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Research Article

Heat Transfer Analysis of Generalized Nanofluid with MHD and Ramped Wall Temperature Using Caputo–Fabrizio Derivative Approach

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The aim of the present work is to apply the fractional derivative to the heat transformation of a nanofluid along with ramped wall temperature. The flow is analyzed under the effect of magnetohydrodynamic together with heat transfer. A nanofluid under the application of fractional-order differential equations by Caputo–Fabrizio derivatives with respect to time has the ability to explain the behavior of nanofluid under the influence of memory concept. For the same purpose, Caputo–Fabrizio time-fractional derivative is applied to investigate the behavior of nanoparticles on the thermal conductivity of a fluid. Appropriate nondimensional variables are engaged in the equation which governs the problem and guides us to obtain the exact solutions for the fields of velocity and temperature. These obtained solutions for the nondimensional set of governing equations are found by extracting them from the governing equations by applying Laplace transform techniques along with Caputo–Fabrizio time-fractional derivative. The influence of the fractional variable on the velocity, temperature, and Nusselt number is graphically exposed and discussed. The velocity for the state of wall temperature as ramped falls down with the enlarging values of the fractional parameter. Variation in Nusselt number is shown in the tabular form. Solutions are visualized graphically to make an analysis of how the variation is taking place in the physical behavior of the nanofluid flow with respect to the change in distinct physical parameters. The obtained results here will have useful industrial and engineering implementations. It is found that fluid velocity in the flow direction decreases with the increase in the magnetic parameter. The relationship of fractional parameter with the velocity and temperature of the nanofluid is found as direct proportional for a smaller time. However, this direct proportionality converts into inverse proportionality for larger values of time. It is observed that the increase in the nanoparticles volume fraction causes an increase in temperature distribution. It is due to lower specific heat of nanoparticles and its higher thermal conductivity than that of the base fluid.

1. Introduction

Generally, when one speaks about free convection flow, it can be discovered by the discussion and observed practically that heat when provided to the lower part of the container containing liquid reaches to the top of the liquid by the bulk motion of particles. The question here is how the motion of particles of liquid takes place when it is heated up. Actually, the heating effect lowers down the density of particles of the liquid due to which they receive more buoyant force than those of the above particles of the liquid and hence rises up. This leads us to say that the natural (free) convection is just because of the density differences during the cooling or heating activities. Similarly, the buoyant force is also a result of density differences. Any fluid contained in a component with none zero concentration gradient. This component of the fluid travels from the state of higher concentration to the state of lower concentration till the concentration gradient fall and eventually becomes zero. This process of transformation of mass due to the concentration gradient is termed as mass transfer by free convection. The knowledge about the interaction between the fields of magnetism and moving fluids that conducts these magnetic electric lines of forces (fluids containing charged particles) is termed as magnetohydrodynamics (MHD). Many natural and man-made flows are under the effect of magnetic fields. MHD has wide applications in industries to pump in, heat, stir, etc. The magnetic field produced by the fluid motion in the Earth's core is known as the terrestrial magnetic field and is mainly used in finding directions and locations on the Earth. The fluid in motion must contain the charged particles that are to be electrically conducting like molten metal, plasma whose temperature is around $10^8 k$ which is a result of highly ionized gases and strong electrolytes. MHD has also an important use in the nuclear reactor which is to remove hot plasma away from the reactor walls. Hannes Alfven (1908–1995) was the first who introduced the word magnetohydrodynamics and was dignified with the Nobel Prize for his esteemed work in the same field of study [1].

The first person who provided an exact solution for the free convection effect on the flow of fluid with nonzero viscosity and constant density because of an impulse-oriented infinite vertical plate was Soundalgekar [2]. The investigation of the natural mode of heat transfer by bulk motion of fluid particles of a viscoelastic fluid through an accelerated plate placed which bears an infinite slope under the effect of the magnetic field was done by Raptis and Singh [3]. The effects due to the natural mode of heat transfer by bulk motion of fluid particles on an exponentially accelerated plate placed which bear an infinite slope were searched by both Singh and Kumar [4]. The problem of nondimensional governing equations has been solved analytically as well as by numerical techniques. Several time-dependent flow of free convection passing through a vertical plate introduced by a variety of sets of heat flow problem restrictions at the plate with finite boundaries have been solved [5–7]. The demands for many numbers of applied problems are considered to be the wall condition they are arbitrary or nonuniform. Hayday et al. studied the free convention from a vertical flat plate with step discontinuities in surface temperature [8]. The investigation of laminar flows which are fully expanded is provided by Schetz [9]. It was Schetz who provided the approximation to the analytical solutions for the same bidimensional flow of fluid. This procedure was considered under the effect of heat. Later on, Kao [10], Kelleher [11], Lee and Yovanovich [12], refined analytically the previous solutions for the verification of various types of wall temperatures. The solution for the fluid which remains exact having a constant density in the case of the free mode of heat transfer near the vertical walls was for the first time found and further extended by Soundalgekar [2, 13]. The examination of mass transfer effects on the accelerated vertical plate in a rotating fluid with first-order chemical reaction is accomplished by Muthucumaraswamy et al. [14]. The same work about the conventional mode of transfer of heat is also investigated by Raptis and Singh [3]. They have investigated the effects of magnetic field on viscous fluid when in motion. Due to this investigation, a new direction of research has been provided to the scientists to work on many other problems related to heat in transit in the presence of magnetic fields. A solution of the vibrating flow of fluid with nonzero viscosity is obtained. It was the time during which Khaled and Vafai [15] were working on the fluid with nonzero viscosity under the effect of slip boundary conditions.

The density gradient which causes the buoyant forces induces the free convection flow in the fluid. Because of the wide range of applications, most of the physicists and chemists use the knowledge of free convection flow in the study of nature and industries. For example, it is applied in the liquefaction of crystals, the process of cooling down of electronic components, dilute acids suspensions, and chemicals when they are in the state of higher temperature [16]. Scientist faces two types of boundary restrictions in various flow-related problems namely slip boundary conditions and no-slip boundary condition on the slippage of the boundary. If there does not exist corresponding motion between the fluid and wall (plane of the plate facing the fluid), then it is termed as no-slip boundary restrictions (condition). In order to simplify a complicated situation, this condition is preferred more but it has also some hurdles. The wall slip effects, fractional derivative, Maxwell fluid, oscillating vertical plate, and the heat transfer flow all are studied with the help of a new definition of Caputo–Fabrizio fractional derivatives by Thahir et al. [17]. As it is understood, the no-slip condition is of no use to be applied in the capillaries [18] although the restrictions which are introduced by Navier in his initial effort [19] are made easy under the application of the slip condition. This condition is also termed as the Navier condition. The slip condition has many practical implementations in many fields of practical life as medical science, extrusion, lubrication, predominantly in flows through a medium having small holes in it, the process of polishing of heart valves made artificially, microfluids and nanofluid, research of friction of various surfaces and fluids related to the life cycle [20, 21]. In order to have natural convectional flow in the case of viscous fluids, one needs slip conditions which are used by many authors [22–24]. Ghara et al. [25] investigated the natural convection flow which is composed of electrically charged particles flowing along a vertically moving plate with temperature as ramped. Seth et al. [26] searched for the plate which drives fluids due its sudden change in it under the observation of Soret along with Hall effects on MHD flows and the suction knowledge in such flows throughout a plate with small holes in it is provided by Prabhakar Reddy [27]. The investigation of the fluid of second grade for MHD free and bulk motion of particles of the fluid is fulfilled by Samiulhaq et al. [28]. Radiation effects on free convection flow near a moving vertical plate with newtonian heating is investigated by Narahari and Ishaq [29]. Rajesh and Varma [30] studied the influence of radiation on MHD flow through a porous medium with a nonuniform degree of temperature. Heat transfer over an inclined sheet has been explored by Kimura et al. [31].

As in the literature it is mentioned that the nanofluid has a higher heat flow rate as compared to the common fluid that is why the nanofluid is considered for the research. Here, the base fluid we have used is water while the nanoparticles used are copper (Cu) and silver (Ag). It is assumed that at the boundary, there are no-slip conditions. The solutions of the nondimensional governing equation in its generalized form, i.e., temperature and velocity, are obtained by using the integral transform method (Laplace transform) and fractional-order derivatives which we find in the fractional calculus. During the time of seventeenth century when Newton formed the base of calculus related to the derivatives along with integral calculus, Leibniz also did a lot in the foundation of integral calculus. It was Leibniz who gave the method how to show typically the *m*th derivative of an arbitrary function g. But in 1695, when he brought into notice about this symbolic representation of the higher order derivatives in a notification to de l'Hospital, de l'Hospital thought and asked about the meaning of nth order of a derivative of a function if $n = ½$. This letter from de l'Hospital is accepted nowadays in common as the first incident of what we today call a fractional-order derivative [32].

To describe the viscoelastic behavior of the material, the calculus of fractional-order derivatives has not been avoided. In recent times, the calculus of fractional-order derivatives has been expanded in miscellaneous directions, particularly in the study of fluids in motion under the action of forces, neurons model in biological science, chemistry related to the electrolysis of the fluids like sodium chloride solutions hydrochloric acids, etc. and bioengineering. Derivatives are implemented on the large scale to formulate mathematical models of problems related to the real life. Truly speaking, the fractional-order derivative of certain significant problems is much more suitable than a derivative having integer order. All of it is because of two most important reasons firstly, no one is bound to choose a particular order for the derivative of a dependent variable and the operators of integration, and not be limited to integer order only. Secondly, the fractional-order derivatives depend on the local conditions as well as on the past, and it is also useful for the long-term memory of the system. It has numerous applications not only in the physics of modern studies but also has a wide range of applications in other sciences such as Chemistry of fluids, Ecology, Geology, Biology, etc. [33].

Flat and oscillating porous plate with time-varying surface conditions, exact and statistical computations, of radiated, convective, noncoaxial rotating nanofluids nonlinear mixed convective flow, Casson and hybrid nanofluids corresponding to heat transfer of Cu-Al_2O_3-H_2O over an infinite vertical plate in the presence of heat and radioactive sources with polar particles suspension are investigated by [34–42].

The purpose of the work in hand is to elaborate MHD in generalized form in a free mode of heat transfer from one to another place by bulk motion of particles of fluids passed a vertical plate with an infinite length, time-dependent motion under the effect of wall temperature as ramped and stepped where generalized MHD means that fractional derivative operator is applied to the constitutive equation which turns it into the generalized form, and hence, the MHD is termed as the generalized MHD.

2. Mathematical Formulation and Solution of the Problem

Consider the unsteady-free convection flow of incompressible nanofluid past, an infinite vertical plate situated in (x, y) plane of Cartesian coordinates x, y, and z. The x-axis is taken along the vertical plate, and the y-axis is taken normal to the plate. The fluid is assumed electrically conducting under influence of a uniform transverse magnetic field of strength B_0 applied parallel to the y-axis as depicted in Figure 1. The motion of the fluid is induced under the condition that no slippage of fluid is taking place over the face of the plate. In order to minimize the induced magnetic field, the magnetic Reynolds number is chosen negligibly small, and also, the external electric field is taken negligible because of the absence of applied and polarization voltage [31]. Just before getting the start of observations, the fluid and plate both are assumed to be at rest with uniform (ambient) temperature T_∞ at the boundary, and the motion of plate with oscillating velocity is mathematically given by

$$U_1(0, t_1) = U_0 f(t_1), \tag{1}$$

where U_0 represents the amplitude of the vibratory motion of the plate and $f(t) = \sin \omega t$, $f(t) = \cos \omega t$ (The case when wall temperature is kept as ramped) $f(t_1) = 1$ (The case when wall temperature is kept as stepped) where ω represents the frequency of vibration of the plate. Meanwhile, the plate is heated up or cooled down to T_w which is thereafter kept fixed. The geometrical interpretation of the problem is shown in Figure 1.

Keeping in view the above assumptions, under the application of Boussinesq approximation [43], the momentum and energy-governing equations for the time-dependent free convection flow of nanofluid for a Newtonian fluid can be expressed as follows:

$$\left(\rho_{nf}\right)\frac{\partial u_1(y_1, t_1)}{\partial t_1} = \mu_{nf}\frac{\partial^2 u_1(y_1, t_1)}{\partial y_1^2} + (\rho\beta)_{nf} g\left(T_1 - T_\infty\right) - \sigma_{nf} B_0^2 u_1(y_1, t_1), y_1, t_1 > 0. \tag{2}$$

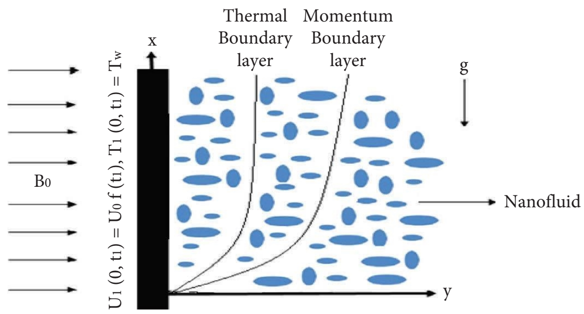


Figure 1: Geometry of the problem.

The constitutive equation for Newtonian fluid is as follows:

$$\tau_1(y_1,t_1)=\mu_{nf}\frac{\partial u_1}{\partial y_1}. \quad (3)$$

The corresponding energy equation can be expressed as follows:

$$\left(\rho C_p\right)_{nf}\frac{\partial T_1}{\partial t_1}=\kappa_{nf}\frac{\partial^2 T_1}{\partial {y_1}^2};\, t_1>0. \quad (4)$$

Assuming that, the no-slip condition exists between the plate and the fluid. The associated initial and boundary conditions are

$$\begin{aligned}
&u_1(y_1,0)=0, T_1(y_1,0)=T_\infty;\, y_1\ge 0,\\
&u_1(0,t_1)=U_1 f(t_1),\\
&T_1(0,t_1)=\begin{cases} T_\infty+(T_w-T_\infty)\dfrac{t_1}{t_0};\ 0<t_1\le t_0,\\ T_w;\ t_1>t_0,\end{cases}\quad u_1(y_1,t_1)\longrightarrow 0, T_1(y_1,t_1)\longrightarrow 0 \text{ as } y_1\longrightarrow\infty.
\end{aligned} \quad (5)$$

Here, "$u_1(y_1,t_1)$" is the velocity of the fluid, "T_1" is the fluid temperature, "T_w" is the fluid temperature at the boundary, "T_∞" is the ambient temperature, "g" is the acceleration due to gravity, "U_0" is the constant velocity of the fluid, "ρ_{nf}" is density of nanofluid, "μ_{nf}" is the dynamic viscosity of the nanofluid, "κ_{nf}" is thermal conductivity of nanofluid, "β_{nf}" is the thermal expansion coefficient of the nanofluid, "$(C_p)_{nf}$" is the specific heat of nanofluid at constant pressure, "β_f" is the thermal expansion coefficient of the base fluid. "$(\rho C_p)_{nf}$" is the heat required to produce a degree rise in temperature of the nanofluid, and "κ_{nf}" and "σ_{nf}" are the thermal conductivity and electrical conductivity of nanofluid, respectively.

Some physical properties of base fluids and nanoparticles, **Ag** and **Cu**, are shown in the form of Table 1 [43–45].

Writing in dimensionless form by using the following dimensionless parameters in equation (2), and neglecting subscripts notation, the governing equation and conditions for the velocity field become

$$\begin{aligned}
&t=\frac{U_0^2 t_1}{v_f}, y=\frac{U_0 y_1}{v_f}, u=\frac{u_1}{U_0}, \theta=\frac{T_1-T_\infty}{T_w-T_\infty}, \tau=\frac{v_f\tau_1}{\mu_{nf}U_0^2}, \omega=\frac{v_f}{U_0^2}\omega_1,\\
&J_1\rho_f\frac{U_0^3}{v_f}\frac{\partial u}{\partial t}=J_2\frac{U_0^3\mu_f}{v_f^2}\frac{\partial^2 u}{\partial y^2}+J_3(\rho\beta)_f g(T-T_\infty)\theta(y,t)-J_4\sigma_f B_0^2 U_0 u(y,t);\, y,t>0,\\
&J_1\frac{\partial u}{\partial t}=J_2\frac{\partial^2 u}{\partial y^2}+J_3 Gr\theta(y,t)-J_4 Mu(y,t);\, y,t>0,
\end{aligned} \quad (6)$$

where each J_1-J_5 is the function of the thermophysical properties of the nanofluid. The expressions of these parameters are given as follows [46]:

Table 1: Physical properties of base fluid and nanoparticles.

Physical properties	Base fluid			Nanoparticles	
	Water	Kerosene	Ethylene glycol	Cu	Ag
ρ (kg/m^3)	997.1	780	1115	8933	10500
c_p (J/kg·K)	4179	2090	2386	385	235
κ (W/m·K)	0.613	0.149	0.2499	401	429
β (1/K)	0.0002	0.00099	0.341	0.167	0.000189
σ (Sm^{-1})	200	—	—	59000000	63000000

$$\begin{aligned}
&J_1 = \frac{\rho_{nf}}{\rho_f} = (1-\varphi) + \varphi\left(\frac{\rho_s}{\rho_f}\right), J_2 = \frac{\mu_{nf}}{\mu_f} = \frac{1}{(1-\varphi)^{2.5}},\\
&J_3 = \frac{(\rho\beta)_{nf}}{(\rho\beta)_f} = (1-\varphi) + \varphi\frac{(\rho\beta)_s}{(\rho\beta)_f}, J_4 = \frac{\sigma_{nf}}{\sigma_f} = \left[1 + \frac{3(\sigma-1)\varphi}{(\sigma+2)-(\sigma-1)\varphi}\right],\\
&M = \frac{\upsilon_f B_0^2 \sigma_f}{U_0^2 \rho_f}, Gr = \frac{g\beta_f (T_w - T_\infty)\upsilon_f}{U_0^3},\\
&J_5 = \frac{\kappa_{nf}}{\kappa_f} = \frac{\kappa_s + 2\kappa_f - 2\varphi(\kappa_f - \kappa_s)}{\kappa_s + 2\kappa_f + \varphi(\kappa_f - \kappa_s)}, \sigma = \frac{\sigma_s}{\sigma_f},
\end{aligned} \tag{7}$$

where φ is the nanoparticle volume fraction, and ρ_s is the density of the solid nanoparticles used. The specific heat at a constant pressure of the base fluid is denoted by c_p. These constants and their expressions are the same as those by Abbas and Magdy [38]. σ_{nf} is electrical conductivity of the nanofluid, B_0 is magnetic field strength, υ_f is the kinematic viscosity of base fluid, and T_w is the plate temperature. Equation (3) in its dimensionless form can be written as follows:

$$\tau(y,t) = \frac{\partial u(y,t)}{\partial y}. \tag{8}$$

This equation (8) is the constitutive equation in its dimensionless form. Equation (4) in its dimensionless form can be written as follows:

$$\Pr\frac{\partial}{\partial t}\theta(y,t) = J_5\frac{\partial^2\theta}{\partial y^2}, \tag{9}$$

where $\Pr = \mu C_p/k$, $(\rho C_p)_{nf} = (1-\varphi)(\rho C_p)_f + \varphi(\rho C_p)_s$

It is clear from these expressions that equation (9) is a function of nanoparticle volume fraction. The initial and boundary conditions in its nondimensional form are shown in (10) while the generalized fractional constitutive equation is given as follows:

$$\begin{aligned}
&u(y,0) = 0, \theta(y,0) = 0 u(0,t) = U_1 f(t)\theta(0,t) = \begin{cases} t; & 0 < t \le 1, \\ 1; & t > 1, \end{cases} \quad u(0,t) \longrightarrow 0, \theta(y,t) \longrightarrow 0 \text{ as } y \longrightarrow \infty,\\
&\tau(y,t) = {}^{CF}D_t^{\alpha}\frac{\partial u}{\partial y}.
\end{aligned} \tag{10}$$

The generalized fractional momentum and energy governing equations for time-dependent free convective flow of nanofluid for a Newtonian fluid can be expressed as follows:

$$J_1\frac{\partial u}{\partial t} = J_2\frac{\partial}{\partial y}\left[{}^{CF}D_t^{\alpha}\frac{\partial u}{\partial y}\right] + J_3 Gr\theta(y,t) - J_4 M u(y,t); y, t > 0. \tag{11}$$

3. Analytical Solution by Laplace Transform

3.1. Solution for Ramped Temperature. The use of Laplace transformations is an easy and one of the typical mathematical techniques for the solution related to various chemical and engineering problems described which may be ordinary or partial differential equations. If $f(t)$ is a piecewise continuous function in the time domain and is of exponential order, then its Laplace transform $\overline{f}(s)$ is defined as follows:

$$£\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t)\text{dt}. \quad (12)$$

It is to be noted that this approach of Laplace transform is universally not applicable [31].

Its restrictions which occur by nature are obtained by the potential for solving the related equations in the domain of Laplace transform and the successive application of the inverse transformation of the solution obtained while, in many technological problems operated by equations containing ordinary or partial derivatives with unchanging coefficients, it, however, does not create substantial hurdles to obtain the subsequent Laplace domain solution, such solutions are relatively problematical functions depending on, and their inverse transformation analytically is extremely difficult if these are possible. However, the numerical techniques are applied to overcome such type of problems. But for the case here, the inversion of Laplace transforms is effectively obtained. For the mathematical model of the problem in hand to convert into a generalized MHD fractional model, the change is taking place only in the constitutive equation. For this, Caputo–Fabrizio derivative is applied under the operator of the Laplace transform to equation (8) and then taking the Laplace transform of both sides.

$$£\left\{{}^{\text{CF}}D_t^\alpha f(t)\right\} = \frac{1}{1-\alpha}\int_0^\infty e^{-pt}\int_0^t \dot{f}(\tau)\, e^{-\alpha(\text{t}-\tau)/1-\alpha} d\tau\, dt. £\left\{{}^{\text{CF}}D_t^\alpha f(t)\right\} = \frac{1}{1-\alpha}£\left\{\dot{f}(\tau)\right\}£\left\{e^{-\alpha(\text{t}-\tau)/1-\alpha}\right\}. \quad (13)$$

Simplifying by using initial and boundary conditions with replacement of $f(t)\, by \quad u(\text{y},\text{t})$ and then plug it into equation (8) which takes the form

$$\tau(\text{y},\text{s}) = \frac{s}{\alpha + s(1-\alpha)}\frac{d}{dy}u(\text{y},\text{s}). \quad (14)$$

This equation is in agreement with the one obtained by Samiulhaq et al. [47].

Applying Laplace transforms to equation (11) and using the above result (11), it takes the form as follows:

$$J_1 su(\text{y},\text{s}) = \frac{J_2 s}{[\alpha + s(1-\alpha)]}\frac{d^2}{dy^2}u(\text{y},\text{s}) + J_3 Gr\theta(y,\text{s}) - J_4 Mu(y,\text{s}); \quad y,t>0. \quad (15)$$

Applying Laplace transform to equation (9) and writing it in its simplified form.

$$\theta(\text{y},\text{s}) = \frac{1-e^{-s}}{s^2}e^{-y\sqrt{s\frac{\text{Pr}}{J_5}}}. \quad (16)$$

The transformation of equation (16) back to the parameter "t" by using the inverse operator of Laplace transforms which lead us to the solution for temperature distribution is obtained as follows:

$$\theta(\text{y},\text{t}) = \left(\frac{\text{Pr}/J_5 y^2}{2} + t\right)erfc\left(\frac{y\sqrt{\text{Pr}/J_5}}{2\sqrt{t}}\right) - \sqrt{\frac{\text{Prt}}{\pi}}ye^{-\left(\frac{\text{Pr}/J_5 y^2}{4t}\right)}$$
$$-\left(\left(\frac{\text{Pr}/J_5 y^2}{2} + t - 1\right)erfc\left(\frac{\sqrt{\text{Pr}/J_5}y}{2\sqrt{(t-1)}}\right) - \sqrt{\frac{\text{Pr}/J_5(\text{t}-1)}{\pi}}ye^{-\left(\frac{\text{Pr}/J_5 y^2}{4(t-1)}\right)}\right)H(t-1). \quad (17)$$

Put equation (16) in equation (15) after simplification and using the Laplace transform applied to the Caputo–Fabrizio fractional derivative as under

$$£\left[{}^{\mathrm{CF}}D_t^{\alpha} f(t)\right] = \frac{s£[\mathrm{f}(\mathrm{t})] - \mathrm{f}(0)}{s + \alpha(1-s)}. \tag{18}$$

Equation (15) becomes

$$\begin{aligned} \mathrm{u}(\mathrm{y},\mathrm{s}) &= \mathrm{u}_1 f(\mathrm{s}) \mathrm{e}^{-ya\sqrt{\frac{(\mathrm{s}+J_4/J_1 M)(\mathrm{s}+\mathrm{b})}{s}}} \\ &\quad - J_3 J_5 Gr \frac{[\alpha + s(1-\alpha)](1-e^{-s})}{\left[\mathrm{Pr} J_2 \mathrm{s}^2 - J_1 J_5 (\mathrm{s}+J_4/J_1 M)(\mathrm{s}+\mathrm{b})(1-\alpha)\right]\mathrm{s}^2} \mathrm{e}^{-ya\sqrt{(\mathrm{s}+J_4/J_1 M)(\mathrm{s}+\mathrm{b})/s}} \\ &\quad + J_3 J_5 Gr (1-\alpha) \frac{(\mathrm{s}+\mathrm{b})(1-e^{-s})}{\left[\mathrm{Pr} J_2 \mathrm{s}^2 - (\mathrm{s}+J_1 J_3 M)(\mathrm{s}+\mathrm{b})\right]\mathrm{s}^2} \mathrm{e}^{-y\sqrt{s\mathrm{Pr}/J_5}}, \end{aligned} \tag{19}$$

where $a = \sqrt{J_1(1-\alpha)/J_2}$, $b = \alpha/1-\alpha\ 0<\alpha<1$. Where "$\alpha$" is a fractional order of the Caputo–Fabrizio derivative.

The compound function on the right-hand side of equation (19) is

$$\mathrm{A}(\mathrm{y},\mathrm{w}(\mathrm{s})) = \mathrm{e}^{-ya\sqrt{(\mathrm{s}+J_4/J_1 M)(\mathrm{s}+\mathrm{b})/s}} = \mathrm{e}^{-ya\sqrt{w(\mathrm{s})/s}}, \tag{20}$$

where $w(\mathrm{s}) = (\mathrm{s}+J_4/J_1 M)(\mathrm{s}+\mathrm{b})$.

Laplace inverse transform of a compound function $\mathrm{A}(\mathrm{y},\mathrm{w}(\mathrm{s}))$ implies

$$\begin{aligned} £^{-1}[A(y,w(s))] &= f(t) * A(y,t), \\ A(y,t) &= £^{-1}[A(y,w(s))] \\ &= e^{-u\left(b+(J_4/J_1)M\right)} \int_0^{\infty} \frac{ay}{2u\sqrt{\pi u}} e^{-(a^2y^2/4u)} \left[\delta(t-u) + \int_0^t \delta(t-z-u)\sqrt{\frac{ub((J_4/J_1)M)}{z}} I_1\left(2\sqrt{ub\left(\frac{J_4}{J_1}M\right)z}\right)\mathrm{d}z\right]\mathrm{d}u, \end{aligned} \tag{21}$$

where the "$*$" sign represents the convolution of two functions.

Equation (19) can be written in a simplified form as

$$\begin{aligned} \mathrm{u}(\mathrm{y},\mathrm{s}) &= \mathrm{u}_1 f(\mathrm{s}) A(\mathrm{y},\mathrm{w}(\mathrm{s})) + \frac{a_2 Gr}{a_{02}(c+1)}\left[\frac{a_7}{s} + \frac{a_8}{s+a_5} + \frac{a_9}{s+a_6} - \left(\frac{a_7}{s} + \frac{a_8}{s+a_5} + \frac{a_9}{s+a_6}\right)\mathrm{e}^{-s}\right] A(\mathrm{y},\mathrm{w}(\mathrm{s})) \\ &\quad + \frac{a_2 bGr}{a_{02}(c+1)}\left[\frac{a_{10}}{s} + \frac{a_{11}}{s^2} + \frac{a_{12}}{s+a_5} + \frac{a_{13}}{s+a_6} - \left(\frac{a_{10}}{s} + \frac{a_{11}}{s^2} + \frac{a_{12}}{s+a_5} + \frac{a_{13}}{s+a_6}\right)\mathrm{e}^{-s}\right] A(\mathrm{y},\mathrm{w}(\mathrm{s})) \\ &\quad + \frac{a_2 Gr}{a_{02}(c+1)}\left[\frac{a_7}{s} + \frac{a_8}{s+a_5} + \frac{a_9}{s+a_6} - \left(\frac{a_7}{s} + \frac{a_8}{s+a_5} + \frac{a_9}{s+a_6}\right)\mathrm{e}^{-s}\right] \mathrm{e}^{-y\sqrt{\mathrm{Pr}/J_5}} \\ &\quad + \frac{a_2 Grb}{a_{02}(c+1)}\left[\frac{a_{10}}{s} + \frac{a_{11}}{s^2} + \frac{a_{12}}{s+a_5} + \frac{a_{13}}{s+a_6} - \left(\frac{a_{10}}{s} + \frac{a_{11}}{s^2} + \frac{a_{12}}{s+a_5} + \frac{a_{13}}{s+a_6}\right)\mathrm{e}^{-s}\right] \mathrm{e}^{-y\sqrt{s\mathrm{Pr}/J_5}}, \end{aligned} \tag{22}$$

where $\mathrm{c} = J_2\mathrm{Pr}/a_2$, $a_0 = \sqrt{J_1(1-\alpha)/J_2}$, $a_1 = \frac{J_4}{J_1}$, $a_2 = -J_3 J_5(1-\alpha)$, $a_{02} = -J_1 J_5(1-\alpha)$, $a_3 = a_1 M + b/2(c+1)$, $a_4 = \sqrt{(a_1 M + b)^2 - 4a_1 Mb(c+1)/4(c+1)^2}$, $a_5 = a_3 + a_4$, $a_6 = a_3 - a_4$, $a_7 = 1/a_5 a_6$, $a_8 = 1/a_5(a_5 - a_6)$, $a_9 = 1/a_6(a_6 - a_5)$, $a_{10} = 1/a_5 a_6$, $a_{11} = b/a_5^2(a_6 - a_5)$, $a_{12} = b/a_6^2(a_5 - a_6)$, $a_{13} = b/(a_6 - a_5)(a_6^2 - a_5^2/a_6^2 a_5^2)$

$$f_1(\mathrm{y},\mathrm{s}) = \frac{1}{s}e^{-y\sqrt{s}\sqrt{\Pr/J_5}},\ f_2(\mathrm{y},\mathrm{s}) = \frac{1}{s^2}e^{-y\sqrt{s}\sqrt{\Pr/J_5}},$$
$$f_3(\mathrm{y},\mathrm{s}) = \frac{1}{s+a_5}e^{-y\sqrt{s}\sqrt{\Pr/J_5}},\ f_4(\mathrm{y},\mathrm{s}) = \frac{1}{s+a_6}e^{-y\sqrt{s}\sqrt{\Pr/J_5}}. \tag{23}$$

Taking the inverse Laplace Transform of this equation (22) which results into

$$\begin{aligned}
u(y,t) = {} & f(t) * A(y,t) + \frac{Gra_2}{a_{02}}\left(\frac{a_7}{c+1} + \frac{ba_{10}}{c+1}\right)[c_1(y,t) - c_1(y,t-1)H(t-1)] \\
& + \frac{ba_{11}}{c+1}[c_2(y,t) - c_2(y,t-1)H(t-1)] + \frac{Gra_2}{a_{02}}\left(\frac{a_8}{c+1} + \frac{ba_{12}}{c+1}\right)[c_3(y,t) - c_3(y,t-1)H(t-1)] \\
& + \frac{Gra_2}{a_{02}}\left(\frac{a_9}{c+1} + \frac{ba_{13}}{c+1}\right)[c_4(y,t) - c_4(y,t-1)H(t-1)] \\
& + \frac{Gra_2}{a_{02}}\left(\frac{a_7}{c+1} + \frac{ba_{10}}{c+1}\right)[f_1(y,t) + f_1(y,t-1)H(t-1)] \\
& + \frac{a_2Grba_{11}}{a_{02}(c+1)}[f_2(y,t) + f_2(y,t-1)H(t-1)] \\
& + \frac{Gra_2}{a_{02}}\left(\frac{a_8}{c+1} + \frac{ba_{12}}{c+1}\right)[f_3(y,t) + f_3(y,t-1)H(t-1)] \\
& + \frac{Gra_2}{a_{02}}\left(\frac{a_9}{c+1} + \frac{a_{13}b}{c+1}\right)[f_4(y,t) + f_4(y,t-1)H(t-1)], \\
u(y,t) = {} & f(t) * A(y,t) + a_{14}[c_1(y,t) - c_1(y,t-1)H(t-1)] \\
& + a_{15}[c_2(y,t) - c_2(y,t-1)H(t-1)] + a_{16}[c_3(y,t) - c_3(y,t-1)H(t-1)] \\
& + a_{17}[c_4(y,t) - c_4(y,t-1)H(t-1)] + a_{18}[f_1(y,t) + f_1(y,t-1)H(t-1)] \\
& + a_{19}[f_2(y,t) + f_2(y,t-1)H(t-1)] + a_{20}[f_3(y,t) + f_3(y,t-1)H(t-1)] \\
& + a_{21}[f_4(y,t) + f_4(y,t-1)H(t-1)],
\end{aligned}$$

Where $c_1(y,t) = \pounds^{-1}\left[\frac{1}{s}A(y,w(s))\right] = 1 * A(y,t) = \int_0^t A(y,\xi)\mathrm{d}\xi,$

$$c_2(y,t) = \pounds^{-1}\left[\frac{1}{s^2}A(y,w(s))\right] = t * A(y,t) = \int_0^t (t-\xi)A(y,\xi)\mathrm{d}\xi,$$

$$c_3(y,t) = \pounds^{-1}\left[\frac{1}{s+a_5}A(y,w(s))\right] = e^{-a_5t} * A(y,t) = \int_0^t e^{a_5(t-\xi)}A(y,\xi)\mathrm{d}\xi,$$

$$c_4(y,t) = \pounds^{-1}\left[\frac{1}{s+a_6}A(y,w(s))\right] = e^{a_6t} * A(y,t) = \int_0^t e^{a_6(t-\xi)}A(y,\xi)\mathrm{d}\xi,$$

$$
\begin{aligned}
f_1(y,t) &= erfc\left(\frac{y\sqrt{(\Pr/J_5)}}{2\sqrt{t}}\right),\\
f_2(y,t) &= \left[\frac{1}{\sqrt{t}}erfc\left(\frac{y\sqrt{(\Pr/J_5)}}{2\sqrt{t}}\right) - \frac{e^{-\left(y^2(\Pr/J_5)/4t\right)}}{2\sqrt{\pi}}\right],\\
f_3(y,t) &= \frac{e^{-a_5 t}}{2}\left[e^{\left(-iy\sqrt{(\Pr/J_5)}\sqrt{a_5}\right)}erfc\left(\frac{y\sqrt{(\Pr/J_5)}}{\sqrt{t}} - i\sqrt{a_5 t}\right) + e^{\left(iy\sqrt{(\Pr/J_5)}\sqrt{a_5}\right)}erfc\left(\frac{y\sqrt{(\Pr/J_5)}}{2\sqrt{t}} + i\sqrt{a_5 t}\right)\right],\\
f_4(y,t) &= \frac{e^{-a_6 t}}{2}\left[e^{\left(-iy\sqrt{(\Pr/J_5)}\sqrt{a_6}\right)}erfc\left(\frac{y\sqrt{(\Pr/J_5)}}{\sqrt{t}} - i\sqrt{a_6 t}\right) + e^{\left(iy\sqrt{(\Pr/J_5)}\sqrt{a_6}\right)}erfc\left(\frac{y\sqrt{(\Pr/J_5)}}{2\sqrt{t}} + i\sqrt{a_6 t}\right)\right].\\
a_{14} &= \frac{a_2}{a_{02}}\left(\frac{Gra_7}{c+1} + \frac{Grba_{10}}{c+1}\right),\\
a_{15} &= \frac{ba_{11}}{c+1},\\
a_{16} &= \frac{a_2}{a_{02}}\left(\frac{Gra_8}{c+1} + \frac{Grba_{12}}{c+1}\right),\\
a_{17} &= \frac{a_2}{a_{02}}\left(\frac{Gra_9}{c+1} + \frac{Grba_{13}}{c+1}\right),\\
a_{18} &= \frac{a_2}{a_{02}}\left(\frac{Gra_7}{c+1} + \frac{Grba_{10}}{c+1}\right),\\
a_{19} &= \frac{a_2 Grba_{11}}{a_{02}(c+1)},\\
a_{20} &= \frac{a_2}{a_{02}}\left(\frac{Gra_8}{c+1} + \frac{Grba_{12}}{c+1}\right),\\
a_{21} &= \frac{a_2}{a_{02}}\left(\frac{a_9 Gr}{c+1} + \frac{a_{13}bGr}{c+1}\right).
\end{aligned}
\tag{24}
$$

3.2. Nusselt Number. In this section, the expressions for the Nusselt number are presented when the wall temperature is kept as ramped and are termed as the measures of the heat transfer in unit time on the boundary.

$$
\begin{aligned}
Nu(\mathrm{t}) &= -\frac{\partial\theta(y,\mathrm{t})}{\partial y}\Big|_{y=0};\ t>0,\\
Nu(\mathrm{t}) &= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\Pr}{J_5}}\left[\sqrt{t} - \sqrt{t-1}\,H(\mathrm{t}-1)\right].
\end{aligned}
\tag{25}
$$

3.2.1. Solution for Constant Temperature. In order to explore the temperature of the walls as constant in the presence of fluid on the borders, it is necessary to match up to the motion of the fluid along a plate moving with uniform speed. Under these circumstances, it can be seen that both the temperature and velocity variables in their dimensionless form can be obtained by using Laplace transform operator to equations (6) and (9), and using initial along with boundary condition (10), equations (6) and (9) become

$$
J_1\frac{\partial u}{\partial t} = J_2\frac{\partial}{\partial y}\left[{}^{\mathrm{CF}}D_t^{\alpha}\frac{\partial u}{\partial y}\right] + J_3 Gr\theta(y,t) - J_4 Mu(y,t);\ y,t>0,
\tag{26}
$$

$$
\theta(\mathrm{y},\mathrm{s}) = \frac{1}{s}e^{-y\sqrt{s\Pr/J_5}}.
\tag{27}
$$

The transformation of equation (26) back to the parameter "t" by using the inverse operator of Laplace transforms which lead us to the solution for the distribution of the temperature field is calculated as follows:

$$
\theta(\mathrm{y},\mathrm{t}) = erfc\left(\frac{y}{2\sqrt{t}}\sqrt{\frac{\Pr}{J_5}}\right).
\tag{28}
$$

Put equation (27) in equation (26) after simplification, solving and using

$$
£\left[{}^{\mathrm{CF}}D_t^{\alpha}f(\mathrm{t})\right] = \frac{s£[\mathrm{f}(\mathrm{t})] - \mathrm{f}(0)}{\alpha + s(1-\alpha)}.
\tag{29}
$$

The solution of equation (26) subject to the transform condition of equation (10) is

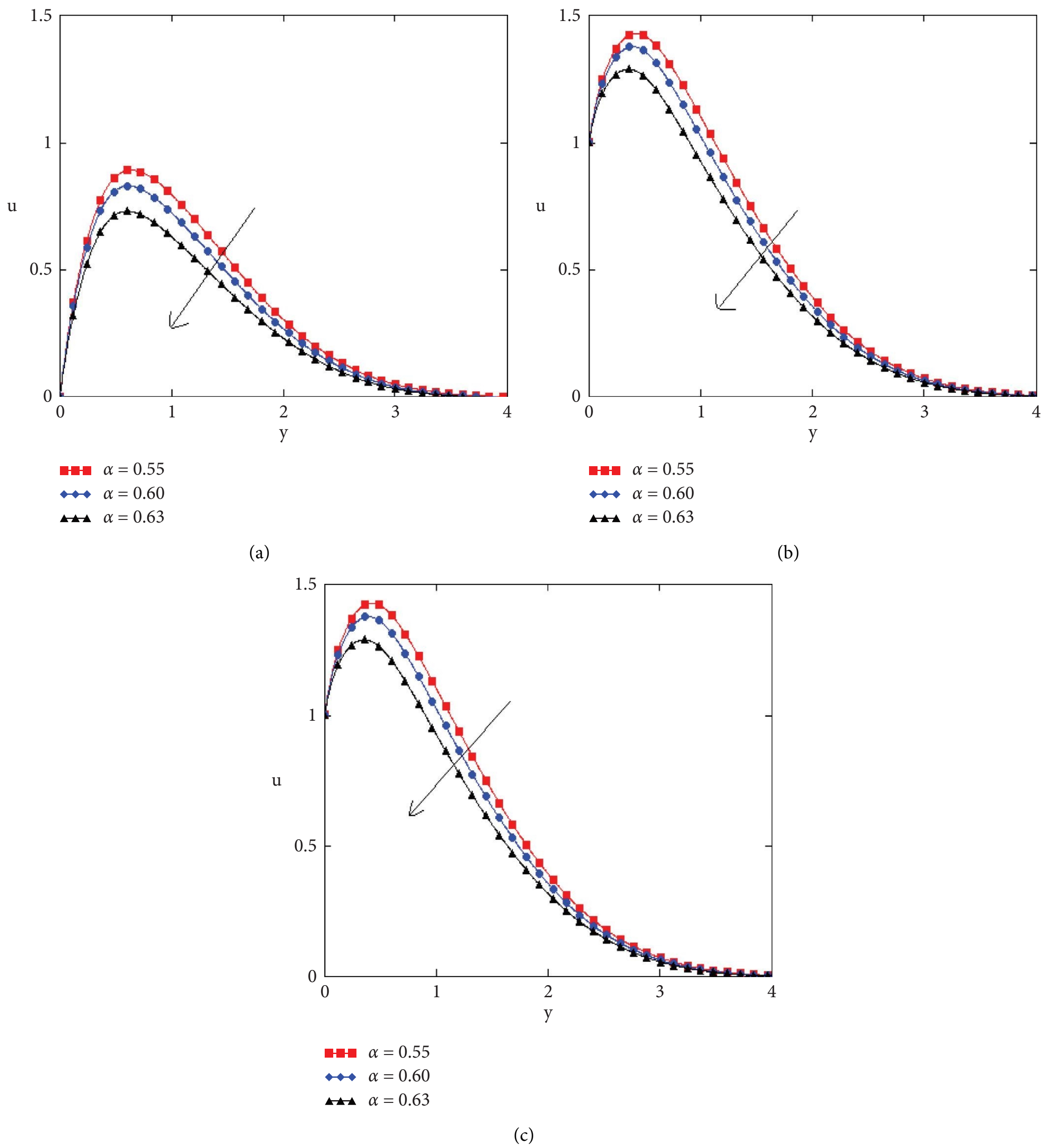


Figure 2: Nondimensional velocity outline for variation in α with $M = 0.4, t = 1$ (a) $f(t) = \sin \omega t$ (b) $f(t) = \cos \omega t$ and (c) $f(t) = 1$ (b) corresponding to ramped temperature of the plate.

$$
\begin{aligned}
\mathrm{u}(\mathrm{y}, \mathrm{s}) = {} & \mathrm{u}_1 f(\mathrm{s}) A(\mathrm{y}, \mathrm{w}(\mathrm{s})) + \frac{a_2 Gr}{a_{02}(c+1)}\left[\frac{a_{22}}{s+a_5} + \frac{a_{23}}{s+a_6}\right] A(\mathrm{y}, \mathrm{w}(\mathrm{s})) \\
& + \frac{a_2 Grb}{a_{02}(c+1)}\left[\frac{a_7}{s} + \frac{a_8}{s+a_5} + \frac{a_9}{s+a_6}\right] A(\mathrm{y}, \mathrm{w}(\mathrm{s})) + \frac{a_2 Gr}{a_{02}(c+1)}\left[\frac{a_{22}}{s+a_5} + \frac{a_{23}}{s+a_6}\right] e^{-y\sqrt{s\mathrm{Pr}/J_5}} \\
& + \frac{a_2 Grb}{a_{02}(c+1)}\left[\frac{a_7}{s} + \frac{a_8}{s+a_5} + \frac{a_9}{s+a_6}\right] e^{-y\sqrt{s\mathrm{Pr}/J_5}},
\end{aligned}
\tag{30}
$$

where $a_{22} = 1/a_6 - a_5, a_{23} = 1/a_5 - a_6$.

Taking the Laplace inverse of the equation (30) takes the form

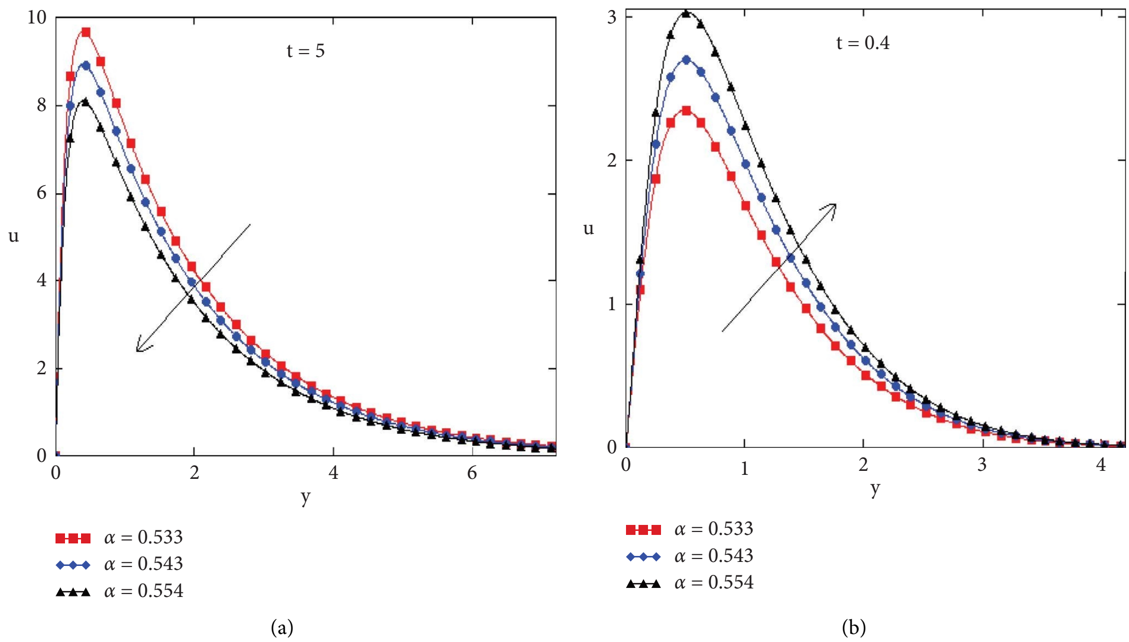


FIGURE 3: Nondimensional velocity outline for α (a) for large time $f(t) = \sin \omega t$ and (b) for small time $f(t) = \sin \omega t$ corresponding to temperature as (a) stepped and (b) ramped of the plate.

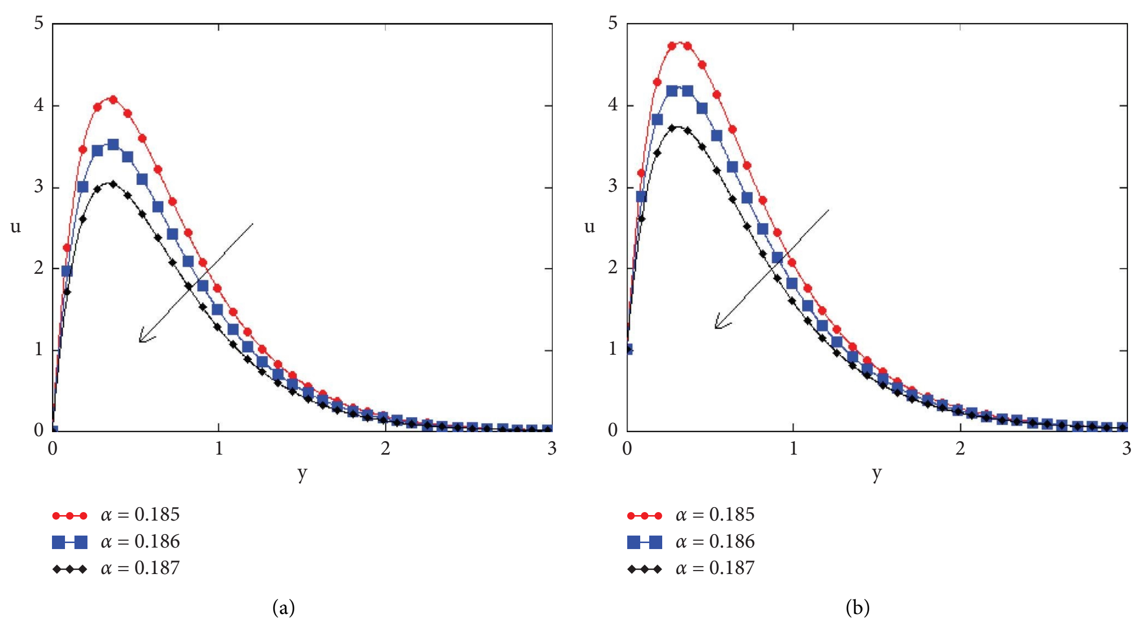
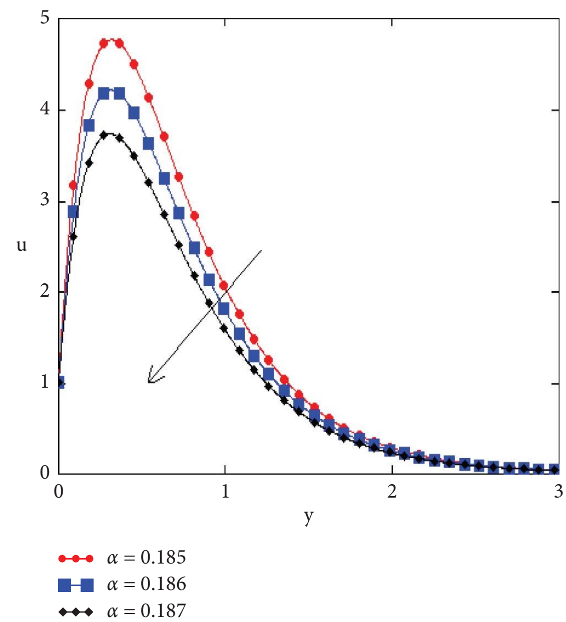
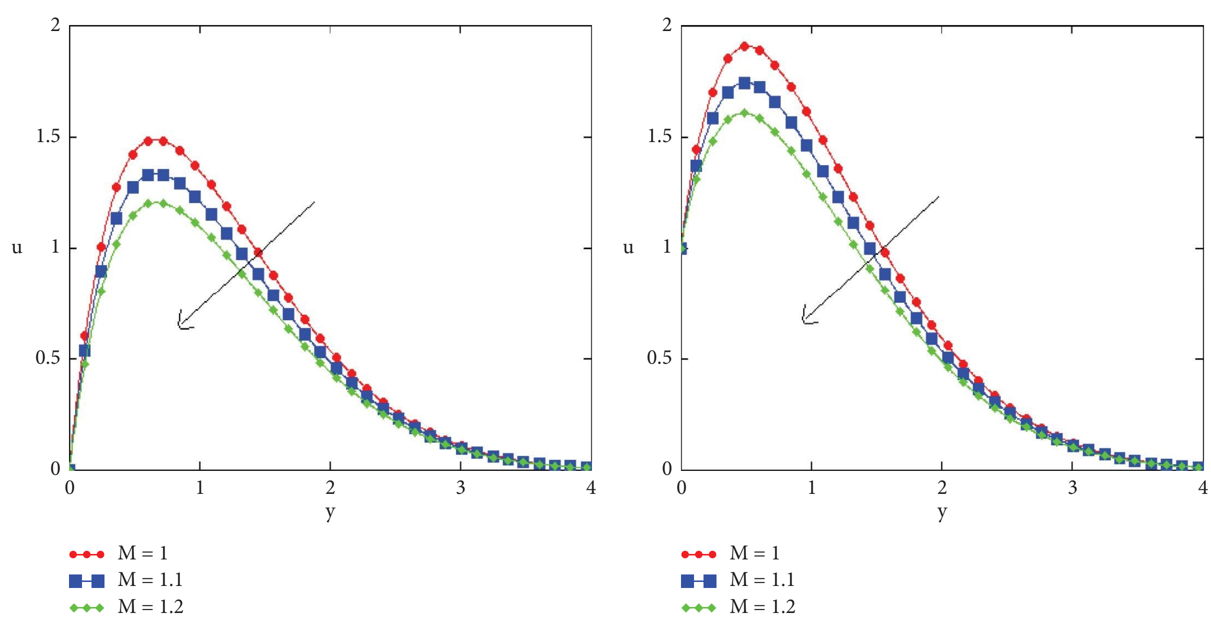


FIGURE 4: Continued.



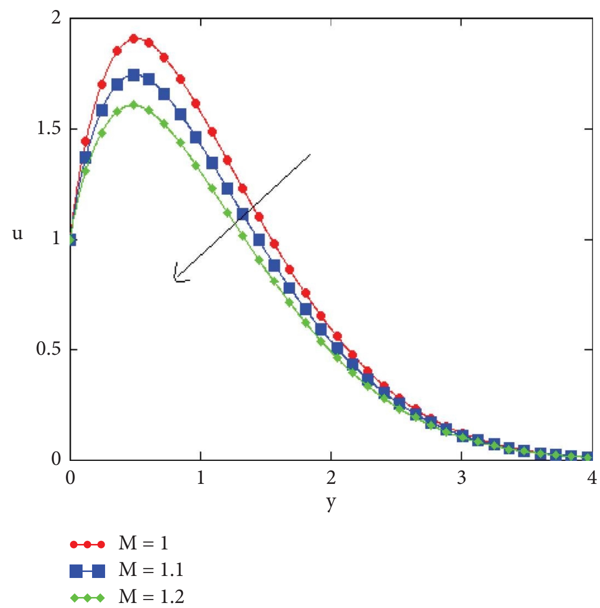
(c)

Figure 4: Nondimensional velocity versus fractional parameter outline for variation in α with $M = 0.6, t = 1$: (a) $f(t) = \sin \omega t$, (b) $f(t) = \cos \omega t$, and (c) $f(t) = 1$ analogous to temperature of the plate as stepped.



(a) (b)

Figure 5: Continued.



(c)

FIGURE 5: Nondimensional velocity outline for variation in M with $t = 1$: (a) $f(t) = \sin \omega t$, (b) $f(t) = \cos \omega t$, and (c) $f(t) = 1$ corresponding to temperature as ramped of the plate.

$$
\begin{aligned}
\mathrm{u}(\mathrm{y},\mathrm{t}) &= f(\mathrm{t}) * \mathrm{A}(\mathrm{y},\mathrm{t}) + \left[\frac{a_2 Gr}{a_{02}(c+1)} a_{22}\{\mathrm{c}_3(\mathrm{y},\mathrm{t}) + f_3(\mathrm{y},\mathrm{t})\} + \frac{a_2 Gr}{a_{02}(c+1)} a_{23}\{\mathrm{c}_4(\mathrm{y},\mathrm{t}) + f_4(\mathrm{y},\mathrm{t})\}\right] \\
&+ \left[\frac{a_2 Grb}{a_{02}(c+1)} a_7\{\mathrm{c}_1(\mathrm{y},\mathrm{t}) + f_1(\mathrm{y},\mathrm{t})\} + \frac{a_2 Grb}{a_{02}(c+1)} a_8\{\mathrm{c}_3(\mathrm{y},\mathrm{t}) + f_3(\mathrm{y},\mathrm{t})\} + \frac{a_2 Grb}{a_{02}(c+1)} a_9\{\mathrm{c}_4(\mathrm{y},\mathrm{t}) + f_4(\mathrm{y},\mathrm{t})\}\right] \\
\mathrm{u}(\mathrm{y},\mathrm{t}) &= f(\mathrm{t}) * \mathrm{A}(\mathrm{y},\mathrm{t}) + a_{24}\{\mathrm{c}_3(\mathrm{y},\mathrm{t}) + f_3(\mathrm{y},\mathrm{t})\} + a_{25}\{\mathrm{c}_4(\mathrm{y},\mathrm{t}) + f_4(\mathrm{y},\mathrm{t})\} \\
&+ a_{26}\{\mathrm{c}_1(\mathrm{y},\mathrm{t}) + f_1(\mathrm{y},\mathrm{t})\} + a_{27}\{\mathrm{c}_3(\mathrm{y},\mathrm{t}) + f_3(\mathrm{y},\mathrm{t})\} + a_{28}\{\mathrm{c}_4(\mathrm{y},\mathrm{t}) + f_4(\mathrm{y},\mathrm{t})\}
\end{aligned} \tag{31}
$$

Where $a_{24} = \frac{a_2 Gr}{a_{02}(c+1)} a_{22}, a_{25} = \frac{a_2 Gr}{a_{02}(c+1)} a_{23}, a_{26} = \frac{a_2 Grb}{a_{02}(c+1)} a_7, a_{27} = \frac{a_2 Grb}{a_{02}(c+1)} a_8, a_{28} = \frac{a_2 Grb}{a_{02}(c+1)} a_9.$

3.3. Nusselt Number. In this section, an expression for an important number named as Nusselt is derived in the case of constant temperature which are the measures of the heat transfer rate at the boundary

$$
\begin{aligned}
Nu(\mathrm{t}) &= -\frac{\partial}{\partial y}\theta(y,\mathrm{t})|_{y=0}\,; t > 0, \\
Nu(\mathrm{t}) &= \sqrt{\frac{\mathrm{Pr}}{J_5 \pi t}}.
\end{aligned} \tag{32}
$$

4. Results and Discussions

The time-dependent MHD corresponding to its free convection flow in the case of nanofluid is calculated analytically by using the technique of the Laplace transform. The effect of different physical parameters such as fractional parameter(α), Prandtl number of nanofluid (Pr), and MHD (M) on the profiles of nanofluid velocity component and temperature along with the number named as Nusselt number (Nu) is existing in Figures 2–8 for ramped and stepped wall temperature. Figure 2 represents the velocity profile for the case of ramped wall temperature and is increasing with the increasing value of the fractional parameter α which is the order of fractional derivative.

Figure 3 represents stepped wall temperature in which as well the velocity increases with the decreasing value of fractional parameter α for large time, and there is a direct relationship between velocity and the parameter α for smaller time. The dual behavior of velocity with time is also presented by Raza et al. [48]. This behavior validates the present work. Figure 4 represents stepped and ramped wall temperature in which as well the velocity increases with the

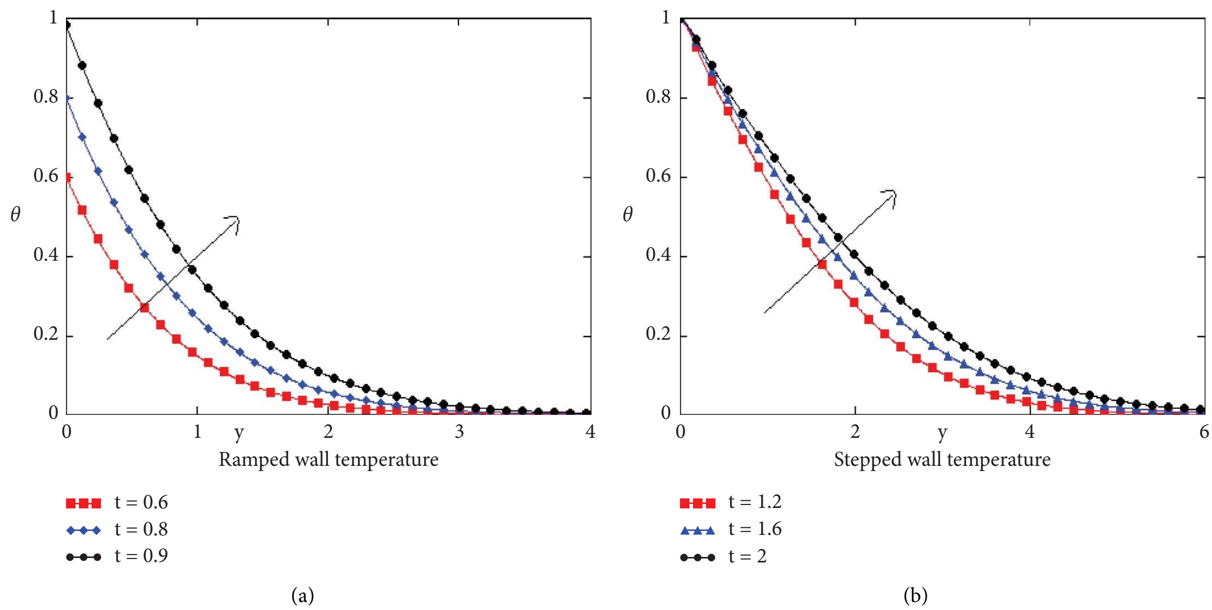


FIGURE 6: Nondimensional temperature outline for variation in t: (a) corresponding to the plate temperature as ramped and (b) corresponding to the plate temperature as stepped.

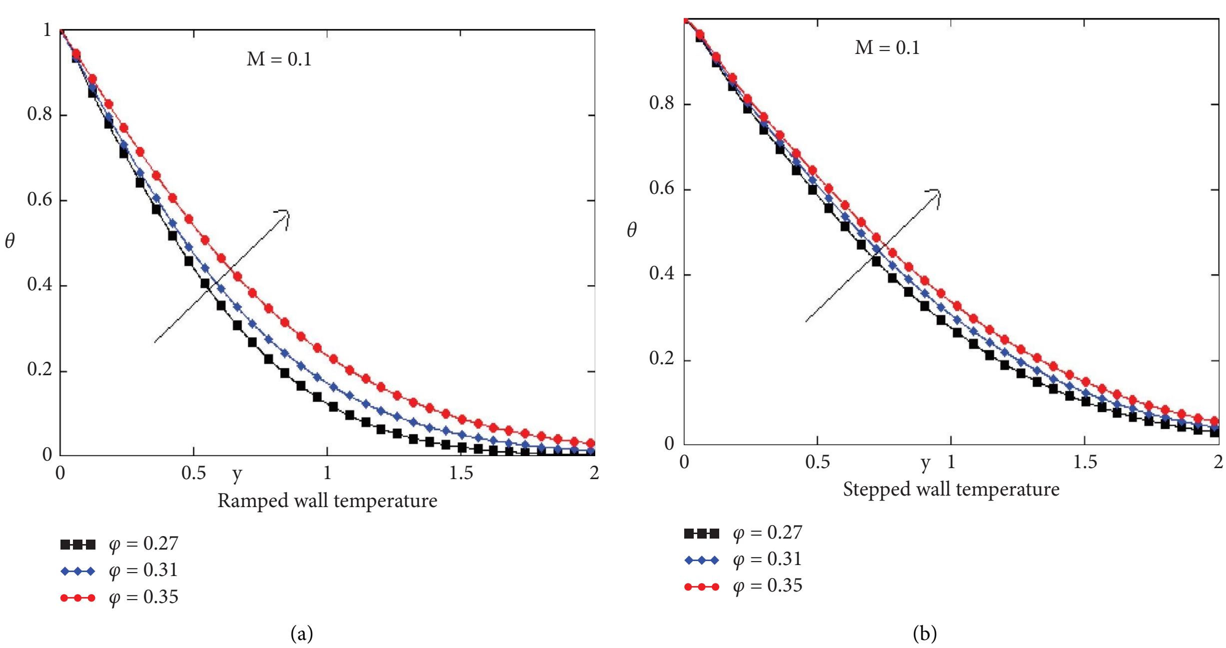


FIGURE 7: Nondimensional temperature θ graph for different values of φ: (a) corresponding to the plate temperature as ramped and (b) corresponding to the plate temperature as stepped.

decreasing value of fractional parameter α. Figure 5 describes that the velocity value increases for ramped wall temperature with the increasing value of MHD parameter M. This behavior of the graph exhibits the memory effect. An interesting note about the fractional parameter α from the graph is as it provides the memory effect which is a property of Newtonian fluids. As the Caputo–Fabrizio fractional-order operator of the derivative with respect to time has local, its kernel which is nonsingular has better effects on memory. This behavior arises because the kernel used in Caputo–Fabrizio with fractional-order differentiation is exponential, which appears obviously in many problems of physical sciences as power law, and for a large time, its behavior is asymptotic. On this same basis, Caputo–Fabrizio fractional derivative is a better tool for the temperature to have memory effect and as well for velocity parameter. It is

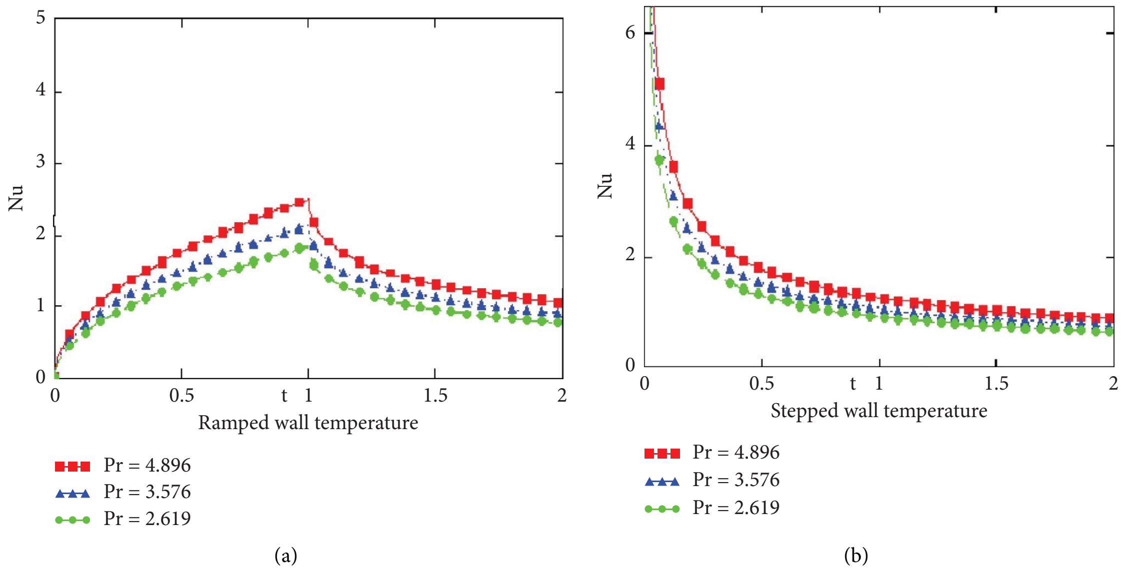


Figure 8: Variation in Pr causes changes in Nusselt number Nu: (a) corresponding to temperature of the plate as ramped and (b) corresponding to temperature of the plate as stepped.

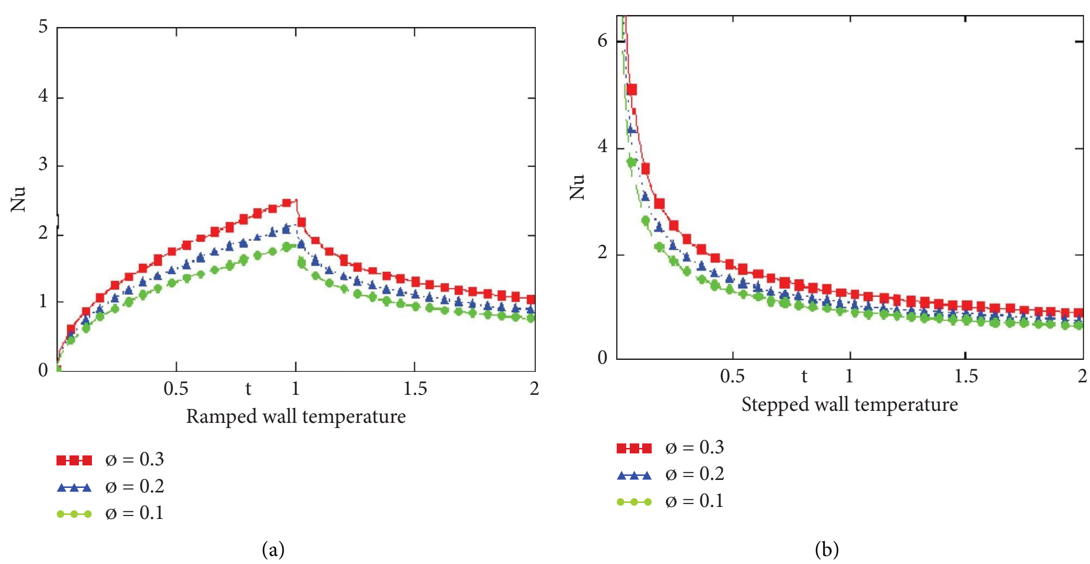


Figure 9: Variation in φ causes changes in Nusselt number Nu: (a) corresponding to temperature of the plate as ramped and (b) corresponding to temperature of the plate as stepped.

evident from Figure 6 that temperature is accelerated (for both the cases ramped wall temperature as well as in stepped wall temperature) for the growing value of time t.

Figure 7 represents the temperature outlines for ramped and stepped wall temperature corresponding to the variation in the nanoparticle volume fraction which shows that the increase in value of ? causes an increase in temperature of the nanofluid. This is because of lower specific heat with much higher thermal conductivity of the nanoparticle than that of the base fluid. This is an important result which increases the validity of the work. In Figure 8, the effect of Prandtl number Pr brought into the light for the velocity profile. The increase in Prandtl number (because of variation in nanoparticles volume fraction) means an increase in viscosity which creates some resistance in the flow of fluid, and fluid velocity decreases by increasing the value of Prandtl number. Figure 9 represents that the increase in nanoparticle volume fraction φ results in the promotion of the Nusselt number for both ramped wall temperature and for isothermal wall. In general, the value of the Nusselt number is inversely proportional to the value of the thermal conductivity of the fluid. This result is also verified in Tables 2 and 3 by using software named as Mathcad.

Table 2: Nusselt number for ramped temperature.

S. no	t	Nu Pr = 4.896 $\varphi = 0.1$	Nu Pr = 3.576 $\varphi = 0.2$	Nu Pr = 2.619 $\varphi = 0.3$	Nu Pr = 1.905 $\varphi = 0.4$
1	0.1	0.782	0.675	0.577	0.492
2	0.2	1.060	0.954	0.817	0.696
3	0.3	1.355	1.169	1.000	0.855
4	0.4	1.565	1.350	1.155	0.985
5	0.5	1.749	1.509	1.291	1.101
6	0.6	1.916	1.653	1.414	1.206
7	0.7	2.070	1.785	1.528	1.303
8	0.8	2.213	1.909	1.633	1.393
9	0.9	2.347	2.024	1.732	1.477

Table 3: Nusselt number for stepped temperature.

S. no	t	Nu Pr = 4.896 $\varphi = 0.1$	Nu Pr = 3.576 $\varphi = 0.2$	Nu Pr = 2.619 $\varphi = 0.3$	Nu Pr = 1.905 $\varphi = 0.4$
1	1.1	1.190	1.017	0.871	0.742
2	1.2	1.140	0.974	0.833	0.711
3	1.3	1.095	0.936	0.801	0.683
4	1.4	1.055	0.902	0.772	0.658
5	1.5	1.019	0.871	0.745	0636
6	1.6	0.987	0.843	0.722	0.616
7	1.7	0.957	0.818	0.700	0.597
8	1.8	0.930	0.795	0.681	0.580
9	1.9	0.906	0.774	0.662	0.565

Table 2 shows that the Nu as Nusselt number is a function of time and prandtl number Pr of the nanofluid. For increasing value of time, the value of Nusselt number also increases. The value of Nusselt number decreases with increasing value of nanoparticles volume fraction for the case of ramped wall temperature. It is because of that the nanoparticles increase the thermal conductivity of the nanofluid. Table 3 provides the evidence that the Nusselt number Nu also depends directly on Prandtl number Pr and accelerates for growing values on Prandtl number Pr. However, the Nu value is decreasing with the increasing value of nanoparticle volume fraction for walls temperature when kept constant.

5. Conclusions

Exact analytical solutions are obtained for the unsteady MHD-free convection flow of generalized nanofluid bounded by an infinite vertical plate. A nanofluid under the application of fractional-order differential equations by Caputo–Fabrizio derivatives with respect to time has the ability to explain the behavior of nanofluid under the influence of memory concept. For the same purpose, Caputo–Fabrizio time-fractional derivative is applied to investigate the behavior of nanoparticles on the thermal conductivity of a fluid. Closed-form solutions for velocity and temperature are obtained using the Laplace transform technique. The variation in both velocity and temperature is studied for different parameters graphically. In this continuation, the velocity profile for a larger time has a quite similar trend over the boundary. The fractional differential equation shows dual behavior for small and large times. This phenomenon for fractional derivatives is due to the fact of the effective role of a singular kernel. In this connection, the profile of nanoparticles for smaller time t and larger time t with Caputo–Fabrizio fractional derivatives can be predicated. Both cases of ramped and isothermal plate temperatures are discussed. The numerical values are evaluated for the Nusselt number and presented in tabular forms for both ramped and stepped temperatures. The following main results are concluded during the solutions of the problem and graphical analysis:

(i) An increase in nanoparticle volume fraction increases the nanofluid temperature, which leads to an increase in the heat transfer rates for both ramped and stepped temperature

(ii) Fractional parameter plays the role of controlling the thermal and momentum boundary layers for different times and values of the fractional parameter

(iii) Velocity depreciates for a large value of MHD parameter in case of ramped wall temperature while the result is reversed in the isothermal case

(iv) The Nusselt number increases with increasing values of Pr and t for ramped wall temperature whereas the Nusselt number associated with isothermal temperature increases with increasing Pr but decreases with increasing t as shown in Tables 1 and 2

Nomenclature

g: Acceleration due to gravity, $[LT^{-2}]\,(ms^{-2})$
ρ: Fluid density, $[ML^{-3}]\,(kgm^{-3})$
μ: Dynamic viscosity, $[ML^{-1}T^{-1}]\,(kgm^{-1}s^{-1})$
u: Fluid velocity, $[LT^{-1}]\,(ms^{-1})$
T: Fluid temperature, $[\theta]\,(K)$
T_∞: Fluid temperature at an infinite distance from the plate, $[\theta]\,(K)$
v: Kinematic viscosity, $[L^2T^{-1}]\,(m^2s^{-1})$
s: Laplace transforms parameter
L: Length of the plate, $[L]\,(m)$
M: MHD parameter
B_0^2: Magnetic field strength, $[T/L^3]\,(T)$
Nu: Nusselt number
ρ_{nf}: $Nano\,f$luid density, $[ML^{-3}]\,(kgm^{-3})$
μ_{nf}: $Nano\,fluid$ dynamic viscosity, $[ML^{-1}T^{-1}]\,(kgm^{-1}s^{-1})$
φ: Nanoparticles volume fraction
Pr: Prandtl number, dimensionless parameter
c_{-p}: Heat capacity at constant pressure, $[ML^2T^{-1}\theta^{-1}]\,(kgm^2s^{-1}K^{-1})$,
t: Time, $[T]\,(s)$
κ: Thermal conductivity of the fluid, $[MLT^{-3}\theta^{-1}]\,(Wm^{-1}K^{-1})$
σ_{nf}: Electrical conductivity of nanofluid, $[MT^{-2}]\,(Sm^{-1})$
σ_f: Electrical conductivity of base fluid, $[MT^{-2}]\,(Sm^{-1})$
σ_s: Electrical conductivity of solid, $[MT^{-2}]\,(Sm^{-1})$
T_w: Temperature of the plate, $[\theta]\,(K)$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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