Hindawi
Mathematical Problems in Engineering
Volume 2023, Article ID 2449502, 6 pages
https://doi.org/10.1155/2023/2449502

Research Article

On Subtractive Derivations of Rℓ-Monoids

Yu Gao and Bingfang Li

Department of Basic Courses, Shaanxi Railway Institute, Wennan 714000, China

Correspondence should be addressed to Yu Gao; gy19890524@163.com

Received 16 September 2022; Revised 19 October 2022; Accepted 25 November 2022; Published 18 February 2023

Academic Editor: Jun Tao Wang

This paper is intended to introduce the subtractive derivations and study some of their algebraic properties on Rℓ-monoids. Also, we give some characterizations of subtractive derivations on the Gödel center. Moreover, Gödel algebras are characterized by a fixed set of subtractive derivations. Finally, we discuss the relationship between subtractive derivations and other derivations for Rℓ-monoids. These results of the paper can provide the common properties of subtractive derivations in the t-norm-based fuzzy logical algebras.

1. Introduction

Residuated lattice ordered monoids (Rℓ-monoids, for short) were introduced by Swamy [1] as a common generalization of Abelian lattice ordered groups and Heyting algebras. Moreover, residuated lattice ordered monoids are in very close connections with algebras of t-norm-based fuzzy logics [2–8]. In particular, BL algebras and MV-algebras can be viewed as particular cases of such algebras. It is worth noting that many properties of BL-algebras are also satisfied in all Rℓ-monoids. In view of this point, Rℓ-monoids could be taken as an algebraic semantics of a more general logic than Hájek's [9] basic fuzzy logic. So, R ℓ-monoids can also play an important role in studying fuzzy logic.

The notion of derivations is instrumental in studying properties and structure in fuzzy logical algebraic structure. Posner [10], in 1957, studied different kinds of derivations in a prime ring and some of their basic algebraic properties. After that, Borzooei et al. [11–13] gave some characterizations of p-semisimple BCI-algebras via derivations with respect to BCI-algebras with derivation. In 2008, Xin et al. [14–16] characterized modular lattices and distributive lattices by isotone derivations with respect to lattices with derivations. Moreover, Alshehri et al. [17–19] derived the derivations on MV-algebras and gave some conditions under which an additive derivation is, in fact, isotone for a linearly ordered MV-algebra. In 2013, Lee et al. [20, 21] introduced and studied derivations and f-derivations on lattice implication algebras and discussed the relations between derivations and filters. In 2016, He et al. [22] investigated the kinds of derivations in residuated lattices and characterized Heyting algebras with respect to the above derivations. In 2017, Hua [23] studied derivations in R_0-algebras, which are equivalent to NM-algebras, and discussed the relation between filters and fixed point set of these derivations. In 2022, Liu studied some results on implicative derivations on MTL-algebras and gave some characterizations of them by these kinds of derivations. The paper is motivated by the following considerations: the previous research about derivations on t-norm-based fuzzy logical algebras is multiplicative derivation and implicative derivations, which are two maps that satisfy.

$$\begin{aligned} d(x \circledast y) &= (d(x) \circledast y) \Cup (x \circledast d(y)), \text{ (multiplicative derivation)}, \\ d(x \circledast y) &= (d(x) \hookrightarrow y) \Cup (x \hookrightarrow d(y)), \text{ (implicative derivation)}. \end{aligned} \tag{1}$$

However, there have been few research studies on derivations defined by $\Cap$ and any other operations on residuated structures so far. But this point is worth exploring since it can be studied in algebraic structures much more thoroughly by other operations. Then, it is interesting to study these kinds of derivations on fuzzy logical algebras.

Based on these considerations, we propose a new type of derivation, subtractive derivation, for Rℓ-monoids and study some of their algebraic properties. The structure of this paper is as follows: In Section 2, we review some basic notions and definitions of Rℓ-monoids. In Section 3, we introduce subtractive derivation on Rℓ-monoids and give some of their characterizations. In Section 4, we discuss the relations between the fixed point set of subtractive derivations and the ideals of Rℓ-monoids. In Section 5, we discuss the relations between subtractive derivations and other derivations, for example, multiplicative derivations and implicative derivations on Rℓ-monoids.

2. Preliminaries

First, some basic notions of Rℓ-monoids and their related algebraic results are presented.

Definition 1 (see [9]). An algebra $(\mathscr{M}, \circledast, \hookrightarrow, \Cap, \Cup, 0, 1)$ is said to be a residuated lattice if

(1) $(\mathscr{M}, \Cap, \Cup, 0, 1)$ is a bounded lattice,

(2) $(\mathscr{M}, \circledast, 1)$ is a commutative monoid,

(3) $u \circledast v \leq w$ iff $u \leq v \hookrightarrow w$, for any $u, v, w \in L$.

By $\mathscr{M}$ we mean that the universe of a residuated lattice $(\mathscr{M}, \circledast, \hookrightarrow, \Cap, \Cup, 0, 1)$. On $\mathscr{M}$, we define

$$u \leq v \text{ iff } u \hookrightarrow v = 1. \tag{2}$$

Then, $\leq$ is a binary partial order on $\mathscr{M}$ and for $u \in \mathscr{M}, 0 \leq u \leq 1$.

A residuated lattice $\mathscr{M}$ is an *Rℓ*-monoid if it satisfies the divisibility equation

$$(\text{DIV}) u \Cap v = u \circledast (u \hookrightarrow v). \tag{3}$$

An *Rℓ*-monoid $\mathscr{M}$ is a Gödel algebra if it satisfies

$$(\text{IDE}) u \circledast u = u. \tag{4}$$

We denote the set $\{u \mid u \circledast u = u\}$ of $\mathscr{M}$ by $\mathscr{I}(\mathscr{M})$.

In every *Rℓ*-monoid, we define the operation as follows:

$$u \boxminus v = u \circledast v^{\star}, \tag{5}$$

where $v^{\star} = v \hookrightarrow 0$.

Proposition 1 (see [1]). *The following hold in Rℓ-monoid $\mathscr{M}$, for all $u, v, w \in \mathscr{M}$:*

(1) $u \boxminus 0 = u, 0 \boxminus u = 0, u \boxminus u = 0, 1 \boxminus u = u^{\star}, u \boxminus 1 = 0$,

(2) $u \hookrightarrow (v \rightharpoonup w) = (u \circledast v) \hookrightarrow w = v \hookrightarrow (u \hookrightarrow w)$,

(3) if $u \leq v$, *then* $v \hookrightarrow w \leq u \hookrightarrow w, w \hookrightarrow u \leq w \hookrightarrow v, u \circledast w \leq v \circledast w, u \boxminus w \leq v \boxminus w, w \boxminus v \leq w \boxminus u$,

(4) $u \circledast v \leq u \Cap v \leq u,\ v \leq u \Cup v \leq u \boxplus v$,

(5) $u \boxminus v \leq u,\ u \boxminus v \leq v$,

(6) $u \circledast u^{\star} = 0, u \circledast 0 = 0$,

(7) $u \leq v$ *iff* $u \circledast v^{\star} = 0$ *iff* $u \boxminus v = 0$,

(8) $(u \boxminus v) \boxminus w = (u \boxminus w) \boxminus v$.

Definition 2 (see [24]). A nonempty subset I of an *Rℓ*-monoid $\mathscr{M}$ is an ideal if it satisfies the following conditions:

(1) if $u \leq v$ and $v \in I$, then $u \in I$,

(2) if $u, v \in I$, then $u \Cup v \in I$.

Definition 3 (see [15]). A self-map d on an *Rℓ*-monoid $\mathscr{M}$ is called a lattice derivation if it satisfies, for any $u, v \in \mathscr{M}$,

$$d(u \Cap v) = (du \Cap v) \Cup u \Cap dy). \tag{6}$$

Definition 4 (see [24]). A self-map d on an *Rℓ*-monoid $\mathscr{M}$ is called a multiplicative derivation if it satisfies, for any $u, v \in \mathscr{M}$,

$$d(u \circledast v) = (du \circledast v) \Cup u \circledast dv). \tag{7}$$

Proposition 2 (see [22]). *A self-map d_a: $\mathscr{M} \longrightarrow \mathscr{M}$*

$$d_a u = a \circledast u. \tag{8}$$

On an Rℓ-monoid $\mathscr{M}$ is a multiplicative derivation.

3. Subtractive Derivations of Rℓ-Monoids

Then, we introduce a new kind of derivations on Rℓ-monoids and give some characterizations of them.

Definition 5. Let $\mathscr{M}$ be an Rℓ-monoid. A mapping d: $L \hookrightarrow L$ is called a subtractive derivation on $\mathscr{M}$ if

$$d(u \boxminus v) = (du \boxminus v) \circledast (u \boxminus dv), \tag{9}$$

for any $u, v \in \mathscr{M}$.

We will denote by $\mathscr{D}(\mathscr{M})$ to be the set of all subtractive derivations of $\mathscr{M}$.

Some examples of subtractive derivations on Rℓ-monoids are presented.

Example 1. Let $\mathscr{M}$ be an Rℓ-monoid. Define a mapping 0_d: $\mathscr{M} \longrightarrow \mathscr{M}$ by

$$0_d(u) = 0, \tag{10}$$

for all $u \in \mathscr{M}$. Then, $0_d \in \mathscr{D}(\mathscr{M})$. Moreover, defining d_1: $\mathscr{M} \longrightarrow \mathscr{M}$ by

$$d_1(u) = u, \tag{11}$$

for all $u \in \mathscr{M}$. Then, $d_1 \in \mathscr{D}(\mathscr{M})$.

Example 2. Let $\mathscr{M} = \{0, u, v, 1\}$ be a chain. Defining operations $\circledast$ and $\hookrightarrow$ as follows (see Table1):

Then, $(\mathscr{M}, \circledast, \hookrightarrow, \Cap, \Cup, 0, 1)$ is an Rℓ-monoid. Now, we define d: $\mathscr{M} \longrightarrow \mathscr{M}$ as follows:

$$d(x) = \begin{cases} 0, & x = 0, u, \\ 1, & x = v, 1. \end{cases} \tag{12}$$

Then, $d \in \mathscr{D}(\mathscr{M})$.

Table 1: The operations of ⊛, and ↪.

$\circledast$	0	u	v	1
0	0	0	0	0
u	0	u	u	u
v	0	u	u	v
1	0	u	v	1
$\hookrightarrow$	0	u	v	1
0	1	1	1	1
u	0	1	1	1
v	0	u	1	1
1	0	u	v	1

Example 3. Let $\mathscr{M}_n$ be the standard n-valued MV-algebra, and hence an Rℓ-monoid, for some $n \geq 2$.

$$d(u) = \begin{cases} \dfrac{1}{n-1}, & u = 1, \\ 0, & u \in \mathscr{M}_n - \{1\}. \end{cases} \tag{13}$$

Then, $d \in \mathscr{D}(\mathscr{M})$.

Remark 1. Considering the subtractive derivation d in Example 3, we have $d(u \circledast v) = \mathrm{d}u = 0 \neq (\mathrm{d}u \circledast v) \Cup (u \circledast \mathrm{d}v)$, which implies that d is not a multiplicative derivation on $\mathscr{M}$. Moreover, $d(u \Cap v) = \mathrm{d}u \neq 0 \Cup u = (\mathrm{d}u \Cap v) \Cup (u \Cap \mathrm{d}v)$, and hence d not a lattice derivation. This all shows that not every subtractive derivation is a multiplicative or lattice derivation on $\mathscr{M}$.

Definition 6. A subtractive derivation d on an Rℓ-monoid $\mathscr{M}$ is called isotone if $u \leq v$ implies $\mathrm{d}u \leq \mathrm{d}v$ for any $u, v \in \mathscr{M}$.

Example 4. The subtractive derivations in Example 2, 3 are all isotone.

Proposition 3. *If $d \in \mathscr{D}(\mathscr{M})$, then for any $u, v \in \mathscr{M}$,*

(1) $\mathrm{d}0 = 0$,
(2) $\mathrm{d}u = du \circledast u$,
(3) $\mathrm{d}u \leq u$,
(4) d is isotone,
(5) $\mathrm{d}u \boxminus v \leq u \boxminus dv$,
(6) $\mathrm{d}u = d1 \circledast u \circledast (d(u^{**})^*$,
(7) $d(u \boxminus v) \leq \mathrm{d}u \boxminus \mathrm{d}v \leq \mathrm{d}u \Cup \mathrm{d}v$.

Proof

(1) $\mathrm{d}0 = d(0 \boxminus 0) = (\mathrm{d}0 \boxminus 0) \circledast (0 \boxminus \mathrm{d}0) = 0$.
(2) $\mathrm{d}u = d(u \boxminus 0) = (\mathrm{d}u \boxminus 0) \circledast (u \boxminus \mathrm{d}0) = \mathrm{d}u \circledast u$.
(3) $\mathrm{d}u = \mathrm{d}u \circledast u \leq 1 \circledast u = u$.
(4) If $u \leq v$, then $u = u \Cap v = v \circledast (v \hookrightarrow u)$, and hence

$$\begin{aligned} \mathrm{d}u &= d(u \Cap v) \\ &= d(v \circledast (vu)) \\ &= d(v \boxminus (v \hookrightarrow u)^{**}) \\ &= (\mathrm{d}v \boxminus (v \hookrightarrow u)^{**}) \circledast (v \boxminus d(v \hookrightarrow u)^{**}) \\ &\leq \mathrm{d}v \boxminus (v \hookrightarrow u)^{**} \\ &\leq \mathrm{d}v. \end{aligned} \tag{14}$$

(5) It can be directly obtained from (2) and Proposition 1 (3).
(6) $\mathrm{d}u = d(1 \boxminus u^{**}) = (\mathrm{d}1 \circledast u) \circledast (d(u^{**})^*$.
(7) Obviously from Definition 5 and (3).

We will give some characterizations of subtractive derivations on $\mathscr{I}(\mathscr{M})$, which is a Gödel algebra, and study some of their basic algebraic properties. □

Theorem 1. *Let d: $\mathscr{M} \longrightarrow \mathscr{M}$ be a map on an R ℓ-monoid $\mathscr{M}$. Then, the following are equivalent:*

(1) $d \in \mathscr{D}(\mathscr{I}(\mathscr{M}))$,
(2) $d(u \boxminus v) = \mathrm{d}u \boxminus v$, $\forall u, v \in \mathscr{I}(\mathscr{M})$.

Proof

(1)⇒(2) if $d \in \mathscr{D}(\mathscr{I}(\mathscr{M}))$, then we have

$$\begin{aligned} d(u \boxminus v) &= (\mathrm{d}u \boxminus v) \circledast (u \boxminus \mathrm{d}v) \\ &\geq (\mathrm{d}u \boxminus v) \circledast (\mathrm{d}u \boxminus v) \\ &= \mathrm{d}u \boxminus v. \end{aligned} \tag{15}$$

Conversely, $d(u \boxminus v) = (\mathrm{d}u \boxminus v) \circledast (u \boxminus \mathrm{d}v) \leq \mathrm{d}u \boxminus v$. So $d(u \boxminus v) = \mathrm{d}u \boxminus v$, $\forall u, v \in \mathscr{I}(\mathscr{M})$.

(2)⇒(1) let d be a map on $\mathscr{D}(\mathscr{I}(\mathscr{M}))$ such that $d(u \boxminus v) = \mathrm{d}u \boxminus v$, $\forall u, v \in \mathscr{I}(\mathscr{M})$. Then, $\mathrm{d}0 = d(0 \boxminus \mathrm{d}0) = \mathrm{d}0 \boxminus \mathrm{d}0 = 0$. Furthermore, $0 = d(u \boxminus u) = \mathrm{d}u \boxminus u$, which implies $\mathrm{d}u \leq u$, hence by Proposition 3 (6), we have $\mathrm{d}u \boxminus v \leq u \boxminus \mathrm{d}v$.

So $d(u \boxminus v) = (\mathrm{d}u \boxminus v) \circledast (u \boxminus \mathrm{d}v) = (\mathrm{d}u \boxminus v) \Cap (u \boxminus \mathrm{d}v) = \mathrm{d}u \boxminus v$, $\forall u, v \in \mathscr{I}(\mathscr{M})$. □

Proposition 4. *Let $d \in \mathscr{D}(\mathscr{M})$. Then, the following hold, $\forall u, v \in \mathscr{I}(\mathscr{M})$:*

(1) $\mathrm{d}u = \mathrm{d}1 \circledast u = \mathrm{d}1 \Cap u$,
(2) $d(u \circledast v) = \mathrm{d}u \circledast \mathrm{d}v$,
(3) $d(\mathscr{I}(\mathscr{M})) \subseteq \mathscr{I}(\mathscr{M})$,
(4) $d(u \Cup v) = \mathrm{d}u \Cup dv$,
(5) $d(u \hookrightarrow v) \leq \mathrm{d}u \hookrightarrow dv$,
(6) $u \in [0, d1]$ *iff* $\mathrm{d}u = u$,
(7) $\mathrm{d}1 \leq u$ *iff* $\mathrm{d}u = d1$.

Proof

(1) By Proposition 3 (3), we have $u \leq (d(u^{**})^{*}$, and hence $\mathrm{d}u = \mathrm{d}1 \circledast u \circledast (d(u^{**})^{*} = \mathrm{d}i \circledast u \Cap (d(u^{**})^{*} = \mathrm{d}i \circledast u = d1 \Cap u$.

(2) By (1), we have $d(u \circledast v) = \mathrm{d}1 \Cap (u \circledast v) = \mathrm{d}1 \Cap (u \Cap v) = \mathrm{d}u \Cap \mathrm{d}v$.

(3) If $u \in \mathcal{J}(\mathcal{M})$, then by (2), $d(u) = d(u \circledast u) = \mathrm{d}u \circledast \mathrm{d}u$, which shows $d(\mathcal{J}(\mathcal{M})) \subseteq \mathcal{J}(\mathcal{M})$.

(4) By (1), we have $d(u \Cup v) = \mathrm{d}1 \Cap (u \Cup v) = (\mathrm{d}1 \Cap u) \Cup (\mathrm{d}1 \Cap v) = \mathrm{d}u \Cup \mathrm{d}v$.

(5) By (2), we have $\mathrm{d}u \circledast d(u \hookrightarrow v) = d(u \circledast (u \hookrightarrow v)) = d(u \Cap v) \leq \mathrm{d}v$, and hence $d(u \hookrightarrow v) \leq \mathrm{d}u \longrightarrow \mathrm{d}v$.

(6) and (7) are directly from (1), and hence we omit the proof of them. □

4. The Fixed Point Set of Subtractive Derivations on Rℓ-Monoids

Let $\mathcal{M}$ be an Rℓ-monoid. Define $F_{\mathcal{M}} = \{u \in \mathcal{M} \mid du = u\}$, which is called the fixed point set of subtractive derivation on an Rℓ-monoid $\mathcal{M}$.

Proposition 5. *If $d \in \mathcal{D}(\mathcal{M})$, then $F_{\mathcal{M}} \subseteq \mathcal{J}(\mathcal{M})$.*

Proof. If $u \in F_{\mathcal{M}}$, then by Proposition 3 (2), $\mathrm{d}u = \mathrm{d}u \circledast du$, and hence $u = u \circledast u$, which shows $u \in \mathcal{J}(\mathcal{M})$.

The converse of Proposition 5 is not true in general. □

Example 5. Let $\mathcal{M} = \{0, u, v, 1\}$ be a chain. Defining operations $\circledast$ and $\hookrightarrow$ as follows (see Table 2):

Table 2: The operations of $\circledast$, and $\hookrightarrow$.

$\circledast$	0	u	v	1
0	0	0	0	0
u	0	u	u	u
v	0	0	u	v
1	0	u	v	1
$\hookrightarrow$	0	u	v	1
0	1	1	1	1
u	v	1	1	1
v	u	v	1	1
1	0	u	v	1

Then, $(\mathcal{M}, \circledast, \hookrightarrow, \Cap, \Cup, 0, 1)$ is an Rℓ-monoid. Defining d: $\mathcal{M} \longrightarrow \mathcal{M}$ as follows:

$$d(x) = \begin{cases} 0, & x = 0, 1, \\ 1, & x = u, v. \end{cases} \tag{16}$$

But $F_{\mathcal{M}} = \{0\} \subsetneq \{0, 1\} = \mathcal{J}(\mathcal{M})$ and $d \notin \mathcal{D}(\mathcal{M})$ since $d(u \boxminus v) = \mathrm{d}u = 1 \neq 0 = (\mathrm{d}u \boxminus v) \circledast (u \boxminus \mathrm{d}v)$.

Proposition 6. *The identity map $id_{\mathcal{M}} \in \mathcal{D}(\mathcal{M})$ iff $\mathcal{M}$ is a Gödel algebra.*

Proof. If $id_{\mathcal{M}} \in \mathcal{D}(\mathcal{M})$, then by Proposition 5, $\mathcal{M} = F_{\mathcal{M}} \subseteq \mathcal{J}(\mathcal{M})$, and hence $\mathcal{M} = \mathcal{J}(\mathcal{M})$, which implies that $\mathcal{M}$ is a Gödel algebra.

Conversely, if $\mathcal{M}$ is a Gödel algebra, then $id_{\mathcal{M}} \in \mathcal{D}(\mathcal{M})$. Indeed, $id_{\mathcal{M}}(u \boxminus v) = u \boxminus v = id_{\mathcal{M}} u \boxminus v$, by Theorem 1, $id_{\mathcal{M}} \in \mathcal{D}(\mathcal{M})$.

Proposition 6 shows that the identity map on a Gödel algebra is a subtractive derivation. Then, we give some conditions under which a subtractive derivation is identified. □

Theorem 2. *Let $\mathcal{M}$ be a Gödel algebra and $d \in \mathcal{D}(\mathcal{M})$. Then, the following are equivalent:*

(1) $d = id_{\mathcal{M}}$,

(2) $u \boxminus \mathrm{d}v = \mathrm{d}u \boxminus v$,

(3) d is injective.

Proof

(1)⇒(2) Obviously.

(2)⇒(1) if d satisfies $u \boxminus \mathrm{d}v = \mathrm{d}u \boxminus v$, then by Theorem 1, $\mathrm{d}u = d(u \boxminus 0) = \mathrm{d}u \boxminus 0 = u \boxminus \mathrm{d}0 = u$, and hence $d = id_{\mathcal{M}}$.

(1)⇒(3) Obviously.

(3)⇒(1) if d is injective and for any $u \in \mathcal{M}$, then $d(u \boxminus u) = \mathrm{d}u \boxminus \mathrm{d}u = 0 = d0$, and hence $u \boxminus \mathrm{d}u = 0$, which implies $u \leq \mathrm{d}u$. So $\mathrm{d}u = u$ by Proposition 3 (3). □

Proposition 7. *Let $\mathcal{M}$ be a Gödel algebra and $d \in \mathcal{D}(\mathcal{M})$. Then*

(1) if $u \in \mathcal{M}$ and $v \in F_{\mathcal{M}}$, then $v \boxminus u \in F_{\mathcal{M}}$,

(2) if $v \in F_{\mathcal{M}}$ and $\forall u \in \mathcal{M}$, then $v \Cap u \in F_{\mathcal{M}}$.

Proof

(1) if $u \in \mathcal{M}$ and $v \in F_{\mathcal{M}}$, then $du = d$ and by Theorem 1, $d(v \boxminus u) = \mathrm{d}v \boxminus u = v \boxminus u$, which implies $v \boxminus u \in F_{\mathcal{M}}$.

(2) If $v \in F_{\mathcal{M}}$ and $\forall u \in \mathcal{M}$, then by Proposition 4 (2), $d(v \Cap u) = \mathrm{d}v \circledast \mathrm{d}u = v \Cap u$, which implies $v \Cap u \in F_{\mathcal{M}}$. □

Proposition 8. *Let $\mathcal{M}$ be an Rℓ-monoid. Define a map h_a: $\mathcal{M} \longrightarrow \mathcal{M}, h_a u = u \boxminus a$, $\forall x, a \in \mathcal{M}$, then $h_a \in \mathcal{D}(\mathcal{M})$ iff $h_a(\mathcal{M}) \subseteq \mathcal{J}(\mathcal{M})$.*

Proof. If $h_a(\mathcal{M}) \subseteq \mathcal{J}(\mathcal{M})$, then by Proposition 3 (3),

$$\begin{aligned}(h_a u \boxminus v) \circledast (u \boxminus h_a v) &= (h_a u \circledast u) \circledast (v^\star \circledast h_a v^\star) \\ &= (h_a u \Cap u) \circledast (v \Cup h_a v)^\star \\ &= h_a u \circledast v^\star \\ &= (u \boxminus a) \circledast v^\star \\ &= (u \boxminus a) \boxminus v \\ &= (u \boxminus v) \boxminus a \\ &= h_a (u \boxminus v),\end{aligned} \tag{17}$$

which implies $h_a \in \mathscr{D}(\mathscr{M})$.

Conversely, if $h_a \in \mathscr{D}(\mathscr{M})$, then

$$\begin{aligned}u \boxminus a &= h_a u \\ &= h_a (u \boxminus 0) \\ &= (h_a u \boxminus 0) \circledast (u \boxminus h_a 0) \\ &= (u \boxminus a) \circledast (u \boxminus a), \forall u \in \mathscr{M},\end{aligned} \tag{18}$$

which implies $h_a(\mathscr{M}) \subseteq \mathscr{I}(\mathscr{M})$. □

Theorem 3. *If $d \in \mathscr{D}(\mathscr{M})$ such that d is injective, then $F_{\mathscr{M}}$ is a lattice ideal iff $\mathscr{M}$ is a Gödel algebra.*

Proof. If $d \in \mathscr{D}(\mathscr{M})$ such that d is injective and $\mathscr{M}$ is a Gödel algebra, then by Theorem 2 (3), $d = id_{\mathscr{M}}$, and hence $F_{\mathscr{M}} = \mathscr{M}$, which shows that $F_{\mathscr{M}}$ is a lattice ideal.

Conversely, if $F_{\mathscr{M}}$ is a lattice ideal and $d \in \mathscr{D}(\mathscr{M})$ such that d is injective, then

$$\begin{aligned}d((d1)^\star) &= (1 \boxminus d1) \\ &= (d1 \boxminus d1) \circledast (1 \boxminus dd1) \\ &= 0 \\ &= d0,\end{aligned} \tag{19}$$

that is $d1 = 1$, and hence $\mathscr{M} = F_{\mathscr{M}} \subseteq \mathscr{I}(\mathscr{M})$, which shows that $\mathscr{M}$ is a Gödel algebra. □

Proposition 9. *If $d_a(\mathscr{M}) \subseteq \mathscr{I}(\mathscr{M})$, then $d_a \in \mathscr{D}(\mathscr{M})$.*

Proof. If $d_a(\mathscr{M}) \subseteq \mathscr{I}(\mathscr{M})$, then by Proposition 4 (3), $\forall u, v \in \mathscr{M}$,

$$\begin{aligned}(d_a u \boxminus v) \circledast (u \boxminus d_a v) &= (d_a u \circledast u) \circledast (v^\star \circledast d_a v^\star) \\ &= (d_a u \Cap u) \circledast (v \Cup d_a v)^\star \\ &= d_a u \circledast v^\star \\ &= (a \circledast u) \circledast v^\star \\ &= a \circledast (u \boxminus v) \\ &= d_a (u \boxminus v),\end{aligned} \tag{20}$$

which implies $d_a \in \mathscr{D}(\mathscr{M})$. □

Corollary 1. *If $\mathscr{M}$ is a Gödel algebra, then $d_a \in \mathscr{D}(\mathscr{M})$.*

Proposition 10. *If $\mathscr{M}$ is a Gödel algebra, then the following hold:*

(1) $d1 \in F_{\mathscr{M}}$,

(2) $d(\mathscr{M}) = F_{\mathscr{M}}$.

Proof

(1) It follows from Proposition 4 (1).

(2) It is obvious that $d(\mathscr{M}) \supseteq F_{\mathscr{M}}$. Conversely, if $u \in d(\mathscr{M})$, then there exists $v \in \mathscr{M}$ such that $u = dv$. Since $u = dv \leq d1$ and $d1 \in F_{\mathscr{M}}$, by Theorem 2, $u \in F_{\mathscr{M}}$, and hence $d(\mathscr{M}) \subseteq F_{\mathscr{M}}$. □

Theorem 4. *If $\mathscr{M}$ is a Gödel algebra and I is a lattice ideal with the greatest element, then there exists $d \in \mathscr{D}(\mathscr{M})$ such that $F_{\mathscr{M}} = I$.*

Proof. If $b = \vee_{a \in I} a \in I$ and $d_b \in\in \mathscr{D}(\mathscr{M})$, then $d_b u \leq b$ with $b \in I$, and hence $d_b(\mathscr{M}) \subseteq I$. By Theorem 3, $d_b \in \mathscr{D}(\mathscr{M})$. Moreover, if $u \in I$, then $d_b u = u \Cap b$, and hence $u \in F_{\mathscr{M}}$ with respect to d_b, which implies $I \subseteq F_{\mathscr{M}}$. Furthermore, $F_{\mathscr{M}} = d(\mathscr{M})$, and hence $F_{\mathscr{M}} \subseteq I$ and $F_{\mathscr{M}} = I$. □

5. The Relations between Kinds of Derivations on Rℓ-Monoids

In this section, we will discuss the relations between subtractive derivations and other derivations on Rℓ-monoids. In particular, we discuss the relations among subtractive derivations, lattice derivations, and multiplicative derivations on Rℓ-monoids.

Proposition 11. *Every subtractive derivation is multiplicative on a Gödel algebra $\mathscr{M}$.*

Proof. It follows from Propositions 4 (1) and (3). □

Proposition 12. *If d is a multiplicative derivation on an Rℓ-monoid $\mathscr{M}$ and $d(\mathscr{M}) \subseteq \mathscr{I}(\mathscr{M})$, then $d \in \mathscr{D}(\mathscr{M})$.*

Proof. It follows from Propositions 2 and Corollary 1. □

Proposition 13 (see [22]). *If d is a multiplicative derivation on an Rℓ-monoid $\mathscr{M}$ and $d1 \in \mathscr{I}(\mathscr{M})$, then the following are equivalent:*

(1) d is isotone,

(2) $du = d1 \circledast u$.

Proof. It follows from Propositions 4 (1) and (3). □

Proposition 14 (see [22]). *If d is a lattice derivation on an Rℓ-monoid $\mathscr{M}$, then the following are equivalent:*

(1) d is isotone;

(2) $du = d1 \Cap u$.

Proof. It follows from Propositions 4 (1) and (3). □

Theorem 5. *If d is a map such that $d1 \in \mathcal{F}(\mathcal{M})$ on an $R\ell$-monoid $\mathcal{M}$, then d is a multiplicative derivation iff it is a lattice derivation.*

Proof. It follows from Propositions 13 and 14. □

Proposition 15. *Every subtractive derivation is multiplicative on a Gödel algebra $\mathcal{M}$.*

Proof. It follows from Proposition 11. □

Corollary 2. *If d is a lattice derivation on an $R\ell$-monoid $\mathcal{M}$ and $d(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$, then $d \in \mathcal{D}(\mathcal{M})$.*

Corollary 3. *Subtractive derivations and lattice derivations are equivalent on the Gödel algebra.*

6. Conclusions

The notion of subtractive derivations is beneficial for discussing structures and properties in fuzzy logic algebraic. In order to provide the common properties of subtractive derivations in the t-norm-based logical algebras, we introduce the subtractive derivations on Rℓ-monoids and obtain some characterizations of them. We also discuss the relations between the fixed point set of subtractive derivations and other kinds of derivations on Rℓ-monoids. In the future, we will study some representations of Rℓ-monoids by the algebraic structures of the set of subtractive derivations.

Data Availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was funded by the grants of the Natural Science Foundation of Education Committee of Shannxi Province (grant no. 17JK0170) and the Scientific Research Fund Project of Shaanxi Railway Engineering Vocational and Technical College (grant no. KY2017-20).

References

[1] K. L. N. Swamy, "Dually residuated lattice ordered semigroups," *Mathematische Annalen*, vol. 159, no. 2, pp. 105–114, 1965.
[2] J. T. Wang, P. F. He, and Y. H. She, "Monadic NM-algebras," *Logic Journal of IGPL*, vol. 27, no. 6, pp. 812–835, 2019.
[3] J. T. Wang, X. L. Xin, and P. F. He, "Monadic bounded hoops," *Soft Computing*, vol. 22, no. 6, pp. 1749–1762, 2018.
[4] J. T. Wang, P. F. He, J. Yang, M. Wang, and X. L. He, "Monadic NM-algebras: an algebraic approach to monadic predicate nilpotent minimum logic," *Journal of Logic and Computation*, vol. 32, no. 4, pp. 741–766, 2022.
[5] J. T. Wang and X. L. Xin, "Monadic algebras of an involutive monoidal t-norm based logic," *Iranian Journal of Fuzzy Systems*, vol. 19, pp. 187–202, 2022.
[6] J. T. Wang, A. B. Saeid, and P. F. He, "Similarity MTL-algebras and their corresponding logics," *Journal of Multiple-Valued Logic and Soft Computing*, vol. 32, pp. 607–628, 2019.
[7] J. T. Wang, Y. H. She, P. F. He, and N. N. Ma, "On categorical equivalence of weak monadic residuated distributive lattices and weak monadic c-differential residuated distributive lattices," *Studia Logica*, 2022.
[8] J. T. Wang and P. F. He, "Generalized valuations on MTL-algebras," *Frontiers of Mathematics in China*, vol. 17, no. 4, pp. 521–543, 2022.
[9] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.
[10] E. C. Posner, "Derivations in prime rings," *Proceedings of the American Mathematical Society*, vol. 8, no. 6, pp. 1093–1100, 1957.
[11] R. A. Borzooei and O. Zahiri, "Some results on derivations of BCI-algebras," *Scientiae Mathematicae Japonicae*, vol. 26, pp. 529–545, 2013.
[12] Y. B. Jun and X. L. Xin, "On derivations of BCI-algebras," *Information Sciences*, vol. 159, no. 3-4, pp. 167–176, 2004.
[13] J. M. Zhan and Y. L. Liu, "On f-derivations of BCI-algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 11, pp. 1675–1684, 2005.
[14] X. L. Xin, "The fixed set of aderivation in lattices," *Fixed Point Theory and Applications*, vol. 2012, pp. 218–312, 2012.
[15] X. L. Xin, T. Y. Li, and J. H. Lu, "On derivations of lattices," *Information Sciences*, vol. 178, no. 2, pp. 307–316, 2008.
[16] Y. Çeven and M. A. Öztürk, "ON f-DERIVATIONS OF LATTICESf-derivations of lattices," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 4, pp. 701–707, 2008.
[17] N. O. Alshehri, "Derivations of MV-algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 312027, 7 pages, 2010.
[18] S. Ghorbain, L. Torkzadeh, and S. Motamed, "$\otimes,\oplus$-derivations and $\ominus,\oplus$-derivations on MV-algebras," *Iranian Journal of Mathematical Sciences and Informatics*, vol. 8, pp. 75–90, 2013.
[19] H. Yazarli, "A note on derivations in MV-algebras," *Miskolc Mathematical Notes*, vol. 14, no. 1, pp. 345–354, 2013.
[20] S. D. Lee and K. H. Kim, "On derivations of lattice implication algebras," *Ars Combinatoria*, vol. 108, pp. 279–288, 2013.
[21] Y. H. Yon and K. H. Kim, "On f-derivations of lattice implication algebras," *Ars Combinatoria*, vol. 110, pp. 205–215, 2013.
[22] P. F. He, X. L. Xin, and J. M. Zhan, "On derivations and their fixed point sets in residuated lattices," *Fuzzy Sets and Systems*, vol. 303, pp. 97–113, 2016.
[23] X. J. Hua, "On derivations of R0-algebras," *Computer Engineering and Applications*, vol. 53, pp. 54–57, 2017.
[24] G. Grätzer, *Lattice Theory*, W. H. Freeman and Company, San Francisco, CA, USA, 1979.
[25] Y. K. Omid, E. Esfandiar, and B. S. Arsham, "K-modal BL-algebras," *Kuwait Journal of Science*, vol. 43, pp. 39–60, 2016.
[26] J. Schmid, "Multipliers on distributive lattices and rings of quotients I," *Houston Journal of Mathematics*, vol. 3, pp. 401–425, 1980.
[27] J. X. Liu, Y. J. Li, Y. W. Yang, and J. T. Wang, "On implicative derivations of MTL-algebras," *Axioms*, vol. 11, no. 7, p. 340, 2022.