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Research Article

The Approximate Solutions of Large Deflection of a Cantilever Beam under a Point Load

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By integrating the physical and time domains with the Galerkin method and adopting approximate solutions satisfying the boundary conditions, a set of algebraic equations of nonlinear nature is obtained from the nonlinear differential equation of an elastic beam for the undetermined coefficients of solutions of the deflection. These coefficients are to be used with the approximate basis functions for the asymptotic and explicit solutions of the nonlinear differential equation of flexure of a beam widely known as an elastica. Taking advantage of powerful tools for the symbolic manipulation of algebraic equations, such a novel method and procedure offer a new and efficient approach and option with known linear solutions in dealing with increasingly complex nonlinear problems in practical applications of both static and dynamic nature.

1. Introduction

The large deformation of a beam is frequently encountered in mechanical and structural engineering with critical importance in guaranteeing the safety and proper functioning of such essential components. The analysis based on the beam equations has been done with many methods and techniques for the deformation, stress, and vibration frequencies of an elastica [1–4]. In addition, many techniques have been developed and improved in solving problems with nonlinear equations, and the applications can be found in many fields and subjects with rich histories [5–7]. On the other hand, the deformation and vibration of beams are of great interest to researchers because of the signatory roles and extensive results for comparisons and validation of solution methods and techniques [8–16]. The method and procedure can be used for the analysis of nonlinear wave propagation in homogeneous and functionally graded materials [5, 17].

Particularly, with this classical problem of flexure of a beam under a point load at the tip, the Galerkin method is used with a modification for good approximation and demonstration of the newly proposed and demonstrated extended Galerkin method (EGM) for nonlinear problems with asymptotic solutions [18, 19]. The newly revitalized Galerkin method with such a modification is capable and efficient for both linear and nonlinear problems from structural analysis, vibrations, and wave propagation as a new technique applicable to a wide class of nonlinear problems with a novel, simple, and efficient procedure for approximate and explicit solutions with novel forms. From a comparison, it is clear that the proposed procedure for solutions of nonlinear differential equations should be the preferred choice in solving such problems because of the simple process and less time in the iteration.

2. The Large Deformation of an Elastic Beam

Elastic beams with variations and complications of structure and supports are widely used as structural elements for loading resistance and proper functions with allowed deformation, primarily flexure but can also be combinations of other forms, usually infinitesimal in the magnitude of deformation as the basic assumption of solid mechanics. In this case, the classical beam theory is adequate in the analysis of deformation

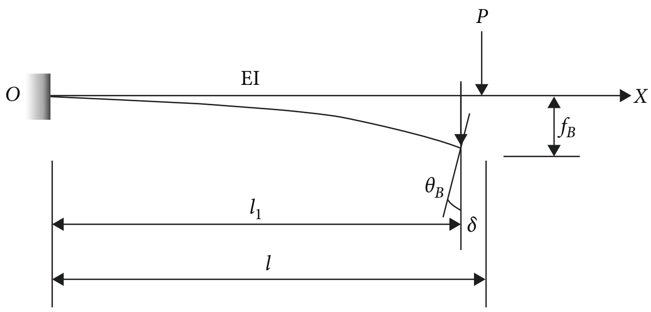

FIGURE 1: A cantilever beam under a point load.

under static and dynamic loadings with the linear formulation, equations, and methods presented in textbooks. In case of large deformation or with special configurations, modifications of the classical beam theory, which is also known as the Euler–Bernoulli beam theory, are available to accommodate additional requirements of analyses. The nonlinear beam theory with the options for both physical and kinematic linearities is certainly the choice for large deformation, but the nonlinear equations have to be solved with approximate methods, as shown in a recent study of the flexure of a cantilever beam under a point load [16, 20–24]. It is highly nonlinear differential equation for the classical problem of an elastica, and the accurate solutions can be given in elliptic functions through extensive studies in history [1–3, 25–27]. To obtain practical solutions to nonlinear differential equations, many approximate techniques have been tried for improved solutions and simpler procedures including the finite-element method and alternative formulation [16, 28–39]. These extensive efforts, including the recent innovative work with the EGM and Rayleigh–Ritz method, demonstrated the needs and efforts in obtaining effective solutions to nonlinear equations arising from many research fields and problems.

As it was solved with the homotopy analysis method (HAM) in earlier studies [40–43], the EGM is now used for the approximate solutions as a new and effective alternative with great potential for the same problem [4, 5, 44]. It is found that using the sine function as the basis function in this study, relatively accurate solutions with explicit expressions of a few lower-order terms can be obtained in a simple and efficient procedure based on the Galerkin method with integration over the physical domain for the coefficients of series solutions with accuracy and efficiency [43, 44]. In this paper, a different basis function is chosen to demonstrate the versatility and options in utilizing the popular Galerkin method with a modification or extension. Particularly, explicit solutions with fewer terms of basis functions would be more intuitive for the evaluation of solutions in comparison with the asymptotic solutions of larger numbers of terms.

With the cantilever beam under a point load at the tip in Figure 1, the rotation angle of the beam is given by [43, 45]

$$\frac{d\theta}{ds} = \frac{P}{\mathrm{EI}}(l_1 - x),\ \theta(0) = 0,\ \theta'(l) = 0, \tag{1}$$

where $\theta, s, P, \mathrm{EI}, l_1, x,$ and$l$ are rotation angle, arc length, force, flexural stiffness, the position of load, coordinate, and length, respectively.

By introducing a new dimensionless variable $z = s/l$, Equation (1) will be modified to

$$\theta'' + \alpha\cos\theta = 0,\ \alpha = \frac{Pl^2}{\mathrm{EI}},\ \theta(0) = 0,\ \theta'(1) = 0\ \cdot \tag{2}$$

The exact solution to Equation (2) can be given in elliptic functions known as the analytical solution of an elastica [26, 27], as has been shown in earlier solutions [43, 45, 46]. The vibrations of such a beam has been studied with the EGM also in a similar procedure [4].

With an infinitesimal θ in Equation (2), the linear solution is

$$\theta(z) = \frac{\alpha}{2}(2 - z)z,\ \theta_B = \theta(1) = \frac{\alpha}{2}. \tag{3}$$

Let the solution of Equation (2) is a power series in the form of

$$\theta(z) = \sum_{n=0}^{\infty} A_n z^{n+1}, \tag{4}$$

with the consideration of boundary conditions $\theta(0) = 0$ and $\theta'(1) = 0$, the expression in Equation (4) will be modified to

$$\theta(z) = \sum_{n=0}^{\infty} A_n[z^{n+1} - (n+2)]z, \tag{5}$$

where coefficients A_n are to be determined. Apparently, this is a basis function which has not been tried, but another study with the sine functions also yield good results [45].

By applying the standard Galerkin method

$$\int_0^1 \left(\frac{d^2\theta}{dz^2} + \alpha\cos\theta\right)\delta\theta dz = 0, \tag{6}$$

with $\cos\theta \approx 1 - \frac{\theta^2}{2}$ and Equations (5) and (6) will be written as

$$\int_0^1 \left[\sum_{n=0}^{N} A_n(n+2)(n+1)z^n + \alpha\left(1 - \frac{\theta^2}{2}\right)\right] [z^{m+1} - (m+2)]z dz = 0, N \geq m = 0, 1, 2, \ldots, \tag{7}$$

By evaluating the above integration with different integers m and N, a set of nonlinear algebraic equations can be obtained for the coefficients A_n. Then there are approaches for the determination of these coefficients systematically. One procedure is to set $m = N$, then there will be a set of nonlinear algebraic equations of A_n to be solved simultaneously from the system of undetermined coefficients. Another approach

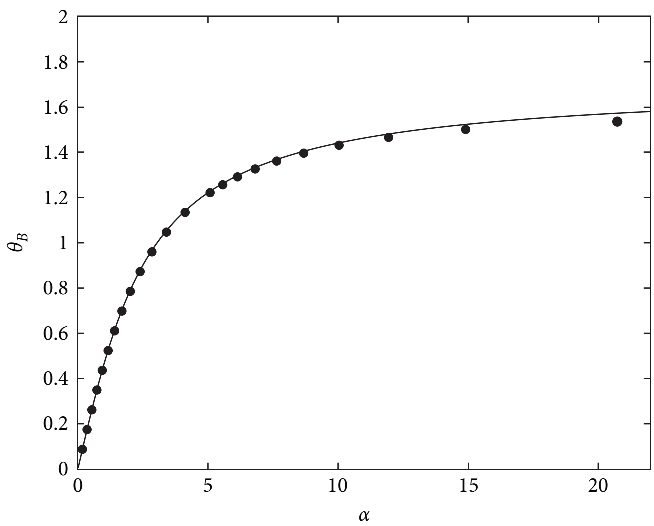

FIGURE 2: A comparison of the first-order approximation and exact solution of rotation angle at the end of the beam.

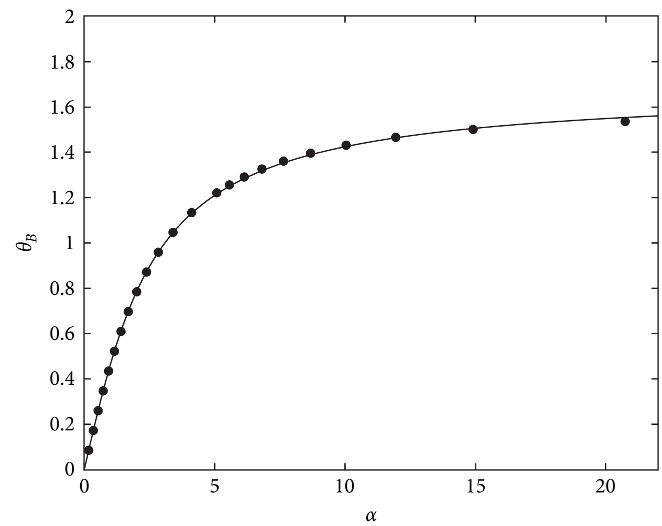

FIGURE 3: A comparison of the second-order approximation and exact solution of rotation angle at the end of the beam.

is the iterative process by the successive determination of coefficients one at each time for the successive approximation. Both approaches have been demonstrated as part of the solution procedures of the EGM [4, 5, 44–46].

In this study, the approximation starts from the linear solution

$$\theta(z) = A_0(z-2)z, \tag{8}$$

with only one equation for the coefficient from Equation (7), it is easily obtained that

$$A_0 = -\frac{35}{24\alpha}\left(3\sqrt{\frac{16\alpha^2}{105}+\frac{4}{9}}-2\right). \tag{9}$$

Then the rotation angle at the end of beam is compared with the exact solution in Figure 2. It is clear that the first-order approximation is quite good with the given parameter α.

The second-order approximation is now

$$\theta(z) = A_0(z-2)z + A_1(z^2-3)z, \tag{10}$$

and with known coefficient A_0, the solution of coefficient A_1 can be obtained from Equation (7) with $m=1$ after integration. The iterative procedure is adopted in this study for the coefficient of the asymptotic solution, and the simultaneous solution to the problem is left to future study. With the nonlinear algebraic equation for the amplitude A_1, the solution is

$$A_1 = -\frac{2}{8253\alpha}\left(2137\alpha A_0 - 6048 + 1260\times\left(\frac{391\alpha A_0}{4200}+\frac{131\alpha^2}{16}-\frac{1193\alpha^2A_0^2}{254016}+\frac{576}{25}\right)\frac{1}{2}\right), \tag{11}$$

and the exact solution of rotation angle at the end of the beam is compared again in Figure 3. It is clear that the approximation is improved significantly with the second-order approximation. The advantages and disadvantages of the procedure here will be left for future studies as the accuracy of this example is satisfactory so far.

For the third-order approximation with

$$\theta(z) = A_0(z-2)z + A_1(z^2-3)z + A_2(z^3-4)z\ . \tag{12}$$

Using the same procedure demonstrated above with Equation (7) by setting $m=2$, the unknown amplitude A_2 in two known coefficients A_0 and A_1 is

$$A_2 = \frac{13}{1849716\alpha}\left(13860\times\left(\frac{84078\alpha^2}{2275}-\frac{1558A_1\alpha}{3185}-\frac{109188A_0\alpha}{175175}-\frac{207741A_0^2\alpha^2}{3853850}-\frac{6746713A_1^2\alpha^2}{227026800}-\frac{1013A_0A_1\alpha^2}{12740}+\frac{5184}{49}\right)^{\frac{1}{2}} \right. \\ \left. +50220A_0\alpha + 97163A_1\alpha - 142560,\right) \tag{13}$$

and the comparison of solutions is shown in Figure 4.

It is clear that the approximate solutions from the Galerkin method are actually very good as comparisons shown in Figures 2–4. The procedure to obtain these accurate coefficients and solutions are simple and straightforward. The procedure is based on the popular Galerkin method and can be modified and optimized with choices of basis functions in conjunction with the equations and boundary conditions, furtherly using the orthogonality property to simplify the calculation. An additional comparison of solution strategies for iterative or systematic determination of the coefficients is largely dependent on the algorithms and computing cost, and the preferences can be made with a detailed investigation of both approaches with Equation (7).

Furthermore, additional properties of the beam deformation and stress can be obtained accordingly with the solutions and essential equations from the beam theory.

For the cantilever beam under a point load at the tip, as shown in Figure 1, the vertical displacement, or the deflection, of the beam at the free end is given by

$$\frac{f_{\mathrm{B}}}{l} = \int_0^1 \sin\theta \, dz. \tag{14}$$

Returning to Equation (1), it is clear

$$\frac{d}{ds}\left(\frac{d\theta}{ds}\right) = -\frac{P}{\mathrm{EI}}\frac{dx}{ds} = -\frac{P}{\mathrm{EI}}\cos\theta. \tag{15}$$

Then multiplying both sides of Equation (15) by $d\theta$ and integrating over z will give

$$\frac{1}{2}\left(\frac{d\theta}{dz}\right)^2 = -\alpha\sin\theta + C, \tag{16}$$

where C is a constant to be determined. With the consideration of boundary condition $\theta'(1) = 0$, the integral constant is obtained as

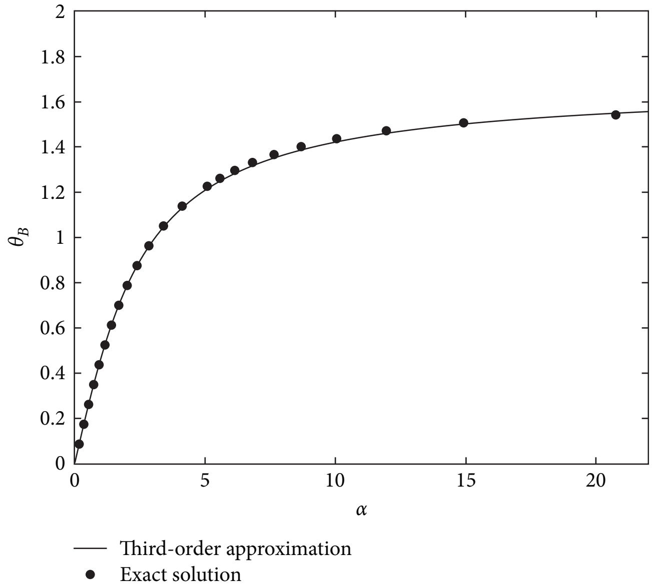

FIGURE 4: A comparison of the third-order approximation and exact solution of rotation angle at the end of the beam.

$$C = \alpha\sin\theta_B, \tag{17}$$

where θ_B is the angle of rotation at the free end of the cantilever beam. Finally with Equations (14), (16), and (17) will be rewritten as

$$\frac{f_{\mathrm{B}}}{l} = \int_0^1 \sin\theta dz = \int_0^1 \left(\sin\theta_B - \frac{1}{2\alpha}\left(\frac{d\theta}{dz}\right)^2\right) dz. \tag{18}$$

Approximating $\sin\theta_{\mathrm{B}} \approx \theta_{\mathrm{B}} - \frac{\theta_{\mathrm{B}}^3}{6}$ for a simple evaluation, Equation (18) will be written as

$$\frac{f_{\mathrm{B}}}{l} = \int_0^1 \left(\theta_{\mathrm{B}} - \frac{\theta_{\mathrm{B}}^3}{6} - \frac{1}{2\alpha}\left(\frac{d\theta}{dz}\right)^2\right) dz, \tag{19}$$

then it is obtained with the third-order approximation that

$$\begin{aligned}\frac{f_{\mathrm{B}}}{l} = {} & \frac{A_0^3}{6} + A_0^2A_1 + \frac{3A_0^2A_2}{2} - \frac{2A_0^2}{3\alpha} + 2A_0A_1^2 + 6A_0A_1A_2 - \frac{5A_0A_1}{2\alpha} \\ & + \frac{9A_0A_2^2}{2} - \frac{18A_0A_2}{5\alpha} - A_0 + \frac{4A_1^3}{3} + 6A_1^2A_2 - \frac{12A_1^2}{5\alpha} \\ & + 9A_1A_2^2 - \frac{7A_1A_2}{\alpha} - 2A_1 + \frac{9A_2^3}{2} - \frac{36A_2^2}{7\alpha} - 3A_2.\end{aligned} \tag{20}$$

Finally, the comparison of deflection at the end of the beam with an exact solution is shown in Figure 5. Again, the accuracy of such an approximation is satisfactory.

For the cantilever beam under a point load at the tip as shown in Figure 1, the horizontal length of the beam at free end is given by

$$\frac{l_1}{l} = \int_0^1 \cos\theta(z) dz, \tag{21}$$

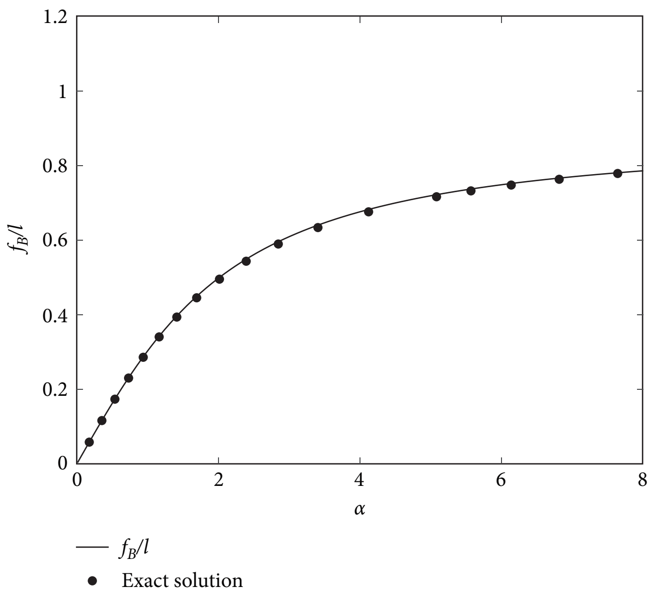

FIGURE 5: A comparison of deflection of the approximation and exact solution of deflection at the end of the beam.

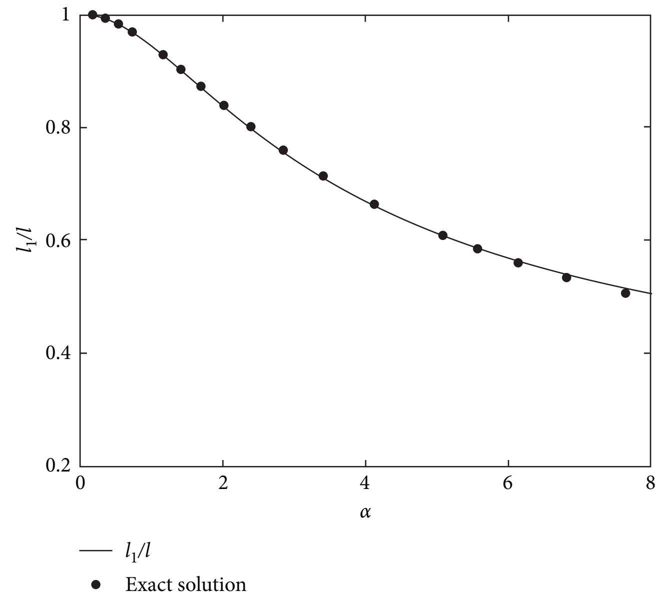

FIGURE 6: A comparison of the approximation and exact solution of horizontal length at the end of the beam.

and with $\cos\theta \approx 1 - \frac{\theta^2}{2}$, Equation (21) will be written as

$$\frac{l_1}{l} = \int_0^1 \left(1 - \frac{\theta^2}{2}\right) dz, \tag{22}$$

then from Equation (22) it can be obtained with the third-order approximation that

$$\frac{l_1}{l} = 1 - \frac{4A_0^2}{15} - \frac{61A_0A_1}{60} - \frac{31A_0A_2}{21} - \frac{34A_1^2}{35} - \frac{113A_1A_2}{40} - \frac{37A_2^2}{18}. \tag{23}$$

Finally, the comparison of horizontal length at the end of beam with an exact solution is shown in Figure 6.

The above procedure completed the full calculation of the deflection of a cantilever beam under a point load at the free end including the movement of the free end. The asymptotic solutions with the terms satisfying the boundary conditions are reasonably accurate in comparison with the exact solutions by other methods. It demonstrates that the Galerkin method can be effective in solving the static and dynamic deformation of elastic beams.

3. Results and Discussion

An approximate procedure based on the Galerkin method has been demonstrated with the analysis of the large deformation of an elastic beam with the nonlinear differential equation, which is also widely known as the elastica problem. By solving the equation with the Galerkin method with basis functions satisfying the boundary conditions, a set of nonlinear equations of coefficients of basic functions are solved with the aid of symbolic software tools such as Matlab® for an efficient evaluation by an iterative procedure. Of course, it can be done by solving the nonlinear system of algebraic equations also. Eventually, the procedure as an extension of the classical Galerkin method is capable to provide accurate solutions to nonlinear differential equations with an efficient computational procedure. It is also apparent that such a procedure is more versatile and flexible in accommodating the variety of differential equations arising from scientific and engineering problems. The Galerkin method and the equivalent Rayleigh–Ritz method have been modified for a series of typical nonlinear equations of vibrations and wave propagation as the EGM and Rayleigh–Ritz method [4–6, 14, 44–46]. It is obvious that using the extension of both Galerkin and Rayleigh–Ritz methods, the nonlinear analysis of structures under static and dynamic loadings can be done with the popular methods widely covered in textbooks and utilized in calculations. The effort will unify the popular and powerful Galerkin method for more convenient implementations and applications with more nonlinear differential equations. The advantage of such a procedure is the quick convergence with appropriate basis functions. Consequently, the beam deflection and elongation from the approximate solutions of successive orders also show great accuracy. Clearly, the simple procedure demonstrates a practical and reliable technique for accurate solutions of some nonlinear equations. In fact, the advantage of the Galerkin method in solving nonlinear differential equations is the coefficients which are the combination of many terms of power series. The current process is reduced significantly with the combination in comparison to the single term of the iterative procedure with other methods and algorithms. A possible improvement with the optimal choices of the coefficients in a global sense with the Galerkin method can also be tried for comparisons with the results from this study.

4. Conclusions

The advantages of the Galerkin method for nonlinear equations can be clearly seen from the simple procedure that produces accurate solutions more efficiently in comparison with some known popular asymptotic methods. Since the symbolic tools of mathematical manipulation are widely available and powerful in solving nonlinear algebraic equations, the adoption of the Galerkin method should be more preferable to avoid the tedious iterative procedure in solving the nonlinear equations. The procedure shown in this study is definitely more attractive in finding the solutions to nonlinear problems with similar features through the adoption of the Galerkin method and solving nonlinear algebraic equations with powerful symbolic mathematical tools.

Data Availability

The data of this study are available from the authors and the author's website.

Disclosure

After conducting our research, we presented an earlier version of this manuscript as a preprint to disseminate our findings and gather feedback from the scientific community. The preprint is available on ResearchGate [47].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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