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Research Article

Novel Results on Finite-Time Stability of Solutions for Stochastic Ψ-Hilfer Fractional System

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This article studies a new kind of Ψ-Hilfer fractional system driven by m-dimensional Brownian motion. By utilizing the generalized Laplace transform and its inverse, the contraction mapping principle, and the properties of a semigroup, we establish the uniqueness of the solution. In addition, finite-time stability is investigated by means of the properties of norm and inequalities scaling technique. As verification, an example is given to show the deduced conclusions.

1. Introduction

Actually, the fractional derivative can be expressed as a differential-integral convolution operator, which is nonlocal. The integral term defined by it reflects the historical dependence of system development well. The fractional time derivative operator has long-range correlation and memory. Therefore, fractional calculus has been widely used in the research of viscoelastic material, abnormal diffusion, fluid mechanics, biomedicine, chaos and turbulence, control theory, and many other fields. One can refer to the monographs [1–4] for more information.

The existence and uniqueness theorems for solutions are a primary subject of fractional system. Various methods were used to obtain the existence and uniqueness results in [5, 6] and the following literatures. As for the non-Lipschitz condition, Abouagwa et al. [7, 8] established the existence theorem for solutions by applying the Carathéodory approximation. Under global Lipshitz conditions, with the aid of Picard iteration method and contradiction method, Moghaddam et al. [9] and Umamaheswari et al. [10] deduced the existence and uniqueness results. In [11–13], the monotone iterative method was used to obtain the existence theorem of mild solutions. In addition, various fixed point theorems were proposed to show the existence and uniqueness results for solutions in [14–16]. Jleli et al. [17] researched the uniqueness of solutions for a kind of coupled system by applying Perov's fixed point theorem together with a type of Lyapunov inequality. In [18], Baghani derived the uniqueness result for the Langevin equation with two orders by establishing a new type of norm in a Banach space and combining the contraction principle. In [19], the uniqueness theorem was deduced by establishing a new type of α-ψ-contractive mapping. In [20, 21], Krasnoselskii-type fixed point theorem and the extended Krasnoselskii's fixed point theorem were used to deduce the existence and uniqueness theorem of the considered system.

The finite-time stability analysis of a system has the following two situations. The first case is to research the transient performance of the system over a fixed finite time domain, that is, in a finite time domain the state of the system remains within a given boundary, which is independent of Lyapunov stability. The second case is to study steady-state performance over an infinite time domain; that is, the system converges to equilibrium over finite time within the category of Lyapunov stability. The second case is known as finite time convergent stable. In the present article, we study the first kind. Finite-time stability analysis plays an indispensable role in many practical problems, for example, the launch of a rocket, automatic active suspension system, traffic flow node control, and satellite sliding mode control Amato et al. [22, 23].

Since the groundbreaking work of Dorato [24], subsequently, the basic definition of finite-time stability in stochastic system was first proposed by Kushner [25]. The Grönwall approach was used to establish the finite-time stability of stochastic fractional system in [26–29]. Based on the properties of nabla difference for Riemann–Liouville-type and the generalized Grönwall inequality, Lu et al. [26] researched finite-time stability in the mean for the fractional difference equations with the nabla operator, in which contains uncertain term. With the aid of the Laplace transform and its inverse, Luo et al. [27] researched two kinds of stochastic fractional delay systems, and the finite-time stability results were established by applying the generalized Henry–Grönwall delay inequality. Under some assumptions, Mathiyalagan and Balachandran [28] researched the finite-time stability of fractional stochastic singular delay system driven by white noise by using the Laplace transform and its inverse and based on the Grönwall method. Mchiri et al. [29] studied the finite-time stability of stochastic fractional linear delay system, in which the analysis method was the generalized Grönwall inequality. The analysis of finite-time stability for various systems had been investigated by applying different methods in [30] and the follows. By applying the Lyapunov functions approach, Luo et al. [31] derived finite-time stability results. In [32–34], based on a delayed Mittag–Leffler-type matrix, Li et al. deduced finite-time stability results of different systems. Moreover, the delayed exponential matrix method was also a useful tool to study finite-time stability in [35–37]. In [38], with the aid of variation of constants method and fractional order cosine and sine delayed matrices, Liang et al. obtained the representation of the solution, and finite-time stability results were subsequently deduced by norm estimates and Caputo derivative properties. Zhang and Wang [16] studied a kind of Hadamard-type fractional nonlinear system, in which finite-time stability result was derived by means of Hadamard-type impulsive Grönwall inequality.

At the same time, the Hilfer-type fractional system was also favored by many scholars. Harikrishnan et al. researched a class of Ψ-Hilfer fractional system under boundary conditions in [39] and coupled differential equations in the sense of Ψ-Hilfer fractional derivative in [40]. With the aid of Grönwall approach, Luo and Luo [41] researched the finite-time stability of Ψ-Hilfer fractional impulsive delay system. In addition, Zhou et al. [42] and Luo et al. [43] established the existence of solutions and stability results for Ψ-Hilfer fractional system. Under the non-Lipschitz assumption, using the Laplace transform and its inverse, Luo et al. [44] considered a kind of stochastic Hilfer-type fractional system. Under non-local conditions, Gou [12] studied a kind of Hilfer fractional system. For more knowledge about Hilfer fractional calculus, one can refer to [1, 45]. Compared with the references [41–44], in this article, we will investigate the stochastic Ψ-Hilfer fractional system. According to all the studies we known, there are few results on Ψ-Hilfer fractional system driven by random process. In addition, we are particularly interested in the difficulties arising from considering the Ψ function in the analysis of stochastic Hilfer-type fractional system. These provide the main motivations for us to find a new method to investigate the stochastic Ψ-Hilfer fractional system.

Based on the discussions above, in the present article, we will study the following stochastic Ψ-Hilfer fractional system:

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\beta;\Psi}X(t) = AX(t) + \sigma\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right) \\ +g\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right)\dfrac{\mathrm{d}W(t)}{\mathrm{d}t}, t \in J = [0,T], \\ I_{0^+}^{1-\gamma;\Psi}X(0) = 0, \end{cases} \tag{1}$$

where ${}^{H}D_{0^+}^{\alpha,\beta;\Psi}(\cdot)$ and ${}^{H}D_{0^+}^{p,q;\Psi}(\cdot)$ are the fractional Ψ-Hilfer derivative operators with order $\alpha - p > 1/2$, $0 < \alpha, p < 1$ and type $0 \leq \beta, q \leq 1$. $I_{0^+}^{1-\gamma;\Psi}(\cdot)$ represents the fractional Ψ-Hilfer integral operator with order $1-\gamma$, where $\gamma = \alpha + \beta - \alpha\beta$. $A \in \mathbb{R}^n \times \mathbb{R}^n$, $\sigma: J \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and $g: J \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ are continuously differentiable functions. $\{\Omega, \mathcal{F}, \mathbb{P}\}$ is the complete probability space, and $W(t)$ denotes m-dimensional Brownian motion on it.

Up to now, there exist a lot of literatures using the Laplace transform and its inverse to solve Caputo fractional differential equations [27] and the Hilfer fractional system [44, 46], but few literatures have used this kind of technique to solve Ψ-Hilfer fractional system. In this article, we apply a new type generalized Laplace transform and its inverse to solve this kind of stochastic Ψ-Hilfer fractional system. The main contributions and innovations of this article are at least as follows:

(1) Compared with [42], the proposed model in present manuscript is more generalized, in which the random term is considered in Ψ-Hilfer fractional system. There are few literatures available for solving this type of considered system.

(2) By applying the generalized Laplace transform and its inverse, we make the first attempt to construct the form of solutions for stochastic Ψ-Hilfer fractional system. This method is essentially new.

(3) In order to estimate equation $|{}^{H}D_{0^+}^{p,q;\Psi}\mathfrak{R}X(t) - {}^{H}D_{0^+}^{p,q;\Psi}\mathfrak{R}Y(t)|$ in the process of proving the uniqueness of the solution, we construct a Ψ-Riemann–Liouville fractional integral for $\mathfrak{R}X(t)$ at the first step, and then skillfully use its semigroup properties, which greatly simplify our proof.

The vein of this article is developed as follows: In Section 2, some basic definitions and their properties are introduced, which play an indispensable role in the subsequent derivation. Section 3 mainly proves the existence and finite-time stability results for our investigated system. As verification, an example is given to expound the derived conclusions in Section 4.

2. Essential Definitions and Lemmas

For the convenience of reading and for the smooth derivation, some basic definitions and related lemmas are introduced.

Definition 1 (see [45]). If $\Psi(t)$ is a positive and monotonically increasing function on $(a, b]$, $\Psi'(t) \in \mathbb{C}(a, b)$. The fractional ψ-Riemann–Liouville integral can be written as

$$I_{a^+}^{\alpha;\Psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} f(s)\Psi'(s)\mathrm{d}s. \tag{2}$$

Remark 1. Fractional integral operator has the following semigroup property, for $\alpha > 0$ and $\beta > 0$

$$I_{a^+}^{\alpha;\Psi} I_{a^+}^{\beta;\Psi} f(t) = I_{a^+}^{\alpha+\beta;\Psi} f(t). \tag{3}$$

Definition 2 (see [45]). If $f, \Psi \in \mathbb{C}^n([a, b], \mathbb{R})$, Ψ is increasing and $\Psi'(t) \neq 0$, for $\forall t \in [a, b]$. The fractional Ψ -Hilfer derivative of order α and type $0 \leq \beta \leq 1$ can be written as

$$^{H}D_{a^+}^{\alpha,\beta;\Psi} f(t) = I_{a^+}^{\beta(n-\alpha);\Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{\mathrm{d}t} \right)^n I_{a^+}^{(1-\beta)(n-\alpha);\Psi} f(t), n-1 < \alpha < n, n \in \mathbb{N}. \tag{4}$$

Remark 2

(1) If $f \in \mathbb{C}^1[a, b]$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, $(1-\alpha)(1-\beta) = 1-\gamma$, we have

$$I_{a^+}^{\alpha;\Psi} HD_{a^+}^{\alpha,\beta;\Psi} f(t) = f(t) - \frac{(\Psi(t) - \Psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{(1-\beta)(1-\alpha);\Psi} f(a). \tag{5}$$

(2) If $f \in \mathbb{C}^1[a, b]$, $\alpha > 0$ and $0 \leq \beta \leq 1$, then $^{H}D_{a^+}^{\alpha,\beta;\Psi} I_{a^+}^{\alpha;\Psi} f(t) = f(t)$.

Definition 3 (see [47]). We assume $f\colon [0, (\infty) \longrightarrow \mathbb{R}$, and $\Psi(\cdot)$ is a non-negative increasing function satisfying $\Psi(0) = 0$. The generalized the Laplace transform of f can be written as

$$\mathscr{L}_\Psi\{f(t)\} = F(s) = \int_0^\infty e^{-s\Psi(t)} \Psi'(t) f(t)\mathrm{d}t. \tag{6}$$

Definition 4 (see [47]). We assume f and g are piecewise continuous and Ψ-exponential order functions in $[0, T]$. The Ψ-convolution with respect to f and g is defined as

$$(f *_\Psi g)(t) = \int_0^t \left(\Psi^{-1}(\Psi(t) - \Psi(\tau))\right) f g(\tau)\Psi'(\tau)\mathrm{d}\tau. \tag{7}$$

Lemma 1 (see [47]). *We assume $\alpha > 0, n = \lfloor \alpha \rfloor + 1, 0 \leq \beta \leq 1$, and a Ψ-exponential order function $f(\cdot)$ satisfying $f(t)$, $D_{0^+}^{i;\Psi} I_{0^+}^{(1-\beta)(n-\alpha);\Psi} f(t) \in \mathbb{C}[0, \infty)$, where $i = 0, 1, 2, \ldots, n-1$, while $^{H}D_{0^+}^{\alpha,\beta;\Psi} f(t)$ is piecewise continuous on $[0, \infty)$. Then,*

$$\mathscr{L}_\Psi\left\{^{H}D_{0^+}^{\alpha,\beta;\Psi} f(t)\right\} = s^\alpha \mathscr{L}_\Psi\{f(t)\} - \sum_{i=0}^{n-1} s^{n(1-\beta)+\alpha\beta-i-1} \left(I_{0^+}^{(1-\beta)(n-\alpha)-i;\Psi} f \right)(0). \tag{8}$$

Lemma 2 (see [47]). *Let $\alpha > 0$, a Ψ-exponential order function of f which is piecewise continuous on $[0, T]$. Then,*

$$\mathscr{L}_\Psi\left\{\left(I_{0^+}^{\alpha;\Psi} f\right)(t)\right\} = s^{-\alpha} \mathscr{L}_\Psi\{f(t)\}. \tag{9}$$

Lemma 3 (Jensen's inequality [48]). *Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ be real and nonnegative numbers. Then,*

$$\left(\sum_{i=1}^n a_i\right)^p \leq n^{p-1} \sum_{i=1}^n a_i^p, \text{for } p > 1. \tag{10}$$

3. Main Results

In the present section, we shall deduce the existence and uniqueness of solutions for system (1) by applying the contraction mapping principle. Furthermore, finite-time stability results are obtained by means of the properties of norm and inequalities scaling technique. We define the following space:

$$\aleph = \left\{X \mid X(t) \in \mathbb{C}^1(J, \mathbb{R}^n), {}^{H}D_{0^+}^{p,q;\Psi} X(t) \in \mathbb{C}^1(J, \mathbb{R}^n)\right\}, \tag{11}$$

with norm

$$\|X\|_{\aleph} = \max\left\{\mathbb{E}\left(\sup_{t\in J}|X(t)|^2\right), \mathbb{E}\left(\sup_{t\in J}\ |{}^{H}D_{0^+}^{p,q;\Psi}X(t)\ |^2\right)\right\}, \tag{12}$$

where $\mathbb{E}$ is the mathematical expectation. We can readily verify that $(\aleph, \|\cdot\|_{\aleph})$ is a Banach space, and see [49, 50] for more details.

Definition 5 (see [22]). Assuming that there exist positive constants T, δ, ε with $\delta < \varepsilon$, then system (1) is finite-time stable if $\|X_0\|_{\aleph} \le \delta$ implies $\|X\|_{\aleph} \le \varepsilon$ for $\forall t \in [0, T]$, where $X_0 = X(0)$.

Before starting our proof of the main conclusions, we make the following assumptions on the coefficients of system (1):

$[(H_1)]$ As for any $X_i, Y_i \in \mathbb{R}^n$, there exists a positive bounded function $L_1(\cdot)$ satisfying

$$\begin{aligned} &\left|\sigma(t, X_1, Y_1) - \sigma(t, X_2, Y_2)\right|^2 \vee \left|g(t, X_1, Y_1) - g(t, X_2, Y_2)\right|^2 \\ &\quad \le L_1(t)\left(\left|X_1 - Y_1\right|^2 + \left|X_2 - Y_2\right|^2\right). \end{aligned} \tag{13}$$

$[(H_2)]$ We assume that there is a positive constant M such that $\sup_{t\in J}[\Psi'(t)]^2 \le M$.

$[(H_3)]$ For $\forall t \in J$, let $\Xi = \sup_{s\in[0,t]}|E_{\alpha,\alpha}A(\Psi(t) - \Psi(s))^{\alpha}|$, where $E_{\alpha,\alpha}(\cdot)$ denotes the two-parameters Mittag–Leffler function, and see [44] for details.

$[(H_4)]$ As for any $X, Y \in \mathbb{R}^n$, there exists a bounded positive function $L_2(\cdot)$ satisfying

$$|\ \ |^2 \vee |g(t, X, Y)|^2 \le L_2(t)\left(1 + |X|^2 + |Y|^2\right). \tag{14}$$

$|\cdot|$ is the norm of $\mathbb{R}^n$, $a \vee b = \max\{a, b\}$, $\sup_{t\in J} L_i(t) = L_i$, $i = 1, 2$, $t \in J$.

Theorem 1. *We assume that hypotheses* (H_1)–(H_3) *hold, then system (1) has a unique solution in* $(\aleph, \|\cdot\|_{\aleph})$*, if there is a constant C,* $0 < C < 1$*, and*

$$\max\left\{\frac{(T+4)4M\Xi^2 L_1[\Psi(T) - \Psi(0)]^{2\alpha-1}}{2\alpha - 1}, \frac{(T+4)4M\Xi^2 L_1[\Psi(T) - \Psi(0)]^{2(\alpha-p)-1}}{2(\alpha - p) - 1}\right\} \le C. \tag{15}$$

Proof. Taking the generalized Laplace transform on (1), we get the following with the aid of Lemma 1:

$$\mathscr{L}_{\Psi}\{X(t)\} = \frac{\mathscr{L}_{\Psi}\left\{\sigma\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right)\right\}}{s^{\alpha} - A} + \frac{\mathscr{L}_{\Psi}\left\{g\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right)dW(t)/dt\right\}}{s^{\alpha} - A}. \tag{16}$$

Subsequently, using the generalized inverse Laplace transform, we obtain the solution of system (1) is

$$\begin{aligned} X(t) &= E_{\alpha,\alpha}\left(A(\Psi(t))^{\alpha}\right)(\Psi(t))^{\alpha-1} *_{\Psi} \sigma\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right) \\ &\quad + E_{\alpha,\alpha}\left(A(\Psi(t))^{\alpha}\right)(\Psi(t))^{\alpha-1} *_{\Psi} g\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right)\frac{dW(t)}{dt} \\ &= \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t) - \Psi(s))^{\alpha}\right)(\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}s \\ &\quad + \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t) - \Psi(s))^{\alpha}\right)(\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}W(s). \end{aligned} \tag{17}$$

We define operator $\mathfrak{R}\colon (\aleph, \|\cdot\|_{\aleph}) \longrightarrow (\aleph, \|\cdot\|_{\aleph})$ as

$$\begin{aligned}\mathfrak{R}X(t) = &\int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}s \\ &+ \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}W(s).\end{aligned} \tag{18}$$

According to the properties of fractional Ψ-Hilfer derivative and fractional Ψ-Riemann–Liouville integral, it is readily to verify that above operator $\mathfrak{R}$ is well-defined. Moreover, we need to deduce that the operator $\mathfrak{R}$ is a contraction mapping on $\aleph$ for all $X, Y \in \aleph$. For $\forall t \in J$, we get the following by (H_3)

$$\begin{aligned}|\mathfrak{R}X(t) - \mathfrak{R}Y(t)| \le &\left|\int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\cdot\left[\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right) - \sigma\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right]\mathrm{d}s\right| \\ &+ \left|\int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\cdot\left[g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\right.\right. \\ &\left.\left.- g\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right]\mathrm{d}W(s)\right| \\ \le &\,\Xi\left|\int_0^t (\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\cdot\left[\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right) - \sigma\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right]\mathrm{d}s\right| \\ &+ \Xi\left|\int_0^t (\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\cdot\left[g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right) - g\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right]\mathrm{d}W(s)\right|.\end{aligned} \tag{19}$$

Then, by means of Jensen's inequality, Hölder inequality, and Doob's martingale inequality, we get for all $t \in J$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le u\le t}|\mathfrak{R}X(u) - \mathfrak{R}Y(u)|^2\right) \le &\, 2\Xi^2\mathbb{E}\left(\sup_{0\le u\le t}\left|\int_0^u (\Psi(u)-\Psi(s))^{\alpha-1}\Psi'(s)\cdot\left[\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right) - \sigma\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right]\mathrm{d}s\right|^2\right) \\ &+ 2\Xi^2\mathbb{E}\left(\sup_{0\le u\le t}\left|\int_0^u (\Psi(u)-\Psi(s))^{\alpha-1}\Psi'(s)\right.\right. \\ &\left.\left.\cdot\left[g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right) - g\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right]\mathrm{d}W(s)\right|^2\right) \\ \le &\, 2\Xi^2\mathbb{E}\left(\sup_{0\le u\le t}\int_0^u \left[\Psi'(s)\right]^2(\Psi(u)-\Psi(s))^{2\alpha-2}\mathrm{d}s\right. \\ &\left.\cdot\int_0^u \left|\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right) - \sigma\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right|^2\mathrm{d}s\right) \\ &+ 8\Xi^2\mathbb{E}\left(\int_0^t (\Psi(t)-\Psi(s))^{2\alpha-2}\left[\Psi'(s)\right]^2\cdot\left|g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right) - g\left(s, Y(s), {}^{H}D_{0^+}^{p,q;\Psi}Y(s)\right)\right|^2\mathrm{d}s\right) \\ \coloneqq &\, I_1 + I_2.\end{aligned} \tag{20}$$

By assumptions (H_1)-(H_2), we obtain

$$\begin{aligned} I_1 &\leq \frac{2M\Xi^2[\Psi(t)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\cdot\int_0^t L_1(s)\mathbb{E}\Big(|X(s)-Y(s)|^2+ \\ &\quad |{}^H D_{0^+}^{p,q;\Psi}X(s)-{}^H D_{0^+}^{p,q;\Psi}Y(s)|^2\Big)\mathrm{d}s \\ &\leq \frac{2M\Xi^2L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\Big(\sup_{0\leq s_1\leq s}|X(s_1)-Y(s_1)|^2\Big)+ \\ &\quad \mathbb{E}\Big(\sup_{0\leq s_1\leq s}|{}^H D_{0^+}^{p,q;\Psi}X(s_1)-{}^H D_{0^+}^{p,q;\Psi}Y(s_1)|^2\Big)\mathrm{d}s \\ &\leq \frac{2M\Xi^2L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}2T\|X-Y\|_{\aleph}. \end{aligned} \tag{21}$$

Similarly, one can obtain

$$\begin{aligned} I_2 &\leq 8\Xi^2\mathbb{E}\bigg(\int_0^t(\Psi(t)-\Psi(s))^{2\alpha-2}\Big[\Psi'(s)\Big]^2L_1(s) \\ &\quad \cdot\mathbb{E}\Big(|X(s)-Y(s)|^2+|{}^H D_{0^+}^{p,q;\Psi}X(s)-{}^H D_{0^+}^{p,q;\Psi}Y(s)|^2\Big)\mathrm{d}s\bigg) \\ &\leq 8\Xi^2L_1\int_0^t(\Psi(t)-\Psi(s))^{2\alpha-2}\Big[\Psi'(s)\Big]^2\Big[\mathbb{E}\Big(\sup_{0\leq s_1\leq s}|X(s_1)-Y(s_1)|^2\Big) \\ &\quad +\mathbb{E}\Big(\sup_{0\leq s_1\leq s}|{}^H D_{0^+}^{p,q;\Psi}X(s_1)-{}^H D_{0^+}^{p,q;\Psi}Y(s_1)|^2\Big) \\ &\leq 8\Xi^2L_1\int_0^t(\Psi(t)-\Psi(s))^{2\alpha-2}\Big[\Psi'(s)\Big]^2 2\|X-Y\|_{\aleph}\mathrm{d}s \\ &\leq \frac{8M\Xi^2L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}2\|X-Y\|_{\aleph}. \end{aligned} \tag{22}$$

Then, it is easy to obtain

$$\begin{aligned} &\mathbb{E}\Big(\sup_{0\leq u\leq t}|\mathfrak{R}X(u)-\mathfrak{R}Y(u)|^2\Big) \\ &\leq \frac{4TM\Xi^2L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\|X-Y\|_{\aleph} \\ &\quad +\frac{16M\Xi^2L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\|X-Y\|_{\aleph} \\ &= \frac{(T+4)4M\Xi^2L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\|X-Y\|_{\aleph} \\ &\leq C\|X-Y\|_{\aleph}. \end{aligned} \tag{23}$$

On the contrary, we have

$$
\begin{aligned}
\mathfrak{R}X(t) \leq \Xi\Gamma(\alpha)\left[\frac{1}{\Gamma(\alpha)}\int_0^t (\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s) \mid \sigma\left(s, X(s), {}^H D_{0^+}^{p,q;\Psi} X(s)\right) \mid \mathrm{d}s\right] \\
+ \Xi\Gamma(\alpha)\left[\frac{1}{\Gamma(\alpha)}\int_0^t (\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s) \mid g\left(s, X(s), {}^H D_{0^+}^{p,q;\Psi} X(s)\right) \mid \mathrm{d}W(s)\right] \\
= \Xi\Gamma(\alpha) I_{0^+}^{\alpha;\Psi} \mid \sigma\left(t, X(t), {}^H D_{0^+}^{p,q;\Psi} X(t)\right) \mid + \Xi\Gamma(\alpha) I_{0^+}^{\alpha;\Psi} \mid g\left(t, X(t), {}^H D_{0^+}^{p,q;\Psi} X(t)\right) \mid \frac{\mathrm{d}W(t)}{\mathrm{d}t}.
\end{aligned}
\tag{24}
$$

Then, with the aid of semigroup property introduced in Remark 1, we can readily derive the following:

$$
\begin{aligned}
&\mid {}^H D_{0^+}^{p,q;\Psi} \mathfrak{R}X(t) - {}^H D_{0^+}^{p,q;\Psi} \mathfrak{R}Y(t) \mid \\
&= \mid {}^H D_{0^+}^{p,q;\Psi}\Bigg(\int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s) \\
&\quad \cdot \left[\sigma\left(s, X(s), {}^H D_{0^+}^{p,q;\Psi} X(s)\right) - \sigma\left(s, Y(s), {}^H D_{0^+}^{p,q;\Psi} Y(s)\right)\right]\mathrm{d}s \\
&\quad + \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s) \\
&\quad \cdot \left[g\left(s, X(s), {}^H D_{0^+}^{p,q;\Psi} X(s)\right) - g\left(s, Y(s), {}^H D_{0^+}^{p,q;\Psi} Y(s)\right)\right]\mathrm{d}W(s)) \mid \\
&\leq \mid {}^H D_{0^+}^{p,q;\Psi}(\Xi)\Gamma(\alpha) I_{0^+}^{\alpha;\Psi}\left[\mid \sigma\left(t, X(t), {}^H D_{0^+}^{p,q;\Psi} X(t)\right) - \sigma\left(t, Y(t), {}^H D_{0^+}^{p,q;\Psi} Y(t)\right) \mid\right] \\
&\quad + \Xi\Gamma(\alpha) I_{0^+}^{\alpha;\Psi}\left[\mid g\left(t, X(t), {}^H D_{0^+}^{p,q;\Psi} X(t)\right) - g\left(t, Y(t), {}^H D_{0^+}^{p,q;\Psi} Y(t)\right) \mid\right]\frac{\mathrm{d}W(t)}{\mathrm{d}t}) \mid \\
&\leq \mid \Xi\Gamma(\alpha) I_{0^+}^{\alpha-p;\Psi}\left[\mid \sigma\left(t, X(t), {}^H D_{0^+}^{p,q;\Psi} X(t)\right) - \sigma\left(t, Y(t), {}^H D_{0^+}^{p,q;\Psi} Y(t)\right) \mid\right] \mid \\
&\quad + \mid \Xi\Gamma(\alpha) I_{0^+}^{\alpha-p;\Psi}\left[\mid g\left(t, X(t), {}^H D_{0^+}^{p,q;\Psi} X(t)\right) - g\left(t, Y(t), {}^H D_{0^+}^{p,q;\Psi} Y(t)\right) \mid\right]\frac{\mathrm{d}W(t)}{\mathrm{d}t} \mid \\
&= \Xi \mid \int_0^t (\Psi(t)\Psi(s))^{(\alpha-p)-1}\Psi'(s) \\
&\quad \cdot \left[\mid \sigma\left(s, X(s), {}^H D_{0^+}^{p,q;\Psi} X(s)\right) - \sigma\left(s, Y(s), {}^H D_{0^+}^{p,q;\Psi} Y(s)\right) \mid\right]\mathrm{d}s \mid \\
&\quad + \Xi \mid \int_0^t (\Psi(t)-\Psi(s))^{(\alpha-p)-1}\Psi'(s) \\
&\quad \cdot \left[\mid g\left(s, X(s), {}^H D_{0^+}^{p,q;\Psi} X(s)\right) - g\left(s, Y(s), {}^H D_{0^+}^{p,q;\Psi} Y(s)\right) \mid\right]\mathrm{d}W(s) \mid .
\end{aligned}
\tag{25}
$$

Similarly, by means of Jensen's inequality, Hölder inequality, and Doob's martingale inequality, we get

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t} \mid {}^{H}D_{0^+}^{p,q;\Psi}\mathfrak{R}X(u)-{}^{H}D_{0^+}^{p,q;\Psi}\mathfrak{R}Y(u)\mid^2\right)\\&\le\frac{(T+4)4M\Xi^2L_1[\Psi(T)-\Psi(0)]^{2(\alpha-p)-1}}{2(\alpha-p)-1}\|X-Y\|_{\aleph}\\&\le C\|X-Y\|_{\aleph}.\end{aligned}\tag{26}$$

Therefore, by the definition of $\|\cdot\|_{\aleph}$, for $\forall X,Y\in\aleph$

$$\|\mathfrak{R}X(t)-\mathfrak{R}Y(t)\|_{\aleph}\le C\|X-Y\|_{\aleph}<\|X-Y\|_{\aleph}, t\in J,\tag{27}$$

then $\mathfrak{R}$ is a contraction mapping within $(\aleph,\|\cdot\|_{\aleph})$, which implies that $\mathfrak{R}$ has a fixed point. Therefore, system (1) has a unique solution. □

Theorem 2. *We assume that the assumptions* (H_2)–(H_4) *hold, and there exist positive constants* δ,ε *satisfying* $\delta<\varepsilon$ *and* $\|X_0\|_{\aleph}\le\delta$. *Then, system (1) is finite-time stable on* $[0,T]$, *provided that*

$$K\le\frac{\varepsilon}{1+2\varepsilon},\tag{28}$$

where

$$K=\max\left\{(T+4)2M\Xi^2L_2\frac{[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1},(T+4)2M\Xi^2L_2\frac{[\Psi(T)-\Psi(0)]^{2(\alpha-p)-1}}{2(\alpha-p)-1}\right\}.\tag{29}$$

Proof. From Theorem 1, system (1) has a unique solution that has the following form:

$$\begin{aligned}X(t)=&\int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\sigma\left(s,X(s),{}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}s\\&+\int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)g\left(s,X(s),{}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}W(s).\end{aligned}\tag{30}$$

By applying Jensen's inequality, Hölder inequality, and Doob's martingale inequality, we have for $\forall t\in J$

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t}|X(u)|^2\right)\\&\le 2\mathbb{E}\left(\sup\nolimits_{0\le u\le t}\left|\int_0^u E_{\alpha,\alpha}\left(A(\Psi(u)-\Psi(s))^{\alpha}\right)(\Psi(u)-\Psi(s))^{\alpha-1}\cdot\Psi'(s)\sigma\left(s,X(s),{}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}s\right|^2\right)\\&\quad+2\mathbb{E}\left(\sup\nolimits_{0\le u\le t}\left|\int_0^u E_{\alpha,\alpha}\left(A(\Psi(u)-\Psi(s))^{\alpha}\right)(\Psi(u)-\Psi(s))^{\alpha-1}\cdot\Psi'(s)g\left(s,X(s),{}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\mathrm{d}W(s)\right|^2\right)\\&\le 2\mathbb{E}\left(\sup\nolimits_{0\le u\le t}\int_0^u E_{\alpha,\alpha}\left(A(\Psi(u)-\Psi(s))^{\alpha}\right)^2(\Psi(u)-\Psi(s))^{2\alpha-2}\left[\Psi'(s)\right]^2\mathrm{d}s\right.\\&\quad\left.\cdot\int_0^u\left|\sigma\left(s,X(s),{}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\right|^2\mathrm{d}s\right)\\&\quad+8\mathbb{E}\left(\int_0^t\left[E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)\right]^2(\Psi(t)-\Psi(s))^{2\alpha-2}\left[\Psi'(s)\right]^2\cdot\left|g\left(s,X(s),{}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)\right|^2\mathrm{d}s\right)\coloneqq I_3+I_4.\end{aligned}\tag{31}$$

By assumptions (H_2)–(H_4), we obtain

$$
\begin{aligned}
I_3 &\le 2M\Xi^2\frac{[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^t |\sigma\left(s, X(s), {}^H D_{0^+}^{p,q;\Psi}X(s)\right)|^2 \mathrm{d}s\right)\\
&\le 2M\Xi^2\frac{[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\int_0^t L_2(s)\mathbb{E}\left(1+|X(s)|^2+|{}^H D_{0^+}^{p,q;\Psi}X(s)|^2\right)\mathrm{d}s\\
&\le 2M\Xi^2 L_2\frac{[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\int_0^t \left(1+2\|X\|_{\aleph}\right)\mathrm{d}s\\
&\le 2TM\Xi^2 L_2\frac{[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\left(1+2\|X\|_{\aleph}\right).
\end{aligned} \tag{32}
$$

Similarly, we obtain

$$
\begin{aligned}
I_4 &\le 8\Xi^2\int_0^t (\Psi(t)-\Psi(s))^{2\alpha-2}\left[\Psi'(s)\right]^2 L_2(s)\mathbb{E}\left(1+|X(s)|^2+|{}^H D_{0^+}^{p,q;\Psi}X(s)|^2\right)\mathrm{d}s\\
&\le 8\Xi^2 L_2\int_0^t (\Psi(t)-\Psi(s))^{2\alpha-2}\left[\Psi'(s)\right]^2\left(1+2\|X\|_{\aleph}\right)\mathrm{d}s\\
&\le 8M\Xi^2 L_2\frac{[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\left(1+2\|X\|_{\aleph}\right).
\end{aligned} \tag{33}
$$

Therefore, we obtain

$$
\mathbb{E}\left(\sup_{0\le u\le t}|X(u)|^2\right)\le (T+4)2M\Xi^2 L_2\frac{[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}\left(1+2\|X\|_{\aleph}\right). \tag{34}
$$

On the other hand, we can readily derive the following:

$$
\mathbb{E}\left(\sup_{0\le u\le t}|{}^H D_{0^+}^{p,q;\Psi}X(u)|^2\right)\le (T+4)2M\Xi^2 L_2\frac{[\Psi(T)-\Psi(0)]^{2(\alpha-p)-1}}{2(\alpha-p)-1}\left(1+2\|X\|_{\aleph}\right). \tag{35}
$$

According to the definition of $\|\cdot\|_{\aleph}$, we have

$$
\|X\|_{\aleph}\le K\left(1+2\|X\|_{\aleph}\right), \tag{36}
$$

then by the conditions of Theorem 2, yields

$$
\|X\|_{\aleph}\le \varepsilon, \tag{37}
$$

which implies that system (1) is finite-time stable on $[0, T]$. □

Remark 3. Taking the initial value $I_{0^+}^{1-\gamma;\Psi}X(0)=0$ is somewhat strict. One can let the initial value to $I_{0^+}^{1-\gamma;\Psi}X(0)=X_0$ or $X(0)=X_0$. However, when performing stability estimation, we will meet the following difficulties.

(1) Taking the initial value to $I_{0^+}^{1-\gamma;\Psi}X(0)=X_0\neq 0$ and applying the generalized Laplace transform and its inverse, we derive that system has the following solution:

$$X(t) = E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(0))^{\alpha}\right)[\Psi(t)-\Psi(0)]^{\gamma-1}X_0 + \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)ds + \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)dW(s). \quad (38)$$

We will find that the first term is a singular function, then the estimation of $\mathbb{E}(\sup_{0\le u\le t} |X(u)|^2)$, for all $t \in J$, will be unbounded.

(2) Taking the initial value to $X(0) = X_0$, similarly, the system has the following solution:

$$X(t) = \frac{X_0}{\Gamma(\gamma)\Gamma(2-\gamma)}E_{\alpha}\left(A(\Psi(t)-\Psi(0))^{\alpha}\right) + \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\sigma\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)ds + \int_0^t E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)g\left(s, X(s), {}^{H}D_{0^+}^{p,q;\Psi}X(s)\right)dW(s). \quad (39)$$

In order to estimate $\mathbb{E}(\sup_{0\le u\le t} |{}^{H}D_{0^+}^{p,q;\Psi}X(u)|^2)$, we need to take the Ψ-Hilfer fractional derivative of the first term, which will produce a singular function $\Lambda[\Psi(t)-\Psi(0)]^{-\alpha}$, where Λ represents a constant. This prevents us from considering the stability analysis.

4. Example

We consider the following stochastic Ψ-Hilfer fractional system.

Example 1

$$\begin{cases} {}^{H}D_{0^+}^{0.8,0.5;\sqrt{0.1}t}X(t) = AX(t) + 0.1e^{-t}X(t) + 0.1\,{}^{H}D_{0^+}^{0.2,0.5;\sqrt{0.1}t}X(t) \\ +\left[0.1\cos(t)X(t) + 0.1\sin(t)\,{}^{H}D_{0^+}^{0.2,0.5;\sqrt{0.1}t}X(t)\right]\dfrac{dW(t)}{dt}, t \in [0,10], \\ I_{0^+}^{0.1;\sqrt{0.1}t}X(0) = 0. \end{cases} \quad (40)$$

We know that

$$\begin{aligned} \sigma\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right) &= 0.1e^{-t}X(t) + 0.1\,{}^{H}D_{0^+}^{0.2,0.5;\sqrt{0.1}t}X(t), \\ g\left(t, X(t), {}^{H}D_{0^+}^{p,q;\Psi}X(t)\right) &= 0.1\cos(t)X(t) + 0.1\sin(t)\,{}^{H}D_{0^+}^{0.2,0.5;\sqrt{0.1}t}X(t). \end{aligned} \quad (41)$$

We can readily calculate that $L_1 = 0.02$, $L_2 = 0.02$. Taking $M = 0.1$, $C = 0.8$, $\delta = 0.1$, $\varepsilon = 1.5$, and $A = 0.2\mathcal{I}$, where $\mathcal{I}$ denotes the identity matrix, using mathematical software, we calculate the following results:

$$\begin{aligned}&\Xi = \sup_{s\in[0,t]}\left|E_{\alpha,\alpha}\left(A(\Psi(t)-\Psi(s))^{\alpha}\right)\right| = 1.119, t \in [0,10],\\ &\cdot \max\left\{\frac{(T+4)4M\Xi^2 L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}, \frac{(T+4)4M\Xi^2 L_1[\Psi(T)-\Psi(0)]^{2(\alpha-p)-1}}{2(\alpha-p)-1}\right\} = 0.7012 < C,\\ &K = 0.3506, K < \frac{\varepsilon}{1+2\varepsilon} = 0.375.\end{aligned} \tag{42}$$

Then, it can be easily verified that all the conditions in Theorem 1 and 2 are satisfied. Therefore, system (40) has a unique solution, and the norm of solution will not exceed the given bound $\varepsilon = 1.5$ over the finite time interval [0, 10]. Then, we can conclude that system (40) is finite-time stable on [0, 10].

Remark 4. When the system evolves beyond this given finite time domain, for example, let us consider $T = 12$. By simple calculation, we can obtain $\Xi = 1.167$ and $C = 0.95$ and the other values are the same as in Example 1, and the following results are obtained:

$$\max\left\{\frac{(T+4)4M\Xi^2 L_1[\Psi(T)-\Psi(0)]^{2\alpha-1}}{2\alpha-1}, \frac{(T+4)4M\Xi^2 L_1[\Psi(T)-\Psi(0)]^{2(\alpha-p)-1}}{2(\alpha-p)-1}\right\} = 0.904 < C. \tag{43}$$

It is verified that system (40) has a unique solution, according to Theorem 1. However, $K = 0.452 > \varepsilon/1 + 2\varepsilon = 0.375$; then, with the aid of Theorem 2, we conclude that system (40) is not finite time stable on [0, 12].

5. Conclusion

In this article, we study a new kind of stochastic Ψ-Hilfer fractional system and apply the generalized Laplace transform and its inverse to solve this kind of system. We have established existence and uniqueness theories as well as finite-time stability results for the solutions of the considered problem. The nonlinear analysis method we used is essentially new, and yet there are few literatures available for solving this type of considered system. The obtained results have been expounded via an example.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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