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Research Article

Numerical Approximation of One-Dimensional Transport Model Using an Hybrid Approach in Finite Volume Method

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A finite volume method is a well-known and appropriate approach for numerical approximation of governing problems. This article proposes a finite volume approach to simulate the one-dimensional convection–diffusion transport problem. New expressions for interface approximation are constructed by combining the assumption of step-wise profile and piece-wise linear profile. In addition, a new numerical technique is developed based on these new interface approximations. This new numerical algorithm produces consistent results for numerical approximation of the governing problem and gives second-order accuracy along space and time. In order to justify the effectiveness of our numerical technique, numerical experiments are conducted for various magnitudes of convection and diffusion coefficients. The numerical results of the proposed algorithm are also compared to the finite volume method and the finite difference method. Based on this comparison, our numerical scheme presents stable and highly accurate results compared to the alternatives.

1. Introduction

The convection–diffusion equation is a basic governing problem to represent the transport phenomena of any property. These transport phenomena have been observed in a variety of fields, including industry, agriculture, biology, meteorology, and petrochemistry. The numerical solution of convection–diffusion model involves the mathematical modeling and simulation of pollutants and suspended matter in water and soil [1–7]. Mathematical models representing any transport phenomena are given by systems of second-order partial differential equations that contain terms describing convection and diffusion of a fluid in medium. The convection is defined by first-order terms, while the second-order term describes the diffusion of property. The convection–diffusion equation can be driven straightforwardly from the equation of continuity, which states that the rate of change for a scalar quantity in a differential control volume is given by convection and diffusion in and out along with any source or sinks inside the control volume [8, 9]. These governing equations cannot be solved using any analytical approach. Due to this reason, it is essential to apply different numerical methods to find out the numerical solution for such problems. The finite difference, finite element, and finite volume approaches are the three most popular and widely used numerical techniques for approximation of such governing problems.

The finite difference method (FDM) studied by [10–16] is a domain discretizing technique that converts the governing problem into a difference equation. The functional values are approximated at the nodes of the network. In [17], finite difference approach was applied to simulate the basic conservation equations. It concludes that, convergence rate of finite difference approach is based over the conditions in liquid region. An efficient parallel characteristic scheme of finite difference method was proposed for numerical simulation of convection–diffusion equation (18). The advantage of the numerical approach of this study is that minimum iterations are required to get the optimal accuracy at each time level. In [19], the finite difference method is applied to a system of simultaneous nonlinear model representing the transport phenomena. This study concludes that a constraint is needed on the physical parameters of the problem to achieve the existence of a solution to the steady-state problem. A finite difference approach with high order accuracy is proposed for solving the convection–diffusion equation in [20]. For a limiting case where mesh-size and viscosity approach zero, a numerical algorithm based on the finite difference approach is discussed for numerical simulation of singularly perturbed convection problem [21]. A numerical simulation is discussed to study the convection and diffusion mechanism using the finite difference technique. Based on this study, the transport phenomena in the cylinder husk furnace is convection-dominated as treated by [22]. A compact form of finite difference is applied to solve the time-dependent convection–diffusion model [23]. In this new approach, the convection–diffusion equation is first transformed into the reaction-diffusion equation, and then it is solved by a proposed unconditionally stable numerical method. As we have studied in [24], a nine-point compact discretization strategy has been used in conjunction with the multigrid approach to get more accurate numerical approximation of the convection–diffusion equation. Finite difference techniques using spreadsheets are applied to approximate the one-dimensional advection-diffusion equation (see [25] the details). The numerical approximation of different finite difference schemes is compared with the analytical solution of the problem.

The finite element approach, which is discussed in [26–30] is another numerical method used for the numerical approximation of the governing problems. For the approximation of transport problem using the finite element approach, local error estimates are derived [31]. In the study [32] with polynomials of degree $n \geq 2$, an explicit form of the finite element technique is presented for the time-dependent convection–diffusion problem. The proposed method employs space–time elements and allows the numerical solution to be computed one element at a time. A technique based on the finite element method is introduced for hyperbolic problems, which is also extended to steady-state convection-dominated problem [29]. The constructed method can be applied explicitly, which is stable for smaller mesh-size (h) and complements to Galerkin method in the convection-dominant regime. For the one-dimensional transport problem, convergence of finite element approach is discussed [33]. Linear and nonlinear problems are focused. For linear elements, optimal rates of convergence order with different norms are studied. The finite element technique is applied to discretized unsteady transport equation [34]. The resultant matrix system is solved by using the conjugate gradient method, which improve computational time and memory size. A form of weak finite element method is applied to the approximate convection–diffusion equation implemented [35]. The optimal error estimates are discussed in different discrete norms. Finite element approximation [36] for a singularly perturbed one-dimensional transport model is constructed. For the convection–diffusion boundary value problem, Galerkin piecewise linear finite element approach is considered, which is second-order accurate. By considering boundary layer character of solution, special formulae are applied to generate a discrete linear algebraic system [37]. A new finite volume approach with some novel feature for all variables is discussed [38]. The formulation proposed in this study shows flexibility in geometry and boundary conditions. The results obtained from this finite element formulation illustrate its ability to accurately predict the fluid properties in both forced and free convection flows. In [39], a finite element formulation for numerical modeling of porous medium flow is presented. In this study, both the semi- and quasi-implicit schemes are used to solve many problems. Mostly, the results presented are highly accurate and consistent with the experimental and other numerical data.

In flow simulation, it is essential to maintain conservation laws, which is not guaranteed in the finite difference method (FDM) and finite element method (FEM). The finite volume method (FVM) [40–44] is evolved from the finite differences method, which is flexible and widely employed in fluid dynamics. The finite volume technique is an integration-based approach in which volume integrals with divergence terms are converted to surface integrals. This integration is a basic key step to ensure conservation of relevant property at each control volume level. This control volume integration gives a semidiscretize equation that involves interface fluxes which are needed to approximate. This interface approximation converts partial differential equations into a set of algebraic equations. In the past, based on different choices to these interface approximations, several numerical techniques of finite volume method (FVM) formulation are developed [45–50]. The convection–diffusion equation can be either convection-dominated or diffusion-dominated its depending on the rate of convection and diffusion. Convection-dominated diffusion equations govern many phenomena in engineering and science. In many applications based on convection-dominated problems, diffusion rate will be smaller than that of convection. In such cases, standard numerical methods give unstable solution. To deal with this unstability issue, multiple numerical approaches have been proposed [51–53]. An exponentially fitted form of finite volume technique is applied to approximate the convection-dominated diffusion problem [54]. After simple coordinate transformation, the resulting problem is approximated by using the exponentially fitted finite volume method. The adaptive mesh method is also a valuable technique for solving convection-dominated problems. Adaptive mesh refinement method and moving mesh refinement method are the most popular adaptive mesh approaches. A self-adaptive mesh is proposed to address the convection dominancy [55].

In our previous research article [45], Taylor series formulation is applied for the approximation of variable at the interface of the computational domain, and a new upwind numerical formulation is proposed for numerical simulation of the convection–diffusion problem. In the current study, new expressions are obtained using Lagrange interpolation for the approximation of variables at spatial interfaces of the control volume. Subsequently, a numerical scheme based on the upwind approach formulation is constructed using these interface approximations. One of the most common and widely used techniques, the Crank–Nicolson method, is applied for temporal approximation. The numerical results produced by our new algorithm are compared with some other numerical schemes of finite volume approach and finite difference technique. The obtained numerical results indicate that our new upwind approach gives highly accurate and stable results compared to other numerical algorithms.

The format of this article is organized as follows: A brief about the mathematical model of convection–diffusion and domain discretization using finite volume method is shown in Section 2. In Section 3, some numerical experiments are carried out to validate our theoretical algorithm, and the conclusion of this study is presented in Section 4.

2. Finite Volume Based Formulation

The basic principle of all finite volume methods is to study the differential equation in conservative form, integrate it over small regions (called cells or control volumes), and convert each such integral into an integral over the boundary of the cell using Gauss's theorem. In order to have the nodes of the grid at the centres of cells, we introduced a new rectangular grid whose nodes are the cell centres of the original grid, which is shown in Figure 1. Nodal points are used within these control volumes for interpolating the field variable. Usually, the single node at the center of the control volume is used for each control volume. The finite volume method is a discretization of the governing equation in integral form, in contrast to the finite difference method, which is unusually applied to the governing equation in differential form.

In order to obtain a finite volume discretization, the domain Ω will be subdivided to M subdomains $\Omega_i, i = 1, 2, \ldots, M$ such that the collection of all those subdomains forms a partition of Ω, i.e., as in [56],

(i) Each of the domain Ω_i is an open, connected, and bounded set without slits.

(ii) There is no common point between each subdomains (*i.e.*, $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$).

(iii) The union of all the subdomains gives the domain of the region (*i.e.*, $\cup_{i=1}^{M} \Omega_i = \Omega$). These subdomains Ω_i are called control volumes (CVs) or control domains (see Figure 2). In the finite volume formulation the first step is to divide the whole computational domain into finite number of subdomain also called control volumes Ω_i. When we are talking about one-dimensional problems, we are considering the CVs as subintervals of the problem and the nodes can be the midpoints or the edges of the subintervals (see Figure 3).

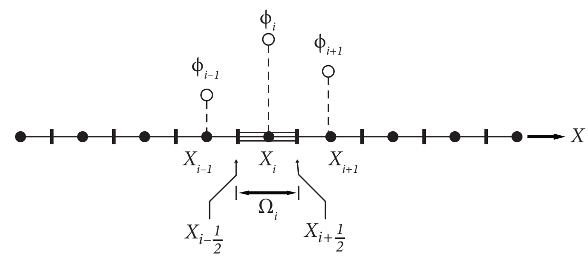

Figure 1: Adjustment of interfaces and computational node inside control volume.

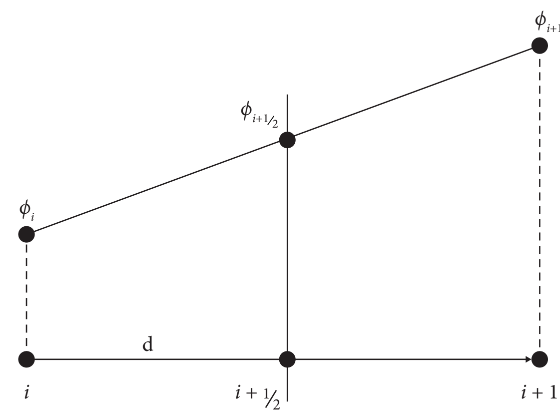

Figure 2: Profile of central differencing scheme.

Assume that $\Omega \subset R^d, d \geq 1$ be a bounded polygonal domain and let $T > 0$ be given, then the following initial-boundary value problem is written as follows.

The analysis and discretization of these problem is treated by using upwind finite volume method. The work is discussed in the analysis of finite volume method for coupled systems in which transport similar to equation play a great role. The one-dimensional convection–diffusion transport phenomena are governed by the following equation:

$$\begin{cases} \dfrac{\partial \phi(x,t)}{\partial t} + v \dfrac{\partial \phi(x,t)}{\partial x} = k \dfrac{\partial^2 \phi(x,t)}{\partial x^2} + Q(x,t), (x,t) \in \Omega \times (0,T], \\ \phi(x,0) = f(x), 0 \leq x \leq L, \\ \phi(0,t) = \phi_1(t), u(L,t) = \phi_2(t), 0 \leq t \leq T, \end{cases} \tag{1}$$

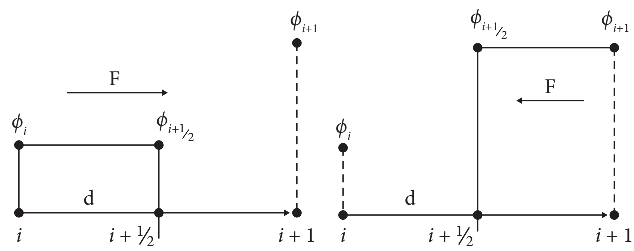

Figure 3: Profile of upwind differencing scheme.

where $k \geq 0, v \geq 0$ represents the coefficients of diffusion and convective velocity, respectively, where as $\phi(x,t)$ is unknown field variable, which needs to compute over one-dimensional domain $\Omega = (0, L)$.

Governing equation (1) is the mathematical model for the convection–diffusion physical transport phenomena. The terms $v\partial\phi/\partial x$ and $k\partial^2\Phi/\partial x^2$ represents convection and diffusion process, respectively.

We have restricted to cell centred and uniform discretization of computational domain $\Omega = (0, L)$. For each control volume, the grid point or computational node is adjusted at the centre x_i of the control volume.

The interface Γ or common boundary of i^{th} control volume with its adjacent control volumes Ω_{i-1} and Ω_{i+1} are mostly positioned at the midposition between their centres, i.e., $\Gamma_{i-1/2} = x_i + x_{i-1}/2$ and $\Gamma_{i+1/2} = x_i + x_{i+1}/2$. The spatial domain $[0, L]$ is divided into M equal control volumes of uniform length h. The partition of spatial and temporal domain are, respectively, define as $\Delta x = h = x_{i+1} - x_i$ and $\Delta t = t_{j+1} - t_j$. By integrating governed equation (1) over the control volume $[x_{i-\Delta x/2}, x_{i+\Delta x/2}] \times [t_{j-1}, t_j]$ around node x_i, we get the following equation.

$$\int_{i-\Delta x/2}^{i+\Delta x/2}\int_{j-\Delta t/2}^{j+\Delta t/2}\frac{\partial\Phi}{\partial t}dt\,\mathrm{d}x + v\int_{j-\Delta t/2}^{j+\Delta t/2}\int_{i-\Delta x/2}^{i+\Delta x/2}\frac{\partial\Phi}{\partial x}\mathrm{d}x\,\mathrm{d}t = k\int_{j-\Delta t/2}^{j+\Delta t/2}\int_{i-\Delta x/2}^{i+\Delta x/2}\frac{\partial^2\Phi}{\partial^2 x}\mathrm{d}x\,\mathrm{d}t. \quad (2)$$

The first term is approximated by using trapezoidal rule for numerical integration, as follows:

$$\begin{aligned}&\frac{\Delta x}{2}\left(\phi_{i+\Delta x/2,j+\Delta t/2} + \Phi_{i-\Delta x/2,j+\Delta t/2}\right) - \frac{\Delta x}{2}\left(\phi_{i+\Delta x/2,j-\Delta t/2} + \Phi_{i-\Delta x/2,j-\Delta t/2}\right)\\ &\quad + v\int_{j-\Delta t/2}^{j+\Delta t/2}\left(\Phi_{i+\Delta x/2,j} - \Phi_{i-\Delta x/2,j}\right)\mathrm{d}t,\\ &= k\int_{j-\Delta t/2}^{j+\Delta t/2}\left(\left(\frac{\partial\Phi}{\partial x}\right)_{i+\Delta x/2,j} - \left(\frac{\partial\Phi}{\partial x}\right)_{i-\Delta x/2,j}\right)\mathrm{d}t.\end{aligned} \quad (3)$$

Expression (3) represents our semidiscretized equation. In the past, based on different interface approximation techniques, numerous numerical schemes have been constructed. In the finite volume method, the most and well known interface approximation approaches are central differencing approach and upwind differencing approach.

The interface approximation by central differencing approach is defined as follows:

$$\Phi_f = f_x\Phi_{i-1} + (1 - f_x)\Phi_i. \quad (4)$$

Here, for uniform discretization of domain $f_x = 1/2$ and the interface approximation becomes as follows

$$\Phi_f = \begin{cases}\dfrac{\phi_i + \phi_{i+1}}{2}, \text{at right interface } x_{i+\Delta x/2},\\[2ex] \dfrac{\phi_{i-1} + \phi_i}{2} \quad \text{at left interface } x_{i-\Delta x/2}.\end{cases} \quad (5)$$

Figure 2 represents the central differencing profile for the interface approximation computed as follows.

Remark 1. The Peclet number $(\mathrm{Pe} = v\Delta x/k)$ is a dimensionless number, it helps to calculate the transportiveness property of any numerical approach. The central differencing ignores the transportiveness property and gives unstable solution for greater values of Pe.

While calculating convective flux at the interfaces, central differencing scheme is equally treated with the neighbouring nodes of the interface. This approach do not consider the flow direction and just takes the average of neighbouring nodes for interface approximation. This central differencing scheme has the major disadvantage of ignoring the influence of convection at the interfaces while approximating the variable.

$$\phi_f = \Phi_{i-\Delta x/2} = \begin{cases}\Phi_{i-1} & \text{for } v > 0,\\ \Phi_i & \text{for } \quad v < 0,\end{cases} \quad (6)$$

$$\Phi_f = \Phi_{i+\Delta x/2} = \begin{cases}\Phi_i & \text{for } v > 0,\\ \Phi_{i+1} & \text{for } v < 0.\end{cases} \quad (7)$$

The above expressions (6) and (7) represent the upwind approach based approximation of convective flux at the interfaces. In the upwind approach, the interface approximation depends on flow direction. The value of variable at any interface is equal to the value at its upstream node. Figure 3 shows the interface approximation profile of upwind approach. The upwind approach addresses the transportiveness of any property in a better way. So, in this proposed study, the blending factor (α) is introduced to merge the central differencing approach and upwind approach. In our proposed approach, we have combined the central differencing and upwind approach for the approximation of convective flux at the interfaces of control volume. The introduced blending factor (α) take values $\alpha \in (0,1)$ and combine both interface approximation approaches as follows:

$$\Phi_f = \Phi_{i+\Delta x/2} = \alpha\left(\frac{\phi_i + \phi_{i+1}}{2}\right) + (1-\alpha)\phi_i. \tag{8}$$

Expression (8) represents our new expression for approximation of convective flux at the right interface ($\Phi_{i+\Delta x/2}$) of the control volume. The new expression for the approximation at left interface ($\Phi_{i-\Delta x/2}$) will be as follows:

$$\Phi_f = \Phi_{i-\Delta x/2} = \alpha\left(\frac{\phi_i + \phi_{i-1}}{2}\right) + (1-\alpha)\phi_{i-1}. \tag{9}$$

For $\alpha = 0$ and $\alpha = 1$, this new approach gives same approximation expressions as upwind approach and central differencing approach, respectively.

Based on the central differencing approach, the gradients at the east and west interfaces of the control volume are approximated as follows:

$$\left(\frac{\partial\phi}{\partial x}\right)_{i+\Delta x/2} = \left(\frac{\phi_{i+1} - \phi_i}{\Delta x}\right) + O\left(\Delta x^2\right), \tag{10}$$

$$\left(\frac{\partial\phi}{\partial x}\right)_{i-\Delta x/2} = \left(\frac{\phi_i - \phi_{i-1}}{\Delta x}\right) + O\left(\Delta x^2\right). \tag{11}$$

Now by substituting equations (8)–(11) in equation (3), it becomes as follows:

$$\begin{aligned} &\frac{\Delta x}{2}\left(\left(1-\frac{\alpha}{2}\right)\phi_{i-1,j+1} + \phi_{i,j+1} + \frac{\alpha}{2}\phi_{i+1,j+1}\right) - \frac{\Delta x}{2}\left(\left(1-\frac{\alpha}{2}\right)\phi_{i-1,j} + \phi_{i,j} + \frac{\alpha}{2}\phi_{i+1,j}\right) \\ &\quad + \nu\int_{j-\Delta t/2}^{j+\Delta t/2}\left(\left(\frac{\alpha}{2}-1\right)\phi_{i-1,j} + (1-\alpha)\phi_{i,j} + \frac{\alpha}{2}\phi_{i+1,j}\right)dt = \frac{k}{\Delta x}\int_{j-\Delta t/2}^{j+\Delta t/2}\left(\phi_{i-1} - 2\phi_i + \phi_{i+1}\right)dt, \end{aligned} \tag{12}$$

$$\begin{aligned} &\frac{\Delta x}{2}\left(\left(1-\frac{\alpha}{2}\right)\phi_{i-1,j+1} + \phi_{i,j+1} + \frac{\alpha}{2}\phi_{i+1,j+1}\right) \\ &\quad + \left(\frac{\nu\alpha}{2} - \frac{k}{\delta x}\right)\int_{j-\Delta t/2}^{j+\Delta t/2}\phi_{i-1,j}dt + \left(\nu(1-\alpha) + \frac{2k}{\delta x}\right)\int_{j-\Delta t/2}^{j+\Delta t/2}\phi_{i,j}dt + \left(\frac{\nu\alpha}{2} - \frac{k}{\delta x}\right)\int_{j-\Delta t/2}^{j+\Delta t/2}\phi_{i+1,j}dt \\ &= -\frac{\Delta x}{2}\left(\left(1-\frac{\alpha}{2}\right)\phi_{i-1,j} + \phi_{i,j} + \frac{\alpha}{2}\phi_{i+1,j}\right). \end{aligned} \tag{13}$$

Now, θ scheme is applied to evaluate the temporal integral as follows:

$$\int_{j-\Delta t/2}^{j+\Delta t/2}\phi_{i,j}dt = \left(\theta\phi_{i,j+1} + (1-\theta)\phi_{i,j}\right)\Delta t, \text{ where, } \theta \in (0,1). \tag{14}$$

Applying this temporal approximation (14) over each spatial node in (13) and taking $\theta = 1/2$ for Crank-Nicolson temporal approximation, the expression (13) becomes as follows:

$$\begin{aligned} &\left(\frac{(2-\alpha)}{4} + \frac{\nu\Delta t}{2\Delta x}\left(\frac{\alpha}{2}-1\right) - \frac{k\Delta t}{2\Delta x^2}\right)\phi_{i-1,j+1} + \left(\frac{1}{2} + \frac{\nu\Delta t}{2\Delta x}(1-\alpha) + \frac{k\Delta t}{\Delta x^2}\right)\phi_{i,j+1} + \left(\frac{\alpha}{4} + \frac{\nu\alpha\Delta t}{4\Delta x} - \frac{k\Delta t}{2\Delta x^2}\right)\phi_{i+1,j+1} \\ &= \left(\frac{(2-\alpha)}{4} - \frac{\nu\Delta t}{2\Delta x}\left(\frac{\alpha}{2}-1\right) + \frac{k\Delta t}{2\Delta x^2}\right)\phi_{i-1,j} + \left(\frac{1}{2} - \frac{\nu\Delta t}{2\Delta x}(1-\alpha) - \frac{k\Delta t}{\Delta x^2}\right)\phi_{i,j} + \left(\frac{\alpha}{4} - \frac{\nu\alpha\Delta t}{4\Delta x} + \frac{k\Delta t}{2\Delta x^2}\right)\phi_{i+1,j}. \end{aligned} \tag{15}$$

Expression (15) represents our fully discretized equation valid for interior nodes from node 2 to node $n-1$.

Remark 2. This new proposed algorithm utilises consistent expressions to calculate fluxes through cell interfaces.

Therefore it can be easily shown that this new formulation is conservative. This new discretised equation is diagonally dominant and satisfies the requirements for boundedness of the solution.

2.1. Boundary Conditions. The discretization of the problem at left boundary node $\varphi_{1,j}$ and right boundary node $\varphi_{n,j}$. There is no node at the left of $\varphi_{1,j}$ and also at right side of node $\varphi_{n,j}$. For the treatment of boundary nodes, we have adjusted imaginary nodes or mirror nodes out side the computational domain at a distance of $\Delta x/2$ as shown in Figure 4.

The approximation of left imaginary node $\varphi_{i-\Delta x,j}$ is as follows:

$$\Phi_{1-\Delta x/2} = \Phi_L = \frac{\phi_1 + \phi_{1-\Delta x}}{2} \Longrightarrow \phi_{1-\Delta x} = 2\Phi_L - \Phi_1, \quad (16)$$

where Φ_L is left boundary of computational domain and it coincides with the left interface $(\Phi_{1-\Delta x/2})$ of first control volume around node 1. With this approximation of left imaginary node the discretized expression at node 1 becomes as follows:

$$\begin{aligned}
&\left(\frac{\alpha}{4} + \frac{v\Delta t}{2\Delta x}\left(2 - \frac{3\alpha}{2}\right) + \frac{3k\Delta t}{2\Delta x^2}\right)\varphi_{1,j+1} + \left(\frac{\alpha}{4} + \frac{v\alpha\Delta t}{4\Delta x} - \frac{k\Delta t}{2\Delta x^2}\right)\varphi_{2,j+1} \\
&= \left(\frac{\alpha}{4} - \frac{v\Delta t}{2\Delta x}\left(2 - \frac{3\alpha}{2}\right) - \frac{3k\Delta t}{2\Delta x^2}\right)\varphi_{1,j} + \left(\frac{\alpha}{4} - \frac{v\alpha\Delta t}{4\Delta x} + \frac{k\Delta t}{2\Delta x^2}\right)\varphi_{2,j} \\
&+ \left(\left(\frac{\alpha}{2} - 1\right) - \frac{v\Delta t}{2\Delta x}(\alpha - 2) + \frac{k\Delta t}{\Delta x^2}\right)\varphi_{L,j+1} + \left(\left(1 - \frac{\alpha}{2}\right) - \frac{v\Delta t}{2\Delta x}(\alpha - 2) + \frac{k\Delta t}{\Delta x^2}\right)\varphi_{L,j}.
\end{aligned} \quad (17)$$

Similarly, the approximation of right imaginary node $\varphi_{n+1,j}$ is given as follows:

$$\Phi_{n+\Delta x/2} = \Phi_R = \frac{\phi_n + \phi_{n+\Delta x}}{2} \Longrightarrow \phi_{n+\Delta x} = 2\Phi_R - \Phi_n. \quad (18)$$

Now, the discretized expression for node n is given as follows:

$$\begin{aligned}
&\left(\frac{2-\alpha}{4} + \frac{v\Delta t}{2\Delta x}\left(\frac{\alpha}{2} - 1\right) - \frac{k\Delta t}{2\Delta x^2}\right)\varphi_{n-1,j+1} + \left(\frac{2-\alpha}{4} + \frac{v\Delta t}{2\Delta x}\left(1 - \frac{3\alpha}{2}\right) + \frac{3k\Delta t}{2\Delta x^2}\right)\varphi_{n,j+1} \\
&= \left(\frac{2-\alpha}{4} - \frac{v\Delta t}{2\Delta x}\left(\frac{\alpha}{2} - 1\right) + \frac{k\Delta t}{2\Delta x^2}\right)\varphi_{n-1,j} + \left(\frac{2-\alpha}{4} - \frac{v\Delta t}{2\Delta x}\left(1 - \frac{3\alpha}{2}\right) - \frac{3k\Delta t}{2\Delta x^2}\right)\varphi_{n,j} \\
&+ \left(-\frac{\alpha}{2} - \frac{v\alpha\Delta t}{2\Delta x} + \frac{k\Delta t}{\Delta x^2}\right)\varphi_{R,j+1} + \left(\frac{\alpha}{2} - \frac{v\alpha\Delta t}{2\Delta x} + \frac{k\Delta t}{\Delta x^2}\right)\varphi_{R,j}.
\end{aligned} \quad (19)$$

The above expressions (17) and (19) represents fully discretized expression at node 1 and node n, respectively.

3. Numerical Experiments

Experiment 1. Consider one-dimensional convection–diffusion problem over the domain $0 \le x \le 1, 0 \le t \le 1$. The exact solution is

$\phi(x,t) = 1/\sqrt{4t+1}e^{(-(x-1-vt)^2)/k(4t+1)}$ with $v \ge 0$ as the convection velocity to the $x-$ direction and $k > 0$ as the diffusion coefficient [57, 58]. Now, the proposed scheme (15) is applied to approximate $\varphi(x,t)$, for different cases of convection dominance and diffusion dominance. The initial and boundary values are calculated from an exact solution by taking $t = 0$, $x = 0$, and $x = 1$,

The above Figures 5–7 represent the approximation for $\alpha = 1$, $\alpha = 0.5$, and $\alpha = 0$, respectively. In this approximation the value of diffusion coefficient (k) and convective velocity (v) is taken as 0.03 and 0.02, respectively. The magnitude of maximum error for $\alpha = 1, 0.5$ and $\alpha = 0$ is 5.9113×10^{-4}, 0.0020 and 0.0042, respectively. Figures 8–10 show the approximation at different time levels (t) for

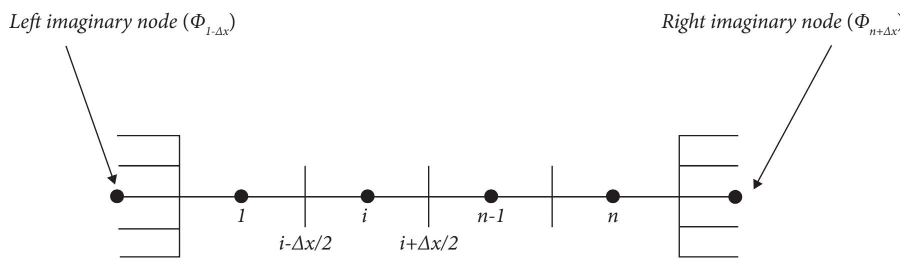

FIGURE 4: Adjustment of mirror nodes at the left and right side of computational domain.

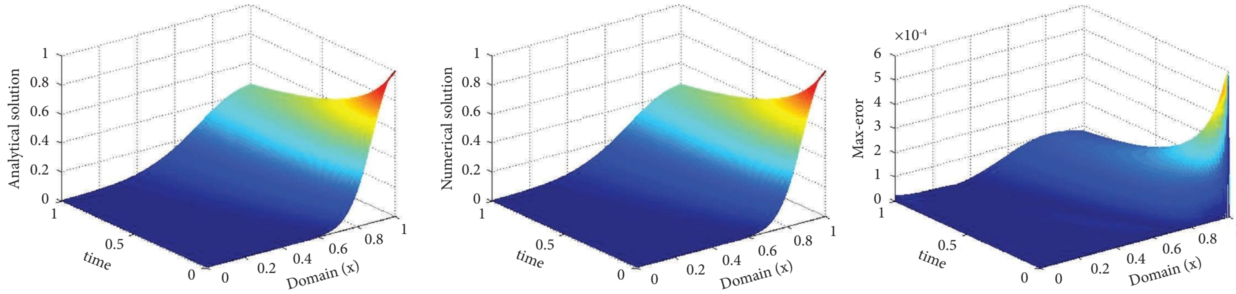

FIGURE 5: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.03$ and $v = 0.002$ gives max − error $= 5.9113 \times 10^{-4}$.

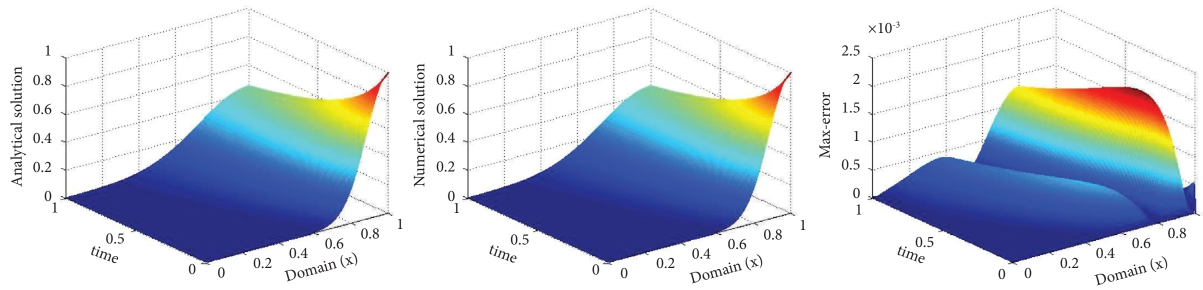

FIGURE 6: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.03$ and $v = 0.002$ gives max − error = 0.0020.

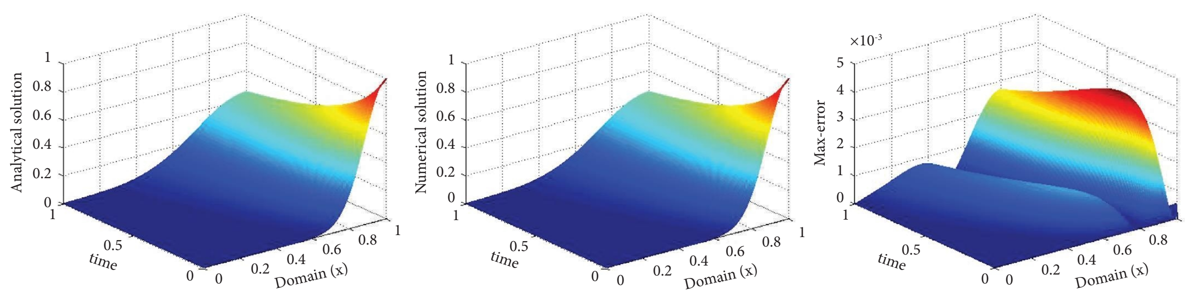

FIGURE 7: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.03$ and $v = 0.002$ gives max − error = 0.0042.

different values of diffusion and convection coefficients. Results show that, in each case, the numerical approximation is stable and consistent.

Experiment 2. Consider a homogeneous one-dimensional convection–diffusion problem over a computational domain $0 \le x \le 1, 0 \le t \le 1$; its exact solution is given as follows: $\phi(x,t) = \sqrt{20}/\sqrt{t+20}\,e^{(-(x-0.5-vt)^2)/4k(t+20)}$ with $v \ge 0$ as the convection velocity to the $x-$ direction and $k > 0$ as the diffusion coefficient.

Figure 11 represents the influence of blending factor over the numerical approximation. Taking $a = 1, 0.5$ and $a = 0$ a diffusion dominant problems is approximated taking $k = 2$ fixed where as the convective velocity is taken

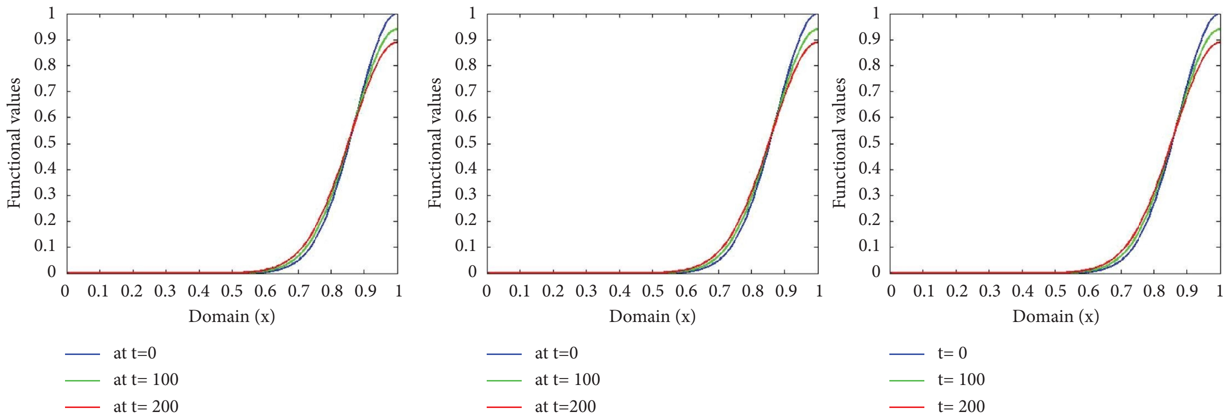

Figure 8: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.03$ and $v = 0.002$.

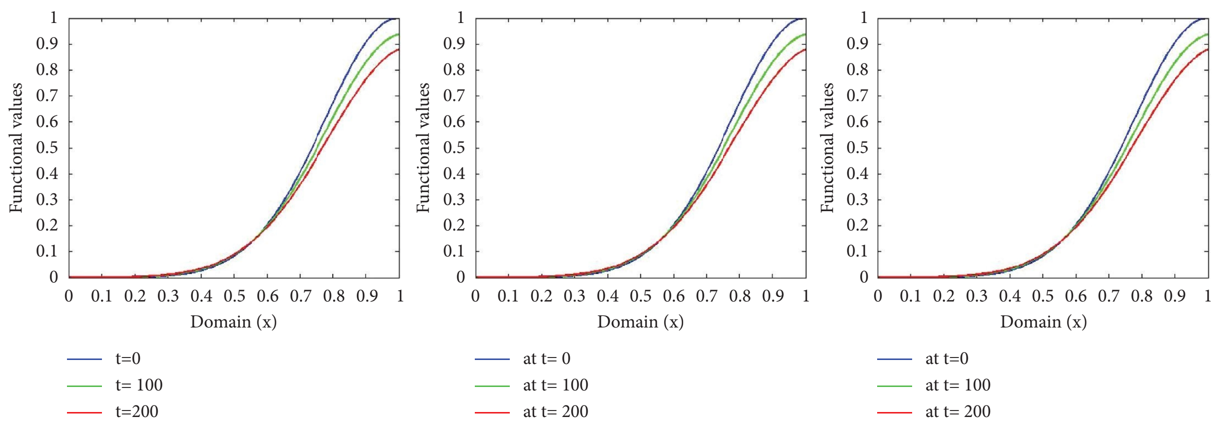

Figure 9: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.1, v = 0.6$, and $Pe = 0.0375$.

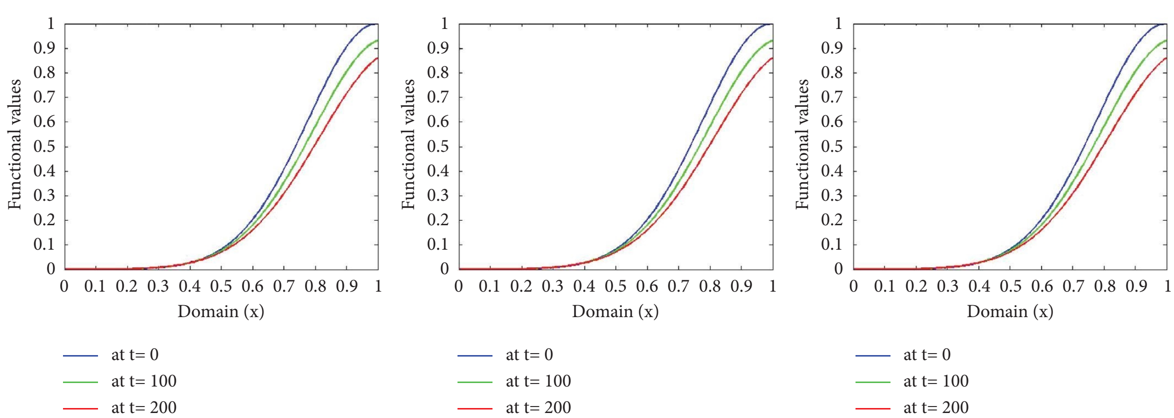

Figure 10: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.1, v = 1$, and $Pe = 0.0625$.

as $v = 0.1, 0.01$ and $v = 0.001$ from left to right figure, respectively. Δx and Δt is taken as 1/15 and 1/1000, respectively. The results shown in Figure 11 represent that, in each diffusion dominant cases for high values of blending factor (a), the results are stable whereas for small vales of blending factor (a), it gives solution with high oscillations.

Figure 12 represents the numerical solution for different values of Peclect number (pe). Here, taking diffusion coefficient $k = 0.02$ and convective velocity $v = 1$ results for convection-dominant case are presented in Figure 12(a). Figure 12(b) represents the results for $k = 0.01$ and $v = 0.8$. Figure 12(c) at the right side represents the approximation for $k = 0.003$ and $v = 0.6$. In Figure 12, numerical results indicate that the convection-dominant problem are highly unstable for low value of factor a. In every figure we can see that, as we decrease the values of factor a, the the amplitude of oscillation goes down.

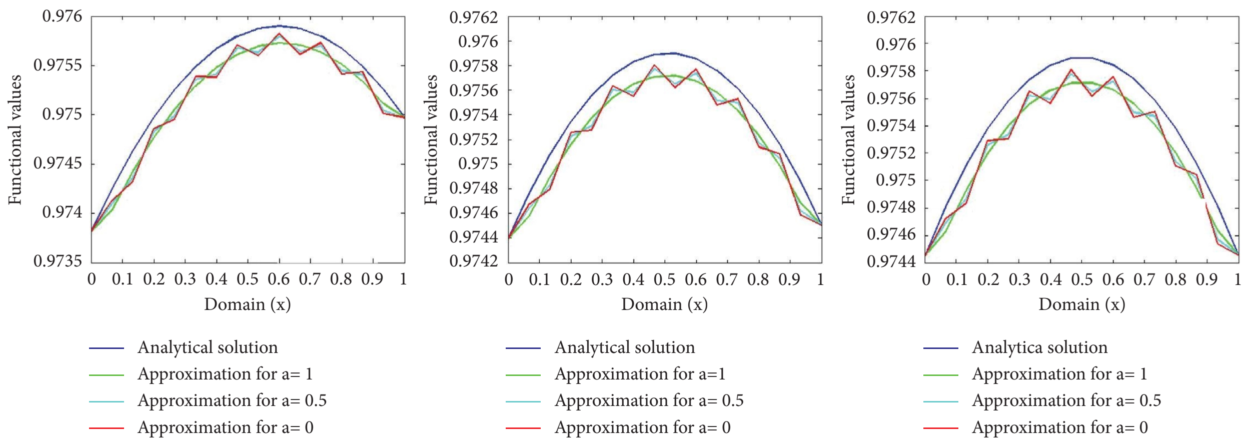

FIGURE 11: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 2, \nu = 0.1, 0.01$, and $\nu = 0.001$.

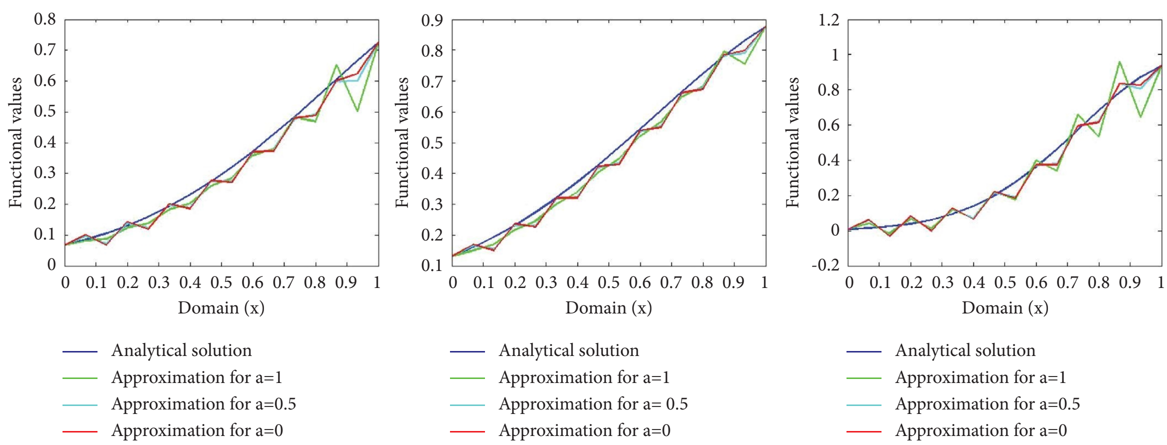

FIGURE 12: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 2, \nu = 0.1, 0.01$, and $\nu = 0.001$.

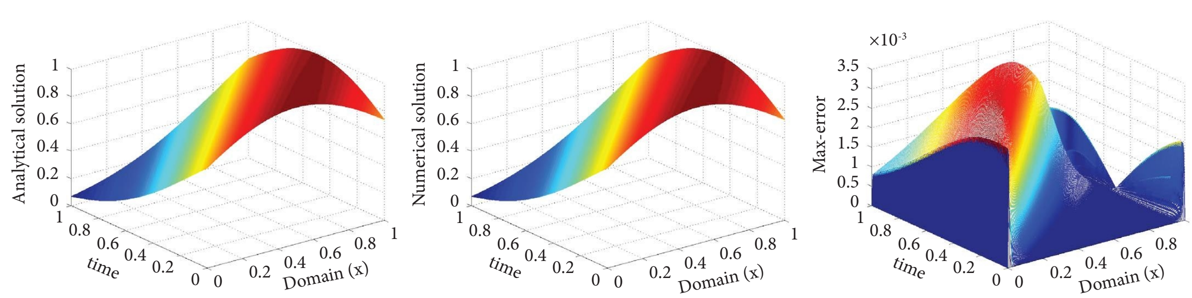

FIGURE 13: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.01, \nu = 1$, and $a = 1$ max − error = 0.0031.

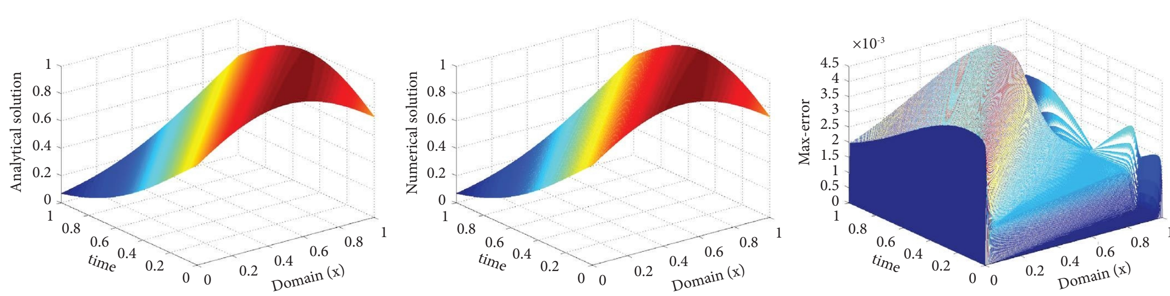

FIGURE 14: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.01, \nu = 1$, and $a = 0.5$ max − error = 0.0042.

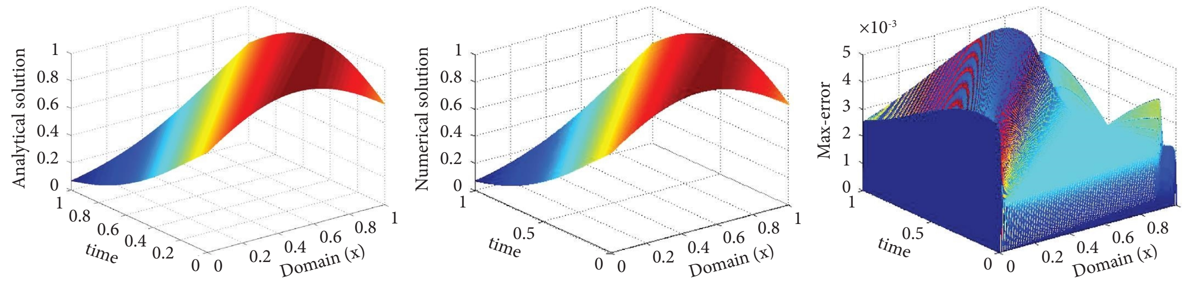

FIGURE 15: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.01, \nu = 1$, and $a = 0$ gives max − error = 0.0047.

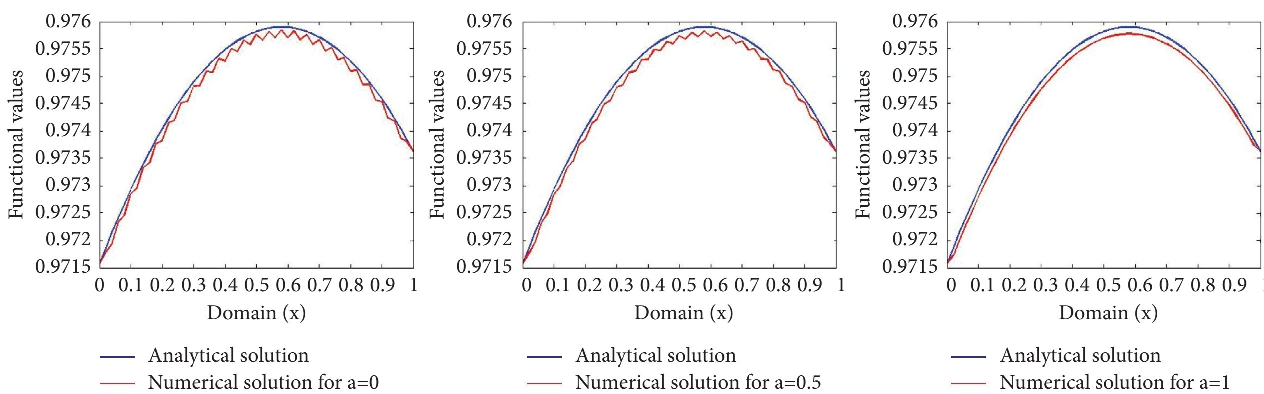

FIGURE 16: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.9, \nu = 0.08$, and $\Delta x = 0.02$.

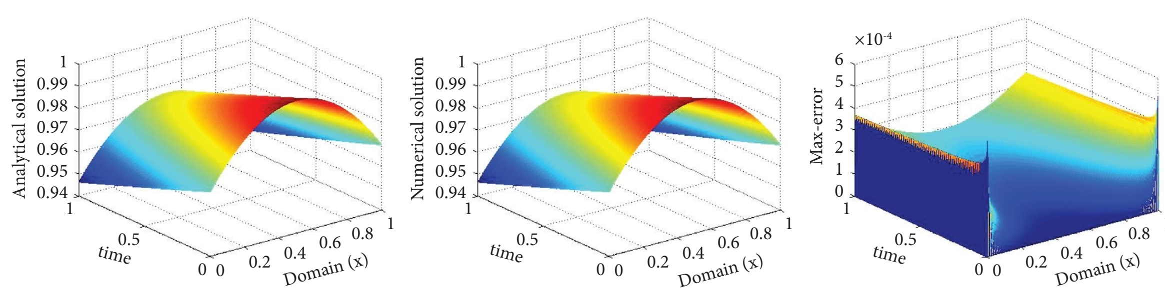

FIGURE 17: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.1, \nu = 0.01$, and $a = 1$ gives max − error = 5.2215×10^{-4}.

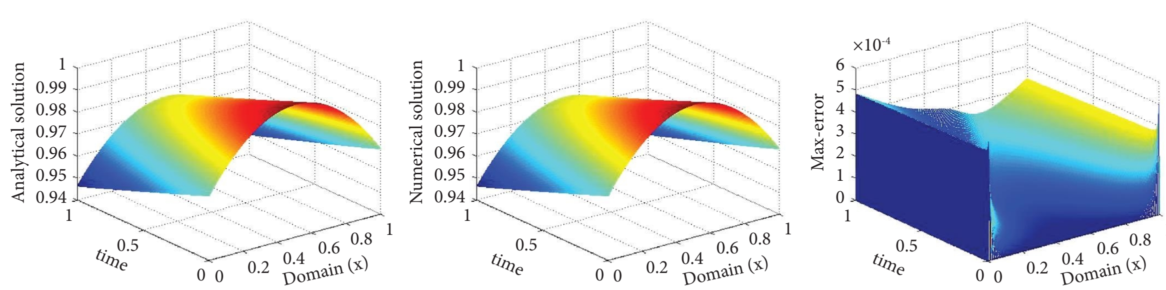

FIGURE 18: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.1, \nu = 0.01$, and $a = 0.5$ gives max − error = 5.3943×10^{-4}.

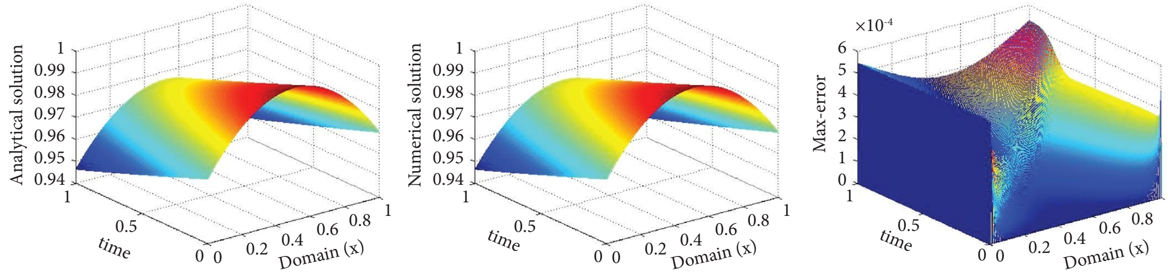

FIGURE 19: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.1, v = 0.01$, and $a = 0$ gives −error $= 5.5642 \times 10^{-4}$.

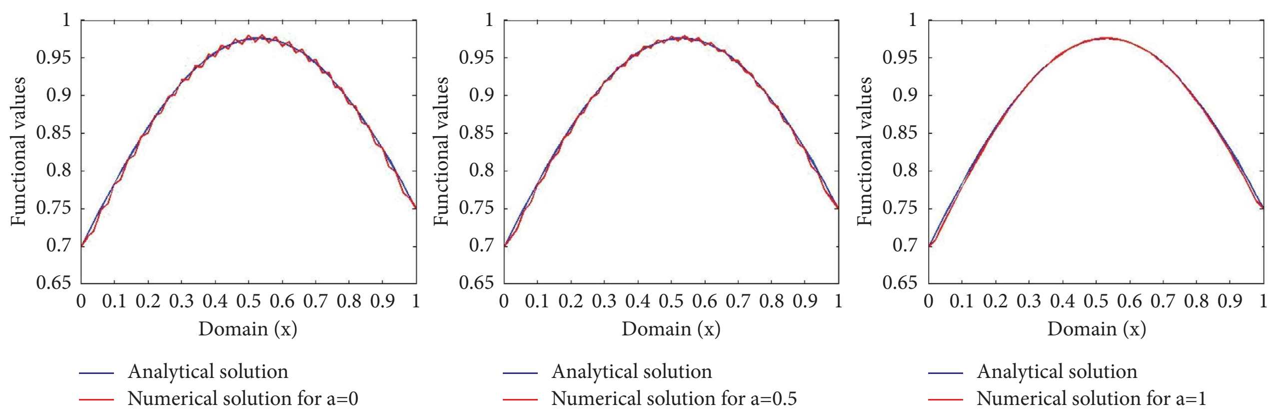

FIGURE 20: Approximation of convection–diffusion problem by numerical scheme (15) taking $k = 0.01$ and $v = 0.03$.

Figures 13–15 show the simulation of the case $k = 0.01$ and convective velocity $(v) = 1$. The numerical results are consistent with the analytical solution for each value of blending factor (a). In Figure 16, numerical results for diffusion coefficient $K = 0.9$ and convective velocity $v = 0.08$ are presented.

Now, Figures 17–19 represent the simulation of the problem for diffusion dominant problem. Taking the diffusion coefficient $(k) = 1$ and convective velocity $(v) = 0.1$ the below results are obtained. Figures 17–19 show results for blending factor $a = 1, a = 0.5$, and $a = 0$. Figure 20 shows results for $k = 0.01$ and $v = 0.03$.

4. Conclusion

In this article, a numerical algorithm based on the finite volume method is proposed. A blending factor (a) is introduced to merge the central differencing and upwind approach, and a new numerical scheme is constructed. Our proposed new algorithm is applied to simulate the convection–diffusion transport problem for different values of diffusion coefficient (k), convective velocity (v), and blending factor (a). The existing central differencing approach gives poor results for convection-dominant problems. Therefore, we have combined the central differencing approach and up-winding approach to simulate convection-dominant and diffusion-dominant problems. Numerical results obtained by the proposed algorithm are consistent with the analytical solution.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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