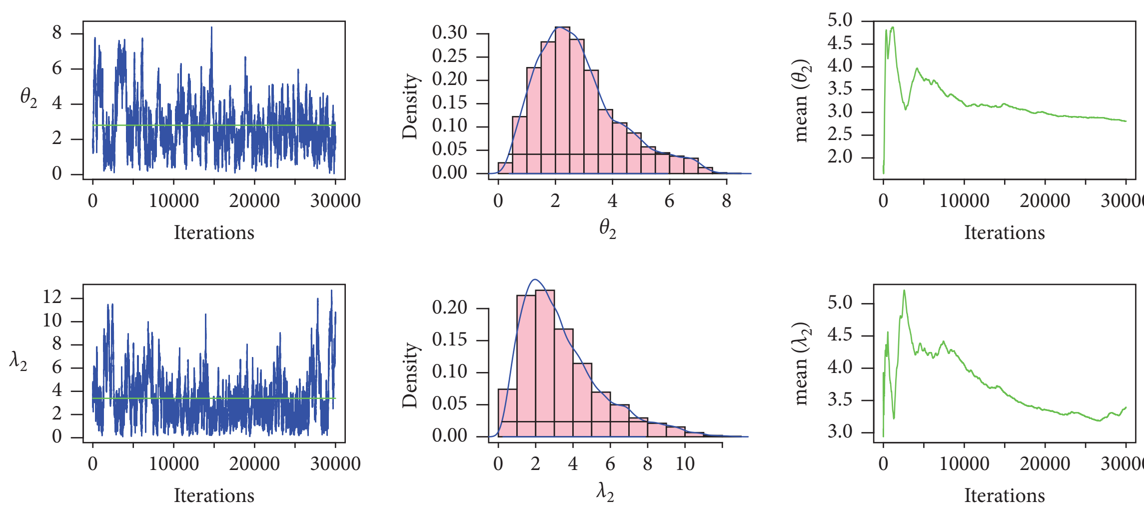

FIGURE 4: ACF, histogram, and trace plot of λ_2 and θ_2.

For the MLE method, we used Newton–Raphson algorithm and for the Bayesian estimation method, we used MCMC by MH algorithm. Confidence intervals of Bayesian estimation method is performed using credible intervals algorithm of MCMC results when 10000 loop is performed.

In a $(1-\xi)100\xi\%$ confidence interval, we get the length of the interval (L.CI) when $\xi = 5\%$. The Biases and MSEs are used to compare MLE and Bayesian estimation methods. The results are obtained in Tables 1–3 after 10000 loops.

The following concluding remakes are noticed based on these Tables:

(1) As sample size increases with fixing all other values of model, the bias, MSE, and L.CI associated with the parameter of the GP distribution based on JPC estimates decrease for all methods of estimation.

(2) As the number of failed units increases (r) with fixing all other values of the model, the bias, MSE, and L.CI associated with the parameter of the GP distribution based on JPC estimates decrease for all methods of estimation.

(3) The Bayesian estimation method is the best estimation method to estimate the parameters of the GP distribution based on JPC, since it has the smallest bias and MSE values and shortest L.CI.

(4) The bootstrap confidence interval is the shortest interval length for estimation of the GP parameters based on JPC.

(5) When comparing the asymptotic confidence intervals with the credible confidence intervals, we can realize that the latter have shorter interval lengths.

5.2. Application of Data. Abu-Zinadah [32] used this data to inference the jointly Type-II censored samples from two Pareto distributions. The data are as follow: $X = 0.152$, 0.548, 0.759, 0.778, 0.916, 0.976, 1.017, 1.433, 1.558, 1.822, 1.888, 2.395, 3.066, 3.901, 5.489, 5.809, 17.886, 21.829, 43.239, and

Table 4: MLE and Bayesian estimation methods for GP based on JPC scheme with binomial removal.

P		0.1				0.5			
m		MLE		Bayesian		MLE		Bayesian	
		Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE
20	θ_1	0.5393	0.6240	0.9504	0.6121	0.1229	0.4495	0.8500	0.3281
	λ_1	3.6581	2.4356	3.6370	2.2474	3.1396	1.2869	2.8928	1.2789
	θ_2	2.9656	1.1532	2.3175	1.0591	1.7832	1.6354	2.8058	1.4668
	λ_2	2.4019	1.6005	3.0646	1.4616	2.8372	1.9272	3.3960	1.8141
25	θ_1	0.8031	0.4999	1.0957	0.3366	0.7766	0.5568	0.8263	0.3387
	λ_1	2.9710	1.3029	2.2852	0.6248	3.2473	1.8360	3.1524	0.9213
	θ_2	1.1402	1.0574	2.1494	0.9079	2.6532	0.9443	2.4127	0.3564
	λ_2	3.4690	2.4049	3.2225	0.5395	2.7533	1.7404	3.1848	0.2771
30	θ_1	1.1172	0.5119	1.8141	0.3047	0.6859	0.4335	0.8834	0.3192
	λ_1	2.4175	1.0644	1.7678	0.5406	3.2102	1.4751	2.5594	0.6514
	θ_2	1.6383	0.7780	2.0479	0.4506	3.3062	0.9110	2.8932	0.3247
	λ_2	2.8376	1.6038	1.8831	0.5966	2.2662	1.6233	1.6953	0.2339
35	θ_1	1.0690	0.4798	0.8663	0.2643	0.9730	0.4341	1.1057	0.3050
	λ_1	2.4427	1.0600	2.3529	0.4543	2.5322	1.0979	2.2959	0.6252
	θ_2	3.2396	0.9787	3.4954	0.3715	2.9439	0.9221	3.6897	0.2512
	λ_2	2.2195	1.3993	1.9277	0.5021	2.7814	1.7635	3.9422	0.2770

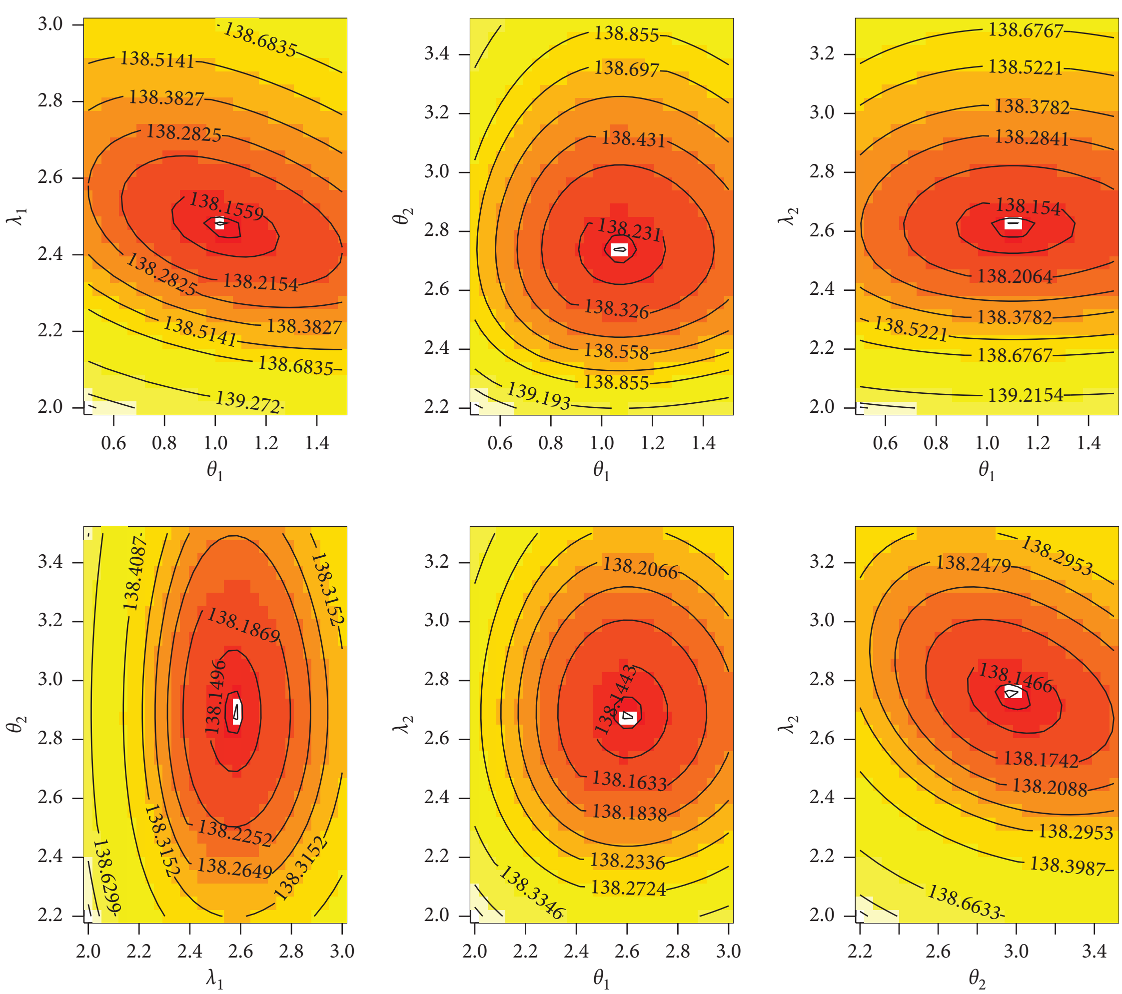

Figure 5: Contour plot of log-likelihood function with different values of parameters; $m = 35$ and $p = 0.5$.

90.793. $Y = 0.006$, 0.383, 0.489, 0.925, 1.25, 1.337, 1.448, 1.976, 2.426, 5.484, 8.611, 9.430, 16.120, 37.360, 41.090, 49.276, 152.313, 442.915, 12510.900, and 63621.000.

The empirical and fitted distribution functions, CDF, and PP-plots are presented in Figures 1 and 2. The graphical tools such as trace plots and auto-correlation function (ACF) plots are used to check the convergence of MCMC. Figures 3 and 4 show the trace and ACF plots for θ_1, λ_1, θ_2, and λ_2 of a chain of different number of iterations. The ACF plots for θ_1, λ_1, θ_2, and λ_2 show that the chains have a low autocorrelation. Also, they indicate a rapid convergence of the MCMC subject to the normal distribution. For more information about convergence of MCMC one may refer to Freitas et al. [33]. From Table 4, it is clear that Bayesian estimation performs better than MLE for different number of failures and different binomial probabilities, this is because it has less mean squared error (SE). Figure 5 shows the Contour plots of log-likelihood function with different parameter values, the MLE results of model with $m = 35$ and $p = 0.5$ are unique and attain their maximum points.

6. Conclusions

In this article, we considered point and interval estimation for two joint populations with generalized Pareto lifetimes under progressive Type-II censoring schemes. Classical and nonclassical estimation methods were proposed and numerical methods were implemented to evaluate the performance of the different methods of estimation, it was shown through a real data example that Bayesian methods were superior to the classical method (MLE). While comparing the confidence intervals it was realized that Bootstrap confidence interval has the shortest interval lengths compared to asymptotic and credible confidence intervals.

Data Availability

The data used to support this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Project no. GRANT2550). The authors, therefore, acknowledge technical and financial support of the Deanship of Scientific Research at KFU.

References

[1] A. Rasouli and N. Balakrishnan, "Exact likelihood inference for two exponential populations under joint progressive type-II censoring," *Communications in Statistics - Theory and Methods*, vol. 39, no. 12, pp. 2172–2191, 2010.

[2] A. P. Basu, "On a generalized Savage statistic with applications to life testing," *The Annals of Mathematical Statistics*, vol. 39, no. 5, pp. 1591–1604, 1968.

[3] R. A. Johnson and K. G. Mehrotra, "Locally most powerful rank tests for the two-sample problem with censored data," *The Annals of Mathematical Statistics*, vol. 43, no. 3, pp. 823–831, 1972.

[4] G. K. Bhattacharyya and K. G. Mehrotra, "On testing equality of two exponential distributions under combined Type-II censoring," *Journal of the American Statistical Association*, vol. 76, no. 376, pp. 886–894, 1981.

[5] G. K. Bhattacharyya, "Inferences under two-sample and multi-sample situations," in *The Exponential Distribution: Theory, Methods and Applications*, N. Balakrishnan and A. P. Basu, Eds., pp. 93–118, Gordon and Breach, Newark, NJ, USA, 1995.

[6] N. Balakrishnan and A. Rasouli, "Exact likelihood inference for two exponential populations under joint Type-II censoring," *Computational Statistics and Data Analysis*, vol. 52, no. 5, pp. 2725–2738, 2008.

[7] S. Ashour and O. Eraki, "Parameter estimation for multiple Weibull populations under joint type-II censoring," *International Journal of Advanced Statistics and Probability*, vol. 2, no. 2, pp. 102–107, 2014.

[8] S. Parsi, M. Ganjali, and N. S. Farsipour, "Conditional maximum likelihood and interval estimation for two Weibull populations under joint Type-II progressive censoring," *Communications in Statistics - Theory and Methods*, vol. 40, no. 12, pp. 2117–2135, 2011.

[9] M. Doostparast, M. V. Ahmadi, and J. Ahmadi, "Bayes estimation based on joint progressive type II censored data under LINEX loss function," *Communications in Statistics - Simulation and Computation*, vol. 42, no. 8, pp. 130102064423005–1886, 2013.

[10] N. Balakrishnan, F. Su, and K. Y. Liu, "Exact likelihood inference for k exponential populations under joint progressive Type-II censoring," *Communications in Statistics - Simulation and Computation*, vol. 44, no. 4, pp. 902–923, 2015.

[11] S. Mondal and D. Kundu, "Inference on Weibull parameters under a balanced two-sample type II progressive censoring scheme," *Quality and Reliability Engineering International*, vol. 36, no. 1, pp. 1–17, 2020.

[12] A. Algarni, M. Elgarhy, A. M Almarashi, A. Fayomi, and A. R El-Saeed, "Classical and bayesian estimation of the inverse weibull distribution: using progressive type-I censoring scheme," *Advances in Civil Engineering*, vol. 2021, Article ID 5701529, 15 pages, 2021.

[13] M. Shrahili, A. R. El-Saeed, A. S. Hassan, I. Elbatal, and M. Elgarhy, "Estimation of entropy for log-logistic distribution under progressive type II censoring," *Journal of Nanomaterials*, vol. 2022, Article ID 2739606, 10 pages, 2022.

[14] N. Alotaibi, I. Elbatal, E. M. Almetwally, S. A. Alyami, A. S. Al-Moisheer, and M. Elgarhy, "Truncated cauchy power weibull-G class of distributions: bayesian and non-bayesian inference modelling for COVID-19 and carbon fiber data," *Mathematics*, vol. 10, no. 9, Article ID 10091565, 2022.

[15] J. Martín, M. I. Parra, M. M. Pizarro, and E. L. Sanjuán, "Baseline methods for the parameter estimation of the generalized Pareto distribution," *Entropy*, vol. 24, no. 2, Article ID 24020178, 2022.

[16] C. Huang, X. Zhao, W. Cheng, Q. Ji, Q. Duan, and Y. Han, "Statistical inference of dynamic conditional generalized Pareto distribution with weather and air quality factors," *Mathematics*, vol. 10, no. 9, Article ID 10091433, 2022.

[17] P. L. Shui, P. J. Zou, and T. Feng, "Outlier-robust truncated maximum likelihood parameter estimators of generalized Pareto distributions," *Digital Signal Processing*, vol. 127, Article ID 103527, 2022.
[18] Y. He, L. Peng, D. Zhang, and Z. Zhao, "Risk analysis via generalized Pareto distributions," *Journal of Business and Economic Statistics*, vol. 40, no. 2, pp. 852–867, 2022.
[19] H. Mahgoun, A. Taieb, B. Azmedroub, and B. Souissi, "Generalized Pareto distribution exploited for ship detection as a model for sea clutter in a Pol-SAR application," in *Proceedings of the 2022 7th International Conference on Image and Signal Processing and their Applications (ISPA)*, pp. 1–5, IEEE, Mostaganem, Algeria, May, 2022.
[20] D. V. Lindley, "Approximate bayesian methods," *Trabajos de Estadistica y de Investigacion Operativa*, vol. 31, no. 1, pp. 223–245, 1980.
[21] R. L. Karandikar, "On the Markov chain Monte Carlo (MCMC) method," *Sadhana*, vol. 31, no. 2, pp. 81–104, 2006.
[22] G. Andrew, B. C. John, S. S. Hal, B. D. David, V. Aki, and B. R. Donald, *Bayesian Data Analysis*, FL: CRC press, Boca Raton, FL, USA, 2014.
[23] H. Z. Muhammed and E. M. Almetwally, "Bayesian and non-Bayesian estimation for the bivariate inverse weibull distribution under progressive type-II censoring," *Annals of Data Science*, vol. 10, no. 2, pp. 481–512, 2020.
[24] E. S. A. El-Sherpieny, E. M. Almetwally, and H. Z. Muhammed, "Bayesian and non-bayesian estimation for the parameter of bivariate generalized Rayleigh distribution based on clayton copula under progressive type-II censoring with random removal," *Sankhya*, pp. 1–38, 2021.
[25] A. C. Davison and D. V. Hinkley, *Bootstrap Methods and Their Applications*, Cambridge University Press, Cambridge, UK, 1997.
[26] B. Efron and R. J. Tibshirani, *An Introduction to the Bootstrap*, New York Chapman and Hall, New York, NY, USA, 1993.
[27] P. Hall, "Theoretical comparison of bootstrap confidence intervals," *Annals of Statistics*, vol. 16, no. 3, p. 927, 1988.
[28] B. Efron, "Bootstrap methods: another look at the jackknife," *Annals of Statistics*, vol. 7, pp. 1–26, 1979.
[29] B. Efron, "Censored data and the bootstrap," *Journal of the American Statistical Association*, vol. 76, no. 374, pp. 312–319, 1981.
[30] N. Balakrishnan, "Progressive censoring methodology: an appraisal," *Test*, vol. 16, no. 2, pp. 211–259, 2007.
[31] N. Balakrishnan and E. Cramer, *The Art of Progressive Censoring*, Springer, New York, NY, USA, 2014.
[32] H. H. Abu-Zinadah, "Statistical inferences with jointly type-II censored samples from two Pareto distributions," *Open Physics*, vol. 15, no. 1, pp. 557–565, 2017.
[33] N. D. P. Freitas, H. Sørensen, M. Jordan, and S. Russell, "Variational MCMC," in *Proceedings of the Seventeenth Conference on Uncertainty in Artificial Intelligence*, vol. 70, pp. 120–27, San Francisco, CA, USA, Augest 2001.