Hindawi
Mathematical Problems in Engineering
Volume 2023, Article ID 7184528, 14 pages
https://doi.org/10.1155/2023/7184528

Research Article

Bayesian Estimation of a Geometric Life Testing Model under Different Loss Functions Using a Doubly Type-1 Censoring Scheme

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Received 16 September 2022; Revised 17 October 2022; Accepted 25 November 2022; Published 11 April 2023

Academic Editor: Tahir Mehmood

In this article, we consider the doubly type-1 censoring scheme that researchers frequently use in clinical trials and lifetime experiments. The Bayesian paradigm will be used to estimate the parameters of the Geometric Lifetime Model (GLTM) using a doubly type-I censoring scheme. Bayes estimators and their associated Bayes risks are examined in terms of closed-form algebraic expressions. This research also includes a strategy for eliciting hyperparameters based on prior prediction distributions. To evaluate the strength and effectiveness of the suggested estimating approach, thorough simulation studies as well as real-life data analysis are presented. The results depict that Squared Error Loss Function (SELF) is more efficient, and the Beta prior is suitable while estimating the parameter of GLTM.

1. Introduction

It is customary in clinical or biological investigations to use censoring schemes while assessing the worth of new procedures. Among others, the doubly censoring scheme is widely used in clinical and other lifetime investigations. The worth of censoring schemes is not hidden in literature, and many authors have focused their research study based on different censoring schemes. Bravo and De Fuentes [1] derived maximum likelihood estimates by considering the doubly type-II-censored exponential scenario. Fauzy et al. [2] constructed intervals to estimate the parameters of an exponential distribution under the doubly type-II censoring scheme. Krishna and Malik [3] considered reliability estimation for the doubly type-II censored Maxwell distribution. Algarni et al. [4] considered type-I censoring while estimating the parameters of the Chen distribution. Feroze [5] discussed the application of doubly censored data from a 2-component mixture of inverse Weibull distributions . Ghosh and Nadarajah [6] described the Bayesian inference of Kumaraswamy distributions based on censored samples. Long [7] estimated the parameters of the Rayleigh distribution based on double type-I hybrid censored data.

While investigating lifetime phenomenon, discrete life testing models got less attention in the literature because the mathematics required in dealing with discrete lifetime model is difficult to handle though their significance in many fields needs no depiction. Among others, the geometric distribution is specifically important in many sectors of biological, social, and life-testing experiments. The geometric distribution has been considered as a lifetime model in reliability theory by Yaqub and Khan [8], Bhattacharya and Kumar [9], Krishna and Jain [10], Sarhan and Kundu [11], and many others. These researchers developed Bayes estimators for reliability measures of the individual components in a multicomponent geometric lifetime model using disguised systems of life testing data.

The literature of estimation lacks, to the best of our knowledge, analyzing through Bayesian approach the Geometric Lifetime Model (GLTM) while considering doubly type-I censoring scheme. Hence, the current study is devoted to provide and analyze doubly type I censored GLTM using Bayesian estimation tools. For the unknown parameter of GLTM, informative and uninformative priors are examined under the Square Error Loss Function (SELF), DeGroot Loss Function (DLF), Quadratic Loss Function (QLF), Precautionary Loss Function (PLF) and Simple Asymmetric Precautionary Loss Function (SAPLF). Bayes estimators and Bayes risks for the unknown parameter of GLTM under doubly type-I censoring scheme are derived for the aforementioned priors and loss functions. A simulation and real-lifedata-based analyzes are carried out to evaluate the suggested model's strength and utility..

2. Methodology

A random variable X is said to follow a discrete geometric distribution if its probability distrition funcion of X can be written as

$$f_X(x) = \theta(1-\theta)^x, x = 0, 1, \ldots . \tag{1}$$

Geometric distribution having the parameter θ. The cumulative distribution function for a random variable X, the geometric distribution function is given by:

$$F(x) = 1-(1-\theta)^{x+1}, x = 0, 1, \ldots . \tag{2}$$

2.1. The Likelihood Function and Posterior Distributions. Consider n items are placed in a life testing experiment and we begin studying these items after time $T_1 = t_1$ and continue to observing them until time $T_2 = t_2$. The observations are assumed to come from the geometric distribution having the parameter θ; it is assumed that left sided observations m_1 are censored in experiment with fixed time $T_1 = t_1$. Let r be the number of observed failures from the observations and experiment will proceed from time $T_1 = t_1$ up to time $T_2 = t_2$, and m_2 items will be censored after $T_2 = t_2$. Hence, total $m_1 + m_2$ items are censored. Therefore, the likelihood function of the geometric distribution the under doubly type-I censoring scheme can be derived as

$$L(\theta) \propto [F(t_1)]^{m_1} \prod_{i=1}^{r} f(x_i)[1-F(t_2)]^{m_2},$$
$$L(\theta) = \sum_{j=0}^{m_1} \binom{m_1}{j} (-1)^j \theta^r (1-\theta)^{j(t_1+1)+\sum_{i=1}^{r} x_i + m_2(t_2+1)}. \tag{3}$$

At first, it is assumed that the parameter follows Uniform prior i.e. $\theta \sim U(0,1)$

$$p_U(\theta) \propto 1, 0<\theta<1. \tag{4}$$

Combining likelihood function (2) and prior probability density (3), the posterior density of θ is

$$p_{1U}(\theta|\underset{-}{x}) = \frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+1-1}(1-\theta)^{j(t_1+1)+\sum_{i=m_1}^{r} x_i+m_2(t_2+1)+1-1}}{\int_0^1\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+1-1}(1-\theta)^{j(t_1+1)+\sum_{i=m_1}^{r} x_i+m_2(t_2+1)+1-1}d\theta} = \frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+1-1}(1-\theta)^{\delta_{1U}-1}}{K_{1U}}, 0<\theta<1, \tag{5}$$

where

$$\delta_{1U} = j(t_1+1) + \sum_{i=m_1}^{r} x_i + m_2(t_2+1) + 1,$$
$$K_{1U} = \sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+1, \delta_{1U}). \tag{6}$$

Uniform prior is vital in situations where no prior information is available and only the current/sample information is in hand. This prior is widely used by the data analysts while analyzing the data through the Bayesian approach (Yaqub and Khan [8], Bhattacharya and Kumar [9], Krishna and Jain [10], and references cited therein).

In situations where prior information is available, the Bayesian analysts suggest using informative priors, which enhance the efficiency of the estimation techniques. Among many informative priors, the Beta prior is considered a better informative prior as it is a natural conjugate prior for proportion (probability success) (Van de Schoot [12]. Keeping in view the importance of Beta prior, it is assumed that the parameter of GLTM follows Beta prior distribution with hyperparameters ω_1 *and* ω_2, i.e. $\theta \sim \beta(\omega_1, \omega_2)$.

Table 1: BEs and BRs SELF.

Priors	Bayes estimators	Bayes risks
Uniform	$1/K_{1U}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1U})$	$1/K_{1U}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1U})-(BE)^2$
Beta	$1/K_{1\beta}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+1,\delta_{1\beta})$	$1/K_{1\beta}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1t^j)B\left(r+\omega_1 t+n2q,h\delta_{1\beta}\right)-(BE)^2$
Kumaraswamy	$1/K_{1KS}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1KS})$	$1/K_{1KS}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1KS})-(BE)^2$

$$p_B(\theta)\propto\theta^{\omega_1-1}(1-\theta)^{\omega_2-1},0<\theta<1,\quad \omega_1,\omega_2>0, \tag{7}$$

where to be proper density, we must have $\omega_1>0$ and $\omega_2>0$. Combining the likelihood function (2) and prior probability density (6), the posterior density of θ becomes

$$p_{1\beta}\left(\theta|x\right)=\frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+\omega_1-1}(1-\theta)^{j(t_1+1)+\sum_{i=1}^{r}x_i+m_2(t_2+1)+\omega_2-1}}{\int_0^1\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+\omega_1-1}(1-\theta)^{j(t_1+1)+\sum_{i=1}^{r}x_i+m_2(t_2+1)+\omega_2-1}d\theta},$$
$$p_{1\beta}\left(\theta|x\right)=\frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+\omega_1-1}(1-\theta)^{j(t_1+1)+\sum_{i=1}^{r}x_i+m_2(t_2+1)+\omega_2-1}}{K_{1\beta}},0<\theta<1,K_{1\beta}>0, \tag{8}$$

where

$$\delta_{1\beta}=j(t_1+1)+\sum_{i=1}^{r}x_i+m_2(t_2+1)+\omega_2, \tag{9}$$

and

$$K_{1\beta}=\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B\left(\omega_1+r,\delta_{1\beta}\right). \tag{10}$$

Another informative prior considered in this study is the well-known Kumaraswamy distribution, which has significant rule in distribution theory (Dey et al. [13]). Therefore, we also assume a special type of Kumaraswamy distribution as prior for the parameter of GLTM, i.e. $\theta\sim K(1,\omega_3)$.

$$p_K\propto\omega_3\,\theta^{\omega_3-1},0<\theta<1. \tag{11}$$

The posterior distribution for this prior is derived as

$$p_{1KS}\left(\theta|x\right)=\frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+1-1}(1-\theta)^{j(t_1+1)+\sum_{i=1}^{r}x_i+m_2(t_2+1)+\omega_3-1}}{\int_0^1\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+1-1}(1-\theta)^{j(t_1+1)+\sum_{i=1}^{r}x_i+m_2(t_2+1)+\omega_3-1}d\theta},$$
$$p_{1KS}\left(\theta|x\right)=\frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j\theta^{r+1-1}(1-\theta)^{j(t_1+1)+\sum_{i=1}^{r}x_i+m_2(t_2+1)+\omega_3-1}}{K_{KS}},0<\theta<1,K_{1KS}>0, \tag{12}$$

where

$$\delta_{1KS} = j(t_1+1) + \sum_{i=1}^{r} x_i + m_2(t_2+1) + \omega_3,$$
$$K_{1KS} = \sum_{j=0}^{m_1} \binom{m_1}{j} (-1)^j B(r+1, \delta_{1KS}). \tag{13}$$

2.2. Elicitations of Hyperparameters of Informative Priors. To elicit the hyperparameters of the informative (Beta and Kumaraswamy) priors, according to Garthwaite et al. [14]; elicitation of hyperparameters as a method is used to convert an expert's prior knowledge and professional judgment about unknown quantities of interest. To elicit the hyperparameters of the informative (Beta and Kumaraswamy) priors, the method suggested by Aslam [15] is employed in this study.

Let $Y = X_{n+1}$ be future value of a random variable X, then the prior predictive distribution of Y is defined as

$$f(y) = \int_\theta p(\theta) g(y\theta|) \mathrm{d}\theta. \tag{14}$$

The prior predictive distributions of Y for Beta and Kumaraswamy priors are, respectively, derived in the following:

$$f_\beta(y) = \frac{\beta(\omega_1+1, \omega_2+y)}{\beta(\omega_1,\omega_2)}, y = 0,1,\cdots f_{KS}(y) = \beta(2, \omega_3+y), y = 0,1,\cdots. \tag{15}$$

For Beta and Kumaraswamy priors, the elicited values of the hyperparameters are obtained by solving the following equations simultaneously in "Mathematica 10."

$$\sum_{y=0}^{2} \frac{\beta(\omega_1+1,\omega_2+y)}{\beta(\omega_1,\omega_2)} = 0.60,$$
$$\sum_{y=6}^{8} \frac{\beta(\omega_1+1,\omega_2+y)}{\beta(\omega_1,\omega_2)} = 0.10. \tag{16}$$

The resultant values of the hyperparameters of Beta prior are given in the following:

$$\omega_1 = 0.7793,$$
$$\omega_2 = 1.434. \tag{17}$$

The resultant value of the hyperparameter of Kumaraswamy prior is given in the following:

$$\omega_3 = 1.241. \tag{18}$$

2.3. Loss Functions. In this section, we derive the Bayes estimators (BEs) and Bayes risks (BRs) using uninformative prior (UP) and two informative priors (IP) for five different loss functions (SELF, DLF, QLF, PLS, and SAPLF).

2.4. BEs and BRs Using UP and IPs under SELF. For a parameter θ with a BE θ^*, SELF is defined as

$$L(\theta,\theta^*) = (\theta-\theta^*)^2. \tag{19}$$

The BE and BR under SELF are

$$\theta^* = E_{\theta|x}(\theta)\rho(\theta,\theta^*) = E_{\theta|x}(\theta)^2 - \left(E_{\theta|x}(\theta)\right)^2. \tag{20}$$

The BEs and BRs under SELF using different priors are shown in Table 1.

2.5. BEs and BRs Using UP and IPs under DLF. For a parameter θ with a BE θ^*, DLF is defined as

$$L(\theta,\theta^*) = \left(\frac{\theta-\theta^*}{\theta^*}\right)^2. \tag{21}$$

The BE and BR under DLF are

$$\theta^* = \frac{E_{\theta|x}(\theta^2)}{E_{\theta|x}(\theta)},$$
$$\rho(\theta,\theta^*) = 1 - \frac{\left\{E_{\theta|x}(\theta)\right\}^2}{E_{\theta|x}(\theta^2)}. \tag{22}$$

The BEs and BRs under DLF using different priors are shown in Table 2.

2.6. BEs and BRs Using UP and IPs under QLF. For a parameter θ with a BE θ^*, QLF is defined as

$$L(\theta,\theta^*) = \left(\frac{\theta-\theta^*}{\theta}\right)^2. \tag{23}$$

The BE and BR under QLF are

$$\theta^* = \frac{E_{\theta|x}(\theta^{-1})}{E_{\theta|x}(\theta^{-2})},$$
$$\rho(\theta,\theta^*) = 1 - \frac{\left(E_{\theta|x}(\theta^{-1})\right)^2}{E_{\theta|x}(\theta^{-2})}. \tag{24}$$

The BEs and BRs under QLF using different priors are shown in Table 3.

2.7. BEs and BRs Using UP and IPs under PLF. For a parameter θ with a BE θ^*, PLF is defined as

Table 2: Bayes estimators and Bayes risk of θ under DLF.

Priors	Bayes estimators	Bayes risks
Uniform	$\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1U})/\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1U})$	$1-\left(\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1U})\right)^2/\sum_{j=0}^{m_1}K_{1U}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1U})$
Beta	$\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+2,\delta_{1\beta})/\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+1,\delta_{1\beta})$	$1-\left(\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+1,\delta_{1\beta})\right)^2/\sum_{j=0}^{m_1}K_{1\beta}\binom{m_1}{j}(-1)^j B(r+\omega_1+2,\delta_{1\beta})$
Kumaraswamy	$\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1KS})/\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1KS})$	$1-\left(\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1KS})\right)^2/K_{1KS}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1KS})$

Table 3: Bayes estimators and Bayes risks of θ under QLF.

Priors	Bayes estimators	Bayes risks
Uniform	$\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1U})/\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r-1,\delta_{1U})$	$1-\left(\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1U})\right)^2/K_{1U}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r-1,\delta_{1U})$
Beta	$\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1-1,\delta_{1\beta})/\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1-2,\delta_{1\beta})$	$1-\left(\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1-1,\delta_{1\beta})\right)^2/K_{1\beta}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1-2,\delta_{1\beta})$
Kumaraswamy	$\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1KS})/\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r-1,\delta_{1KS})$	$1-\left(\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1KS})\right)^2/K_{1KS}\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r-1,\delta_{1KS})$

Table 4: Bayes estimators and posterior risk of θ under PLF.

Priors	Bayes estimators	Posterior risks
Uniform	$\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1U})/K_{1U}}$	$2*\left(\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1U})/K_{1U}}-\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1U})/K_{1U}\right)$
Beta	$\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(\mathrm{r}+\omega_1+2,\delta_{1\beta})/K_{1\beta}}$	$2*\left(\begin{array}{c}\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+2,\delta_{1\beta})/K_{1\beta}}\\ -\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+1,\delta_{1\beta})/K_{1\beta}\end{array}\right)$
Kumaraswamy	$\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(\mathrm{r}+3,\delta_{1\mathrm{KS}})/K_{1KS}}$	$2*\left(\begin{array}{c}\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+3,\delta_{1KS})/K_{KS}}-\\ \sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1KS})/K_{1KS}\end{array}\right)$

Table 5: Bayes estimators and Bayes risks of θ under SAPLF.

Priors	Bayes estimators	Bayes risks
Uniform	$\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1U}) / \sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1U})}$	$2\left(1/K_U * \sqrt{\frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1U})}{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1U})}} - 1\right)$
Beta	$\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+1,\delta_{1\beta}) / \sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1-1,\delta_{1\beta})}$	$2\left(1/K_\beta * \sqrt{\frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1+1,\delta_{1\beta})}{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+\omega_1-1,\delta_{1\beta})}} - 1\right)$
Kumaraswamy	$\sqrt{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1KS}) / \sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1KS})}$	$2\left(1/K_{KS} * \sqrt{\frac{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r+2,\delta_{1KS})}{\sum_{j=0}^{m_1}\binom{m_1}{j}(-1)^j B(r,\delta_{1KS})}} - 1\right)$

Table 6: BEs and BRs under SELF for $T_1 = 1$ and $T_2 = 5$.

Priors	n	$\theta = 0.2$		$\theta = 0.3$		$\theta = 0.4$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	**0.207096**	**0.0021942**	**0.301308**	**0.0034322**	**0.388416**	**0.004677**
	25	**0.207059**	**0.001748**	**0.299782**	**0.0027536**	**0.385362**	**0.0037448**
	30	**0.2034**	**0.001441**	**0.297211**	**0.0022826**	**0.38400**	**0.0031303**
	35	**0.203117**	**0.0012329**	**0.295931**	**0.0019533**	**0.383534**	**0.0026925**
Beta	*20*	*0.203669*	*0.002160*	*0.296724*	*0.0033886*	*0.382422*	*0.0046268*
	25	*0.204332*	*0.001727*	*0.296113*	*0.002725*	*0.380593*	*0.0037117*
	30	*0.201133*	*0.0014268*	*0.294171*	*0.0022629*	*0.380029*	*0.0031070*
	35	*0.201179*	*0.0012218*	*0.293331*	*0.0019388*	*0.380131*	*0.0026746*
Kumaraswamy	20	**0.206407**	**0.002181**	**0.300077**	**0.0034101**	**0.386485**	**0.0046455**
	25	**0.206515**	**0.0017408**	**0.298804**	**0.002739**	**0.38384**	**0.0037241**
	30	**0.202956**	**0.0014363**	**0.296407**	**0.0022728**	**0.382738**	**0.0031158**
	35	**0.202738**	**0.0012288**	**0.295247**	**0.0019461**	**0.382455**	**0.0026818**

The bold and italic values in this table shows that these are the best results as compared with counterparts.

Table 7: BEs and BRs under SELF for $T_1 = 2$ and $T_2 = 6$.

Priors	n	$\theta = 0.2$		$\theta = 0.3$		$\theta = 0.4$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	**0.205722**	**0.002060**	**0.296696**	**0.003384**	**0.385981**	**0.004996**
	25	**0.203043**	**0.001623**	**0.294495**	**0.002700**	**0.381512**	**0.0039456**
	30	**0.200658**	**0.001336**	**0.2924**	**0.0022328**	**0.378989**	**0.0032758**
	35	**0.200393**	**0.001144**	**0.291126**	**0.001907**	**0.377455**	**0.002772**
Beta	*20*	*0.202503*	*0.002028*	*0.29216*	*0.003335*	*0.379564*	*0.0049200*
	25	*0.20049*	*0.001604*	*0.29088*	*0.002669*	*0.37648*	*0.003897*
	30	*0.198545*	*0.001323*	*0.289418*	*0.002211*	*0.374834*	*0.003242*
	35	*0.19857*	*0.001136*	*0.28859*	*0.001888*	*0.37398*	*0.001626*
Kumaraswamy	20	**0.205078**	**0.002048**	**0.29548**	**0.0033608**	**0.383893**	**0.0049526**
	25	**0.20254**	**0.001616**	**0.29354**	**0.002685**	**0.379905**	**0.003918**
	30	**0.200247**	**0.0013317**	**0.291618**	**0.002226**	**0.37767**	**0.0032571**
	35	**0.20004**	**0.0011411**	**0.290463**	**0.0018996**	**0.37634**	**0.0027591**

The bold and italic values in this table shows that these are the best results as compared with counterparts.

Table 8: BEs and BRs under DLF for $T_1 = 1$ and $T_2 = 5$.

Priors	n	$\theta = 0.2$		$\theta = 0.3$		$\theta = 0.4$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.218961	0.049091	0.314383	0.0368	0.399544	0.030527
	25	0.213284	0.040270	0.307983	0.030161	0.396177	0.0248181
	30	0.21086	0.033943	0.304373	0.025458	0.392915	0.020950
	35	0.208516	0.029401	0.301914	0.022044	0.390254	0.018126
Beta	20	0.215542	0.049962	0.309797	0.037493	0.393622	0.0311323
	25	0.210572	0.040844	0.304343	0.030589	0.391427	0.0252114
	30	0.208603	0.034348	0.301353	0.025759	0.388958	0.021227
	35	0.206587	0.029702	0.299326	0.022268	0.38687	0.0183309
Kumaraswamy	20	0.218243	0.049132	0.31311	0.036902	0.397598	0.0306161
	25	0.21273	0.040296	0.306993	0.030201	0.394631	0.0248749
	30	0.210404	0.033962	0.303561	0.025486	0.391637	0.020990
	35	0.208131	0.029415	0.301224	0.022064	0.389168	0.0181552

Table 9: BEs and BRs under DLF for $T_1 = 2$ and $T_2 = 6$.

Priors	n	$\theta = 0.20$		$\theta = 0.30$		$\theta = 0.40$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.216098	0.046884	0.309001	0.037001	0.398567	0.032172
	25	0.210847	0.038354	0.304013	0.030193	0.391206	0.026295
	30	0.207652	0.032374	0.300045	0.025472	0.387677	0.022193
	35	0.205584	0.028002	0.297452	0.022066	0.385144	0.019196
Beta	20	0.21288	0.047654	0.304467	0.037608	0.392179	0.032755
	25	0.208298	0.038860	0.30041	0.030590	0.386194	0.026672
	30	0.20554	0.032731	0.29707	0.025752	0.383526	0.022458
	35	0.203777	0.027996	0.294831	0.022325	0.381602	0.019343
Kumaraswamy	20	0.215427	0.046916	0.307751	0.037050	0.396442	0.032243
	25	0.210329	0.038375	0.303037	0.0302248	0.389573	0.026339
	30	0.20723	0.0323885	0.299251	0.025494	0.386338	0.022223
	35	0.205227	0.0280131	0.296779	0.022079	0.38401	0.019218

Table 10: BEs and BRs under QLF for $T_1 = 1$ and $T_2 = 5$.

Priors	n	$\theta = 0.20$		$\theta = 0.30$		$\theta = 0.40$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.18684	0.056444	0.279705	0.041549	0.365181	0.034162
	25	0.188813	0.044834	0.280356	0.033191	0.367446	0.027222
	30	0.190382	0.037170	0.282396	0.0275329	0.368954	0.022615
	35	0.190762	0.031799	0.283383	0.023535	0.370556	0.01933
Beta	20	0.183395	0.057522	0.275064	0.042311	0.359123	0.0348465
	25	0.186072	0.045509	0.27668	0.0336779	0.362624	0.0276581
	30	0.188104	0.037638	0.279308	0.027965	0.36494	0.0229597
	35	0.188823	0.032107	0.283922	0.014167	0.36703	0.0169626
Kumaraswamy	20	0.186204	0.0564876	0.27852	0.0416134	0.363304	0.0342536
	25	0.188304	0.0448618	0.279421	0.0332318	0.365951	0.0272802
	30	0.189957	0.0371898	0.281615	0.027561	0.367714	0.0226554
	35	0.1904	0.031813	0.282715	0.023556	0.369484	0.0193791

Table 11: BEs and BRs under QLF for $T_1 = 2$ and $T_2 = 6$.

Priors	n	$\theta = 0.20$		$\theta = 0.30$		$\theta = 0.40$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.186214	0.048090	0.274456	0.038107	0.357976	0.0309210
	25	0.187461	0.042203	0.276961	0.032107	0.359825	0.0273087
	30	0.188345	0.035088	0.277473	0.027448	0.363487	0.0239400
	35	0.189434	0.029950	0.278577	0.023466	0.363513	0.020498
Beta	20	0.183214	0.0540906	0.271896	0.042107	0.352873	0.0369971
	25	0.184867	0.043038	0.273554	0.033207	0.356822	0.0293456
	30	0.18558	0.037298	0.27471	0.0299844	0.359108	0.032243
	35	0.186326	0.046331	0.276975	0.094055	0.357368	0.0283035
Kumaraswamy	20	0.184214	0.0483090	0.274786	0.040107	0.355980	0.034090
	25	0.186989	0.0422212	0.275961	0.034107	0.358850	0.0283897
	30	0.187954	0.035094	0.276734	0.027395	0.362022	0.0244519
	35	0.189155	0.0294862	0.277971	0.023110	0.361899	0.0217438

Table 12: BEs and BRs under PLF for $T_1 = 1$ and $T_2 = 5$.

Priors	n	$\theta = 0.20$		$\theta = 0.30$		$\theta = 0.40$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.202749	0.054398	0.296873	0.0398347	0.383912	0.032633
	25	0.201822	0.043437	0.294862	0.0320422	0.382337	0.0262331
	30	0.200899	0.036215	0.294126	0.0267557	0.380959	0.0219364
	35	0.199831	0.031079	0.293107	0.0229976	0.381811	0.0188069
Beta	20	0.199169	0.055494	0.292127	0.0406026	0.377767	0.0333162
	25	0.198972	0.044129	0.291085	0.032534	0.377436	0.0266733
	30	0.198532	0.036690	0.290979	0.027098	0.376888	0.0222421
	35	0.197803	0.031515	0.290423	0.0231847	0.378305	0.0190534
Kumaraswamy	20	0.20207	0.0544441	0.29565	0.0399006	0.381996	0.0327255
	25	0.201286	0.043465	0.293899	0.0320838	0.380819	0.0262918
	30	0.200458	0.036234	0.293327	0.0267844	0.379705	0.0219768
	35	0.199456	0.031095	0.292427	0.0230187	0.380731	0.0188369

Table 13: BEs and BRs under PLF for $T_1 = 2$ and $T_2 = 6$.

Priors	n	$\theta = 0.20$		$\theta = 0.30$		$\theta = 0.40$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.200886	0.051252	0.293527	0.039751	0.380843	0.030219
	25	0.199996	0.040956	0.291028	0.031986	0.379843	0.027801
	30	0.198067	0.034293	0.289526	0.0267119	0.376152	0.0232654
	35	0.197243	0.029406	0.28887	0.0229312	0.375122	0.0199946
Beta	20	0.197536	0.052191	0.288838	0.0404667	0.373807	0.0325091
	25	0.197331	0.041528	0.287311	0.0324475	0.371807	0.0282648
	30	0.195743	0.0326921	0.286323	0.0269002	0.371911	0.0236417
	35	0.194931	0.022441	0.285706	0.0209342	0.371143	0.022923
Kumaraswamy	20	0.200253	0.051287	0.29232	0.0398029	0.372807	0.032478
	25	0.199497	0.040978	0.290085	0.0320191	0.370190	0.0281220
	30	0.197658	0.034302	0.288754	0.0266975	0.374843	0.0232954
	35	0.196892	0.029294	0.288228	0.0227868	0.373992	0.0201647

Table 14: BEs and BRs under SAPLF for $T_1 = 1$ and $T_2 = 5$.

Priors	n	$\theta = 0.20$		$\theta = 0.30$		$\theta = 0.40$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.202783	0.054331	0.298158	0.0397708	0.383034	0.032667
	25	0.201992	0.043375	0.295599	0.032017	0.381829	0.0262514
	30	0.200954	0.036166	0.294419	0.0267549	0.381843	0.0219061
	35	0.199881	0.031061	0.293139	0.0229676	0.379994	0.0188673
Beta	20	0.199204	0.055425	0.293388	0.0405395	0.376908	0.033354
	25	0.19914	0.044065	0.291806	0.0325091	0.376937	0.0266917
	30	0.198588	0.036642	0.291263	0.0271171	0.377759	0.0222155
	35	0.197861	0.031502	0.290545	0.0239676	0.376519	0.0190731
Kumaraswamy	20	0.202104	0.0543774	0.296925	0.0398373	0.381126	0.032759
	25	0.201456	0.043404	0.294628	0.032058	0.380315	0.0263099
	30	0.200513	0.036186	0.293618	0.0267838	0.380583	0.0219466
	35	0.199506	0.031074	0.29246	0.0229882	0.378925	0.0188966

Table 15: The BEs and BRs under SAPLF for $T_1 = 2$ and $T_2 = 6$.

Priors	n	$\theta = 0.20$		$\theta = 0.30$		$\theta = 0.40$	
		BEs	BRs	BEs	BRs	BEs	BRs
Uniform	20	0.202021	0.051039	0.292749	0.0398214	0.378952	0.0319251
	25	0.199826	0.040989	0.291018	0.0319509	0.377098	0.0269082
	30	0.198023	0.034265	0.289264	0.0267448	0.375387	0.0232795
	35	0.1973	0.029381	0.288741	0.0229466	0.375258	0.0199925
Beta	20	0.198657	0.051968	0.28807	0.0405492	0.374925	0.0352015
	25	0.197143	0.041821	0.287305	0.0324082	0.372518	0.0288082
	30	0.195737	0.038075	0.286179	0.0270362	0.371089	0.0242001
	35	0.194955	0.0284212	0.285636	0.0240927	0.3710013	0.0209829
Kumaraswamy	20	0.201382	0.0510753	0.291547	0.0398724	0.378123	0.0322510
	25	0.199327	0.0410119	0.290076	0.031983	0.3757328	0.0270023
	30	0.197615	0.0342877	0.288489	0.0267729	0.374084	0.0233108
	35	0.196949	0.0295333	0.288281	0.0230170	0.374143	0.0200129

$$L(\theta, \theta^*) = \frac{(\theta^* - \theta)^2}{\theta}. \quad (25)$$

The BE and BR under PLF are

$$\theta^* = \sqrt{E_{\theta|x}(\theta^2)},$$
$$\rho(\theta^*, \theta) = 2\left(\sqrt{E_{\theta|x}(\theta^2)} - E_{\theta|x}(\theta)\right). \quad (26)$$

The BEs and BRs under PLF using different priors are shown in Table 4.

2.8. BEs and BRs Using UP and IPs under SAPLF. For a parameter θ with a BE θ^*, PLF is defined as

$$L(\theta, \theta^*) = \frac{(\theta - \theta^*)^2}{\theta\theta^*}. \quad (27)$$

The BE and BR under PLF are

$$\theta^* = \sqrt{\frac{E_{\theta|x}(\theta)}{E_{\theta|x}(\theta^{-1})}},$$
$$\rho(\theta^*, \theta) = 2\left(\sqrt{E_{\theta|x}(\theta)E_{\theta|x}(\theta^{-1})} - 1\right). \quad (28)$$

The BEs and BRs under SAPLF using different priors are shown in Table 5.

3. Simulations Study

In this section, a thorough simulation study is carried out to check the efficiency of the Geometric Lifetime Model under the doubly type-I censoring scheme. Random samples of different sizes with various combinations of the test termination times ($T_1 = t_1$, $T_2 = t_2$) and various parametric settings are drawn from GLTM under doubly type-II censoring. The BEs and BRs are determined using the resulting mathematical expressions under various loss functions and priors, the simulation process is performed 10,000 times, and the average of the results are obtained and showcased in Tables 6–15.

The results displayed in Tables 6–15 depict that BEs approach to the true parametric values as the sample size increases. Increasing sample size has negative association with BR as it follows a decreasing trend with increasing sample size. The decreased test termination time T_1 and increased test termination time T_2 result in a smaller BR, which is obvious. While comparing the performance of different priors, it is evident from the numerical results that Beta prior outperforms the rest of the priors as it yields smaller BR. On the other hand, SELF stands higher among its competitors on the shoulder of its minimum BR.

Table 16: BEs and BRs for lung cancer data.

Prior		Square Error Loss Function		DeGroot Loss Function		QL Function		PLF		SAPL Function	
		$T_1 = 1, T_2 = 18$	$T_1 = 2, T_2 = 20$	$T_1 = 1, T_2 = 18$	$T_1 = 2, T_2 = 20$	$T_1 = 1, T_2 = 18$	$T_1 = 2, T_2 = 20$	$T_1 = 1, T_2 = 18$	$T_1 = 2, T_2 = 20$	$T_1 = 1, T_2 = 18$	$T_1 = 2, T_2 = 20$
Uniform	BEs	**0.10943995**	**0.100084949**	0.1107861	0.10974813	0.10427726	0.098640289	0.104889993	0.089123324	0.102084301	0.083242000
	BRs	0.0000873699	**0.00007212**	0.00643429	0.00617676	0.00740415	0.0072753742	0.00081908	0.000735071	0.007037100	0.007014600
Beta	BEs	**0.105619032**	**0.100059687**	0.1104137	0.1099447	0.10407617	0.09844240	0.10467841	0.088939603	0.100866000	0.083050400
	BRs	**0.000081952**	**0.00007039**	0.00635068	0.00619107	0.00742716	0.007316421	0.00081909	0.000735073	0.007053930	0.0070170370
Kumaraswamy	BEs	**0.105809484**	**0.100065142**	0.11067158	0.1093628	0.10425561	0.0986343259	0.104866822	0.089105958	0.102370630	0.083257000
	BRs	**0.000082003**	**0.000077045**	0.00653454	0.00617693	0.00740433	0.0073049555	0.00081892	0.000734940	0.007038341	0.007055840

The bold and italic values in this table shows that these are the best results as compared with counterparts.

4. Applications

To further strengthen the utility of the GLTM, a real-life data is analyzed. The data originally discussed by Krishna and Goel [16] is about the remission periods in months of 137 lung cancer patients. The numerical results for this data set are presented in Table 16.

The numerical results displayed in Table 16 for the lung cancer patients cement the findings of the simulation study.

5. Conclusion

This paper presents an estimation technique for the GLTM parameter under the doubly type-1 censoring scheme. Five different loss functions (SELF, DLF, QLF, PLF, and SAPLF) and three priors (Uniform, Beta, and Kumaraswamy priors) are considered for the estimation strategy. The strength of the estimation technique is tested through the simulation study and a real-data analysis. The numerical results, obtained for different settings, depict that BR tends to decrease when larger sample size is considered. Also, the lower termination time (t_1) and the BR are positively correlated, while the correlation between the upper termination time (t_2) and BR is negative. While overviewing the performance of different priors, Beta prior cement itself as better one among its competitor by yielding a smaller BR under all the loss functions. On the other side, SELF performs efficiently in comparison to the rest of the loss functions as it gives lower BR under all the three priors. Hence, Beta prior and SELF are suggested for estimating the parameter of GLTM while modeling real-life phenomena that are based on doubly type-AI censoring.

Data Availability

Data are available upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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