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Research Article

A New Study for the Investigation of Nonlinear Fractional Drinfeld–Sokolov–Wilson Equation

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In this paper, we present an idea of a new iterative procedure (NIP) to examine the approximate solution of the nonlinear fractional Drinfeld–Sokolov–Wilson (DSW) equation. We first use Mohand transform (MT) to the problem and obtain a recurrence relation without any assumption or restrictive variable. This relation is now very easy to handle and suitable for the study of the homotopy perturbation method (HPM). We observe that HPM produces the iterations in the form of convergence series that becomes very close to the precise solution. The fractional derivative is considered in the Caputo sense. We also demonstrate the graphical solution to show that NIP is a very simple, straightforward, and efficient tool for nonlinear problems of fractional derivatives.

1. Introduction

Many phenomena in science and engineering are studied through fractional calculus. These phenomena arise in various fields such as physics, mathematical biology, signal processing, finance, social science, and many more. Therefore, this research is more interesting but difficult in the presence of fractional order derivatives [1–3]. In physics and mathematics, the process for the evaluation of the exact or approximate results of nonlinear partial differential equations (PDE) is still important. Most of the equations have some difficulties to solve, whereas the difficulty of the solution increases if the nonlinear systems have a fractional order. Drinfeld and Sokolov [4] and Wilson [5] presented the Drinfeld–Sokolov–Wilson (DSW) system for dispersive water waves that perform a significant part in fluid dynamics. In this paper, we assume the coupled DSW system as follows [6]:

$$\begin{aligned} &\Psi_{\varsigma} + a\Phi\Phi_{\varphi} = 0, \\ &\Phi_{\varsigma} + b\Phi_{\varsigma\varsigma\varsigma} + c\Psi\Phi_{\varphi} + d\Psi_{\varphi}\Phi = 0, \end{aligned} \tag{1}$$

where $\Psi(\varphi, \varsigma)$ and $\Phi(\varphi, \varsigma)$ are the functions of time ς with space φ and they designate the amplitude of the wave modes, and a, b, c, and d are nonzero elements. Specially, for $a = 3$, $b = 2$, $c = 2$, $d = 1$. The above system plays a key part in the different branches of hydrodynamic fields. Jawad [7] demonstrated a theory of traveling wave and the new solitary wave solutions of the DSW problem. Singh et al. [6] presented a different approach to evaluate the approximate results of the fractional DSW problem that occurs in dispersive water waves. Gao et al. [8] implemented q-homotopy analysis transform scheme to find the solution for fractional DWS equation. This approach has designed such that Laplace transform technique is coupled with q-homotopy analysis method, whereas fractional derivative defined with Atangana–Baleanu operator. Sahoo and Ray [9] studied the double-periodic solutions of fractional DSW equation in shallow water waves via Jacobi elliptical function method. Srivastava and Saad [10] studied Adomian decomposition method to find the approximate solution of time-fractional DSW system. Jaradat et al. [11] used residual power series method to find the analytical results of nonlinear fractional DSW problem.

The homotopy perturbation method (HPM) [12] has been introduced to solve the ordinary and PDE. Various approaches have been studied to solve the differential problems such as differential transform scheme [13], Adomian decomposition scheme [14], finite difference methods [15], homotopy analysis approach [16], Sumudu transformation method [17, 18], Elzaki transform [19, 20], Aboodh transformation method [21], ZZ transform [22], Chebyshev collection method [23] and Haar wavelet approach [24], Mohand homotopy perturbation transform [25], and so on [26, 27]. Later, Elzaki transform is combined with HPM to obtain the solution of nonlinear PDE and showed that HPM is very effective and simple approach. Ganji and Sadighi [28] implemented variational iteration method with HPM to achieve the approximate results of nonlinear transfer and porous media problems.

Mahgoub [29] introduced a new scheme known as Mohand transform that provides the results without restriction variable. Mohand transform is very easy to implement to the differential problems than variational iteration method (VIM), Laplace transform, and homotopy analysis method (HAM). Since VIM involves the integration and produces constant of integration, Laplace transform involves the convolution theorem and HAM considered some assumptions. Due to these restrictions of variables, it is not easy to handle the solution of the problems, whereas Mohand transform has been applied in a direct way, we do not need any integration, convolution theorem, and any other assumption. This significance makes Mohand transform unique and different. In this paper, we combine Mohand transform with HPM to develop a new iterative procedure (NIP) for evaluating the approximate results of nonlinear fractional DSW equation. The rest of paper is designed as: in Section 2, we define the basic idea of Mohand transform with basic propositions. In Section 3, we construct the idea of NIP starting with a nonlinear differential problem. We implement this strategy to a numerical problem in Section 4 and discuss some results in Section 5 and then the conclusion is presented in Section 6.

2. Basic Definitions of Fractional Calculus and Mohand Transform

This segment describes few basic properties of fractional calculus and Mohand transform that are very helpful in the formulation of this scheme.

Definition 1. If α is the fractional order of a function $\Psi(\varsigma)$, then [30]:

$$\frac{\partial^\alpha \Psi}{\partial \varsigma^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\varsigma} \int_0^\varsigma (\varsigma - w)^{-\alpha} \Psi(w) dw, \; 0 < \alpha < 1. \tag{2}$$

Definition 2. Consider $\Psi(\varsigma) = \varsigma^\alpha$, then:

$$\mathbb{M}\left[D_\varphi^{n\alpha} \Psi(\varphi, \varsigma)\right] = s^{n\alpha} F(s) - \sum_{k=0}^{n-1} s^{n\alpha - k - 1} \Psi_\varphi^{(k)}(0, \varsigma), \; n - 1 < \alpha \le n. \tag{3}$$

Definition 3. The Caputo fractional derivative of the function Ψ with respect to ς of the order α, where $\alpha > 0$, is defined as [2]:

$$D_\tau^\alpha \Psi(\varphi, \varsigma) = \frac{1}{\Gamma(n-\alpha)} \int_0^\varsigma (\varsigma - \varphi)^{n-\alpha-1} \Psi^n(\varphi) \partial\varphi, \quad n - 1 < \alpha < n. \tag{4}$$

Definition 4. Mahgoub [29] developed a scheme for solving the differential problems. The Mohand transform is defined as:

$$\mathbb{M}\{\Psi(\varsigma)\} = R(w) = w^2 \int_0^\varsigma \Psi(\varsigma) e^{-v\varsigma} d\varsigma, \quad k_1 \le w \le k_2. \tag{5}$$

Similarly, If $R(w)$ is the Mohand transform of $\Psi(\varsigma)$, then $\Psi(\varsigma)$ is called the inverse of $R(w)$, i.e.,

$$\mathbb{M}^{-1}\{R(w)\} = \Psi(s), \tag{6}$$

where $\mathbb{M}^{-1}$ is called inverse Mohand transform.

Definition 5. Mohand transform with fractional derivative is defined as [29]:

$$\mathbb{M}\{\Psi^\alpha(\varsigma)\} = w^\alpha R(w) - \sum_{k=0}^{n-1} \frac{\Psi^k(0)}{w^k - (\alpha + 1)}, \; 0 < \alpha \le n. \tag{7}$$

Definition 6. Mohand transform in case of a differential function $\Psi(\varsigma)$ is defined as:

(a) $\mathbb{M}\{\Psi'(\varsigma)\} = wR(w) - w^2 \Psi(0)$
(b) $\mathbb{M}\{\{\Psi''(\varsigma)\} = w^2 R(w) - w^3 \Psi(0) - w^2 \Psi'(0)$
(c) $\mathbb{M}\{\Psi^n(\varsigma)\} = w^n R(w) - w^{n+1} \Psi(0) - w^n \Psi'(0) - \cdots - w^n \Psi^{n-1}(0)$.

3. Development of NIP

This section presents the formulation of the NIP to obtain the approximate solution of fourth-order PDE with fractional derivative. Therefore, we start this idea with consideration of a fractional differential equation such as:

$$D_\varsigma^\alpha \Psi(\varphi, \varsigma) + R\Psi(\varphi, \varsigma) + N\Psi(\varphi, \varsigma) = g(\varphi, \varsigma), \tag{8}$$

$$\Psi(\varphi, 0) = h(\varphi), \tag{9}$$

where $D_\varsigma^\alpha = \partial^\alpha / \partial \varsigma^\alpha$ represents the operator function of fractional order α. The function Ψ is considered in the direction of φ and ς known as spatial and time, respectively, R is called linear and N be the nonlinear differential operators and $g(\varphi, \varsigma)$ is the known function. Applying MT in Equation (8):

$$\mathbb{M}\left[D_{\varsigma}^{\alpha}\Psi(\varphi,\varsigma)+R\Psi(\varphi,\varsigma)+N\Psi(\varphi,\varsigma)\right]=\mathbb{M}[g(\varphi,\varsigma)]\ . \tag{10}$$

Applying the definition of MT, we obtain:

$$w^{\alpha}[R(w)-w\Psi(0)]=-\mathbb{M}[R\Psi(\varphi,\varsigma)+N\Psi(\varphi,\varsigma)]+\mathbb{M}[g(\varphi,\varsigma)], \tag{11}$$

that can come

$$R(w)=wu(0)-\frac{1}{w^{\alpha}}\mathbb{M}[R\Psi(\varphi,\varsigma)+N\Psi(\varphi,\varsigma)+g(\varphi,\varsigma)]. \tag{12}$$

With the help of Equation (9), we get:

$$R(w)=wh(\varphi)-\frac{1}{w^{\alpha}}\mathbb{M}[R\Psi(\varphi,\varsigma)+N\Psi(\varphi,\varsigma)+g(\varphi,\varsigma)]. \tag{13}$$

Using the inverse MT, we get:

$$\Psi(\varphi,\varsigma)=G(\varphi,\varsigma)-\mathbb{M}^{-1}\left[\frac{1}{w^{\alpha}}\mathbb{M}[R\Psi(\varphi,\varsigma)+N\Psi(\varphi,\varsigma)]\right]\ . \tag{14}$$

Thus, Equation (14) is known as the recurrence relation of $\Psi(\varphi,\varsigma)$, here:

$$G(\varphi,\varsigma)=\mathbb{M}^{-1}[wh(\varphi)+\mathbb{M}\{g(\varphi,\varsigma)\}]. \tag{15}$$

Let us consider the approximate solution of Equation (9) that can be stated as:

$$\Psi(\varphi,\varsigma)=\sum_{n=0}^{\infty}p^{n}\Psi_{n}(\varphi,\varsigma), \tag{16}$$

and

$$N\Psi(\varphi,\varsigma)=\sum_{n=0}^{\infty}p^{n}H_{n}\Psi(\varphi,\varsigma), \tag{17}$$

where $p\in[0,1]$ is homotopy component and assumed as a slightly parameter, whereas $\Psi_0(\varphi,\varsigma)$ is an initial guess of Equation (8). To obtain the He's polynomials, we may implement the following formula:

$$\begin{aligned}H_{n}(\Psi_0+\Psi_1+\cdots+\Psi_n)\\ =\frac{1}{n!}\frac{\partial^{n}}{\partial p^{n}}\left(N\left(\sum_{i=0}^{\infty}p^{i}\Psi_{i}\right)\right)_{p=0}.\ n=0,1,2,\cdots\ .\end{aligned} \tag{18}$$

Combining Equations (16) and (17), Equation (14) can be written as:

$$\sum_{n=0}^{\infty}p^{n}\Psi_{n}(\varphi,\varsigma)=G(\varphi,\varsigma)-p\mathbb{M}^{-1}\left[\frac{1}{w^{\alpha}}\mathbb{M}\left\{R\left(\sum_{n=0}^{\infty}p^{n}\Psi_{n}(\varphi,\varsigma)\right)+\sum_{n=0}^{\infty}p^{n}H_{n}\Psi_{n}(\varphi,\varsigma)\right\}\right]. \tag{19}$$

Equating the similar components of p, we get:

$$\begin{aligned}p^{0}:\Psi_{0}(\varphi,\varsigma)&=G(\varphi,\varsigma),\\ p^{1}:\Psi_{1}(\varphi,\varsigma)&=-\mathbb{M}^{-1}\left[\frac{1}{w^{\alpha}}\mathbb{M}\{R\Psi_{0}(\varphi,\varsigma)+H_0\}\right],\\ p^{2}:\Psi_{2}(\varphi,\varsigma)&=-\mathbb{M}^{-1}\left[\frac{1}{w^{\alpha}}\mathbb{M}\{R\Psi_{1}(\varphi,\varsigma)+H_1\}\right],\\ p^{3}:\Psi_{3}(\varphi,\varsigma)&=-\mathbb{M}^{-1}\left[\frac{1}{w^{\alpha}}\mathbb{M}\{R\Psi_{2}(\varphi,\varsigma)+H_2\}\right].\\ &\vdots\end{aligned} \tag{20}$$

Using the similar process, we can derive the following results such as:

$$\Psi(\varphi,\varsigma)=\Psi_{0}(\varphi,\varsigma)+p^{1}\Psi_{1}(\varphi,\varsigma)+p^{2}\Psi_{2}(\varphi,\varsigma)++p^{3}\Psi_{3}(\varphi,\varsigma)+\cdots. \tag{21}$$

The analytical solution of Equation (8) for $p=1$ is given as follows:

$$\Psi(\varphi,\varsigma)=\lim_{N\to\infty}\sum_{n=0}^{N}\Psi_{n}(\varphi,\varsigma). \tag{22}$$

Theorem 1. *Assume that φ and χ be two Banach spaces such that $I:\varphi\rightarrow\chi$ is a nonlinear operator. Therefore, $\Psi;\Psi^{*}\in;\varphi$, $\|I(\Psi)-I(\Psi^{*})\|\leq K\|\Psi-\Psi^{*}\|$, $0<K<1$. The Banach contraction states that I denoted as a unique fixed point Ψ, i.e., $I\Psi=\Psi$. So, rewrite the Equation (22) that is given as follows:*

$$\Psi(\varphi,\varsigma)=\lim_{N\to\infty}\sum_{n=0}^{N}\Psi_{n}(\varphi,\varsigma), \tag{23}$$

and assume that $\varphi_0=\Psi_0\in\mathcal{S}_p(\Psi)$, where $\mathcal{S}_p(\Psi)=\{\Psi^{*}\in\varphi:\|\Psi-\Psi^{*}\|<p\}$ then, we have:

$$\begin{aligned}&(B_1)\varphi_n\in\mathcal{S}_p(\Psi),\\ &(B_2)\lim_{n\to\infty}\varphi_n=\Psi.\end{aligned} \tag{24}$$

Proof. (B_1) In the light of mathematical induction at $n = 1$, we have:

$$\|\varphi_1 - \Psi_1\| = \|T(\varphi_0 - T(\Psi))\| \leq K\|\Psi_0 - \Psi\|. \quad (25)$$

Let the result is accurate for $n = 1$, thus:

$$\|\varphi_{n-1} - \Psi\| \leq K^{n-1}\|\Psi_0 - \Psi\|. \quad (26)$$

Therefore,

$$\|\varphi_n - \Psi\| = \|T(\varphi_{n-1} - T(\Psi))\| \leq K\|\varphi_{n-1} - \Psi\| \leq K^n\|\Psi_0 - \Psi\|. \quad (27)$$

Hence, using (B_1), we have:

$$\|\varphi_n - \Psi\| \leq K^n\|\Psi_0 - \Psi\| \leq K^n p < p, \quad (28)$$

which shows $\varphi_n \in \mathcal{S}_p(\Psi)$.

B_2: Since $\|\varphi_n - \Psi\| \leq^{Kn} \|\Psi_0 - \Psi\|$ and as a $\lim_{n\to\infty} K^n = 0$. Therefore, we have $\lim_{n\to\infty} \|\Psi_n - \Psi\| = 0 \Rightarrow \lim_{n\to\infty} \Psi_n = \Psi$. □

4. Applications of NIP

In this segment, we implement the formulation of new strategy and show that this approach is very relatable, straightforward, and simple. We also demonstrate the physical understanding of these approximate solutions obtained by this scheme together with the exact solutions. Graphical results show that only few iteration is enough to obtain the approximate solution that converges to the exact solution at $\alpha = 1$.

4.1. Example. Consider the nonlinear fractional DSW equation [31]:

$$\begin{aligned} &\frac{\partial^\alpha \Psi}{\partial \varsigma^\alpha} + 3\Phi\frac{\partial \Phi}{\partial \varphi} = 0, \\ &\frac{\partial^\alpha \Phi}{\partial \varsigma^\alpha} + 2\frac{\partial^3 \Phi}{\partial \varphi^3} + 2\Psi\frac{\partial \Phi}{\partial \varphi} + \Phi\frac{\partial \Psi}{\partial \varphi} = 0, \end{aligned} \quad (29)$$

with the initial conditions:

$$\begin{aligned} \Psi(\varphi, 0) &= 3\,\mathrm{sech}^2(\varphi), \\ \Phi(\varphi, 0) &= 2\,\mathrm{sech}(\varphi). \end{aligned} \quad (30)$$

Applying MT on the system of Equation (29), we get:

$$\begin{aligned} \mathbb{M}\left[\frac{\partial^\alpha \Psi}{\partial \varsigma^\alpha}\right] &= \mathbb{M}\left[-3\Phi\frac{\partial \Phi}{\partial \varphi}\right], \\ \mathbb{M}\left[\frac{\partial^\alpha \Phi}{\partial \varsigma^\alpha}\right] &= \mathbb{M}\left[-2\frac{\partial^3 \Phi}{\partial \varphi^3} - 2\Psi\frac{\partial \Phi}{\partial \varphi} - \Phi\frac{\partial \Psi}{\partial \varphi}\right]. \end{aligned} \quad (31)$$

Now, using the MT properties, we obtain:

$$\begin{aligned} w^\alpha[R(w) - w\Psi(0)] &= \mathbb{M}\left[-3\Phi\frac{\partial \Phi}{\partial \varphi}\right], \\ w^\alpha[R(w) - w\Phi(0)] &= \mathbb{M}\left[-2\frac{\partial^3 \Phi}{\partial \varphi^3} - 2\Psi\frac{\partial \Phi}{\partial \varphi} - \Phi\frac{\partial \Psi}{\partial \varphi}\right]. \end{aligned} \quad (32)$$

Solving above system, we get:

$$\begin{aligned} R(w) &= w\Psi(0) - \frac{1}{w^\alpha}\mathbb{M}\left[3\Phi\frac{\partial \Phi}{\partial \varphi}\right], \\ R(w) &= w\Phi(0) - \frac{1}{w^\alpha}\mathbb{M}\left[-2\frac{\partial^3 \Phi}{\partial \varphi^3} - 2\Psi\frac{\partial \Phi}{\partial \varphi} - \Phi\frac{\partial \Psi}{\partial \varphi}\right]. \end{aligned} \quad (33)$$

Taking inverse MT, we get:

$$\begin{aligned} \Psi(\varphi, 0) &= \Psi(0) - \mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbb{M}\left\{3\Phi\frac{\partial \Phi}{\partial \varphi}\right\}\right], \\ \Phi(\varphi, 0) &= \Phi(0) - \mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbb{M}\left\{-2\frac{\partial^3 \Phi}{\partial \varphi^3} - 2\Psi\frac{\partial \Phi}{\partial \varphi} - \Phi\frac{\partial \Psi}{\partial \varphi}\right\}\right]. \end{aligned} \quad (34)$$

Now, apply HPM to obtain He's polynomials that are given as follows:

$$\begin{aligned} \sum_{i=0}^{\infty} p^i\Psi_i &= \Psi(0) - \mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbb{M}\left\{3\sum_{i=0}^{\infty} p^i\Phi_i\frac{\partial}{\partial \varphi}\sum_{i=0}^{\infty} p^i\Phi_i\right\}\right], \\ \sum_{i=0}^{\infty} p^i\Phi_i &= \Phi(0) - \mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbb{M}\left\{-2\frac{\partial^3}{\partial \varphi^3}\sum_{i=0}^{\infty} p^i\Phi_i - 2\sum_{i=0}^{\infty} p^i\Psi\frac{\partial}{\partial \varphi}\sum_{i=0}^{\infty} p^i\Phi_i - \sum_{i=0}^{\infty} p^i\Phi\frac{\partial}{\partial \varphi}\sum_{i=0}^{\infty} p^i\Psi_i\right\}\right]. \end{aligned} \quad (35)$$

On comparing the similar results of p, we get:
For p^0:

$$\begin{aligned}\Psi_0(\varphi,\varsigma)&=\Psi(\varphi,0)=3\,\text{sech}^2\,(\varphi),\\ \Phi_0(\varphi,\varsigma)&=\Phi(\varphi,0)=2\,\text{sech}(\varphi).\end{aligned}\tag{36}$$

For p^1:

$$\begin{aligned}\Psi_1(\varphi,\varsigma)&=-\mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbb{M}\left\{3\Phi_0\frac{\partial\Phi_0}{\partial\varphi}\right\}\right],\\ \Phi_1(\varphi,\varsigma)&=-\mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbb{M}'\left\{-2\frac{\partial^3\Phi_0}{\partial\varphi^3}-2\Psi_0\frac{\partial\Phi_0}{\partial\varphi}-\Phi_0\frac{\partial\Psi_0}{\partial\varphi}\right\}\right].\end{aligned}\tag{37}$$

For p^2:

$$\begin{aligned}\Psi_1(\varphi,\varsigma)&=-\mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbf{M}\left\{3\Phi_0\frac{\partial\Phi_1}{\partial\varphi}+3\Phi_1\frac{\partial\Phi_0}{\partial\varphi}\right\}\right],\\ \Phi_1(\varphi,\varsigma)&=-\mathbb{M}^-\left[\frac{1}{w^\alpha}\mathbb{M}\left\{-2\frac{\partial^3\Phi_1}{\partial\varphi^3}-2\Psi_0\frac{\partial\Phi_1}{\partial\varphi}-2\Psi_1\frac{\partial\Phi_0}{\partial\varphi}-\Phi_0\frac{\partial\Psi_1}{\partial\varphi}-\Phi_1\frac{\partial\Psi_0}{\partial\varphi}\right\}\right],\end{aligned}\tag{38}$$

$$\begin{aligned}\Psi_0(\varphi,\varsigma)&=3\,\text{sech}^2\,(\varphi),\\ \Phi_0(\varphi,\varsigma)&=2\,\text{sech}'\,(\varphi),\end{aligned}\tag{39}$$

and

$$\begin{aligned}\Psi_1(\varphi,\varsigma)&=12\,\text{sech}^2\,(\varphi)\tanh\,(\varphi)\frac{\varsigma^\alpha}{\Gamma(1+\alpha)},\\ \Phi_1(\varphi,\varsigma)&=4\,\text{sech}\,(\varphi)\tanh\,(\varphi)\frac{\varsigma^\beta}{\Gamma(1+\beta)},\end{aligned}\tag{40}$$

and

$$\begin{aligned}\Psi_2(\varphi,\varsigma)&=24\,\text{sech}^4\,(\varphi)(\cosh\,(2\varphi)-2)\frac{\varsigma^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)},\\ \Phi_2(\varphi,\varsigma)&=\text{sech}^5\,(\varphi)(67-52\cosh\,(2\varphi)+\cosh\,(4\varphi))\frac{\varsigma^{2\beta}}{\Gamma(1+2\beta)}\\ &\quad+24\,\text{sech}^5\,(\varphi)(2\cosh\,(2\varphi)-3)\frac{\varsigma^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}.\end{aligned}\tag{41}$$

In the similar way, we can consider the approximate series such as:

$$\begin{aligned}\Psi(\varphi,\varsigma)&=\Psi_0(\varphi,\varsigma)+\Psi_1(\varphi,\varsigma)+\Psi_2(\varphi,\varsigma)+\Psi_3(\varphi,\varsigma)+\Psi_4(\varphi,\varsigma)+\cdots,\\ \Phi(\varphi,\varsigma)&=\Phi_0(\varphi,\varsigma)+\Phi_1(\varphi,\varsigma)+\Phi_2(\varphi,\varsigma)+\Phi_3(\varphi,\varsigma)+\Phi_4(\varphi,\varsigma)+\cdots,\end{aligned}\tag{42}$$

$$\begin{aligned}\Psi(\varphi,\varsigma)&=3\,\text{sech}^2(\varphi)+12\,\text{sech}^2(\varphi)\tanh\,(\varphi)\frac{\varsigma^\alpha}{\Gamma(1+\alpha)}+24\,\text{sech}^4(\varphi)(\cosh(2\varphi)-2)\frac{\varsigma^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)},\\ \Phi(\varphi,\varsigma)&=2\,\text{sech}\,(\varphi)+4\,\text{sech}\,(\varphi)\tanh\,(\varphi)\frac{\varsigma^\beta}{\Gamma(1+\beta)}+\text{sech}^5(\varphi)(67-52\cosh(2\varphi)+\cosh(4\varphi))\frac{\varsigma^{2\beta}}{\Gamma(1+2\beta)}\\ &\quad+24\,\text{sech}^5(\varphi)(2\cosh(2\varphi)-3)\frac{\varsigma^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}+\cdots.\end{aligned}\tag{43}$$

The above series converges to the exact solution at $\alpha=1$ that is given as follows:

$$\begin{aligned}\Psi(\varphi,\varsigma)&=\frac{3h}{2}\text{sech}^2\left(\sqrt{\frac{h}{2}}(\varphi-h\varsigma)\right),\\ \Phi(\varphi,\varsigma)&=\text{sech}\left(\sqrt{\frac{h}{2}}(\varphi-h\varsigma)\right).\end{aligned}\tag{44}$$

5. Results and Discussion

This section contains the results and discussion with the help of tables and some graphical representations obtained by NIP. We demonstrate the graphical solution of Equations (43) and (44) in Ψ and Φ forms, respectively. Figures 1 and 2 represent the approximate solution to the 4th iteration of the fractional differential problem using NIP with different values of $\alpha=0.25, 0.50, 0.75$, and 1. We consider $0\le\varphi\le10$ and $\theta=0.5$ with $\varsigma=0.5$ for these surface plots.

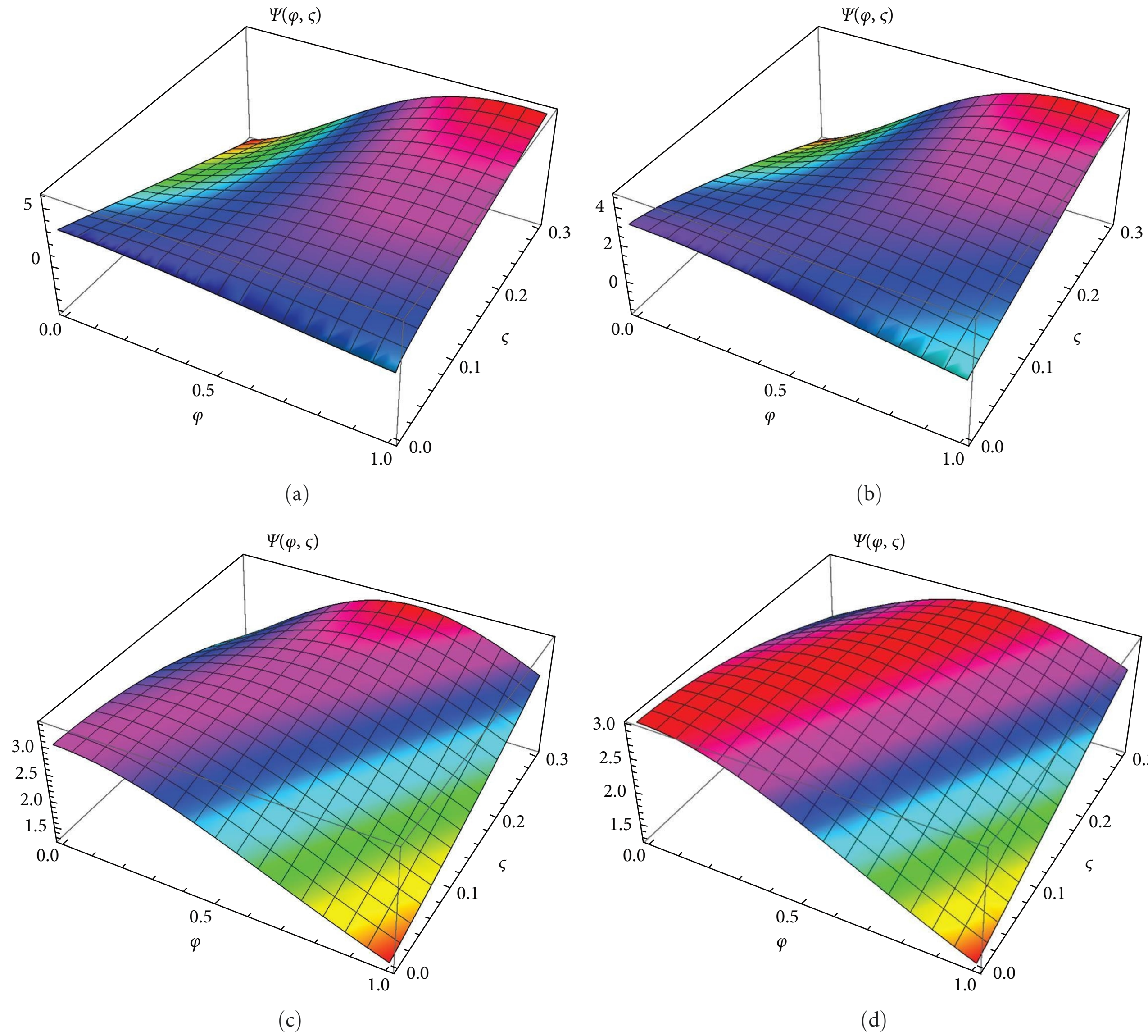

FIGURE 1: The surfaces solution of $\Psi(\varphi, \varsigma)$ for different norms of α: (a) surface plot of $\Psi(\varphi, \varsigma)$ at $\alpha = 0.25$; (b) surface plot of $\Psi(\varphi, \varsigma)$ at $\alpha = 0.50$; (c) surface plot of $\Psi(\varphi, \varsigma)$ at $\alpha = 0.75$; (d) surface plot of $\Psi(\varphi, \varsigma)$ at $\alpha = 1$.

Furthermore, Figure 1(c) demonstrates the graphical representations of the approximate solution, whereas Figure 1(d) represents the graphical representations of the exact solution of Equation (29) in Ψ form. Similarly, Figure 2(c) demonstrates the graphical representations of the approximate solution, whereas Figure 2(d) represents the graphical representations of the exact solution of Equation (29) in Φ form. Figures 3 and 4 show the graphical error between the approximate and the exact solutions of Equations (43) and (44) in Ψ and Φ forms; the green line represents the approximate solution, and the black dot shows the exact solution that coincides with each others at $\alpha = 1$. It shows that this approach is very simple and reliable in finding the solution of fourth-order PDE with fractional derivative. The numerical results, as shown in Tables 1 and 2, reveal that the approximate values are very close to the exact values when the fractional order α for various values of ς is increased. The absolute error in these tables reduces to a small value and becomes zero with high iterations.

6. Conclusion and Future Interact

In this paper, we have successfully developed a strategy of NIP for obtaining the approximate solution of fractional DSW equation with fractional derivative. The procedure of MT is very simple to implement in dealing the recurrence relation and hence it makes very easy to get the iterative results. HPM is very capable to handle the nonlinear terms and presents the series with successive iteration. Using initial condition, these iterations are very simple to obtain the approximate solution that is very close to the exact solution.

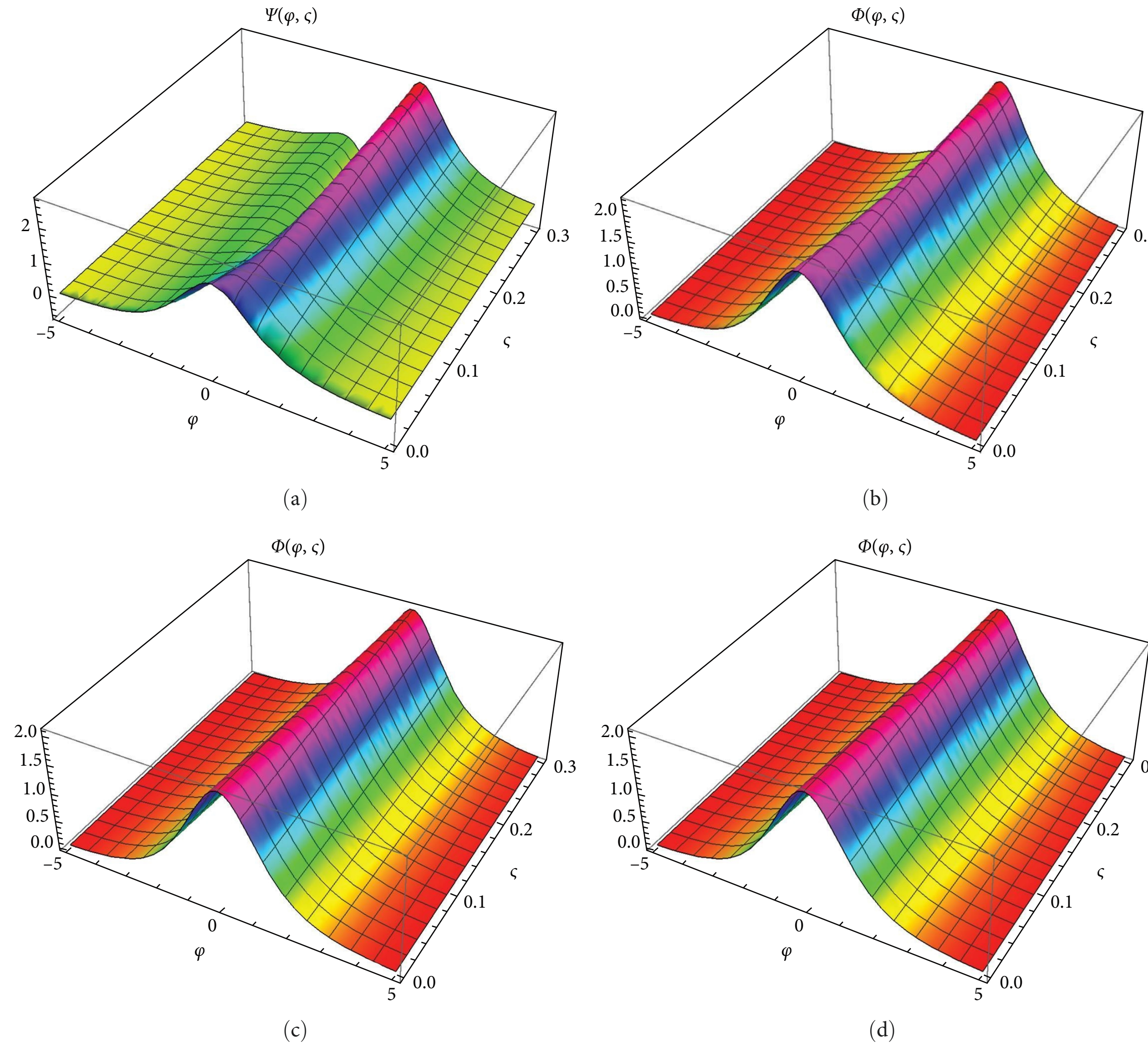

FIGURE 2: The surfaces solution of $\Phi(\varphi,\varsigma)$ for different norms of α: (a) surface plot of $\Phi(\varphi,\varsigma)$ at $\alpha=0.25$; (b) surface plot of $\Phi(\varphi,\varsigma)$ at $\alpha=0.50$; (c) surface plot of $\Phi(\varphi,\varsigma)$ at $\alpha=0.75$; (d) surface plot of $\Phi(\varphi,\varsigma)$ at $\alpha=1$.

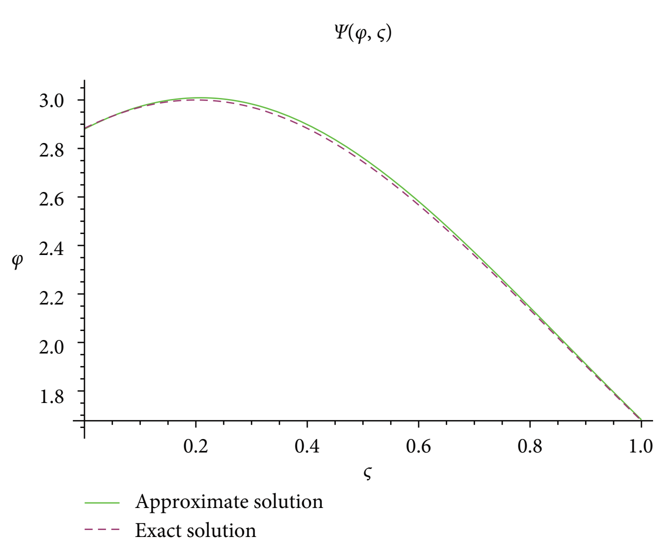

FIGURE 3: 2D Plot distribution for $\Psi(\varphi,\varsigma)$ with various fractional order α.

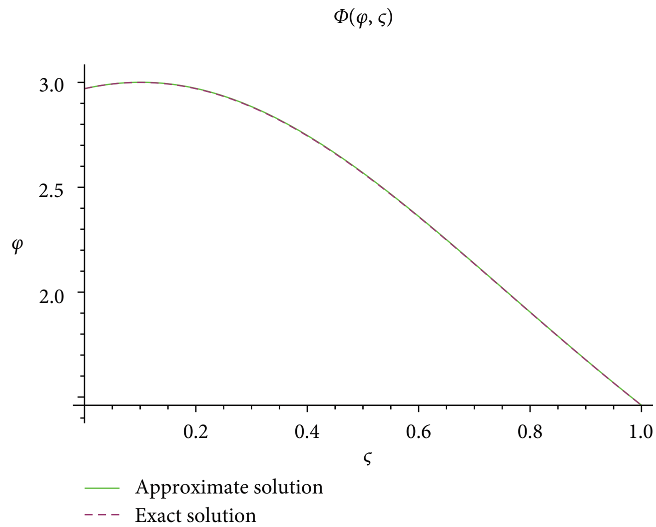

FIGURE 4: 2D Plot distribution for $\Phi(\varphi, \varsigma)$ with various fractional order α.

TABLE 1: The comparison of approximate solutions of Equation (43) and exact solutions of Equation (44) for different fractional order.

ς	φ	$\alpha = \beta = 0.5$	$\alpha = \beta = 0.75$	Approximate solution at $\alpha = \beta = 1$	Exact solution $\alpha = \beta = 1$
	−4	0.00285071	0.00351788	0.00386526	0.00286522
	−3	0.0209706	0.0258956	0.0284434	0.0284431
0.01	−2	0.150047	0.186111	0.203931	0.203929
	−1	0.901424	1.13347	1.22191	1.22192
	0	2.76	2.98195	2.9988	2.9988
	−4	0.00274148	0.003229	0.00371409	0.00371375
	−3	0.0201301	0.237737	0.0273353	0.0273329
0.02	−2	0.142165	0.171108	0.196213	0.196199
	−1	0.796625	1.05369	1.18465	1.18467
	0	2.52	2.94894	2.9952	2.99521

TABLE 2: The comparison of approximate solutions of Equation (43) and exact solutions of Equation (44) for different fractional order.

ς	φ	$\alpha = \beta = 0.5$	$\alpha = \beta = 0.75$	Approximate solution at $\alpha = \beta = 1$	Exact solution $\alpha = \beta = 1$
	−4	0.0596427	0.0684213	0.0717888	0.0717887
	−3	0.161835	0.185639	0.194741	0.194741
0.01	−2	0.434209	0.497711	0.521446	0.521446
	−1	1.08164	1.2288	1.27641	1.27641
	0	1.92	1.99398	1.9996	1.9996
	−4	0.0557228	0.0653893	0.0703689	0.0703681
	−3	0.151146	1.77436	0.190905	0.190903
0.02	−2	0.404563	0.476178	0.511467	0.511467
	−1	0.997665	1.18363	1.25681	1.25681
	0	1.84	1.98298	1.9984	1.9984

The present approach has the advantage of requiring smaller computation and yields a better efficiency. It is worth mentioned that NIP results are very efficient and relatable than other approaches. In future work, we extend this approach to get more different formulas for various nonlinear evolution equations to check its ability and power in science and engineering phenomena.

Data Availability

This article contains all the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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