Research Article

Generalized Heronian Mean Operators Based on Archimedean T-Norms of the Complex Picture Fuzzy Information and Their Application to Decision-Making

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Abstract

Complex picture fuzzy set is a special picture fuzzy set where the truth, abstinence, and falsity grades are shown by a complex number and can effortlessly illustrate the problem and inconsistency in the genuine world. T-norm and t-conorm play an essential and beneficial role in the environment of fuzzy set theory; similarly, Archimedean t-norm and t-conorm have massive flexibility and dominance in the information fusion procedure. The major contribution of this analysis is to explore the algebraic, Einstein, Hamacher, and Frank operational laws under the complex picture fuzzy set. Moreover, the principle of complex picture fuzzy Archimedean Heronian aggregation operator and complex picture fuzzy weighted Archimedean Heronian aggregation operator are also elaborated by using Archimedean t-norm and t-conorm. Additionally, by using the elaborated operators, a multiattribute decision-making technique is presented to elaborate the consistency and reliability of the explored works. Finally, many examples are illustrated for discussing the advantages and sensitive analysis and graphical representation of the investigated works.

1. Introduction

The multiattribute decision-making (MADM) technique is the subpart of the decision-making which is a massive significant technique for choosing the best optimal from the family of alternatives. The process of the MADM technique has been widely employed in all sorts of regions. In genuine life dilemmas, it is massive significant how we show the attribute value more dominant and efficient. Because of inconsistency, it is enough to state the values of the attributes by exact values. To address it, Atanassov [1] modified the principle of the fuzzy set (FS) [2] to initiate the intuitionistic FS (IFS). Under the truth grade (TG) \( M_{\Xi} \), and falsity grade (FG) \( N_{\Xi} \), the well-known result of an IFS is that \( 0 \leq M_{\Xi} + N_{\Xi} \leq 1 \). After the appearance of IFS, researchers widely utilized it in the different regions and presented different methods such as complete ranking technique [3], Dombi aggregation operators [4], generalized entropy measures [5], distance measure [6], failure mode, and their affection [7], TODIM technique [8], hamming distance [9], aggregation of infinite chains [10], parameterized interval-valued intuitionistic soft sets [11], telecom services givers [12], multiattribute border approximation area comparison approach [13], and distance measures [14] to solve the decision-making problems.

In various instances, it is massive challenging to manage three sorts of data in the shape of singleton sets. For
illustration, if an individual faces data in the shape of yes, abstention, and no, then the IFS has been neglected. For this, the principle of picture FS (PFS) was initiated by Cuong [15, 16] which covers the TG $\mathcal{M}$, abstinence grade (AG) $\mathcal{A}$ and FG $\mathcal{N}$ with a well-known principle of $0 \leq \mathcal{M} + \mathcal{A} + \mathcal{N} \leq 1$. IFS and FS are the specific cases of the PFS, and several persons have employed them in the region of distinct areas such as geometric aggregation operators [17], aggregation operators [18], Dombi aggregation operators [19], weighted aggregation operators [20], Hamacher aggregation operators [21], evidence theory [22], picture fuzzy soft set [23], analytical hierarchy processes [24], and distance measures [25].

Alkouri and Salleh [26] revised the assumption of complex FS (CFS) [27] is to introduce the complex IFS (CIFS). The CIFS is massively preferable to employ it in the region of information measures [28], correlation measures [29], complex fuzzy soft set [30], decision making [31], distance measures [32], power aggregation operators [33], complex intuitionistic fuzzy preference relation [34], and aggregation operator [35]. In different occurrences, the principle of CIFS is neglected. For illustration, if a person gives data in the shape of complex-valued TG, AG, and FG, then the theory of CIFS has been unsuccessful. For this, the principle of complex PFS (CPFS) was initiated by Akram et al. [36] which covers the TG $M_{\Xi}^c = (x_i)e^{\frac{i\pi}{2}n_{\Xi}(x_i)}$, abstinence grade (AG) $\mathcal{A}_{\Xi}^c = (x_i)e^{\frac{i\pi}{2}n_{\Xi}(x_i)}$, and FG $N_{\Xi}^c = (x_i)e^{\frac{i\pi}{2}n_{\Xi}(x_i)}$ with a well-known principle of $0 \leq M_{\Xi}^c = (x_i) + A_{\Xi}^c = (x_i) + N_{\Xi}^c = (x_i) \leq 1$ and $0 \leq M_{\Xi}^c = (x_i) + A_{\Xi}^c = (x_i) + N_{\Xi}^c = (x_i) \leq 1$. IFS, CIFS, FS, and CFS are the specific cases of the CPFS.

Hamy mean (HM) operators [37] are massive consistent and dominant to initiate the order among any element of items. The simple averaging aggregation and geometric aggregation operators are the specific cases of the investigated HM operators. Keeping the supremacy of the HM operators, several individuals have employed it in the region of separated areas. For illustration, HM operators for IFSs [38], geometric HM operators for IFSs [39], Dombi HM operators for IFSs [40], HM operators for interval-valued IFSs [41], Heronian aggregation operators for IFSs by using Archimedean t-norm and t-conorm [42], power geometric HM operators for IFSs [43], power HM operators for interval-valued IFSs [44], Frank HM operators for IFSs [45], HM operators for PFSs [46], Dombi HM operators for PFSs [47], and interactional partitioned HM operator for PFSs. Prevaling operators were invented in the availability of the existing theories of IFS, CIFS, and PFS due to their weak and complicated structure. In the CPFS, TG, AG, and FG are in the shape of the polar form and stated in complex numbers. The amplitude object consisting of the TG (AG and FG) includes the extent of supporting (neutral and not supporting) of a term in a CPFS and the phase/periodic object consisting of the TG (AG and FG) includes the extra, morally stating the periodicity in the terms of supporting (neutral and not supporting) of a term in a CPFS. The prevailing theories have a lot of ambiguity because some ideas cannot include the FG, and some cannot include the phase terms. Due to this reason, we faced to lose a lot of data during decision-making strategy, for instance, numerous enterprises are wanted to novel information processing and analysis software. For this, the enterprise refers to an intellectual who provides the data concerning (i) dissimilar features of the software and (ii) diagnosed data of software. The enterprise wants to suggest the massive beneficial optimal(s) of software with its new and latest version continually. Here, the dilemma is two-dimensional, called to choose the beneficial optimal of the software and their new version. This dilemma cannot be addressed efficiently in the presence of the prevailing IFS and PFS theories. So, the beneficial path to express all the data given by the intellectual is by using the prevailing IFS and PFS. The amplitude objects in CPFS may be utilized to provide enterprises decision-making concerning alternatives of software, and the phase object may be stated in the form of the version of the software. Moreover, the CPFS is massive general than the prevailing IFSs, CIFSs, and PFSs to manage inconstant and awkward data in genuine life dilemmas. Keeping the supremacy of the CPFS and HM operators, in this analysis, we combine these two ideas. The major contribution of this analysis is initiated as follows.

1. To explore the algebraic, Einstein, Hamacher, and Frank operational laws under the CPFS.

2. To elaborate the principle of complex picture fuzzy Archimedean Heronian aggregation (CPFWAHA) operator and complex picture fuzzy weighted Archimedean Heronian aggregation (CPFWWAHA) operator is also elaborated by using Archimedean t-norm (TN) and t-conorm (TCN).

3. To explore a MADM technique by using the initiated operators is to elaborate the consistency and reliability of the explored works.

4. To illustrate many numerical examples for discussing the advantages, sensitive analysis, and graphical representation of the investigated works.

The rest of the paper is organized as follows: in Section 2, we briefly describe the principles of CPFSs. In Section 3, we explored the several norm operational laws under the CPFS. In Section 4, the principle of CPFWAHA operator and CPFWWAHA operator is also elaborated by using Archimedean TN and TCN. In Section 5, by using the elaborated operators, a MADM technique is presented to elaborate the consistency and reliability of the explored works. Finally, many examples are illustrated for discussing the advantages, sensitive analysis, and graphical representation of the investigated works. The ending of the evaluation of this analysis is examined in Section 6.
2. Preliminaries

In this analysis, we revised the prevailing principles of CPFSs and their algebraic laws and their operational laws.

In this study, we denoted \( \mathbb{M}_C \) (\( x_i \)) = \( \mathbb{M}_C \) (\( x_i \)) \( e^{2\pi i (M_x (x_i))} \), \( \mathbb{A}_C \) (\( x_i \)) = \( \mathbb{A}_C \) (\( x_i \)) \( e^{2\pi i (A_x (x_i))} \), and \( \mathbb{N}_C \) (\( x_i \)) = \( \mathbb{N}_C \) (\( x_i \)) \( e^{2\pi i (N_x (x_i))} \) as TD, AD, and FD and the universal is invented by \( \mathbb{X}_\text{uni} \). Here, the terms \( R \) and \( I \) in \( \mathbb{E}_R, \mathbb{E}_I \) stated the real and imaginary terms of TD, AD, and FD.

**Definition 1** (see [36]). A CPFS \( \mathbb{E}_C \) is invented by

\[
\mathbb{E}_C = \left\{ \mathbb{M}_C (x_i), \mathbb{A}_C (x_i), \mathbb{N}_C (x_i) : x_i \mathbb{X}_\text{uni} \right\},
\]

where \( \mathbb{M}_C \) (\( x_i \)) = \( \mathbb{M}_C \) (\( x_i \)) \( e^{2\pi i (M_x (x_i))} \), \( \mathbb{A}_C \) (\( x_i \)) = \( \mathbb{A}_C \) (\( x_i \)) \( e^{2\pi i (A_x (x_i))} \), and \( \mathbb{N}_C \) (\( x_i \)) = \( \mathbb{N}_C \) (\( x_i \)) \( e^{2\pi i (N_x (x_i))} \), with \( 0 \leq \mathbb{M}_C \) (\( x_i \)) + \( \mathbb{A}_C \) (\( x_i \)) + \( \mathbb{N}_C \) (\( x_i \)) \leq 1 and \( 0 \leq \mathbb{M}_C \) (\( x_i \)) + \( \mathbb{A}_C \) (\( x_i \)) + \( \mathbb{N}_C \) (\( x_i \)) + \( \mathbb{N}_C \) (\( x_i \)) \leq 1. The structure \( \mathbb{E}_C = \mathbb{E}_C \) (\( x_i \)) \( e^{2\pi i (R_x (x_i))} \) \( e^{2\pi i (R_x (x_i))} \) \( e^{2\pi i (R_x (x_i))} \) conveyed the neutral grade.

The CPFs are mentioned by \( \mathbb{E}_{C,0} = \mathbb{E}_{C,0} \) (\( x_i \)), \( \mathbb{E}_{C,1} = \mathbb{E}_{C,1} \) (\( x_i \)), \( \mathbb{E}_{C,2} = \mathbb{E}_{C,2} \) (\( x_i \)), and \( \mathbb{E}_{C,\theta} = \mathbb{E}_{C,\theta} \) (\( x_i \)), \( \theta = 1, 2, \ldots, \Gamma \).

**Definition 2** (see [36]). Choose any two CPFNs

\[
\begin{align*}
\mathbb{E}_{C,1} &= \left( \mathbb{M}_C + \mathbb{A}_C + \mathbb{N}_C \right) \left( \begin{array}{c}
\mathbb{M}_C \\
\mathbb{A}_C \\
\mathbb{N}_C 
\end{array} \right)
\quad \text{(1)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(2)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} \mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(3)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} \mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(4)}
\end{align*}
\]

where \( \mathbb{M}_C \) (\( x_i \)) = \( \mathbb{M}_C \) (\( x_i \)) \( e^{2\pi i (M_x (x_i))} \), \( \mathbb{A}_C \) (\( x_i \)) = \( \mathbb{A}_C \) (\( x_i \)) \( e^{2\pi i (A_x (x_i))} \), and \( \mathbb{N}_C \) (\( x_i \)) = \( \mathbb{N}_C \) (\( x_i \)) \( e^{2\pi i (N_x (x_i))} \), then

\[
\begin{align*}
\mathbb{E}_{C,1} \mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(5)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} \mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(6)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} \mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(7)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} \mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(8)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} \mathbb{E}_{C,2} &= \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C 
\quad \text{(9)}
\end{align*}
\]

**Definition 3** (see [36]). Choose any CPFN

\[
\begin{align*}
\mathbb{E}_{C,1} &= \left( \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C \right) \left( \begin{array}{c}
\mathbb{M}_C \\
\mathbb{A}_C \\
\mathbb{N}_C 
\end{array} \right)
\quad \text{(10)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} &= \left( \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C \right) \left( \begin{array}{c}
\mathbb{M}_C \\
\mathbb{A}_C \\
\mathbb{N}_C 
\end{array} \right)
\quad \text{(11)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} &= \left( \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C \right) \left( \begin{array}{c}
\mathbb{M}_C \\
\mathbb{A}_C \\
\mathbb{N}_C 
\end{array} \right)
\quad \text{(12)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} &= \left( \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C \right) \left( \begin{array}{c}
\mathbb{M}_C \\
\mathbb{A}_C \\
\mathbb{N}_C 
\end{array} \right)
\quad \text{(13)}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_{C,1} &= \left( \mathbb{M}_C \mathbb{N}_C \mathbb{A}_C \right) \left( \begin{array}{c}
\mathbb{M}_C \\
\mathbb{A}_C \\
\mathbb{N}_C 
\end{array} \right)
\quad \text{(14)}
\end{align*}
\]

the score value (SV) is invented by

\[
\begin{align*}
\mathbb{SV}(\mathbb{E}_{C,1}) &= \frac{1}{3} \left( \mathbb{M}_C + \mathbb{A}_C - \mathbb{N}_C \right)
\quad \text{(15)}
\end{align*}
\]
where $\Xi_{SV} (L_{C-1}) \in [-1, 1]$.

\[
\Xi_{AV} (L_{C-1}) = \frac{1}{3} \left( M_{+} e^{i2\pi L_{C-1}}, A_{-} e^{i2\pi L_{C-1}}, N_{+} e^{i2\pi L_{C-1}} \right),
\]

where $\Xi_{AV} (L_{C-1}) \in [0, 1]$.

**Definition 5** (see [36]). Choose any two CPFNs $\Xi_{C-1} = (M_{+} e^{i2\pi L_{C-1}}, A_{-} e^{i2\pi L_{C-1}}, N_{+} e^{i2\pi L_{C-1}})$ and $\Xi_{C-2} = (M_{+} e^{i2\pi L_{C-2}}, A_{-} e^{i2\pi L_{C-2}}, N_{+} e^{i2\pi L_{C-2}})$, then by using equations (6) and (7), we initiate

1. If $\Xi_{SV}(\Xi_{C-1}) > \Xi_{SV}(\Xi_{C-2})$, then $\Xi_{C-1} > \Xi_{C-2}$
2. If $\Xi_{SV}(\Xi_{C-1}) < \Xi_{SV}(\Xi_{C-2})$, then $\Xi_{C-1} < \Xi_{C-2}$

\[
\Xi_{SV}(\Xi_{C-1}) = \Xi_{SV}(\Xi_{C-2})
\]

**Definition 4** (see [36]). Choose any CPFN $\Xi_{C-1} = (M_{+} e^{i2\pi L_{C-1}}, A_{-} e^{i2\pi L_{C-1}}, N_{+} e^{i2\pi L_{C-1}})$, then the accuracy value (AV) is invented by

\[
\Xi_{AV}(\Xi_{C-1}) = \frac{2}{G(1 + 1)} \sum_{\theta=1}^{G} \sum_{k=1}^{G} \Xi_{PV}^{(k)} \Xi_{PV}^{(k)} = \left( \frac{1}{G(1 + 1)} \right)^{\theta},
\]

where $\Xi_{PV}^{(k)} \geq 0$.

Moreover, we review numerous sorts of prevailing TNs and TCNs, which are illustrated as follows:

(a) Algebraic TN and TCN and their additive generators [42]:

\[
\Xi_{AV}(\Xi_{C-1}) = \Xi_{AV}(\Xi_{C-2}),
\]

(i) If $\Xi_{AV}(\Xi_{C-1}) > \Xi_{AV}(\Xi_{C-2})$, then $\Xi_{C-1} > \Xi_{C-2}$

(ii) If $\Xi_{AV}(\Xi_{C-1}) < \Xi_{AV}(\Xi_{C-2})$, then $\Xi_{C-1} < \Xi_{C-2}$

(iii) If $\Xi_{AV}(\Xi_{C-1}) = \Xi_{AV}(\Xi_{C-2})$, then $\Xi_{C-1} = \Xi_{C-2}$

\[
\Xi_{SV}(\Xi_{C-1}) = \Xi_{SV}(\Xi_{C-2})
\]

**Definition 6** (see [37]). Choose any group of positive integers $\Xi_{PV}, \theta = 1, 2, \ldots, G$, then the HM operator is invented by

\[
\Xi_{AV}(\Xi_{C-1}) = \Xi_{AV}(\Xi_{C-2}),
\]

\[
\Xi_{SV}(\Xi_{C-1}) = \Xi_{SV}(\Xi_{C-2})
\]

(b) Einstein TN and TCN and their additive generators [42]:

\[
\Xi_{AV}(\Xi_{C-1}) = \Xi_{AV}(\Xi_{C-2}),
\]

\[
\Xi_{SV}(\Xi_{C-1}) = \Xi_{SV}(\Xi_{C-2}),
\]

(c) Hamacher TN and TCN and their additive generators [42]:

\[
\Xi_{AV}(\Xi_{C-1}) = \Xi_{AV}(\Xi_{C-2}),
\]

\[
\Xi_{SV}(\Xi_{C-1}) = \Xi_{SV}(\Xi_{C-2})
\]
\[
\overline{T}_{TN}(x_i, y_i) = \frac{x_i y_i}{\delta_{SC} + (1 - \delta_{SC})(x_i + y_i - x_i y_i)} \rightarrow t - \text{norm (additive generator):}
\]

\[
\overline{\eta} (\ell) \ln \left( \frac{\delta_{SC} + (1 - \delta_{SC})^\ell}{\ell} \right),
\]

\[
\overline{S}_{TN}(x_i, y_i) = \frac{x_i + y_i - x_i y_i}{1 - (1 - \delta_{SC})x_i y_i} \rightarrow t - \text{conorm (additive generator):}
\]

\[
\overline{\mu} (\ell) = \ln \left( \frac{\delta_{SC} + (1 - \delta_{SC})(1 - \ell)}{1 - \ell} \right).
\]

(d) Frank TN and TCN and their additive generators [42]:

\[
\overline{T}_{TN}(x_i, y_i) = \log_{\delta_{SC}} \left( 1 + \frac{(\delta_{SC} - 1)(\delta_{SC} - 1)}{(\delta_{SC} - 1)} \right) \rightarrow t - \text{norm (additive generator):}
\]

\[
\overline{T}_{TN}(x_i, y_i) = \log_{\delta_{SC}} \left( 1 + \frac{(\delta_{SC} - 1)(\delta_{SC} - 1)}{(\delta_{SC} - 1)} \right) \rightarrow t - \text{norm (additive generator):}
\]

\[
\overline{S}_{TN}(x_i, y_i) = 1 - \log_{\delta_{SC}} \left( 1 + \frac{(\delta_{SC} - 1)(\delta_{SC} - 1)}{(\delta_{SC} - 1)} \right)
\]

\[
\overline{S}_{TN}(x_i, y_i) = 1 - \log_{\delta_{SC}} \left( 1 + \frac{(\delta_{SC} - 1)(\delta_{SC} - 1)}{(\delta_{SC} - 1)} \right) \rightarrow t - \text{(conorm (additive generator))}
\]

where the general shapes of the TN and TCN are discussed as follows:

\[
\overline{T}_{TN}(x_i, y_i) = \frac{\delta_{SC} - 1}{\delta_{SC} - 1} \rightarrow t - \text{norm (additive generator):}
\]

\[
\overline{T}_{TN}(x_i, y_i) = \frac{\delta_{SC} - 1}{\delta_{SC} - 1} \rightarrow t - \text{norm (additive generator):}
\]

\[
\overline{\eta} (\ell) = -\ln(\ell), \text{if } \delta_{SC} = 1; \overline{\eta} (\ell) = -\ln(\frac{\delta_{SC} - 1}{\delta_{SC} - 1}), \text{if } \delta_{SC} \neq 1.
\]

\[
\overline{\mu} (\ell) = -\ln(1 - \ell), \text{if } \delta_{SC} = 1; \overline{\mu} (\ell) = -\ln(\frac{\delta_{SC} - 1}{\delta_{SC} - 1}), \text{if } \delta_{SC} \neq 1.
\]

\[
3. \text{Operational Laws for CPFSs under the Different TN and TCNs}
\]

Keeping the benefits of the prevailing laws like algebraic, Einstein, Hamacher, and Frank, in this analysis, we initiated these laws for CPFS is to determine the flexibility of the presented works.

\[
\text{Definition 7. For two CPFNs } \Xi_{C-1} = (M_{\Xi_{B-1}}, e^{i2\pi (\alpha_{B-1} z_{B-1})}, \mathbb{E}_{B-1}), \text{ and } \Xi_{C-2} = (M_{\Xi_{B-2}}, e^{i2\pi (\alpha_{B-2} z_{B-2})}, \mathbb{E}_{B-2}), \text{ then}
\]

Where \( \overline{\mu} (\ell) = \overline{\eta} (1 - \ell). \)
By using the TN and TCN in the shape of equations (9) and (10), then equations (19) to (22) are changed to equations (2) to (5), which invented the algebraic operational laws. Additionally, if we chose the TN and TCN in the shape of equations (11) and (12), then equations (19) to (22) are changed to equations...
(23) to (26), which invented the Einstein operational laws.

\[
\Xi_{C-1} \otimes \Xi_{C-2} = \left( \frac{\left( M_{c_i} \right)_{B-2} + \left( M_{c_i} \right)_{B-1}}{1 + M_{c_i}} \right) e^{i \theta} \left( \frac{M_{c_i}}{\left( 1 - M_{c_i} \right)} \right),
\]

\[
\Xi_{C-1} \otimes \Xi_{C-2} = \left( \frac{A_{c_i}}{B_{c_i}} \right) e^{i \theta} \left( \frac{A_{c_i}}{B_{c_i}} \right),
\]

\[
\Xi_{C-1} \otimes \Xi_{C-2} = \left( \frac{N_{c_i}}{B_{c_i}} \right) e^{i \theta} \left( \frac{N_{c_i}}{B_{c_i}} \right),
\]

\[
\Xi_{SC} = \Xi_{C-1} = \left( \frac{2A_{c_i}}{B_{c_i}} \right) e^{i \theta} \left( \frac{2A_{c_i}}{B_{c_i}} \right),
\]

\[
\Xi_{SC} > 0,
\]
When we choose the TN and TCN in the shape of equations (11) and (12), then equations (19) to equations (22) are changed to equations (27) to (30), which invented the Hamacher operational laws.

For $\gamma_{SC} > 0$, we have
For the value of $\delta_{SC} = 1$, equations (27) to (30) are changed to equations (2) to (5) and if we choose the value of $\delta_{SC} = 2$, equations (27) to (30) are changed to equations (23) to (26). When we chose the TN and TCN in the shape of equations (13) and (14), then equations (19) to (22) are changed to equations (31) to (34), which invented the Frank operational laws.
4. General HM Operators under the CIFSs
   Based on TN and TCNs

**Definition 8.** For CPFNs $\Xi_{C-\theta} = (M_{x, \theta}, N_{x, \theta})$, $A_{2\pi (N_{x, \theta})}$, $C_{x, \theta}$, $M_{x, \theta}$, $N_{x, \theta}$, and $\theta = 1, 2, \ldots, \Gamma$, the CPFAHA operator is invented by

$$
CPFAHA^\Gamma f \Xi = \left( \frac{2}{\Gamma + 1} \sum_{\theta=1}^{\Gamma} \sum_{k=1}^{\Gamma} \Xi_{C-\theta} \otimes \Xi_{C-k} \right)^{(1/f + g)}
$$

where $f, g \geq 0$. 
Theorem 1. For CPFNs \( \Xi_{C-\theta} = (M_{k,\theta}, e^{\Omega t(M_{k,\theta})}) \), \( \Xi_{1-\theta} = e^{\Omega t(a_{n+1})}, \), \( \Xi_{2-\theta} = e^{\Omega t(a_{n+2})} \), \( \theta = 1, 2, \ldots, \Gamma \), and by equation (35), we get

\[
CPFAH_{\theta} \Xi_{C-\theta} = \Xi_{C-1} \Xi_{C-2} \cdots \Xi_{C-\Gamma} = \left( \begin{array}{c}
\frac{1}{\eta + g} \eta (\frac{1}{\mu + \eta} (\frac{2}{\Gamma + \eta} (\sum_{b=1}^{\Gamma} \sum_{k=0}^{\eta} \mu (f \eta (M_{b} - \Xi_{k,\theta}) + g \eta (M_{b} - \Xi_{k,\theta}))))))) \\
\frac{1}{\mu + \mu} (\frac{1}{\eta + g} \eta (\frac{2}{\Gamma + \eta} (\sum_{b=1}^{\Gamma} \sum_{k=0}^{\eta} \mu (f \eta (M_{b} - \Xi_{k,\theta}) + g \eta (M_{b} - \Xi_{k,\theta}))))))) \\
\frac{1}{\mu + \mu} (\frac{1}{\eta + g} \eta (\frac{2}{\Gamma + \eta} (\sum_{b=1}^{\Gamma} \sum_{k=0}^{\eta} \mu (f \eta (M_{b} - \Xi_{k,\theta}) + g \eta (M_{b} - \Xi_{k,\theta}))))))) \\
\frac{1}{\mu + \mu} (\frac{1}{\eta + g} \eta (\frac{2}{\Gamma + \eta} (\sum_{b=1}^{\Gamma} \sum_{k=0}^{\eta} \mu (f \eta (M_{b} - \Xi_{k,\theta}) + g \eta (M_{b} - \Xi_{k,\theta})))))))
\end{array} \right)
\]

(36)

Proof. Consider
\[
\begin{align*}
\frac{1}{2} \int_{\mathbb{R}^2} \left( \nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \right) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\nabla^2 (\psi (u_n)) - \nabla^2 (\psi (v_n)) \\
\n\end{align*}
\]
Under equations (35) and (36), we employed some properties such as idempotency, monotonicity, and boundedness.

\[ CPFAHA^{\theta, \varphi}(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}) = \Xi_C. \] (38)

**Property 1.** Choose any group of CPFNs \( \Xi_{C-\theta} = (M_{k=0}^\Xi, \Xi_{\theta}) e^{i2\pi(\Xi_{\theta})}, A_{E_{k=0}}^\Xi e^{i2\pi(\Xi_{\theta})}, N_{E_{k=0}}^\Xi e^{i2\pi(\Xi_{\theta})}, \theta = 1, 2, \ldots, \Gamma. \) If \( \Xi_{C-\theta} = \Xi_C = (M_{k=0}^\Xi, \Xi_{\theta}) e^{i2\pi(\Xi_{\theta})}, A_{E_{k=0}}^\Xi e^{i2\pi(\Xi_{\theta})}, N_{E_{k=0}}^\Xi e^{i2\pi(\Xi_{\theta})}, \theta = 1, 2, \ldots, \Gamma. \), then

\[
CPFAHA^{\theta, \varphi}(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}) = \left( \frac{2}{\Gamma (1 + 1)} \right)^{(1/1 + \theta)} \sum_{\theta=1}^{\Gamma} \sum_{k=1}^{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma}. \]

(39)

**Proof.** By hypothesis \( \Xi_{C-\theta} = \Xi_C = (M_{k=0}^\Xi, \Xi_{\theta}) e^{i2\pi(\Xi_{\theta})}, \theta = 1, 2, \ldots, \Gamma. \), then

\[
CPFAHA^{\theta, \varphi}(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}) = \Xi_C. \]

\[ \sum_{\theta=1}^{\Gamma} \sum_{k=1}^{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \leq \sum_{\theta=1}^{\Gamma} \sum_{k=1}^{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma}. \]

(40)

then,

\[
\frac{2}{\Gamma (1 + 1)} \left( \sum_{\theta=1}^{\Gamma} \sum_{k=1}^{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \right) \leq \frac{2}{\Gamma (1 + 1)} \left( \sum_{\theta=1}^{\Gamma} \sum_{k=1}^{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \right) \]

\[ \leq \frac{2}{\Gamma (1 + 1)} \left( \sum_{\theta=1}^{\Gamma} \sum_{k=1}^{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \Xi_{\theta} \Xi_{\varphi} \Xi_{\Gamma} \right), \]

(41)
therefore,

\[
\tilde{\eta}^{-1}\left(\frac{1}{f + g} \tilde{\eta} \tilde{\eta}^{-1}\left(\frac{2}{\Gamma(\Gamma + 1)} \sum_{b=1}^{\Gamma} \sum_{k=0}^{\Gamma} \mu \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right) \right) + g \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right)\right)\right)\right) \right)\right) \right)\right) \right)
\]

\[
\leq \frac{1}{f + g} \tilde{\eta} \tilde{\eta}^{-1}\left(\frac{2}{\Gamma(\Gamma + 1)} \sum_{b=1}^{\Gamma} \sum_{k=0}^{\Gamma} \mu \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right) + g \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right)\right)\right) \right)\right) \right)\right) \right)\right) \right)\right) \right)
\]

Similarly, we investigate for an unreal term, such that

\[
\tilde{\eta}^{-1}\left(\frac{1}{f + g} \tilde{\eta} \tilde{\eta}^{-1}\left(\frac{2}{\Gamma(\Gamma + 1)} \sum_{b=1}^{\Gamma} \sum_{k=0}^{\Gamma} \mu \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right) \right) + g \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right)\right)\right) \right)\right) \right)\right) \right)\right) \right)
\]

Moreover, for real and unreal terms of AD, we have

\[
\tilde{\eta}^{-1}\left(\frac{1}{f + g} \tilde{\eta} \tilde{\eta}^{-1}\left(\frac{2}{\Gamma(\Gamma + 1)} \sum_{b=1}^{\Gamma} \sum_{k=0}^{\Gamma} \mu \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right) \right) + g \tilde{\eta} \left(\mu_{\mu_{b,k}}\right)\right)\right)\right) \right)\right) \right)\right) \right)\right) \right)\right) \right)
\]

Moreover, for real and unreal terms of FD, we have
Choose any group of CPFNs \( \Xi \). If 

\[
\mathbf{\Xi} = \mathbf{\Xi}_{\mathbf{C}}(\mathbf{\Xi}_{\mathbf{C}}) = \mathbf{\Xi}_{\mathbf{C}}(\mathbf{\Xi}_{\mathbf{C}}), \quad \mathbf{\Xi}_{\mathbf{C}} = \mathbf{\Xi}_{\mathbf{C}}(\mathbf{\Xi}_{\mathbf{C}}), \quad \mathbf{\Xi}_{\mathbf{C}} = \mathbf{\Xi}_{\mathbf{C}}(\mathbf{\Xi}_{\mathbf{C}}), \quad \mathbf{\Xi}_{\mathbf{C}} = \mathbf{\Xi}_{\mathbf{C}}(\mathbf{\Xi}_{\mathbf{C}}), \quad \mathbf{\Xi}_{\mathbf{C}} = \mathbf{\Xi}_{\mathbf{C}}(\mathbf{\Xi}_{\mathbf{C}}), \quad \mathbf{\Xi}_{\mathbf{C}} = \mathbf{\Xi}_{\mathbf{C}}(\mathbf{\Xi}_{\mathbf{C}}),
\]

By using equation (6), we easily get the terms

\[
\text{CPFAHA}^e_g(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}) = \text{CPFAHA}^e_g(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}).
\]

Property 3: Choose any group of CPFNs \( \Xi_{C-\theta} = (M_{\theta}, \mathbf{\Xi}_{R-\theta}) \), \( \mathbf{\Xi}_{R-\theta} = \mathbf{\Xi}_{R-\theta}(\mathbf{\Xi}_{R-\theta}) \), \( \mathbf{\Xi}_{R-\theta} = \mathbf{\Xi}_{R-\theta}(\mathbf{\Xi}_{R-\theta}) \), \( \mathbf{\Xi}_{R-\theta} = \mathbf{\Xi}_{R-\theta}(\mathbf{\Xi}_{R-\theta}) \), \( \theta = 1, 2, \ldots, \Gamma \). If \( \Xi = (\min M_{\theta}, e_{\theta}) \), \( \max M_{\theta}, \min \mathbf{\Xi}_{R-\theta}, \min \mathbf{\Xi}_{R-\theta}, \max \mathbf{\Xi}_{R-\theta} \), then

\[
\text{CPFAHA}^e_g(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}) = \text{CPFAHA}^e_g(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}) \leq \text{CPFAHA}^e_g(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}).
\]

Moreover, by using the value of the parameters \( \mathbf{\Xi} \) and \( \mathbf{\Xi} \), we employed distinct cases of the elaborated works.

(1) For \( \mathbf{\Xi} = 0 \), equation (36) is changed to

\[
\text{CPFAHA}^e_g(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\Gamma}) = \left( \eta^{-1} \left( \frac{2}{\Gamma (1 - \theta) + 1} \left( \sum_{\theta=1}^{\Gamma} \left( \Gamma + 1 - \theta \right) \mu \left( \eta^{-1} \mathbf{\Xi} \left( M_{\theta} \right) \right) \right) \right) \right)_{e}.
\]
It is stated the CPF generalized heavy weighted averaging operator (CPF-GHWO).

(2) For \( f = 1 \) and \( \bar{g} = 0 \), equation (36) is changed to

\[
\begin{aligned}
\mu^{-1} \left( 1 + \mu \left( \frac{2}{\Gamma (\Gamma + 1)} \sum_{\beta=1}^{\Gamma} (\Gamma + 1 - \theta) \mu \left( \frac{A_{2,\beta}^{\alpha}}{2,\beta} \right) \right) \right) \\
\end{aligned}
\]

(3) For \( f = 0 \), equation (36) is changed to

\[
\begin{aligned}
\eta^{-1} \left( \sum_{\alpha=1}^{\Gamma} (\Gamma + 1 - \theta) \eta \left( \frac{B_{2,\alpha}^{\alpha}}{2,\alpha} \right) \right) \\
\end{aligned}
\]
Mathematical Problems in Engineering

\[ C \Xi = \left( \begin{array}{c} \eta^{-1} \left( \frac{1}{\eta} \left( \frac{2}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \end{array} \right) \] 

(51)

It stated the CPFGHWAO.

(4) For \( \tilde{\ell} = 0 \) and \( \tilde{g} = 1 \), equations (36) is changed to

\[ C \Xi = \left( \begin{array}{c} \eta^{-1} \left( \frac{2}{\Gamma (t+1)} \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \end{array} \right) \] 

(52)

(5) By using equations (9) and (10), equation (36) is changed to

\[ C \Xi = \left( \begin{array}{c} \left( \frac{1}{\eta} \left( \frac{2}{\Gamma (t+1)} \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \\
\omega_{n} \left( \frac{1}{\eta} \left( \frac{2\eta}{\Gamma (t+1)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta^{-1} \eta}{g} \left( b_{n} e_{n} \right) \right) \right) \right) \end{array} \right) \] 

(53)
It is stated as the CPFHM operator.

(6) By using equations (11) and (12), equation (36) is changed to

\[
\text{CPFHA}^\mathcal{F}_\mathcal{R}(\xi_{c-1}, \xi_{c-2}, \ldots, \xi_{c-C}) = \begin{pmatrix} \frac{2(z_R^c + z_R^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} + \frac{2(z_I^c + z_I^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} \\ \frac{2(z_R^c + z_R^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} + \frac{2(z_I^c + z_I^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} \\ \frac{2(z_R^c + z_R^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} + \frac{2(z_I^c + z_I^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} \\ \frac{2(z_R^c + z_R^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} + \frac{2(z_I^c + z_I^e)^{(1/\mathcal{F}_\mathcal{R})}}{\mathcal{F}_\mathcal{R}^2} \end{pmatrix} \quad \text{where } z_R^c \]
It is stated as the CPF Einstein HM operator. 

(7) By using equations (13) and (14), equation (36) is changed to

\[
\text{CPFHA}^{A}\left(\pi_{E_{c_{1}}}^{a}, \pi_{E_{c_{2}}}^{a}, \ldots, \pi_{E_{c_{n}}}^{a}\right) = \left\{ \begin{array}{l}
\frac{\partial_{a}(\bar{z}_{a}^{v} - \bar{z}_{a}^{w})^{(\alpha_{i}^{v} - \alpha_{i}^{w})}}{2} + \frac{\bar{z}_{a}^{v} - \bar{z}_{a}^{w}}{2} \left( \bar{z}_{a}^{v} - \bar{z}_{a}^{w}\right)_{\alpha_{i}^{v} - \alpha_{i}^{w}} \\
\left( \frac{\bar{z}_{a}^{w} - \bar{z}_{a}^{v}}{2} \right)^{(\alpha_{i}^{v} - \alpha_{i}^{w})} + \frac{\bar{z}_{a}^{v} - \bar{z}_{a}^{w}}{2} \left( \bar{z}_{a}^{v} - \bar{z}_{a}^{w}\right)_{\alpha_{i}^{v} - \alpha_{i}^{w}} \\
\end{array} \right. , \text{ where } \pi_{a}^{w} \\
\frac{\partial_{a}(\bar{z}_{a}^{v} - \bar{z}_{a}^{w})^{(\alpha_{i}^{v} - \alpha_{i}^{w})}}{2} + \frac{\bar{z}_{a}^{v} - \bar{z}_{a}^{w}}{2} \left( \bar{z}_{a}^{v} - \bar{z}_{a}^{w}\right)_{\alpha_{i}^{v} - \alpha_{i}^{w}} \\
\left( \frac{\bar{z}_{a}^{w} - \bar{z}_{a}^{v}}{2} \right)^{(\alpha_{i}^{v} - \alpha_{i}^{w})} + \frac{\bar{z}_{a}^{v} - \bar{z}_{a}^{w}}{2} \left( \bar{z}_{a}^{v} - \bar{z}_{a}^{w}\right)_{\alpha_{i}^{v} - \alpha_{i}^{w}} \\
\end{array} \right. , \text{ where } \pi_{a}^{w}
\]

(55)
It is stated the CPF Hamacher HM operator.

\[ \text{CPFHA}^{\tilde{\Delta}} \left( \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n \right) = \frac{2 \log \left( \tilde{z}_1^{(1/y)} + \tilde{z}_2^{(1/y)} \right)}{2 \log \left( \tilde{z}_1 \tilde{z}_2 \right)} \]
is stated as the CPF Frank HM operator.

\[
\text{CPFWAHA}_\theta^\varphi\left(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\ell}\right) = \left(\frac{2}{\Gamma(1+\varphi)} \sum_{i=1}^{\Gamma} \sum_{k=1}^{\Gamma} \left(\Gamma_{\theta}\Xi_{C-\beta}\right)^{\varphi} \times \left(\Gamma_{\theta}\Xi_{C-\kappa}\right)^{\varphi}\right), \quad (57)
\]

where \( \varphi, \theta \geq 0 \), with weight vector \( \omega = (\omega_1, \omega_2, \ldots, \omega_\ell)^T \) with a rule \( \sum_{\theta=1}^{\Gamma} \omega_\theta = 1 \).

\[
\text{CPFWAHA}_\theta^\varphi\left(\Xi_{C-1}, \Xi_{C-2}, \ldots, \Xi_{C-\ell}\right) = \left(\frac{2}{\Gamma(1+\varphi)} \sum_{i=1}^{\Gamma} \sum_{k=1}^{\Gamma} \left(\Gamma_{\theta}\Xi_{C-\beta}\right)^{\varphi} \times \left(\Gamma_{\theta}\Xi_{C-\kappa}\right)^{\varphi}\right), \quad (58)
\]

\[
\text{Definition 9.} \text{ Choose any group of CPFNs } \Xi_{C-\theta} = (M_{\Xi}, e^{i\pi(N_{\Xi})}, A_{\Xi}, e^{i\pi(N_{\Xi})}, N_{\Xi}, e^{i\pi(N_{\Xi})}, \Xi_{C-\theta}) \text{, } \theta = 1, 2, \ldots, \Gamma, \text{ then the CPFWAHA operator is invented by}
\]

\[
\text{Theorem 2.} \text{ Choose any group of CPFNs } \Xi_{C-\theta} = (M_{\Xi}, e^{i\pi(N_{\Xi})}, A_{\Xi}, e^{i\pi(N_{\Xi})}, N_{\Xi}, e^{i\pi(N_{\Xi})}, \Xi_{C-\theta}) \text{, } \theta = 1, 2, \ldots, \Gamma, \text{ then by using equation (57), we initiate}
\]

\[
\text{Proof.} \text{ Omitted.}
\]

5. MADM Technique under the Initiated Operators

A bundle of intellectuals has discussed the MADM technique in the region of PFSs and CPFs, but under the HM operators based on Archimedean TN and TCN for CPFs is not developed up to date. The major contribution of this analysis is to initiate the HM operators are employed in the region of the CPF environment.

5.1. Proposed MADM Procedures. Suppose the \( m \) number of alternatives \( \Xi_{AL} = \Xi_{AL-1}, \Xi_{AL-2}, \ldots, \Xi_{AL-m} \) and \( \Gamma \) number of attributes \( \Xi_{AT} = \Xi_{AT-1}, \Xi_{AT-2}, \ldots, \Xi_{AT-\Gamma} \) with weight \( \omega = (\omega_1, \omega_2, \ldots, \omega_\ell)^T \) with a rule \( \sum_{\theta=1}^{\Gamma} \omega_\theta = 1 \). For this, we investigate a decision matrix that covers the items in the shape of CPF data such that \( \Xi_{C-\delta k} = (M_{\delta k}, e^{i\pi(N_{\delta k})}, A_{\delta k}, e^{i\pi(N_{\delta k})}, N_{\delta k}, e^{i\pi(N_{\delta k})}, \Xi_{C-\delta k}) \).

Stage 1: Standardize the matrix. In genuine decision-making, the characteristic qualities are partitioned into two kinds, i.e., the expense characteristic and the advantage property. To kill the distinction in the characteristic qualities, we need to convert them to the same sort. As a rule, because most properties are the advantage type, we need to change over the expense type into the advantage type. The matrix is standardized by using the following formula:
Stage 2: By choosing the elaborated operators, we accumulate the matrix to a single set.

Stage 3: Elaborate the SV of the accumulated values.

Stage 4: To initiate the best optimal, we rank to all alternatives.

Stage 5: Standardize the matrix. In genuine decision-making, the characteristic qualities are partitioned into two kinds, i.e., the expense characteristic and the advantage property. To kill the distinction in the characteristic qualities, we need to convert them to the same sort. As a rule, because most properties are the advantage type, we need to change over the expense type into the advantage type. The matrix is standardized by using the following formula:

\[
\begin{align*}
= D_{DM} = & \begin{cases} 
\left( \begin{array}{c}
\mathbb{M} = e^{i2\pi \left( \frac{\mathbb{M}}{\mathbb{M}_{\theta}} \right)}
\end{array} \right), & \text{for benefit types data,} \\
\left( \begin{array}{c}
\mathbb{N} = e^{i2\pi \left( \frac{\mathbb{N}}{\mathbb{N}_{\theta}} \right)}
\end{array} \right), & \text{for cost types data.}
\end{cases}
\end{align*}
\]

Since the matrix has covered all the benefit sorts of data, so it is not needed to be standardized.

Stage 6: By choosing the elaborated operators (like equation (44)), we aggregate the matrix, whose values are discussed below for \( f = g = 1 \).
Table 1: Input decision-matrix provided by the expert.

<table>
<thead>
<tr>
<th>$\hat{z}_{AT-1}$</th>
<th>$\hat{z}_{AT-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\underline{z}_{AL-1}$</td>
<td>(0.3$e^{2\pi(0.4)}$, 0.1$e^{2\pi(0.1)}$, 0.2$e^{2\pi(0.2)}$)</td>
</tr>
<tr>
<td>$\underline{z}_{AL-2}$</td>
<td>(0.4$e^{2\pi(0.5)}$, 0.2$e^{2\pi(0.2)}$, 0.1$e^{2\pi(0.1)}$)</td>
</tr>
<tr>
<td>$\underline{z}_{AL-3}$</td>
<td>(0.2$e^{2\pi(0.3)}$, 0.2$e^{2\pi(0.2)}$, 0.2$e^{2\pi(0.2)}$)</td>
</tr>
<tr>
<td>$\underline{z}_{AL-4}$</td>
<td>(0.5$e^{2\pi(0.5)}$, 0.1$e^{2\pi(0.1)}$, 0.1$e^{2\pi(0.1)}$)</td>
</tr>
<tr>
<td>$\underline{z}_{AL-5}$</td>
<td>(0.6$e^{2\pi(0.6)}$, 0.2$e^{2\pi(0.2)}$, 0.1$e^{2\pi(0.1)}$)</td>
</tr>
</tbody>
</table>

Stage 7: Elaborate the SV of the accumulated values, which are discussed as follows:

$\underline{x}_{AL-1} = 0.1718e^{2\pi(0.2307)}$, $0.3848e^{2\pi(0.3848)}$, $0.4941e^{2\pi(0.4941)}$

$\underline{x}_{AL-2} = 0.2307e^{2\pi(0.2928)}$, $0.4941e^{2\pi(0.4941)}$, $0.3848e^{2\pi(0.3848)}$

$\underline{x}_{AL-3} = 0.1149e^{2\pi(0.1149)}$, $0.4941e^{2\pi(0.4941)}$, $0.4941e^{2\pi(0.4941)}$

$\underline{x}_{AL-4} = 0.2928e^{2\pi(0.2928)}$, $0.3848e^{2\pi(0.3848)}$, $0.3848e^{2\pi(0.3848)}$

$\underline{x}_{AL-5} = 0.3598e^{2\pi(0.3598)}$, $0.4941e^{2\pi(0.4941)}$, $0.3848e^{2\pi(0.3848)}$

By choosing the elaborated operators (like equation (44)), we aggregate the matrix, whose values are discussed below for $f = g = 1$.

$\underline{x}_{AL-1} = 0.1718, 0.3848, 0.4941$

$\underline{x}_{AL-2} = 0.2307, 0.4941, 0.3848$

$\underline{x}_{AL-3} = 0.1149, 0.4941, 0.4941$

$\underline{x}_{AL-4} = 0.2928, 0.3848, 0.3848$

$\underline{x}_{AL-5} = 0.3598, 0.4941, 0.3848$

Stage 8: To initiate the best optimal, we rank to all alternatives, which is illustrated as follows:

$\underline{x}_{AL-3} > \underline{x}_{AL-1} > \underline{x}_{AL-2} > \underline{x}_{AL-5} > \underline{x}_{AL-4}$

The best optimal is $\underline{x}_{AL-3}$. Moreover, to investigate the supremacy and consistency of the initiated operators, we choose the intuitionistic fuzzy sorts of data and elaborate it by using the presented operators. The graphical representation of ranking of the alternatives is shown in Figure 1.

If the decision-maker considers only the real component during the initial rating values of Table 1, then this representation is summarized as Table 2.

Elaborate the SV of the accumulated values, which are discussed as follows:

$\underline{x}_{AL-1} = 0.2357$

$\underline{x}_{AL-2} = 0.2161$

$\underline{x}_{AL-3} = 0.2911$

$\underline{x}_{AL-4} = 0.1589$

$\underline{x}_{AL-5} = 0.1730$

To initiate the best optimal, we rank to all alternatives, which is illustrated as follows:
Figure 1: Graphical representation of ranking of alternatives of the input.

Table 2: The original matrix is given by the decision-maker.

<table>
<thead>
<tr>
<th></th>
<th>$\hat{e}_{AT-1}$</th>
<th>$\hat{e}_{AT-2}$</th>
<th>$\hat{e}_{AT-3}$</th>
<th>$\hat{e}_{AT-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{e}_{AL-1}$</td>
<td>(0.3, 0.1, 0.2)</td>
<td>(0.31, 0.11, 0.21)</td>
<td>(0.32, 0.12, 0.22)</td>
<td>(0.3, 0.13, 0.23)</td>
</tr>
<tr>
<td>$\hat{e}_{AL-2}$</td>
<td>(0.4, 0.2, 0.1)</td>
<td>(0.41, 0.21, 0.11)</td>
<td>(0.42, 0.22, 0.12)</td>
<td>(0.43, 0.23, 0.13)</td>
</tr>
<tr>
<td>$\hat{e}_{AL-3}$</td>
<td>(0.2, 0.2, 0.2)</td>
<td>(0.21, 0.21, 0.21)</td>
<td>(0.22, 0.22, 0.22)</td>
<td>(0.23, 0.23, 0.23)</td>
</tr>
<tr>
<td>$\hat{e}_{AL-4}$</td>
<td>(0.5, 0.1, 0.1)</td>
<td>(0.51, 0.11, 0.11)</td>
<td>(0.52, 0.12, 0.12)</td>
<td>(0.53, 0.13, 0.13)</td>
</tr>
<tr>
<td>$\hat{e}_{AL-5}$</td>
<td>(0.6, 0.2, 0.1)</td>
<td>(0.61, 0.21, 0.11)</td>
<td>(0.62, 0.22, 0.12)</td>
<td>(0.63, 0.23, 0.13)</td>
</tr>
</tbody>
</table>

Figure 2: Graphical representation of the alternatives for Table 2 input.
Table 3: Comparative analysis of the proposed and existing works.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Score values</th>
<th>Ranking values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liu and Chen [42]</td>
<td>Cannot be calculated</td>
<td>Cannot be calculated</td>
</tr>
<tr>
<td>Akram et al. [36]</td>
<td>(\hat{g}<em>{AL-1} = 0.3127, \hat{g}</em>{AL-2} = 0.2886, \hat{g}_{AL-3} = 0.4019)</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-2} &gt; \hat{g}<em>{AL-1} &gt; \hat{g}</em>{AL-2} )</td>
</tr>
<tr>
<td></td>
<td>(\hat{g}<em>{AL-4} = 0.2130, \hat{g}</em>{AL-5} = 0.2381)</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}_{AL-2} )</td>
</tr>
<tr>
<td>Equation (44)</td>
<td>(\hat{g}<em>{AL-1} = 0.4518, \hat{g}</em>{AL-2} = 0.4114, \hat{g}_{AL-3} = 0.5822)</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}_{AL-2} )</td>
</tr>
<tr>
<td></td>
<td>(\hat{g}<em>{AL-4} = 0.3178, \hat{g}</em>{AL-5} = 0.3461)</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}_{AL-2} )</td>
</tr>
</tbody>
</table>

Table 4: Impact of the parameter \(\bar{g}\) by fixing the value \(\bar{f} = 1\) on the input given in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Score values</th>
<th>Ranking values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{g} = 1)</td>
<td>0.4518, 0.4114, 0.5822, 0.3178, 0.3461</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 2)</td>
<td>0.3437, 0.2974, 0.4815, 0.204, 0.225</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 3)</td>
<td>0.2771, 0.227, 0.4193, 0.1338, 0.1501</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 5)</td>
<td>0.1981, 0.1434, 0.3456, 0.0504, 0.0608</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 7)</td>
<td>0.1523, 0.095, 0.3027, 0.0018, 0.0087</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 9)</td>
<td>0.1222, 0.0631, 0.2745, 0.0302, 0.0255</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 10)</td>
<td>0.1106, 0.051, 0.2637, 0.0426, 0.0387</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
</tbody>
</table>

Figure 3: Comparison of the rating of the given alternatives.

Table 5: Impact of the parameter \(\bar{g}\) by fixing the value \(\bar{f} = 1\), for the input data in Table 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Score values</th>
<th>Ranking values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{g} = 1)</td>
<td>0.2357, 0.2161, 0.2911, 0.1589, 0.173</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 2)</td>
<td>0.1833, 0.1604, 0.2408, 0.102, 0.1125</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 3)</td>
<td>0.151, 0.1261, 0.2097, 0.0669, 0.075</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 5)</td>
<td>0.1127, 0.0854, 0.1728, 0.0252, 0.0304</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 7)</td>
<td>0.0905, 0.0618, 0.1513, 0.0009, 0.0044</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 9)</td>
<td>0.0758, 0.0463, 0.1372, 0.0151, 0.0128</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
<tr>
<td>(\bar{g} = 10)</td>
<td>0.0702, 0.0404, 0.1318, 0.0213, 0.0193</td>
<td>(\hat{g}<em>{AL-3} &gt; \hat{g}</em>{AL-1} &gt; \hat{g}<em>{AL-2} &gt; \hat{g}</em>{AL-5} &gt; \hat{g}_{AL-4})</td>
</tr>
</tbody>
</table>
Figures 4 and 5: Variation of the score values with $\bar{g}$ by taking $f = 1$ for data in Table 1.

### Table 6: Impact of the parameter $f$ with $\bar{g} = 1$ on ranking of alternatives for Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Score values</th>
<th>Ranking values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f = 1$</td>
<td>0.4518, 0.4114, 0.5822, 0.3178, 0.3461</td>
<td>$\Xi_{AL-3} &gt; \Xi_{AL-1} &gt; \Xi_{AL-2} &gt; \Xi_{AL-5} &gt; \Xi_{AL-4}$</td>
</tr>
<tr>
<td>$f = 2$</td>
<td>0.3383, 0.2921, 0.4772, 0.1977, 0.2199</td>
<td>$\Xi_{AL-3} &gt; \Xi_{AL-1} &gt; \Xi_{AL-2} &gt; \Xi_{AL-5} &gt; \Xi_{AL-4}$</td>
</tr>
<tr>
<td>$f = 3$</td>
<td>0.2697, 0.2197, 0.4134, 0.1252, 0.143</td>
<td>$\Xi_{AL-3} &gt; \Xi_{AL-1} &gt; \Xi_{AL-2} &gt; \Xi_{AL-5} &gt; \Xi_{AL-4}$</td>
</tr>
<tr>
<td>$f = 5$</td>
<td>0.1896, 0.1349, 0.3384, 0.0405, 0.0523</td>
<td>$\Xi_{AL-3} &gt; \Xi_{AL-1} &gt; \Xi_{AL-2} &gt; \Xi_{AL-5} &gt; \Xi_{AL-4}$</td>
</tr>
<tr>
<td>$f = 7$</td>
<td>0.1435, 0.0862, 0.2952, 0.0081, 0.0001</td>
<td>$\Xi_{AL-3} &gt; \Xi_{AL-1} &gt; \Xi_{AL-2} &gt; \Xi_{AL-5} &gt; \Xi_{AL-4}$</td>
</tr>
<tr>
<td>$f = 9$</td>
<td>0.1134, 0.0543, 0.2669, 0.04, 0.0344</td>
<td>$\Xi_{AL-3} &gt; \Xi_{AL-1} &gt; \Xi_{AL-2} &gt; \Xi_{AL-5} &gt; \Xi_{AL-4}$</td>
</tr>
<tr>
<td>$f = 10$</td>
<td>0.1019, 0.0421, 0.2561, 0.0522, 0.0475</td>
<td>$\Xi_{AL-3} &gt; \Xi_{AL-1} &gt; \Xi_{AL-2} &gt; \Xi_{AL-5} &gt; \Xi_{AL-4}$</td>
</tr>
</tbody>
</table>
Table 7: Impact of the parameter \( f \) with \( g = 1 \) on ranking of alternatives for Table 2.

<table>
<thead>
<tr>
<th>Parameters ( f )</th>
<th>Score values</th>
<th>Ranking values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f = 1 )</td>
<td>0.2357, 0.2161, 0.2911, 0.1589, 0.173</td>
<td>( \mathcal{AL}<em>{-3} &gt; \mathcal{AL}</em>{-1} &gt; \mathcal{AL}<em>{-2} &gt; \mathcal{AL}</em>{-5} &gt; \mathcal{AL}_{-4} )</td>
</tr>
<tr>
<td>( f = 2 )</td>
<td>0.1806, 0.1578, 0.2386, 0.0988, 0.1099</td>
<td>( \mathcal{AL}<em>{-3} &gt; \mathcal{AL}</em>{-1} &gt; \mathcal{AL}<em>{-2} &gt; \mathcal{AL}</em>{-5} &gt; \mathcal{AL}_{-4} )</td>
</tr>
<tr>
<td>( f = 3 )</td>
<td>0.1473, 0.1224, 0.2067, 0.0626, 0.0715</td>
<td>( \mathcal{AL}<em>{-3} &gt; \mathcal{AL}</em>{-1} &gt; \mathcal{AL}<em>{-2} &gt; \mathcal{AL}</em>{-5} &gt; \mathcal{AL}_{-4} )</td>
</tr>
<tr>
<td>( f = 5 )</td>
<td>0.1084, 0.0811, 0.1692, 0.0203, 0.0261</td>
<td>( \mathcal{AL}<em>{-3} &gt; \mathcal{AL}</em>{-1} &gt; \mathcal{AL}<em>{-2} &gt; \mathcal{AL}</em>{-5} &gt; \mathcal{AL}_{-4} )</td>
</tr>
<tr>
<td>( f = 7 )</td>
<td>0.0861, 0.0574, 0.1476, 0.0041, 0.00005</td>
<td>( \mathcal{AL}<em>{-3} &gt; \mathcal{AL}</em>{-1} &gt; \mathcal{AL}<em>{-2} &gt; \mathcal{AL}</em>{-5} &gt; \mathcal{AL}_{-4} )</td>
</tr>
<tr>
<td>( f = 9 )</td>
<td>0.0715, 0.0419, 0.1334, 0.02, 0.0172</td>
<td>( \mathcal{AL}<em>{-3} &gt; \mathcal{AL}</em>{-1} &gt; \mathcal{AL}<em>{-2} &gt; \mathcal{AL}</em>{-5} &gt; \mathcal{AL}_{-4} )</td>
</tr>
<tr>
<td>( f = 10 )</td>
<td>0.0659, 0.036, 0.128, 0.0261, 0.0238</td>
<td>( \mathcal{AL}<em>{-3} &gt; \mathcal{AL}</em>{-1} &gt; \mathcal{AL}<em>{-2} &gt; \mathcal{AL}</em>{-5} &gt; \mathcal{AL}_{-4} )</td>
</tr>
</tbody>
</table>

Figure 6: Variation of the score values with \( f \) by taking \( g = 1 \) for data in Table 1.

Figure 7: Variation of the score values with \( f \) by taking \( g = 1 \) for data in Table 2.
5.3. Sensitivity Analysis. The major contribution of this analysis is to compare the initiated works with numerous prevailing operators based on PFSs and CPFSs to investigate the flexibility and dominancy of the elaborated works. The prevailing works are followed as the theory of HM operators for IFS which was developed by Liu and Chen [42]; Akram et al. [36] initiated the Hamacher aggregation operators for CPFSs and proposed works based on CPFSs with weight vectors 0.3, 0.3, 0.3, and 0.1. The sensitive analysis of the explored and prevailing operators is discussed in the shape of Table 3.

By using the data in Table 3, we get the two different sorts of results that show the consistency of the initiated works. Moreover, by using ten values of parameters $\varphi$ and $\psi$, we discussed the fluency of the elaborated work. For $\varphi = 1$, the fluency of the $\psi$ is discussed in the shape of Table 4 by using the data in Table 1. The graphical illustration on the ranking of the alternatives between the proposed and existing approaches is shown in Figure 3.

By changing the value of the parameter, we have gotten the same ranking results; further, for $\varphi = 1$, the fluency of the $\psi$ is discussed in the shape of Table 5 by using the data in Table 2. The graphical representation of the variation of the score values for different values of parameter $\psi$ for the information mentioned in Tables 1 and 2 is shown in Figures 4 and 5, respectively.

However, to show the influence of the parameter $\varphi$ on to the ranking of the alternatives, we perform an analysis with $\psi = 1$, and hence the results are listed in Tables 6 and 7, respectively, for the input information in Tables 1 and 2. Furthermore, the graphical representation of the score values of the given alternatives is shown in Figures 6 and 7.

6. Conclusion

To handle problematic and convoluted data in genuine life dilemmas, the principle of complex picture fuzzy set (CPFS) is a capable and skillful technique to resolve real-life problematic dilemmas. CPFS handle such sort of dilemmas, which covers the three sorts of data such as yes, abstinence, and no in the shape of fuzzy numbers. The major influence obtained from this study is summarized here:

(1) We explored the algebraic, Einstein, Hamacher, and Frank operational laws under the CPFS

$$\Xi_{AL-3} > \Xi_{AL-1} > \Xi_{AL-2} > \Xi_{AL-5} > \Xi_{AL-4}. \quad (66)$$

The best optimal is $\Xi_{AL-3}$. The data are in Table 1; if we choose the unreal part or not, the obtained results are the same. To improve the quality of the presented works, we discussed the sensitive analysis of the elaborated works with some prevailing operators to investigate the supremacy of the proposed works. The graphical representation of such information is given in Figure 2.

(2) The principle of CPFAHA operator and CPFWAHA operator is also elaborated by using Archimedean TN and TCN

(3) By using the elaborated operators, a MADM technique is presented to elaborate the consistency and reliability of the explored works

(4) Many examples are illustrated for discussing the advantages, sensitive analysis, and graphical representation of the investigated works.

From the conducted study, we conclude it is efficient to solve the decision-making problems in an efficient manner. Apart from that, we also observe that the several of the existing studies are considered as a special case of the proposed work. For instance,

(1) If $M_{\Xi_{j-\mathfrak{L}}}$ = $N_{\Xi_{j-\mathfrak{L}}}$ = $\Xi_{\mathfrak{L}-\mathfrak{L}}$ = 0 in CPFSs, then we achieved the PFSs

(2) If $M_{\Xi_{j-\mathfrak{L}}}$ = $N_{\Xi_{j-\mathfrak{L}}}$ = $\Xi_{\mathfrak{L}-\mathfrak{L}}$ = 0 in CPFSs, then we achieved the IFSs

(3) If $M_{\Xi_{j-\mathfrak{L}}}$ = $N_{\Xi_{j-\mathfrak{L}}}$ = 0 in CPFSs, then we achieved the CIFSs

(4) If $M_{\Xi_{j-\mathfrak{L}}}$ = $N_{\Xi_{j-\mathfrak{L}}}$ = $\Xi_{\mathfrak{L}-\mathfrak{L}}$ = 0 in CPFSs, then we achieved the FSs

(5) If $M_{\Xi_{j-\mathfrak{L}}}$ = $N_{\Xi_{j-\mathfrak{L}}}$ = $\Xi_{\mathfrak{L}-\mathfrak{L}}$ = 0 in CPFSs, then we achieved the CFSs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


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