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*Research Article*

# Stabilization and Discretization of the Coupled Heat and Wave Equations

**Kun-Yi Yang [1] and Xu Zhang [2]**

[1] College of Science, North China University of Technology, Beijing 100144, China
[2] School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

Correspondence should be addressed to Kun-Yi Yang; kyy@amss.ac.cn

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In this paper, we consider the stabilization of the coupled heat and wave equations under the static feedback or the dynamic feedback. Moreover, we make the coupled systems discretized by using the finite-volume approach, and then we consider the stabilized properties of the discrete systems. First, for the coupled system under the static feedback, it is shown that the system is exponentially stable by using the Lyapunov method, and then the corresponding discrete system can be shown to be exponentially stable by constucting the discretized Lyapunov function. Second, for the coupled system under the dynamic feedback, we also show that both of the system and its discrete scheme are exponentially stable. Third, numerical simulations are given to show the effectiveness of the stable controllers.

## 1. Introduction

Over the past decades, a few research have been focused on the stabilization and convergence properties of the distributed parameter systems [1]. However, a little progress has been made in the related research. Until recent years, Tebou and Zuazua [2] show that, by adding the vanishing numerical viscosity term, the discretization system of the locally damped wave equation can be shown to be exponentially stable by using the discrete multiplier techniques. Moreover, Tebou and Zuazua [3] solve the problem of stabilization for the discrete wave equation with the boundary dissipation, also by the discrete multiplier method. Recently, Liu and Guo [4] construct the semi-discretized scheme for the Euler–Bernoulli beam equation by using the finite-volume approach, and prove that the discrete system is uniformly exponentially stable by the discrete form of Lyapunov function. Furthermore, for the wave equation, Liu and Guo [5] and Liu and Wu [6] show that the difference scheme also is uniformly exponentially stable by using the reduced-order difference schemes. The discretization and stability results of the Schrödinger equation can be found in [7]. In the paper [8], the authors consider the observability inequality of the reduced-order difference schemes for the Schrödinger equation. By the finite difference approach of order reduction, the exponential stability of the wave equation with dynamical boundary condition has been researched in [9]. Until now, the stability of the discretization scheme for the coupled-partial differential equations has rarely been involved in the existing literature. In [10], the authors consider the coupled heat and wave equations under the static and dynamic feedbacks, where the exponential stability of the coupled system has been proved. Our paper will focus on the stabilization of the coupled system with other kinds of boundary conditions which follows the proof of the results [10]. Moreover, we will analyze the difference scheme of the coupled system and the uniformly exponential stability of the discrete system. And numerical simulations show the effectiveness of the stable static or dynamic feedbacks.

This paper is organized in five sections. In Section 1, introduction is elaborated. In Section 2, for the coupled system of the heat and wave equation under the static feedback, we construct the discrete scheme of the coupled system by the finite-volume method. The exponential stability of the discrete scheme for the coupled system has been shown subsequently. In Section 33, for the coupled system of the heat and wave equation under the dynamic feedback, we construct the discrete scheme of the coupled system by the finite-volume method, and show that the discrete scheme for the coupled system is exponentially stable. In Section 4, numerical simulation is explained. In the last section (Section 5), we simulate the states for both of the coupled systems considered above, and the simulation results show the effectiveness for both of the stable static and dynamic feedbacks.

## 2. The Stability Analysis and Discretization of the Coupled Heat and Wave Equation under the Static Feedback

In this section, we consider the system of the coupled heat and wave equations with the static feedback, considered previously in [10] with the different boundary conditions, which can be described as follows:

$$\begin{cases} v_t(x,t) = v_{xx}(x,t),\ 0<x<1, t\geq 0, \\ w_{tt}(x,t) = w_{xx}(x,t),\ 0<x<1, t\geq 0, \\ v(0,t) = -w_t(0,t),\ t\geq 0, \\ v_x(0,t) = w_x(0,t),\ t\geq 0, \\ v(1,t) = 0,\ t\geq 0, \\ w_x(1,t) = -kw_t(1,t),\ t\geq 0, \end{cases} \tag{1}$$

where $v(x,t)$ and $w(x,t)$ are the states of the heat and wave equations, respectively.

The energy space is

$$X = L^2(0,1)\times H^1(0,1)\times L^2(0,1), \tag{2}$$

with the inner product

$$\begin{aligned}&\langle (h_1, f_1, g_1), (h_2, f_2, g_2)\rangle_X \\ &\quad = \int_0^1 \left( h_1(x)\overline{h_2(x)} + f_1'(x)\overline{f_2'(x)} + g_1(x)\overline{g_2(x)} \right) dx,\end{aligned} \tag{3}$$

which directly results that

$$\|(h,f,g)\|_X^2 = \int_0^1 (|h(x)|^2 + |f'(x)|^2 + |g(x)|^2) dx. \tag{4}$$

The energy of the system (1) is defined as follows:

$$E(t) = \frac{1}{2}\int_0^1 (v(x,t)^2 + w_x(x,t)^2 + w_t(x,t)^2) dx. \tag{5}$$

According to the Lyapunov function method supplied in the paper [10], the exponential stability of the system (1) can be obtained as the following theorem:

**Theorem 1.** *The system (1) is exponentially stable, that is,*

$$E(t) \leq e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E(0), \tag{6}$$

*for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\min\{\frac{2k}{k^2+1}, \frac{1}{2}\}$ and $k>0$.*

*Proof.* For the energy Equation (5), simple computation shows that

$$\dot{E}(t) = -kw_t(1,t)^2 - \int_0^1 v_x(x,t)^2 dx. \tag{7}$$

Define the function as follows:

$$\begin{aligned}\phi(t) = &\int_0^1 \frac{1}{10} v(x,t)^2 + xw_t(x,t)w_x(x,t) \\ &+ \frac{1}{10}(1-x)[w_t(x,t)^2 + w_x(x,t)^2] dx.\end{aligned} \tag{8}$$

It can be obtained that

$$|\phi(t)| \leq 2E(t)\ , \tag{9}$$

and

$$\dot{\phi}(t) \leq -\frac{4}{5}E(t) + \frac{k^2+1}{2} w_t(1,t)^2. \tag{10}$$

Now we define the Lyapunov function below, for the positive constant $\varepsilon$,

$$L(t) = E(t) + \varepsilon\phi(t)\ , \tag{11}$$

which together with Equation (9) simply give that

$$(1-2\varepsilon)E(t) \leq L(t) \leq (1+2\varepsilon)E(t), \tag{12}$$

for the positive constant $\varepsilon$ satisfying that $0<\varepsilon<\frac{1}{2}$.

From Equations (7), (10), and (11), we have

$$\dot{L}(t) \leq -\frac{4\varepsilon}{5}E(t) - \left(k - \frac{k^2+1}{2}\varepsilon\right) w_t(1,t)^2\ , \tag{13}$$

for the positive constant $\varepsilon$ satisfying that $0<\varepsilon<\frac{2k}{k^2+1}$.

Moreover, by using the inequality (12), it is naturally to get that

$$\dot{E}(t) \leq -\frac{4\varepsilon}{5(1-2\varepsilon)}E(t)\ , \tag{14}$$

for the positive constant $0<\varepsilon<\frac{1}{2}$.

Thus, we naturally obtain the exponential stability of the system (1), that is,

$$E(t) \le e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E(0)\ , \tag{15}$$

for the positive constant $\varepsilon$ satisfying that $0<\varepsilon<\min\{\frac{2k}{k^2+1}, \frac{1}{2}\}$ and $k>0$. □

Next, we consider the discretization and stabilization of the system (1). First, we use the finite-volume approach to obtain the following difference schemes for the system (1).

$$\begin{cases} \dfrac{v'_{j+\frac{1}{2}} + v'_{j-\frac{1}{2}}}{2} = \delta_x^2 v_j, \quad j = 1, \cdots, N, \\ \dfrac{w''_{j+\frac{1}{2}} + w''_{j-\frac{1}{2}}}{2} = \delta_x^2 w_j, \quad j = 1, \cdots, N, \\ v_0 = -w'_0, \quad \delta_x v_{\frac{1}{2}} = \delta_x w_{\frac{1}{2}} + \dfrac{h}{2}\left(v'_{\frac{1}{2}} - w''_{\frac{1}{2}}\right), \quad v_{N+1} = 0, \\ \delta_x w_{N+\frac{1}{2}} = -k w'_{N+1} + \dfrac{h}{2} w''_{N+\frac{1}{2}}, \end{cases} \tag{16}$$

where in [5]

$$\begin{cases} u'_i = u_t(x_i, t),\ i = 0, 1, \cdots, N+1, \\ u''_i = u_{tt}(x_i, t),\ i = 0, 1, \cdots, N+1, \\ u_{i+\frac{1}{2}} = \dfrac{u_i + u_{i+1}}{2},\ i = 0, 1, \cdots, N, \\ \delta_x u_{i+\frac{1}{2}} = \dfrac{u_{i+1} - u_i}{h},\ i = 0, 1, \cdots, N, \\ \delta_x^2 u_i = \dfrac{\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}}{h} = \dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2},\ i = 1, \cdots, N. \end{cases} \tag{17}$$

From the studies by Liu and Guo [4, 5, 7, 10], it is shown that there exists an error between the original system and its disretization system which is different from such other discretization methods as the finite-difference method.

Second, for the system (1), we define its discrete energy as follows:

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left(v_{j+\frac{1}{2}}^{\ 2} + \delta_x w_{j+\frac{1}{2}}^{\ 2} + w'^{\ 2}_{j+\frac{1}{2}}\right), \tag{18}$$

which results that

$$\dot{E}_h(t) = -k w'^{\ 2}_{N+1} - h\sum_{j=0}^{N}\delta_x v_{j+\frac{1}{2}}^{\ 2}, \tag{19}$$

from the boundary conditions of the system (16).

Finally, we give the following lemma which can be proved easily by using the boundary conditions of the system (16). Moreover, the exponential stability of the system (1) has been shown in the subsequent theorem.

**Lemma 1.**

$$h\sum_{j=0}^{N} v_{j+\frac{1}{2}}^{\ 2} \le \frac{h}{2}\sum_{j=0}^{N}\delta_x v_{j+\frac{1}{2}}^{\ 2}. \tag{20}$$

**Theorem 2.** *The system (16) is exponentially stable, that is to say,*

$$E_h(t) \le e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E_h(0), \tag{21}$$

*for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\min\{\frac{8k}{5(k^2+1)}, \frac{1}{2}\}$ and $k>0$.*

*Proof.* First, we define the discrete function as follows:

$$\begin{aligned} \phi_h(t) = {} & \frac{h}{10}\sum_{j=0}^{N} v_{j+\frac{1}{2}}^{\ 2} + h\sum_{j=1}^{N} jh \frac{w'_{j+\frac{1}{2}} + w'_{j-\frac{1}{2}}}{2} \frac{\delta_x w_{j+\frac{1}{2}} + \delta_x w_{j-\frac{1}{2}}}{2} + \frac{h}{2} w'_{N+\frac{1}{2}} \delta_x w_{N+\frac{1}{2}} \\ & + \frac{h}{10}\sum_{j=1}^{N}\left\{\left(\delta_x w_{j+\frac{1}{2}}^{\ 2} + w'^{\ 2}_{j+\frac{1}{2}}\right) - jh\left[\left(\frac{\delta_x w_{j+\frac{1}{2}} + \delta_x w_{j-\frac{1}{2}}}{2}\right)^2 + \left(\frac{w'_{j+\frac{1}{2}} + w'_{j-\frac{1}{2}}}{2}\right)^2\right]\right\}. \end{aligned} \tag{22}$$

By simple computation we can get the inequality

$$|\phi_h(t)| \le 2E_h(t), \tag{23}$$

where $E_h(t)$ is defined in Equation (18).

Some other computation procedures can tell us the derivative of the function $\phi_h(t)$ as follows:

$$\begin{aligned} \dot{\phi}_h(t) = {} & -\frac{h}{5}\sum_{j=0}^{N}\delta_x v_{j+\frac{1}{2}}^{\ 2} - \frac{k}{5} w'^{\ 2}_{N+1} - \frac{1}{5} w'_{N+\frac{1}{2}}\delta_x w_{N+\frac{1}{2}} + \frac{h}{5}\sum_{j=0}^{N} w'_{j+\frac{1}{2}}\delta_x w_{j+\frac{1}{2}} \\ & -\frac{h}{2}\sum_{j=0}^{N}\left(\delta_x w_{j+\frac{1}{2}}^{\ 2} + w'^{\ 2}_{j+\frac{1}{2}}\right) - \frac{1}{2}\left(\delta_x w_{N+\frac{1}{2}}^{\ 2} + w'^{\ 2}_{N+\frac{1}{2}}\right) - k w'_{N+1}\delta_x w_{N+\frac{1}{2}} + w'_{N+\frac{1}{2}} w'_{N+1}, \end{aligned} \tag{24}$$

by using the equations and boundary conditions of the system (16).

Moreover, according to Lemma 1, it is easy to get that

$$\begin{aligned}\dot{\phi}_h(t) \leq & -\frac{2h}{5}\sum_{j=0}^{N} v_{j+\frac{1}{2}}{}^2 - \frac{k}{5} w'_{N+1}{}^2 - \frac{1}{5} w'_{N+\frac{1}{2}} \delta_x w_{N+\frac{1}{2}} + \frac{h}{5}\sum_{j=0}^{N} w'_{j+\frac{1}{2}} \delta_x w_{j+\frac{1}{2}} \\ & -\frac{h}{2}\sum_{j=0}^{N}\left(\delta_x w_{j+\frac{1}{2}}{}^2 + w'_{j+\frac{1}{2}}{}^2\right) - \frac{1}{2}\left(\delta_x w_{N+\frac{1}{2}}{}^2 + w'_{N+\frac{1}{2}}{}^2\right) - k w'_{N+1}\delta_x w_{N+\frac{1}{2}} + w'_{N+\frac{1}{2}} w'_{N+1}.\end{aligned} \tag{25}$$

Then by the Cauchy inequality, we conclude that

$$\dot{\phi}_h(t) \leq -\frac{4}{5}E_h(t) + \frac{5(k^2+1)}{8} w'_{N+1}{}^2. \tag{26}$$

Second, for the positive constant $\varepsilon$, we define the function as follows:

$$L_h(t) = E_h(t) + \varepsilon\phi_h(t) \quad , \tag{27}$$

which together with the inequality (23) simply give that

$$(1-2\varepsilon)E_h(t) \leq L_h(t) \leq (1+2\varepsilon)E_h(t), \tag{28}$$

for the positive constant $\varepsilon$ satisfying that $0<\varepsilon<\frac{1}{2}$.

According to Equations (19), (26), and (27), it is easy to deduce that

$$\dot{L}_h(t) \leq -\frac{4\varepsilon}{5}E_h(t) - \left(k - \frac{5(k^2+1)\varepsilon}{8}\right) w'_{N+1}{}^2 \quad , \tag{29}$$

which means that

$$\dot{L}_h(t) \leq -\frac{4\varepsilon}{5}E_h(t) \quad , \tag{30}$$

for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\frac{8k}{5(k^2+1)}$ and $k>0$.

Therefore, from Equation (28), it is shown that

$$\dot{E}_h(t) \leq -\frac{4\varepsilon}{5(1-2\varepsilon)}E_h(t) \quad , \tag{31}$$

for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\min\{\frac{8k}{5(k^2+1)}, \frac{1}{2}\}$ and $k>0$.

In conclusion, it is naturally to obtain

$$E_h(t) \leq e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E_h(0), \tag{32}$$

for $0<\varepsilon<\min\{\frac{8k}{5(k^2+1)}, \frac{1}{2}\}$ and $k>0$. □

## 3. The Stability Analysis and Discretization of the Coupled Heat and Wave Equation under the Dynamic Feedback

In this section, we consider the stability and discretization for the following system of the coupled heat and wave equations under the dynamic feedback.

$$\begin{cases} v_t(x,t) = v_{xx}(x,t), \ 0<x<1, t\geq 0, \\ w_{tt}(x,t) = w_{xx}(x,t), \ 0<x<1, t\geq 0, \\ v(0,t) = -w_t(0,t), \ v_x(0,t) = w_x(0,t), \ v(1,t) = 0, \ t\geq 0, \\ w_x(1,t) = -c^T z(t) - k w_t(1,t), \ t\geq 0, \ k>0, \ c\in R^{n\times 1}, \\ \dot{z}(t) = Az(t) + b w_t(1,t), \ \ A\in R^{n\times n}, \ b\in R^{n\times 1}, \end{cases} \tag{33}$$

where $v(x,t)$ and $w(x,t)$ are the states of the heat and wave equations, respectively, and $z(t)\in R^{n\times 1}$ is related to the dynamic feedback.

The energy of the system (33) is defined as follows:

$$E(t) = \frac{1}{2}\int_0^1 (v(x,t)^2 + w_x(x,t)^2 + w_t(x,t)^2)dx + \frac{1}{2}z^T P z. \tag{34}$$

After simple computation we can get

$$\begin{aligned}\dot{E}(t) = & -\int_0^1 v_x(x,t)^2 dx - \gamma w_t(1,t) \\ & -\frac{1}{2}\left[\sqrt{2(k-\gamma)}w_t(1,t) - z^T q\right]^2 - \frac{\Delta}{2}z^T Q z,\end{aligned} \tag{35}$$

by assuming

$$\begin{cases} A^T P + PA = -qq^T - \Delta Q, \\ Pb - c = \sqrt{2(k-\gamma)}q, \end{cases} \quad , \tag{36}$$

where the signs $A, b, c, k$ are defined in the system (33), and $P, Q, q, \Delta, \gamma$ are defined the same as that in [10]. Then we can acquire the subsequently exponential stability of the system (33).

**Theorem 3.** *The system (33) is exponentially stable expressed as follows:*

$$E(t) \le e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E(0), \tag{37}$$

*for the positive constant* $\varepsilon$ *satisfying* $0<\varepsilon<\frac{1}{2}$ *and the subsequent inequalities*

$$\begin{cases} \gamma - \varepsilon\left(k^2 + \dfrac{1}{2}\right) > 0, \\ \dfrac{\Delta}{2} z^T Q z - \varepsilon\left(\dfrac{2}{5} z^T P z + c^T c|z(t)|^2\right) > 0. \end{cases} \tag{38}$$

*Proof.* First, we define the Lyapunov function as follows:

$$\begin{aligned} \phi(t) = & \int_0^1 \frac{1}{10} v(x,t)^2 + x w_t(x,t) w_x(x,t) \\ & + \frac{1}{10}(1-x)[w_t(x,t)^2 + w_x(x,t)^2] dx. \end{aligned} \tag{39}$$

After some computation we can obtain the following inequality

$$\dot{\phi}(t) \le -\frac{4}{5}E(t) + \frac{2}{5} z^T P z + c^T c |z(t)|^2 + \left(k^2 + \frac{1}{2}\right) w_t(1,t)^2, \tag{40}$$

by using the boundary conditions of the system (33) and the definitions (34) and (39).

Second, for the positive constant $\varepsilon$, we define the following function

$$L(t) = E(t) + \varepsilon\phi(t), \tag{41}$$

on the basis of the definitions (34) and (39).

From the definitions (34) and (41), it is naturally to get the following inequalities:

$$(1-2\varepsilon)E(t) \le L(t) \le (1+2\varepsilon)E(t), \tag{42}$$

for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\frac{1}{2}$.

Third, we can obtain the inequality as follows:

$$\begin{aligned} \dot{L}(t) \le & -\frac{4}{5}\varepsilon E(t) - \left[\gamma - \varepsilon\left(k^2 + \frac{1}{2}\right)\right] w_t(1,t)^2 \\ & - \left[\frac{\Delta}{2} z^T Q z - \varepsilon\left(\frac{2}{5} z^T P z + c^T c |z(t)|^2\right)\right], \end{aligned} \tag{43}$$

by using the boundary conditions of the system (33) and the definitions (34) and (41),

Since the matrices $P$ and $Q$ are positive definite, the positive constant $\varepsilon$ can be chosen as that the following inequalities are satisfied:

$$\begin{cases} \gamma - \varepsilon\left(k^2 + \dfrac{1}{2}\right) > 0, \\ \dfrac{\Delta}{2} z^T Q z - \varepsilon\left(\dfrac{2}{5} z^T P z + c^T c|z(t)|^2\right) > 0, \end{cases} \tag{44}$$

which together with Equation (43) give that

$$\dot{L}(t) \le -\frac{4}{5}\varepsilon E(t). \tag{45}$$

Finally, from Equation (42) it can be shown that

$$\dot{E}(t) \le -\frac{4\varepsilon}{5(1-2\varepsilon)} E(t), \tag{46}$$

for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\frac{1}{2}$ and Equation (44).

Therefore, we obtain the exponential stability of the system (33) stated as follows:

$$E_h(t) \le e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E_h(0), \tag{47}$$

for the constant $\varepsilon$ satisfying that $0<\varepsilon<\frac{1}{2}$ and Equation (44). □

Next we consider the discretization and stability of the system (33). By using the finite-volume approach, we discretize the system (33) as the following discrete system:

$$\begin{cases} \dfrac{v'_{j+\frac{1}{2}} + v'_{j-\frac{1}{2}}}{2} = \delta_x^2 v_j, \quad j = 1, \cdots, N, \\ \dfrac{w''_{j+\frac{1}{2}} + w''_{j-\frac{1}{2}}}{2} = \delta_x^2 w_j, \quad j = 1, \cdots, N, \\ v_0 = -w'_0, \quad \delta_x v_{\frac{1}{2}} = \delta_x w_{\frac{1}{2}} + \dfrac{h}{2}\left(v'_{\frac{1}{2}} - w''_{\frac{1}{2}}\right), \quad v_{N+1} = 0, \\ \delta_x w_{N+\frac{1}{2}} = -c^T z(t) - k w'_{N+1} + \dfrac{h}{2} w''_{N+\frac{1}{2}}, \\ \dot{z}(t) = A z(t) + b w'_{N+1}, \end{cases} \tag{48}$$

where the discretization donates have been the same as that in Equation (17).

For the system (48), we define the discrete energy as follows:

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left({v_{j+\frac{1}{2}}}^2 + \delta_x {w_{j+\frac{1}{2}}}^2 + {w'_{j+\frac{1}{2}}}^2\right) + \frac{1}{2} z^T P z, \tag{49}$$

by which we can compute that

$$\dot{E}_h(t) = -\gamma {w'_{N+1}}^2 - h\sum_{j=0}^{N} \delta_x {v_{j+\frac{1}{2}}}^2 - \frac{1}{2}\left[z^T q - \sqrt{2(k-\gamma)}w'_{N+1}\right]^2 - \frac{\Delta}{2}z^T Qz, \tag{50}$$

according to the boundary conditions of the system (48). Then, we can obtain the subsequently exponential stability of the system (48).

**Theorem 4.** *The system (48) is exponentially stable, that is,*

$$E_h(t) \le e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E_h(0), \tag{51}$$

*for the positive constant $\varepsilon$ satisfying that $0<\varepsilon<\frac{1}{2}$ and the following inequalities*

$$\begin{cases} \gamma - \varepsilon(2k^2+1) > 0, \\ \dfrac{\Delta}{2}z^T Qz - \varepsilon\left(\dfrac{2}{5}z^T Pz + 2c^T c|z(t)|^2\right) > 0. \end{cases} \tag{52}$$

*Proof.* First, we define the function of the discrete form as follows:

$$\begin{aligned}\phi_h(t) = {} & \frac{h}{10}\sum_{j=0}^{N} {v_{j+\frac{1}{2}}}^2 + h\sum_{j=1}^{N} jh \frac{w'_{j+\frac{1}{2}} + w'_{j-\frac{1}{2}}}{2}\frac{\delta_x w_{j+\frac{1}{2}} + \delta_x w_{j-\frac{1}{2}}}{2} + \frac{h}{2}w'_{N+\frac{1}{2}}\delta_x w_{N+\frac{1}{2}} \\ & + \frac{h}{10}\sum_{j=1}^{N}\left\{\left(\delta_x {w_{j+\frac{1}{2}}}^2 + {w'_{j+\frac{1}{2}}}^2\right) - jh\left[\left(\frac{\delta_x w_{j+\frac{1}{2}} + \delta_x w_{j-\frac{1}{2}}}{2}\right)^2 + \left(\frac{w'_{j+\frac{1}{2}} + w'_{j-\frac{1}{2}}}{2}\right)^2\right]\right\}.\end{aligned} \tag{53}$$

It is not difficult to infer the following inequality

$$|\phi_h(t)| \le 2E_h(t), \tag{54}$$

by using the definitions (49) and (53).

Then after some computation, for the function (53), we can obtain that

$$\dot{\phi}_h(t) \le -\frac{4}{5}E_h(t) + \frac{2}{5}z^T Pz + (2k^2+1){w'_{N+1}}^2 + 2c^T c|z(t)|^2, \tag{55}$$

by using the boundary conditions of the system (48).

Second, we define the following function

$$L_h(t) = E_h(t) + \varepsilon\phi_h(t), \tag{56}$$

on the basis of the definitions (49) and (53), for the positive constant $\varepsilon$.

Then from Equation (54) it can be shown that

$$(1-2\varepsilon)E_h(t) \le L_h(t) \le (1+2\varepsilon)E_h(t), \tag{57}$$

for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\frac{1}{2}$.

Altogether with the definition (56) and the results of Equations (50) and (55), we have

$$\dot{L}_h(t) \le -\frac{4}{5}\varepsilon E_h(t) - [\gamma - \varepsilon(2k^2+1)]{w'_{N+1}}^2 - \left[\frac{\Delta}{2}z^T Qz - \varepsilon\left(\frac{2}{5}z^T Pz + 2c^T c|z(t)|^2\right)\right]. \tag{58}$$

Since the matrices $P$ and $Q$ are positive definite, we can choose the positive constant $\varepsilon$ such that the following inequalities are satisfied.

$$\begin{cases} \gamma - \varepsilon(2k^2+1) > 0, \\ \dfrac{\Delta}{2}z^T Qz - \varepsilon\left(\dfrac{2}{5}z^T Pz + 2c^T c|z(t)|^2\right) > 0, \end{cases} \tag{59}$$

which together with Equation (58) show that

$$\dot{L}_h(t) \le -\frac{4}{5}\varepsilon E_h(t). \tag{60}$$

Then from Equation (57), it is shown that

$$\dot{E}_h(t) \le -\frac{4\varepsilon}{5(1-2\varepsilon)}E_h(t), \tag{61}$$

for the positive constant $\varepsilon$ satisfying $0<\varepsilon<\frac{1}{2}$ and Equation (59).

Therefore, we have the exponential stability of the system (48), that is,

$$E_h(t) \le e^{-\frac{4\varepsilon}{5(1-2\varepsilon)}t} E_h(0), \tag{62}$$

for the positive constant $\varepsilon$ satisfying that $0<\varepsilon<\frac{1}{2}$ and Equation (59). □

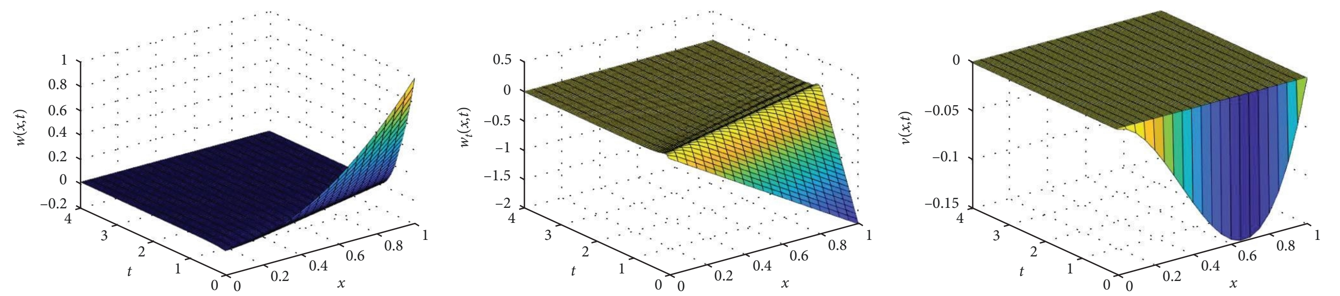


Figure 1: The states of the coupled system (1) under the static feedback.

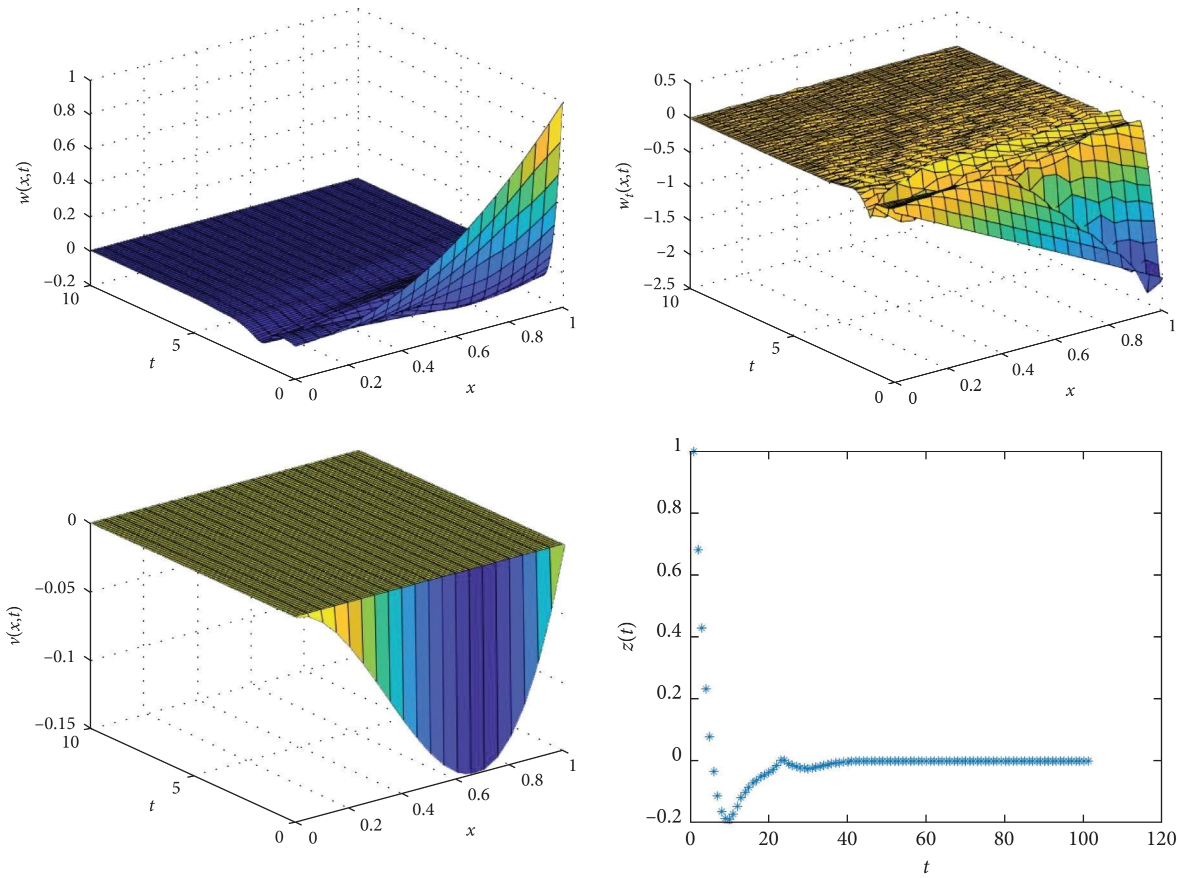


Figure 2: The states of the coupled system (33) under the dynamic feedback.

## 4. Numerical Simulation

In this section, we show some numerical simulation results of the coupled heat and wave equations. And we can find that, the states of the coupled system have been converged to zero in certain time interval. The space step is chosen to be 0.02, and the time step is 0.001. Initial values and parameters have been given as follows:

$$\begin{cases} w_0(x) = x^2, \quad w_1(x) = -x, \quad v_0(x) = x^2 - x^3, \quad z_0 = 1, \\ k = 1, \ c = 1, \ A = -1, \ b = 1. \end{cases} \tag{63}$$

For the coupled systems under the static or dynamic feedback, we have the states of the coupled systems which show that, both of the systems with the static or dynamic feedback are stable illustrated by Figures 1 and 2, respectively.

## 5. Conclusion

In the paper we consider the stabilization and discretization of the coupled heat and wave equations. For the coupled system under the static feedback, we construct the numerical scheme by the finite-volume approach, and then it is shown that the discrete system is exponentially stable proven by the Lyapunov function method. For the coupled system under the dynamic feedback, the discrete scheme is constructed and then is shown to be exponentially stable. Numerical simulations are given to show the effectiveness of the stable controllers. The future direction may be related to the different discrete form of the coupled distributed parameter systems and the stabilized properties.

## Data Availability

The authors confirm that the data supporting the findings of this study are available within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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