Nonpolynomial Spline Method for Singularly Perturbed Time-Dependent Parabolic Problem with Two Small Parameters

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Abstract

This study deals with the numerical solution of parabolic convection-diffusion problems involving two small positive parameters and arising in modeling of hydrodynamics. To approximate the solution, the backward Euler method for time stepping and fitted trigonometric-spline scheme for spatial discretization are considered on uniform meshes. The resulting scheme is shown to be uniformly convergent and its rate of convergence is one in the time variable and two in the space variable. The accuracy and rate of convergence are enhanced by using the Richardson extrapolation. To support the theoretically shown convergence analysis, we have taken some numerical examples and compared the absolute maximum error of the current method with some methods existing in the literature.

1. Introduction

Consider the following time-dependent singularly perturbed one-dimensional parabolic convection-diffusion problem with initial and boundary conditions of Dirichlet type on the domain \( D = \Omega \times (0, T], \Omega = (0, 1) \).

\[
L_{\varepsilon, \mu} u = \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} - \mu a(x, t) \frac{\partial u}{\partial x} + b(x, t)u = f(x, t), (x, t) \in D, \tag{1}
\]

constrained to boundary conditions

\[
u(x, 0) = s(x), x \in \bar{\Omega},
\]

\[
u(0, t) = 0 = u(1, t), \quad t \in [0, T], \tag{2}
\]

where \( 0 < \varepsilon \) and \( \mu \ll 1 \) are two small positive parameters. We assume that the coefficients \( a(x, t), b(x, t), \) and \( f(x, t) \) are sufficiently regular functions with the conditions \( a(x, t) \geq \alpha > 0 \) and \( b(x, t) \geq \beta > 0 \) for all \( (x, t) \in D \). We also assume that the given data satisfy sufficient smoothness and compatibility conditions on the corner of the domains \((0, 0)\) and \((1, 0)\), i.e., \( s(0) = u(0, 0) \) and \( s(1) = u(1, 0) \). These conditions confirm the existence of a unique solution to the problem.

Physical occurrences, where energy, heat, or fluid is transformed inside a physical system due to the movement of molecules within the system (convection) and the spread of particles through the random motion from regions of higher concentration to regions of lower concentration (diffusion) are described by parabolic partial differential equations called convection-diffusion problems. In the case when the spatial derivatives are multiplied by small positive parameters (such as the partial differential equations of hydrodynamics), the problems are said to be singularly perturbed problems (SPPs) [1, 2]. The solution to these problems can have a steep gradient where boundary layers occur [3]. Using an analytic method or asymptotic expansions may be impossible to construct a solution or may fail to simplify the given problem. As a result, numerical approximations are the only option.

Classical numerical methods are inefficient to approximate their solution as the perturbation parameter goes to zero. In recent years, more due attention has been paid by

In this section, we consider some properties of the continuous problem through maximum principle and bounds on the solution and on its derivatives that enable us to see a priori estimates for the solution \( u(x,t) \) and its derivatives.

**2. Properties of the Continuous Problem**

In this section, we consider some properties of the continuous problem through maximum principle and bounds on the solution and on its derivatives that enable us to see a priori estimates for the solution \( u(x,t) \) and its derivatives.

**Lemma 1** (Maximum principle). Let \( z(x,t) \in C^{2,1}(\mathcal{D}) \). If \( z(x,t) \geq 0, \) for all \( (x,t) \) on the boundary \( \partial \mathcal{D} \) of the domain \( D \), and \( b_{ca} \geq 0 \), for all \( (x,t) \in D \) then \( z(x,t) \geq 0 \), for all \( (x,t) \in \mathcal{D} \).

**Proof.** The proof is given in [16]. □

**Lemma 2** (Bounds of the solution). Let \( u(x,t) \) be the solution of equation (1) and \( b(x,t) \geq \beta > 0 \), then we have the estimate

\[
\|u\| \leq \beta^{-1} \|f\| + \max |s(x)|, \tag{3}
\]

where \( \| \cdot \| \) is the maximum norm.

**Proof.** The proof is given in [16]. □

**Lemma 3** (Bounds of the derivatives). The derivatives of the solution \( u(x,t) \) satisfy the following bound for all nonnegative integers \( i, j \) such that \( i + 3j \leq 4 \).

\[
\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\| \leq \begin{cases} \epsilon^{-i/2}, & \mu^2 \leq Ce, \\ \left( \frac{\mu^i}{\epsilon^j} \right), & \mu^2 \geq Ce, \end{cases} \tag{4}
\]

where the constant \( C \) is independent of \( \epsilon \) and \( \mu \) and depends only on the bounded derivatives of the coefficients and the source term.

**Proof.** The proof is given in [25]. □

**3. Construction of the Method**

### 3.1. Temporal Discretization

For the semidiscretization of the IBVP (1), we use the backward Euler method with uniform time-steps \( k = T/M \) in the time-interval \([0,T]\). This gives a time-mesh \( D^h = \{ t_m = k(m-1), m = 1, 2, \ldots, M + 1 \} \) and

\[
\begin{align*}
&u(x,t_m) - u(x,t_{m-1}) - \epsilon \frac{d^2 u(x,t_m)}{dx^2} - \mu a(x,t_m) \frac{du(x,t_m)}{dx} \\
&+ b(x,t_m)u(x,t_m) = f(x,t_m).
\end{align*}
\]

In the \((m)\)th time step, the semidiscrete solution \( u(x,t_m) \) of equation (1) satisfies the following differential equation of the space variable \( x \).
\[ L_{\epsilon}^M u(x, t_m) = -\epsilon \frac{d^2 u(x, t_m)}{dx^2} - \mu a(x, t_m) \frac{du(x, t_m)}{dx} + q(x, t_m)u(x, t_m) = H(x, t_m), \]

where \( H(x, t_m) = f(x, t_m) + 1/ku(x, t_m-1), \quad q(x, t_m) = b(x, t_m) + 1/k. \)

**Lemma 4** (Semidiscrete maximum principle). At mth time level, let \( \theta(x, t_m) \in C^2(\Omega) \). If \( \theta(0, t_m) \geq 0, \theta(1, t_m) \geq 0, \) and \( L_{\epsilon}^M \theta(x, t_m) \geq 0, \forall x \in \Omega, \) then \( \theta(x, t_m) \geq 0, \forall x \in \Omega. \)

**Proof.** Let \( (x^*, t_m) \in \{ (x, t_m): x \in \Omega \} \) such that \( \theta(x^*, t_m) = \min_{x \in \Omega} \theta(x, t_m), \)

suppose that \( \theta(x^*, t_m) < 0. \) Then, we have \( (x^*, t_m) \notin \{ (0, t_m), (1, t_m) \} \) and from derivative tests \( \partial^2 \theta(x^*, t_m)/dx^2 \geq 0. \) Then, \( L_{\epsilon}^M \theta(x^*, t_m) < 0, \) which contradicts the hypothesis. Hence, it follows that \( \theta(x, t_m) \geq 0, \forall x \in \Omega. \)

Now, it is observed that the operator \( L_{\epsilon}^M \) satisfies the semidiscrete maximum principle; hence, by [4] the semidiscrete scheme (6) is stable. Furthermore, in order to analyze the convergence of the semidiscrete scheme, let us see the following error estimate.

**Lemma 5** (Error estimate). Suppose \( u(x, t_m) \) and \( U(x, t_m) \) be solutions of equations (1) and (6), respectively. If \( \| \partial^i u(x, t)/\partial t^i \| \leq C, 0 \leq i \leq 2, \) then the local error in the temporal direction \( e_m(x) = u(x, t_m) - U(x, t_m) \) satisfies

\[ L_{\epsilon}^M U(x, t_m) = L_{\epsilon}^M u(x, t_m) - k \left( \frac{\partial u}{\partial t}(x, t_m-1) + \tau_1 \right) + \int_{t_m}^{t_{m+1}} \frac{\partial u}{\partial t}(x, \xi) d\xi, \]

Simplifying further, we can arrive at

\[ L_{\epsilon}^M e_m(x) = k \tau_1 = \frac{k^2}{2} \frac{\partial^2 u}{\partial t^2} (x, t_m-1), \]

from which we can deduce

\[ \| e_m \| \leq C(k^2). \]

Moreover, the global error is given by

\[ \| e_m(x) \| \leq C(k^2), \]

and the global error satisfies

\[ \| E_m \| = \max m \| e_m \| \leq C(k), \]

with a constant C.

**Proof.** From Taylor’s series expansion we have

\[ \frac{u(x, t_m) - u(x, t_{m-1})}{k} = \partial u(x, t_{m-1}) + \tau_1, \]

\[ \tau_1 = \frac{k}{2} \frac{\partial^2 u}{\partial t^2} (x, t_{m-1}) + O(k^2). \]

Since \( u(x, t_m) \) is also expected to satisfy equation (6), we have

\[ H(x, t_m) = L_{\epsilon}^M u(x, t_m) = L_{\epsilon}^M U(x, t_m) \]

\[ -k \left( \frac{u(x, t_m) - u(x, t_{m-1})}{k} \right) + (u(x, t_m) - u(x, t_{m-1})), \]

and \( U(x, t_m) \) satisfies equation (6), then we can write it as

\[ \| E_m \| = \sum_{i=1}^m \| e_i \| \leq \| e_1 \| + \| e_2 \| + \cdots + \| e_m \| \]

\[ \leq mC_1(k^2), \text{ but } mk \leq T \]

\[ \leq C_1(Tk) \leq C(k). \]

Therefore, the time semidiscretization process is uniformly convergent of order one. \( \square \)
Lemma 6. Let $b(x, t_m)/(x, t_m) \alpha \geq \delta > 0$, then the solution $U(x, t_m)$ of the boundary-value problem equation (6) satisfies the estimate

$$|U^{(j)}(x, t_m)| \leq C \left( 1 + \mu^j \exp \left( -\delta \left( 1 - \frac{x}{\mu} \right) \right) \right), \quad x \in \Omega,$$

for all $1 \leq j \leq 4$.

Proof. For the proof, one can see [26].

3.2. Spatial Discretization. We consider a uniform space mesh $D^n_N$ with nodal point $x_n$ on $[0, 1]$ such that $D^n_N = \{0 = x_1 < x_2 < \ldots, < x_{N+1} = 1 \}$ with $x_n = (n-1) h, h = 1/N, n = 1, 2, \ldots, N + 1$.

Let us introduce a fitting factor $\sigma$ which controls the effect of the perturbation parameter $\epsilon$. Then, at $x = x_n$, equation (6) can be written as

$$\begin{align*}
-\sigma \epsilon \frac{d^2 u(x_n, t_m)}{dx^2} - \mu a_n (x_n, t_m) \frac{d u(x_n, t_m)}{dx} + q(x_n, t_m) u(x_n, t_m) &= H(x_n, t_m),
\end{align*}$$

(17)

Let $U(x_n, t_m)$ be an approximate solution to $u(x_n, t_m)$ of equation (17) obtained by the segment $Q_n(x, t_m)$ passing through the points $(x_n, t_m, U(x_n, t_m))$ and $(x_{n+1}, t_m, U(x_{n+1}, t_m))$. The segment is supposed to satisfy the interpolation condition, and its first-order derivative is continuous at the interior nodes. Assume each segment has the form

$$Q_n(x, t_m) = a_n \sin \left( \tau (x - x_n) \right) + b_n \cos \left( \tau (x - x_n) \right) + c_n (x - x_n) + d_n,$$

(18)

where $a_n, b_n, c_n$ and $d_n$ are constants and $\tau$ is a free parameter used to maintain consistency of the method. Let us denote

$$U^n_n = Q_n(x_n, t_m),$$

$$U^n_{n+1} = Q_n(x_{n+1}, t_m),$$

$$W^n_n = Q''_n(x_n, t_m) = \frac{d^2 U^n_n}{dx^2},$$

$$W^n_{n+1} = Q''_n(x_{n+1}, t_m) = \frac{d^2 U^n_{n+1}}{dx^2}.$$  

(19)

Substituting equations (18) into (19) and assigning $\tau = \tau h$, we can have

$$\begin{align*}
b_n + d_n &= U^n_n, \\
a_n \sin \left( \theta \right) + b_n \cos \left( \theta \right) + c_n h + d_n &= U^n_{n+1}, \\
-a_n \tau^2 \sin \left( \theta \right) - b_n \tau^2 \cos \left( \theta \right) &= W^n_{n+1}.
\end{align*}$$

(20)

Solving equation (20) for the coefficients we get

$$\begin{align*}
a_n &= \frac{h^2 \left( \cos \left( \theta \right) W^n_n - W^n_{n+1} \right)}{\theta^2}, \\
c_n &= \frac{U^n_{n+1} - U^n_n}{h} + \frac{h}{\theta^2} \left( W^n_{n+1} - W^n_n \right), \\
b_n &= -\frac{h^2 W^n_n}{\theta^2}, \\
d_n &= \frac{U^n_{n+1} + \frac{h}{\theta^2} W^n_{n+1}}{h}.
\end{align*}$$

(21)

From the continuity of first derivative, we have

$$Q'_{n-1}(x, t_m) = Q'_{n-1}(x, t_m),$$

where

$$Q_{n-1}(x, t_m) = a_{n-1} \sin \left( \tau (x - x_{n-1}) \right) + b_{n-1} \cos \left( \tau (x - x_{n-1}) \right) + c_{n-1} (x - x_{n-1}) + d_{n-1}. $$

(22)

Then,

$$\tau a_n + c_n = \tau a_{n-1} \cos \left( \theta \right) - \tau b_{n-1} \sin \left( \theta \right) + c_{n-1}. $$

(23)

from which by simple arithmetic manipulation, we get

$$\begin{align*}
U^n_{n-1} - 2U^n_n + U^n_{n+1} &= h^2 \left( \lambda_1 W^n_{n-1} + \lambda_2 W^n_n + \lambda_3 W^n_{n+1} \right), \\
\end{align*}$$

where $\lambda_1 = (\theta - \sin \left( \theta \right))/\theta^2 \sin \left( \theta \right), \lambda_2 = 2 \left( \sin \theta - \theta \cos \theta \right)/\theta^2 \sin \theta$. Applying L'Hospital’s rule as $\theta \to 0$, i.e., $\theta \to 0$ yields a scheme matching that of the ordinary cubic spline in $\Omega$, since

$$\begin{align*}
\lim_{\theta \to 0} \lambda_1 \left( \theta \right) &= \frac{1}{6}, \\
\lim_{\theta \to 0} \lambda_2 \left( \theta \right) &= \frac{4}{6}.
\end{align*}$$

(24)

(25)
The consistency relation for equation (24) leads us to choose \( \lambda_1 \) and \( \lambda_2 \) in such a way that it satisfies the relation
\( 2\lambda_1 + \lambda_2 = 1 \) [19, 20]. Taking the Taylor expansion of \( U^m(x) \) and its first derivative about \( x_n \) only up to second-order gives
\[
U^m_{n+1} = U^m_n + h \frac{dU^m_n}{dx} + \frac{h^2}{2} \frac{d^2U^m_n}{dx^2},
\]
(26)

Similarly,
\[
U^m_{n-1} = U^m_n - h \frac{dU^m_n}{dx} + \frac{h^2}{2} \frac{d^2U^m_n}{dx^2}.
\]

From equation (26), we get
\[
\frac{dU^m_n}{dx} = \frac{U^m_{n+1} - U^m_{n-1}}{2h} \frac{d^2U^m_n}{dx^2} = \frac{U^m_{n+1} - 2U^m_n + U^m_{n-1}}{h^2}.
\]\n(28)

Substituting equations (28) into (27) gives
\[
\frac{dU^m_n}{dx} = \frac{3U^m_{n+1} - 4U^m_n + U^m_{n-1}}{2h} \frac{dU^m_n}{dx} = \frac{-U^m_{n+1} + 4U^m_n - 3U^m_{n-1}}{2h},
\]
(29)

and from equation (17) using spline’s second derivatives, we have
\[
\frac{d^2U^m_n}{dx^2} = \frac{1}{\sigma \varepsilon} \left[ -\mu a^m_n \frac{dU^m_n}{dx} + q^m_n U^m_n - H^m_n \right],
\]
(30)

\[
W^m_n = \frac{1}{\sigma \varepsilon} \left[ -\mu a^m_n \left( \frac{U^m_{n+1} - U^m_{n-1}}{2h} \right) + q^m_n U^m_n - H^m_n \right],
\]
(31)

\[
W^m_{n+1} = \frac{1}{\sigma \varepsilon} \left[ -\mu a^m_n \left( \frac{3U^m_{n+1} - 4U^m_n + U^m_{n-1}}{2h} \right) + q^m_{n+1} U^m_{n+1} - H^m_{n+1} \right],
\]
(32)

\[
W^m_{n-1} = \frac{1}{\sigma \varepsilon} \left[ -\mu a^m_n \left( \frac{-U^m_{n+1} + 4U^m_n - 3U^m_{n-1}}{2h} \right) + q^m_{n-1} U^m_{n-1} - H^m_{n-1} \right].
\]
(33)

Substituting equations (31)-(33) into (24), we arrive at the following difference scheme:
\[
L_{\varepsilon \mu}^{N,M} U^m_n = A^m_n U^m_{n+1} + A^m_n U^m_n + A^m_n U^m_{n-1} = F^m_n, n = 2, \ldots, N - 1, m = 1, \ldots, M + 1,
\]
(34)

where
\[
A^m_n = \frac{\mu h}{2\sigma \varepsilon} (3\lambda_1 a^m_{n-1} + \lambda_2 a^m_{n-1} - \lambda_1 a^m_{n+1}) + \frac{\lambda_1 h^2}{\sigma \varepsilon} q^m_{n-1} - 1,
\]
\[
A^m_n = \frac{2\mu h}{\sigma \varepsilon} (a^m_{n-1} + a^m_{n+1}) + \frac{\lambda_1 h^2}{\sigma \varepsilon} q^m_n + 2,
\]
\[
A^m_n = \frac{\mu h}{2\sigma \varepsilon} (\lambda_1 a^m_{n-1} - \lambda_2 a^m_{n} - 3\lambda_1 a^m_{n+1}) + \frac{\lambda_1 h^2}{\sigma \varepsilon} q^m_{n+1} - 1,
\]
\[
F^m_n = \frac{h^2}{\sigma \varepsilon} (\lambda_1 H^m_{n-1} + \lambda_2 H^m_{n} + \lambda_1 H^m_{n+1}).
\]
(35)

Therefore, the required scheme developed in equation (34) is termed as a fitted operator finite difference method obtained through trigonometric-spline and used to solve the problem in equation (1). It can be solved by the matrix inversion method after determining the value of the fitting factor in the next section.

3.3. Determination of the Fitting Factor \( \sigma \). By fixing the time variable and considering the homogeneous part of equation (6) with constant coefficients that are the minimum values of the corresponding variable coefficients, we have
\[
-\varepsilon \frac{d^2u}{dx^2} - \mu a \frac{du}{dx} + q^* u = 0,
\]
(36)
where \( q(x, t_m) \geq q^* > 0 \). Its characteristic equation on the \( m \)th time level is 
\[-\varepsilon r^2 - \mu ar + q^* = 0, \]
whose solutions are given by the relation
\[ r = \frac{\mu a \pm \sqrt{(\mu a)^2 + 4\varepsilon q^*}}{2\varepsilon}. \] (37)

Equation (1) has two boundary layers which behaves like the reaction-diffusion case \((\mu \approx 0)\) with each of width \( O(\sqrt{\varepsilon}) \) at \( x = 1 \) and \( x = 0 \). Then, the complementary function of equation (6) is
\[ r = \frac{\mu a \pm \sqrt{1 + 4\varepsilon q^*/\mu^2 a^2}}{2\varepsilon} = \frac{\mu a}{2\varepsilon} \left( 1 \pm \sqrt{1 + 4\varepsilon q^*/\mu^2 a^2} \right) = \frac{\mu a}{2\varepsilon} (1 \pm 1). \] (38)

Assume there are two real solutions \( r_1 < 0 \) and \( r_2 > 0 \) that describes the boundary layers at \( x = 0 \) and \( x = 1 \), respectively, based on the following two cases [27]:

1. Case 1: if \( \varepsilon^2/\mu \to 0, \) as \( \varepsilon \to 0, \)
2. Case 2: if \( \varepsilon/\mu^2 \to 0, \) as \( \mu \to 0, \)

where \( C_1 \) and \( C_2 \) are real constant numbers. The current problem is the generalization of problems with one perturbation parameter which is studied intensively as convection-diffusion or reaction-diffusion.

From the theory of singular perturbations given in [2] and using Taylor’s series expansion at layer regions and restriction to their first terms, we get the asymptotic solution as

\[ u(x) = C_1 \exp \left( \frac{q^*}{\varepsilon} x \right) + C_2 \exp \left( \frac{q^*}{\varepsilon} (1-x) \right), \] (39)

where \( C_1 \) and \( C_2 \) are real constant numbers.

By considering \( h \) small enough, the discretized form of equation (42) is

\[ u(x, t_m) = \begin{cases} 
    u_0(x, t_m) + [u(0, t_m) - u_0(0, t_m)]e^{(-\mu a(0, t_m)/\varepsilon)x}, & \text{at left layer,} \\
    u_0(x, t_m) + [u(1, t_m) - u_0(1, t_m)]e^{(\mu a(1, t_m)/\varepsilon)(1-x)}, & \text{at right layer,} 
\end{cases} \] (42)

where \( u_0(x, t_m) \) is the solution of the reduced problem (when \( \varepsilon = 0 \)) of equation (17) that is given by

\[ -\mu a x \frac{du(x, t_m)}{dx} + q(x, t_m)u(x, t_m) = H(x, t_m). \] (43)

Now, use equations (44) into (34) and restrict in each Taylor expansion to the first terms. Then, taking limits both sides as \( h \to 0 \) and solving for \( a \), we get
where $\rho = h/\varepsilon, m = 1, \ldots, M + 1$.

To this end, using equations (34) and (45), it is observable that

$$|A_n^+| \geq |A_n^+| + |A_n^-|.$$  

This shows the coefficient matrix associated to the difference operator $L_{\varepsilon, \mu}^{N,M}$ is irreducibly diagonally dominant.

4. Parameter-Uniform Convergence Analysis

So far, we have shown that the continuous solution and its derivatives are bounded and errors due to the introduction of the discrete approximation in the time variable can be estimated (controlled). To realize the stability and consistency of the developed scheme, we further see the error estimate in the spatial variable and the total discrete scheme.

**Lemma 7** (Discrete maximum principle). Assume the discrete function $\Pi_n^m \geq 0, \Pi_{N+1}^m \geq 0$, and $L_{\varepsilon, \mu}^{N,M} \Pi_n^m \geq 0$ on $D_N^N \times D_M^M$. Then, $\Pi_n^m \geq 0$ at each point of $D_N^N \times D_M^M$.

**Proof.** To follow the proof by contradiction, let there exists a point $(i, m)$ where $i \in \{1, 2, \ldots, N + 1\}$ such that

$$\prod_{i=1}^{N+1} \Pi_n^m,$$

and suppose that $\Pi_n^m < 0$, then we have $i \neq 1, N + 1$. But by using equation (25), the assumptions $a(x, t) \geq a$ and $b(x, t) \geq b$, and the series representation $x \cosh x = 1 + x^2/3 + O(x^3)$ into equation (34) we arrive at

$$L_{\varepsilon, \mu}^{N,M} \Pi_n^m < 0.$$  

This contradicts the assumption $L_{\varepsilon, \mu}^{N,M} \Pi_n^m \geq 0$ on $D_N^N \times D_M^M$, and then it completes the proof.

From Lemma 7, it follows that the discrete operator $L_{\varepsilon, \mu}^{N,M}$ satisfies the discrete maximum principle and then the coefficient matrix associated to it is monotone. Moreover, as it is also irreducibly diagonally dominant, then it is an M-matrix. This guarantees for the existence of unique discrete solution. In the next lemma, we discuss the uniform stability of the discrete solution.

**Lemma 8.** The solution $U_n^m$ of the discrete scheme equation (13) satisfies the following bound

$$|U_n^m| \leq \frac{\|L_{\varepsilon, \mu}^{N,M} U_n^m\|}{q} + \max \{|U_{n+1}^m|, |U_{n+1}^m|\},$$

where $q(x, t) \geq q^\ast > 0$.

**Proof.** Let $\Theta = (\|L_{\varepsilon, \mu}^{N,M} U_n^m\|)/q^\ast + \max \{|U_{n+1}^m|, |U_{n+1}^m|\}$ and define barrier functions as

$$(\theta_c^\ast)^m = \Theta \pm U_n^m.$$  

The values of these barrier functions on the boundary points are

$$\theta_c^1 = \Theta \pm U_1^m = \frac{\|L_{\varepsilon, \mu}^{N,M} U_n^m\|}{q} + \max \{|U_{n+1}^m|, |U_{n+1}^m|\} \pm U_1^m \geq 0,$$

$$\theta_c^{N+1} = \Theta \pm U_{N+1}^m = \frac{\|L_{\varepsilon, \mu}^{N,M} U_n^m\|}{q} + \max \{|U_{n+1}^m|, |U_{n+1}^m|\} \pm U_{N+1}^m \geq 0.$$  

Then, by applying the discrete maximum principle given in Lemma 7, the proof is immediate.

As a result, the method is uniformly stable in $q$.

To establish a parameter-uniform convergence of the discrete scheme equation (34), let the truncation error be given by
The numerical solution obtained by the proposed scheme given in equation (1) at each grid point \( (x_n, t_m) \) is Theorem 9. Let \( u(x_n, t_m) \) be the solution of the problem in equation (1) at each grid point \( (x_n, t_m) \) and \( U(x_n, t_m) \) be its numerical solution obtained by the proposed scheme given in equation (13). Then, the error estimate for the fully discrete scheme is given by

\[
\max_{n,m} | u(x_n, t_m) - U(x_n, t_m) | \leq C(k + h^2). \tag{56}
\]

Proof. Applying the triangular inequality, the proof is immediate from Lemmas 5 and 6. Therefore, the proposed method is convergent independent of the perturbation parameters and its rate of convergence is \( O(k + h^2) \), one in the time variable and two in the space variable.

It is well known that the more refined the step sizes, the better accurate results obtained. But this is computationally tough. So, using the Richardson extrapolation

\[
TE = A_n^r U_{n-1}^m + A_n^r U_n^m + A_n^r U_{n+1}^m - F_n^m
\]

\[
= A_n^r U_{n-1}^m + A_n^r U_n^m + A_n^r U_{n+1}^m - \frac{h^2}{\sigma \epsilon} (\lambda_1 H_{n-1}^m + \lambda_2 H_n^m + \lambda_1 H_{n+1}^m)
\]

\[
= A_n^r U_{n-1}^m + A_n^r U_n^m + A_n^r U_{n+1}^m - \frac{\lambda_1 h^2}{\sigma \epsilon} \left( -\sigma (U^r)_{n-1}^m - \mu a_{n-1}^m (U^r)_{n-1}^m + q_{n-1}^m U_{n-1}^m \right)
\]

Expanding each term by Taylor series method about \( x_n \)

up to third order and simplifying yields

\[
TE = \left\{ A_n^r + A_n^s + \frac{h^2}{\sigma \epsilon} (\lambda_1 q_{n-1}^m + \lambda_2 q_n^m + \lambda_1 q_{n+1}^m) \right\} U_n^m
\]

\[
+ \left\{ h(-A_n^r + A_n^s) + \frac{h^2}{\sigma \epsilon} (\lambda_1 a_{n-1}^m + \lambda_2 a_n^m + \lambda_1 a_{n+1}^m) + \frac{h^3}{\sigma \epsilon} (q_{n-1}^m - q_n^m) \right\} (U^r)_{n-1}^m
\]

\[
+ \left\{ \frac{h^2}{2} (A_n^r + A_n^s) + h^2 (\lambda_1 + \lambda_2) - \frac{h^2}{\sigma \epsilon} (a_{n-1}^m + a_n^m) - \frac{h^4}{2\sigma \epsilon} (q_{n-1}^m + q_n^m) \right\} (U^r)_{n-1}^m
\]

\[
+ \left\{ \frac{h^3}{6} (-A_n^r + A_n^s) - h^3 \lambda_1 - \frac{h^3}{6\sigma \epsilon} (a_{n-1}^m + a_n^m) + \frac{h^5}{6\sigma \epsilon} (q_{n-1}^m - q_n^m) \right\} (U^r)_{n-1}^m
\]

After Taylor expansion of each coefficient up to first term and arithmetic manipulations, we reach at

\[
TE = h^2 (-1 + \lambda_1 + \lambda_2) (U^r)_{n-1}^m + O(h^3). \tag{55}
\]

By Lemma 6, \( (U^r)_{n-1}^m \) is bounded, then the order of the method in the spatial direction is \( O(h^2) \).

Theorem 9. Let \( u(x_n, t_m) \) be the solution of the problem in equation (1) at each grid point \( (x_n, t_m) \) and \( U(x_n, t_m) \) be its numerical solution obtained by the proposed scheme given in equation (13). Then, the error estimate for the fully discrete scheme is given by

\[
\max_{n,m} | u(x_n, t_m) - U(x_n, t_m) | \leq C(k + h^2). \tag{56}
\]

Proof. Applying the triangular inequality, the proof is immediate from Lemmas 5 and 6. Therefore, the proposed method is convergent independent of the perturbation parameters and its rate of convergence is \( O(k + h^2) \), one in the time variable and two in the space variable.

It is well known that the more refined the step sizes, the better accurate results obtained. But this is computationally tough. So, using the Richardson extrapolation

\[
U = U_0 + C_1 \left[ k + h^2 \right] + C_2 \left[ k^2 + h^3 \right] + C_3 \left[ k^3 + h^4 \right] + \ldots,
\]

\[
U = U_1 + C_1 \left[ k \right] + C_2 \left[ k^2 + h^3 \right] + C_3 \left[ k^3 + h^4 \right] + \ldots,
\]

\[
U = U_2 + C_1 \left[ k \right] + C_2 \left[ k^2 + h^3 \right] + C_3 \left[ k^3 + h^4 \right] + \ldots.
\]
The results obtained through the Richardson extrapolation denoted and defined by

\[
U_{\text{Rich}}^{1} = 2U_{1} - U_{0} + O(k^{2} + h^{2}),
\]

\[
U_{\text{Rich}}^{2} = 2U_{2} - U_{1} + O(k^{2} + h^{2}),
\]

better approximate the exact solution \( u \) than \( U \). From equations (60) and (61), we get the error estimate

\[
\max_{n,m} |U_{2\text{Rich}}(x_{n}, t_{m}) - U_{1\text{Rich}}(x_{n}, t_{m})| \leq C(k^{2} + h^{2}).
\]

Furthermore, we have the following error estimate.

**Theorem 10.** Let \( u(x_{n}, t_{m}) \) be the solution of equation (1) and \( U_{1\text{Rich}}(x_{n}, t_{m}) \) be the numerical solution obtained through the Richardson extrapolation as defined in equation (18). Then, the error after extrapolation satisfies
**Table 3:** The maximum point-wise error and rate of convergence comparison for Example 1 where $\mu = 10^{-2}$ and different values of $\epsilon$.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$N = 32$</th>
<th>$N = 64$</th>
<th>$N = 128$</th>
<th>$N = 256$</th>
<th>$N = 512$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 10$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>3.4580e-04</td>
<td>9.1362e-05</td>
<td>2.3475e-05</td>
<td>5.9509e-06</td>
<td>1.4982e-06</td>
</tr>
<tr>
<td>$M = 20$</td>
<td>1.9203</td>
<td>1.9605</td>
<td>1.9799</td>
<td>1.9899</td>
<td></td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>2.4484e-04</td>
<td>6.3016e-05</td>
<td>1.6166e-05</td>
<td>4.1052e-06</td>
<td>1.0353e-06</td>
</tr>
<tr>
<td>$M = 40$</td>
<td>1.9580</td>
<td>1.9628</td>
<td>1.9774</td>
<td>1.9874</td>
<td></td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>2.8997e-04</td>
<td>7.5915e-05</td>
<td>1.9432e-05</td>
<td>6.3214e-06</td>
<td>1.6481e-06</td>
</tr>
<tr>
<td>$M = 80$</td>
<td>1.9334</td>
<td>1.9660</td>
<td>1.6201</td>
<td>1.9394</td>
<td></td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>2.8997e-04</td>
<td>7.5915e-05</td>
<td>1.9432e-05</td>
<td>6.3296e-06</td>
<td>2.1945e-06</td>
</tr>
<tr>
<td>$M = 160$</td>
<td>1.9334</td>
<td>1.9660</td>
<td>1.6182</td>
<td>1.5282</td>
<td></td>
</tr>
<tr>
<td>$10^{-10}$</td>
<td>2.8997e-04</td>
<td>7.5915e-05</td>
<td>1.9432e-05</td>
<td>6.3296e-06</td>
<td>2.1945e-06</td>
</tr>
<tr>
<td>$M = 320$</td>
<td>1.9334</td>
<td>1.9660</td>
<td>1.6182</td>
<td>1.5282</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4:** The maximum point-wise error comparison for Example 2 where $\mu = 2^0$ and different values of $\epsilon$.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$N = 16$</th>
<th>$N = 32$</th>
<th>$N = 64$</th>
<th>$N = 128$</th>
<th>$N = 256$</th>
<th>$N = 512$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 16$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>5.6694e-04</td>
<td>1.2149e-04</td>
<td>2.0567e-05</td>
<td>4.4090e-06</td>
<td>1.0542e-06</td>
<td>2.6087e-07</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>6.2817e-04</td>
<td>1.7672e-04</td>
<td>4.4084e-05</td>
<td>8.9154e-06</td>
<td>1.4965e-06</td>
<td>3.1840e-07</td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td>6.2830e-04</td>
<td>1.7885e-04</td>
<td>4.8782e-05</td>
<td>1.2962e-05</td>
<td>3.3821e-06</td>
<td>8.6187e-07</td>
</tr>
<tr>
<td>$2^{-24}$</td>
<td>6.2830e-04</td>
<td>1.7885e-04</td>
<td>4.8782e-05</td>
<td>1.2962e-05</td>
<td>3.3829e-06</td>
<td>8.7217e-07</td>
</tr>
<tr>
<td>$2^{-26}$</td>
<td>6.2830e-04</td>
<td>1.7885e-04</td>
<td>4.8782e-05</td>
<td>1.2962e-05</td>
<td>3.3829e-06</td>
<td>8.7217e-07</td>
</tr>
<tr>
<td>$2^{-30}$</td>
<td>6.2830e-04</td>
<td>1.7885e-04</td>
<td>4.8782e-05</td>
<td>1.2962e-05</td>
<td>3.3829e-06</td>
<td>8.7217e-07</td>
</tr>
<tr>
<td>$2^{-32}$</td>
<td>6.2830e-04</td>
<td>1.7885e-04</td>
<td>4.8782e-05</td>
<td>1.2962e-05</td>
<td>3.3829e-06</td>
<td>8.7217e-07</td>
</tr>
</tbody>
</table>

**Method in [6]**

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$N = 16$</th>
<th>$N = 32$</th>
<th>$N = 64$</th>
<th>$N = 128$</th>
<th>$N = 256$</th>
<th>$N = 512$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 16$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>3.5638-2</td>
<td>1.3000-2</td>
<td>9.3378-3</td>
<td>8.4218-3</td>
<td>8.3579-3</td>
<td>8.7259-3</td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>5.1972-2</td>
<td>1.4234-2</td>
<td>8.3305-3</td>
<td>7.9579-3</td>
<td>8.1124-3</td>
<td>8.0812-3</td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td>6.7088-2</td>
<td>2.3185-2</td>
<td>8.2129-3</td>
<td>7.6243-3</td>
<td>7.9618-3</td>
<td>8.0662-3</td>
</tr>
</tbody>
</table>

$$\max_{n,m} |u(x_n,t_n) - U_{\text{Rich}}(x_n,t_n)| \leq C(k^2 + h^2). \quad (63)$$

**Proof.** The proof is given in [28].

### 5. Numerical Examples and Results

Under this section, we apply the current method to the following numerical examples with $\lambda_1 = 1/284, \lambda_2 = 282/284$ and verify the theoretical results experimentally.
Example 1. For the problem in [29],
\[
\begin{align*}
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} - \mu (1 + x) \frac{\partial u}{\partial x} + u &= -16x^2 (1-x)^2, \\
\text{subject to } u(x,0) &= 0, \ u(0,t) = 0 = u(1,t).
\end{align*}
\]
(64)

\[ f(x, t) = \exp(-t) \left[ -\exp\left(\frac{1}{\varepsilon}\right) + \left(1 - \exp\left(\frac{-1}{\varepsilon}\right)\right) (1-x) + \exp\left(-\frac{1}{\varepsilon} (1-x)\right) \right], \]
(66)

and the exact solution is given by
\[
\begin{align*}
u_0(x) &= \exp\left(-\frac{1}{\varepsilon}\right) + \left(1 - \exp\left(-\frac{1}{\varepsilon}\right)\right) x + \exp\left(-\frac{1}{\varepsilon} (1-x)\right), \\
\end{align*}
\]
and
\[
\begin{align*}
u(x, t) &= \exp(-t) \left[ \exp\left(\frac{1}{\varepsilon}\right) + \left(1 - \exp\left(\frac{-1}{\varepsilon}\right)\right)x - \exp\left(-\frac{1}{\varepsilon} (1-x)\right) \right].
\end{align*}
\]
(67)

Example 2. For the problem in [6],
\[
\begin{align*}
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} &= f(x, t), \\
\text{subject to } u(x,0) &= u_0(x), \ u(0,t) = 0 = u(1,t).
\end{align*}
\]
where
\[
\begin{align*}
f(x, t) &= \exp(-t) \left[ -\exp\left(\frac{1}{\varepsilon}\right) + \left(1 - \exp\left(\frac{-1}{\varepsilon}\right)\right) (1-x) + \exp\left(-\frac{1}{\varepsilon} (1-x)\right) \right], \\
\end{align*}
\]
(66)

and the exact solution is given by
\[
\begin{align*}
u(x, t) &= \exp(-t) \left[ \exp\left(\frac{1}{\varepsilon}\right) + \left(1 - \exp\left(\frac{-1}{\varepsilon}\right)\right)x - \exp\left(-\frac{1}{\varepsilon} (1-x)\right) \right].
\end{align*}
\]
(67)

It is difficult to get the exact solution of Example 1. Therefore, to illustrate the performance of the proposed scheme through the estimation of the maximum point-wise error, using the double mesh principle is indispensable. We define the absolute maximum errors before the Richardson extrapolation as
and after the Richardson extrapolation as

\[ E_{\text{Rich}}^{N,M} = \max_{1 \leq n \leq N+1, 1 \leq m \leq M+1} |U_{2\text{Rich}}^M - U_{\text{1Rich}}^M|, \]

where \( M \) and \( N \) are the numbers of mesh points in \( t \) and \( x \) directions with \( k \) and \( h \) step sizes, respectively. \( U_{\text{1Rich}}^M \) is the approximate solution obtained using \( M \) and \( N \) number of meshes and \( U_{2\text{Rich}}^M \) is the approximate solution obtained using a double number of meshes \( 2M \) and \( 2N \) with half step sizes. \( U_{\text{1Rich}}^M \) and \( U_{2\text{Rich}}^M \) are given in equations (60) and (61), respectively. But for Example 2, since we have an exact solution, the absolute maximum errors before and after the Richardson extrapolation are defined, respectively, as

\[ E_{\text{ext}}^{N,M} = \max_{1 \leq n \leq N+1, 1 \leq m \leq M+1} |U_N^M - (U_{\text{Ext}})_N^M|, \]

\[ E_{\text{Rich}}^{N,M} = \max_{1 \leq n \leq N+1, 1 \leq m \leq M+1} |U_{\text{1Rich}}^M - (U_{\text{Ext}})_N^M|, \]

where \((U_{\text{Ext}})_N^M\) is the exact solution at the grid points. The rate of convergence for both examples before and after extrapolation is, respectively, obtained by
6. Discussion and Conclusion

We have described a trigonometric-spline method for solving time-dependent singularly perturbed convection-diffusion problems using the fitted operator technique. Applying the backward Euler method, we have first transformed the continuous time-dependent problem into an ordinary differential equation (ODE) of space variable $x$. And then, in turn, using a trigonometric-spline discrete scheme in the space variable, we have obtained a fully discrete scheme. In our numerical experimentation, the calculated maximum point-wise errors before and after extrapolation are presented in Table 1, and the corresponding rates of convergence are given in Table 2. The overall current results for Examples 1 and 2 are shown in Tables 3 and 4, respectively.

The outcomes displayed in Tables 1, 3, and 4 clearly show that, as the perturbation parameter goes to zero, the absolute maximum point-wise error remains constant. This indicates...
the developed scheme given in Equation (34) and the extrapolated technique are insensitive to the perturbation parameters and gives better results than the results in the literature. The rate of convergence of the current method is increased from one to two as a consequence of the Richardson extrapolation supporting the theoretically asserted hypothesis. This is illustrated in Tables 2 and 5. As the number of mesh points increases, the absolute maximum point-wise errors decrease, and the rate of convergence increases. These show us that there is no oscillation or unexpected change in the solution, that is, the effect of the perturbation parameter is controlled by the fitted numerical scheme obtained. From the graphs presented in Figures 1 and 2, one can observe that the problem given in Example 1 has a solution which exhibits a parabolic boundary layer. Besides this, Figure 2(a) indicates that the two boundary layers have equal width, as discussed in Case 1, whereas Figure 2(b) shows the boundary layers have different width ratifying in Case 2. From Figure 3, we can observe the profile of the numerical and exact solution of Example 2. Furthermore, since $\mu = 1$ in Example 2, the boundary layer appears at the right end of the domain depending on the sign of the convection coefficient $a(x,t)$ which is illustrated in Figure 4. We have plotted the maximum point-wise errors in the log-log scale for the sake of revealing the numerical order of convergence before and after the Richardson extrapolation technique in Figure 5, and again it endorses the theoretical order of convergence and the error being constant. Generally, the numerical results obtained by our method confirm the theoretical results. In our future works, we consider problems of the type treated in [32, 33].

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


