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Research Article

Some New Fractional Weighted Simpson Type Inequalities for Functions Whose First Derivatives Are Convex

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The goal of this paper is to establish some weighted Simpson type inequalities for functions whose first derivatives are convex involving Reimann–Liouville integral operators. In order to obtain our results, we first prove a new integral identity as an auxiliary result. Based on this identity we establish some fractional weighted Simpson type inequalities for functions whose modulus of the first derivatives are convex. Several special cases are discussed. Error estimates for some numerical quadrature rules are furnished.

1. Introduction

Convexity is an analytical tool, it represents a basic notion in geometry and widely used in many areas of mathematics such as optimization, calculus of variations, and graph theory. This principle has a closed relationship with theory of inequality. It is clear that the most important inequality directly related to convexity is the so-called HermiteHadamard inequality which can be stated as follows: for any convex function Υ on the interval (υ_1, υ_2), we have the following double inequality:

$$f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(u)du \leq \frac{f(\upsilon_1)+f(\upsilon_2)}{2}. \quad (1)$$

Integral inequalities involving convex functions plays an important role in the different areas of science besides mathematics, such as physics, economics, biology, and engineering sciences, where most of the problems come down by solving integrals in the majority without difficult or impossible to solve directly which leads us to use approximate methods in other words quadrature formulas.

In the last decades, the study of error estimation of quadrature formulas has become a hot and attractive topic in the field of research. Several Newton–Cotes formulas have been studied by many researchers under the different classes of functions. The most important and remarkable three-point Newton–Cotes formula is that of Simpson which can be stated as follows:

$$\left|\frac{1}{6}\left(f(\upsilon_1)+4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+f(\upsilon_2)\right)-7\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(u)du\right| \leq \frac{(\upsilon_2-\upsilon_1)^4}{2,880}\left\|f^{(4)}\right\|_\infty, \quad (2)$$

where f is four times continuously differentiable function on (υ_1, υ_2), and $\|f^{(4)}\|_\infty = \sup_{x\in(\upsilon_1,\upsilon_2)}\left|f^{(4)}(x)\right|$.

In recent years, many researchers have studied the inequality (2), and several papers have been published dealing with refinements, generalizations, extensions as well as analogous versions of (2), for more details we refer the readers to [1–8] and references therein.

Sarikaya et al. [9] gave the following Simpson type inequalities for convex differentiable functions.

Theorem 1.1. Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I and let $\upsilon_1, \upsilon_2 \in I$ such that $\upsilon_1 < \upsilon_2$ and $f' \in L^1[\upsilon_1, \upsilon_2]$. If $|f'|$ is convex, then we have

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right) - 7\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(u)du\right| \leq \frac{5(\upsilon_2-\upsilon_1)}{72}\left(|f'(\upsilon_1)| + |f'(\upsilon_2)|\right). \quad (3)$$

Theorem 1.2. Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I and let $\upsilon_1, \upsilon_2 \in I$ such that $\upsilon_1 < \upsilon_2$ and $f' \in L^1[\upsilon_1, \upsilon_2]$. If $|f'|^q$ is convex where $q > 1$ with $1/p + 1/q = 1$, then we have

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right) - 7\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(u)du\right| \leq \frac{\upsilon_2-\upsilon_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \times\left(\left(\frac{3|f'(\upsilon_1)|^q + |f'(\upsilon_2)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(\upsilon_1)|^q + 3|f'(\upsilon_2)|^q}{4}\right)^{\frac{1}{q}}\right). \quad (4)$$

Theorem 1.3. Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I and let $\upsilon_1, \upsilon_2 \in I^\circ$ such that $\upsilon_1 < \upsilon_2$ and $f' \in L^1[\upsilon_1, \upsilon_2]$. If $|f'|^q$ is convex where $q \geq 1$, then we have

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right) - 7\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(u)du\right| \leq \frac{5(\upsilon_2-\upsilon_1)}{72}\left(\left(\frac{61|f'(\upsilon_1)|^q + 29|f'(\upsilon_2)|^q}{90}\right)^{\frac{1}{q}} + \left(\frac{29|f'(\upsilon_1)|^q + 61|f'(\upsilon_2)|^q}{90}\right)^{\frac{1}{q}}\right). \quad (5)$$

Sarikaya et al. [10] proposed a refinement of the result given in Theorem 1.2 as follows:

Theorem 1.4. Let $f: I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° let $\upsilon_1, \upsilon_2 \in I^\circ$ such that $\upsilon_1 < \upsilon_2$ and $f' \in L^1[\upsilon_1, \upsilon_2]$. If $|f'|^q$ is convex where $q > 1$ with $1/p + 1/q = 1$, then we have

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right) - 7\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(u)du\right| \leq \frac{\upsilon_2-\upsilon_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \times\left(\left(\frac{|f'(\upsilon_1)|^q + |f'(\frac{\upsilon_1+\upsilon_2}{2})|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{\upsilon_1+\upsilon_2}{2})|^q + |f'(\upsilon_2)|^q}{2}\right)^{\frac{1}{q}}\right). \quad (6)$$

Recently, Kashuri et al. [11], established the weighted version of Simpson type inequalities.

Theorem 1.5. Let $f: [\upsilon_1, \upsilon_2] \rightarrow \mathbb{R}$ be a differentiable function on $[\upsilon_1, \upsilon_2]$ such that $f' \in L^1[\upsilon_1, \upsilon_2]$ with $0 \leq \upsilon_1 < \upsilon_2$, and let $w: [\upsilon_1, \upsilon_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\upsilon_1 + \upsilon_2/2$. If $|f'|$ is convex, then we have

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right)\left(\int_{\upsilon_1}^{\upsilon_2} w(u)du\right) - \int_{\upsilon_1}^{\upsilon_2} w(u)f(u)du\right| \leq \frac{(\upsilon_2-\upsilon_1)^2}{324}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(8|f'(\upsilon_1)| + 29\left|f'\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right| + 8|f'(\upsilon_2)|\right). \quad (7)$$

Theorem 1.6. Let $f: [\upsilon_1, \upsilon_2] \rightarrow \mathbb{R}$ be a differentiable function on $[\upsilon_1, \upsilon_2]$ such that $f' \in L^1[\upsilon_1, \upsilon_2]$ with $0 \leq \upsilon_1 < \upsilon_2$, and let $w: [\upsilon_1, \upsilon_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\upsilon_1 + \upsilon_2/2$. If $|f'|^q$ is convex where $q > 1$ with $1/p + 1/q = 1$, then we have

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right)\left(\int_{\upsilon_1}^{\upsilon_2} w(u)du\right) - \int_{\upsilon_1}^{\upsilon_2} w(u)f(u)du\right| \leq \frac{(\upsilon_2-\upsilon_1)^2\|w\|_{[\upsilon_1,\upsilon_2],\infty}}{12}\left(\frac{1+2^{1+p}}{3(1+p)}\right)^{\frac{1}{p}} \times\left(\left(\frac{|f'(\upsilon_1)|^q + |f'(\frac{\upsilon_1+\upsilon_2}{2})|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{\upsilon_1+\upsilon_2}{2})|^q + |f'(\upsilon_2)|^q}{2}\right)^{\frac{1}{q}}\right). \quad (8)$$

Theorem 1.7. Let $f: [\upsilon_1, \upsilon_2] \rightarrow \mathbb{R}$ be a differentiable function on $[\upsilon_1, \upsilon_2]$ such that $f' \in L^1[\upsilon_1, \upsilon_2]$ with $0 \leq \upsilon_1 < \upsilon_2$, and let $w: [\upsilon_1, \upsilon_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\upsilon_1 + \upsilon_2/2$. If $|f'|^q$ is convex where $q \geq 1$, then we have

$$\left|\frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)\left(\int_{v_1}^{v_2}w(u)du\right) - \int_{v_1}^{v_2}w(u)f(u)du\right| \le \frac{5(v_2-v_1)^2\|w\|_{[v_1,v_2],\infty}}{72} \times\left(\left(\frac{16|f'(v_1)|^q+29|f'\left(\frac{v_1+v_2}{2}\right)|^q}{45}\right)^{\frac{1}{q}} + \left(\frac{29|f'\left(\frac{v_1+v_2}{2}\right)|^q+16|f'(v_2)|^q}{45}\right)^{\frac{1}{q}}\right). \tag{9}$$

Nowadays, fractional calculus has become a popular tool for scientists. It has been successfully used in the various fields of science and engineering [12]. Its main strength in describing the memory and genetic properties of the different materials and processes has aroused great interest among researchers from different fields. Regarding some papers dealing with fractional integral inequalities we advise readers to refer to some related studies [11–21].

The main goal of this study is to establish some fractional weighted Simpson type inequalities generalizing some earlier published papers by using a new identity. Several known results can be derived according to the values of the parameter α or the weighted function w. We end the paper by some applications in numerical integration to demonstrate the efficacy of our results.

2. Preliminaries

In this section, we recall certain definitions and a lemma that we will use in the sequel.

Definition 2.1. [22]. A function $f: I \to \mathbb{R}$ is said to be convex, if

$$f(\theta\varrho_1+(1-\theta)\varrho_2) \le \theta f(\varrho_1)+(1-\theta)f(\varrho_2), \tag{10}$$

holds for all $\varrho_1,\varrho_2 \in I$ and all $\theta\in[0,1]$.

Definition 2.2. [23]. A function $w:[\varrho_1,\varrho_2]\to[0,\infty)$ is said to be symmetric with respect to $\varrho_1+\varrho_2/2$, *if* $w(\varrho_1+\varrho_2-x)=w(x)$ *for all* $x\in[\varrho_1+\varrho_2]$.

Definition 2.3. [18]. Let $f\in L^1[a,b]$. The Riemann–Liouville integrals $J^\alpha_{a^+}f$ and $J^\alpha_{b^-}f$ of order $\alpha>0$ with $a\ge 0$ are defined by

$$\begin{aligned} J^\alpha_{a^+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x>a,\\ J^\alpha_{b^-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ b>x, \end{aligned} \tag{11}$$

respectively, where $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$ is the Gamma function and $J^0_{a^+}f(x)=J^0_{b^-}f(x)=f(x)$.

Definition 2.4. [18]. The beta function is defined for all complex numbers x and y with $Re(x)>0$ and $Re(y)>0$ as follows:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{12}$$

Definition 2.5. [18]. The hypergeometric function is defined for $Re(c)>Re(b)>0$ and $|z|<1$, as follows:

$${}_2F_1(a,b,c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \tag{13}$$

where $B(.,.)$ is the beta function.

Lemma 2.1. [24]. For any $0\le a<b$ in $\mathbb{R}$ and $0<\lambda\le 1$, we have

$$b^\lambda - a^\lambda \le (b-a)^\lambda. \tag{14}$$

3. Main Results

Lemma 3.1. Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $v_1,v_2\in I^\circ$ with $v_1<v_2$ and let $w:[v_1,v_2]\to\mathbb{R}$ be a symmetric with respect to $v_1+v_2/2$. If $f', w\in L^1[v_1,v_2]$, then

$$S(v_1,v_2,w,f)=\frac{3(v_2-v_1)^2}{8}\times\left(\int_0^1 K_1(t)f'\left(tv_1+(1-t)\frac{v_1+v_2}{2}\right)dt + \int_0^1 K_2(t)f'\left(t\frac{v_1+v_2}{2}+(1-t)v_2\right)dt\right), \tag{15}$$

where

$$\begin{aligned} S(v_1,v_2,w,f) &= \frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)\\ &\times\left\{\frac{2^{\alpha-1}\Gamma(\alpha)}{(v_2-v_1)^{\alpha-1}}\left(J^\alpha_{v_1^+}w\left(\frac{v_1+v_2}{2}\right)+J^\alpha_{\frac{v_1+v_2}{2}^-}w(v_1)\right)\right\}\\ &-\frac{2^{\alpha-1}\Gamma(\alpha)}{2(v_2-v_1)^{\alpha-1}}\left(J^\alpha_{\frac{v_1+v_2}{2}^+}wf(v_2)+J^\alpha_{v_2^-}wf\left(\frac{v_1+v_2}{2}\right)\right.\\ &\left.+J^\alpha_{v_1^+}wf\left(\frac{v_1+v_2}{2}\right)+J^\alpha_{\frac{v_1+v_2}{2}^-}wf(v_1)\right), \end{aligned} \tag{16}$$

with

$$K_1(t) = \frac{2}{9}\int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1})w\left(sv_1 + (1-s)\frac{v_1+v_2}{2}\right)ds - \frac{1}{3}\int_0^t ((1-s)^{\alpha-1} + s^{\alpha-1})w\left(sv_1 + (1-s)\frac{v_1+v_2}{2}\right)ds \quad (17)$$

and

$$K_2(t) = \frac{1}{9}\int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1})\, w\left(s\frac{v_1+v_2}{2} + (1-s)v_2\right)ds - \frac{1}{3}\int_0^t ((1-s)^{\alpha-1} + s^{\alpha-1})\, w\left(s\frac{v_1+v_2}{2} + (1-s)v_2\right)ds. \quad (18)$$

Proof. Let

$$I_1 = \int_0^1 K_1(t)f'\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right)dt, \quad (19)$$

and

$$I_2 = \int_0^1 K_2(t)f'\left(t\frac{v_1+v_2}{2} + (1-t)v_2\right)dt. \quad (20)$$

Integrating by parts I_1 and changing the variable, we obtain

$$\begin{aligned}
I_1 &= -\frac{2}{v_2-v_1}K_1(t)f\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right)\Big|_{t=0}^{t=1} + \frac{2}{v_2-v_1}\int_0^1 K_1'(t)f\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right)dt \\
&= -\frac{2}{v_2-v_1}K_1(1)f(v_1) + \frac{2}{v_2-v_1}K_1(0)f\left(\frac{v_1+v_2}{2}\right) \\
&\quad - \frac{2}{3(v_2-v_1)}\int_0^1 ((1-t)^{\alpha-1} + t^{\alpha-1})w\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right) \times f\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right)dt \\
&= \frac{2}{9(v_2-v_1)}\left(\int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1})w\left(sv_1 + (1-s)\frac{v_1+v_2}{2}\right)ds\right)f(v_1) \\
&\quad + \frac{4}{9(v_2-v_1)}\left(\int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1})w\left(sv_1 + (1-s)\frac{v_1+v_2}{2}\right)ds\right)f\left(\frac{v_1+v_2}{2}\right) \\
&\quad - \frac{2}{3(v_2-v_1)}\int_0^1 ((1-t)^{\alpha-1} + t^{\alpha-1})w\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right) \times f\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right)dt \\
&= \frac{1}{9}\left(\left(\frac{2}{v_2-v_1}\right)^{\alpha+1}\int_{v_1}^{\frac{v_1+v_2}{2}}\left((u-v_1)^{\alpha-1} + \left(\frac{v_1+v_2}{2}-u\right)^{\alpha-1}\right)w(u)du\right)f(v_1) \\
&\quad + \frac{2}{9}\left(\left(\frac{2}{v_2-v_1}\right)^{\alpha+1}\int_{v_1}^{\frac{v_1+v_2}{2}}\left((u-v_1)^{\alpha-1} + \left(\frac{v_1+v_2}{2}-u\right)^{\alpha-1}\right)w(u)du\right)f\left(\frac{v_1+v_2}{2}\right) \\
&\quad - \frac{1}{3}\left(\frac{2}{v_2-v_1}\right)^{\alpha+1}\int_{v_1}^{\frac{v_1+v_2}{2}}\left((u-v_1)^{\alpha-1} + \left(\frac{v_1+v_2}{2}-u\right)^{\alpha-1}\right)w(u)f(u)dt \\
&= \frac{1}{9}\left(f(v_1) + 2f\left(\frac{v_1+v_2}{2}\right)\right)\left(\frac{2}{v_2-v_1}\right)^{\alpha+1}\Gamma(\alpha)\left(J_{v_1^+}^{\alpha}w\left(\frac{v_1+v_2}{2}\right) + J_{\frac{v_1+v_2}{2}^-}^{\alpha}w(v_1)\right) \\
&\quad - \frac{1}{3}\left(\frac{2}{v_2-v_1}\right)^{\alpha+1}\Gamma(\alpha)\left(J_{v_1^+}^{\alpha}w\left(\frac{v_1+v_2}{2}\right)f\left(\frac{v_1+v_2}{2}\right) + J_{\frac{v_1+v_2}{2}^-}^{\alpha}w(v_1)f(v_1)\right).
\end{aligned} \quad (21)$$

Similarly, we have

$$
\begin{aligned}
I_2 &= -\tfrac{2}{v_2-v_1} K_2(t) f\left(t\tfrac{v_1+v_2}{2} + (1-t)v_2\right)\Big|_{t=0}^{t=1} \\
&\quad + \frac{2}{v_2-v_1}\int_0^1 K_2'(t) f\left(t\frac{v_1+v_2}{2} + (1-t)v_2\right)dt \\
&= \frac{2}{v_2-v_1} K_2(0) f(v_2) - \frac{2}{v_2-v_1} K_2(1) f\left(\frac{v_1+v_2}{2}\right) \\
&\quad - \frac{2}{3(v_2-v_1)}\int_0^1 \left((1-t)^{\alpha-1} + t^{\alpha-1}\right) w\left(t\frac{v_1+v_2}{2} + (1-t)v_2\right) \\
&\quad \times f\left(t\frac{v_1+v_2}{2} + (1-t)v_2\right)dt \\
&= \frac{2}{9(v_2-v_1)}\left(\int_0^1 \left((1-s)^{\alpha-1} + s^{\alpha-1}\right) w\left(s\frac{v_1+v_2}{2} + (1-s)v_2\right)ds\right) f(v_2) \\
&\quad + \frac{4}{9(v_2-v_1)}\left(\int_0^1 \left((1-s)^{\alpha-1} + s^{\alpha-1}\right) w\left(s\frac{v_1+v_2}{2} + (1-s)v_2\right)ds\right) f\left(\frac{v_1+v_2}{2}\right) \\
&\quad - \frac{2}{3(v_2-v_1)}\int_0^1 \left((1-t)^{\alpha-1} + t^{\alpha-1}\right) w\left(t\frac{v_1+v_2}{2} + (1-t)v_2\right) \\
&\quad \times f\left(t\frac{v_1+v_2}{2} + (1-t)v_2\right)dt \\
&= \frac{1}{9}\left(f(v_2) + 2f\left(\frac{v_1+v_2}{2}\right)\right)\left(\frac{2}{v_2-v_1}\right)^{\alpha+1}\Gamma(\alpha)\left(J^{\alpha}_{\frac{v_1+v_2}{2}+} w(v_2) + J^{\alpha}_{v_2^-} w\left(\frac{v_1+v_2}{2}\right)\right) \\
&\quad - \frac{1}{3}\left(\frac{2}{v_2-v_1}\right)^{\alpha+1}\Gamma(\alpha)\left(J^{\alpha}_{\frac{v_1+v_2}{2}+} w(v_2) f(v_2) + J^{\alpha}_{v_2^-} w\left(\frac{v_1+v_2}{2}\right) f\left(\frac{v_1+v_2}{2}\right)\right).
\end{aligned} \tag{22}
$$

From the symmetry of w we have

$$
\begin{aligned}
J^{\alpha}_{\frac{v_1+v_2}{2}-} w(v_1) &= \frac{1}{\Gamma(\alpha)}\int_{v_1}^{\frac{v_1+v_2}{2}} (u-v_1)^{\alpha-1} w(u)\,du \\
&= \frac{1}{\Gamma(\alpha)}\int_{\frac{v_1+v_2}{2}}^{v_2} (v_1+v_2-u-v_1)^{\alpha-1} w(v_1+v_2-u)\,du \\
&= \frac{1}{\Gamma(\alpha)}\int_{\frac{v_1+v_2}{2}}^{v_2} (v_2-u)^{\alpha-1} w(u)\,du = J^{\alpha}_{\frac{v_1+v_2}{2}+} w(v_2),
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
J^{\alpha}_{v_1^+} w\left(\frac{v_1+v_2}{2}\right) &= \frac{1}{\Gamma(\alpha)}\int_{v_1}^{\frac{v_1+v_2}{2}} \left(\frac{v_1+v_2}{2} - u\right)^{\alpha-1} w(u)\,du \\
&= \frac{1}{\Gamma(\alpha)}\int_{\frac{v_1+v_2}{2}}^{v_2} \left(\frac{v_1+v_2}{2} - (v_1+v_2-u)\right)^{\alpha-1} w(v_1+v_2-u)\,du \\
&= \frac{1}{\Gamma(\alpha)}\int_{\frac{v_1+v_2}{2}}^{v_2} \left(u - \frac{v_1+v_2}{2}\right)^{\alpha-1} w(u)\,du = J^{\alpha}_{v_2^-} w\left(\frac{v_1+v_2}{2}\right).
\end{aligned} \tag{24}
$$

Summing Equations (21) and (22), then multiplying the result by $3(v_2 - v_1)^2/8$ and using Equations (23) and (24), we get the desired result. □

In what follows we assume $\|w\|_{[v_1, v_2], \infty} = \sup_{u \in [v_1, v_2]} |w(u)|$.

Theorem 3.1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $v_1, v_2 \in I^\circ$ with $v_1 < v_2$, and let $w : [v_1, v_2] \to \mathbb{R}$ be symmetric with respect to $v_1 + v_2/2$ and $f', w \in L^1[v_1, v_2]$. If $|f'|$ is convex, then we have

$$\begin{aligned} &|S(v_1, v_2, w, f)| \\ &\leq \frac{(v_2 - v_1)^2}{24\alpha} \|w\|_{[v_1,v_2],\infty} \left(\left(\frac{\alpha}{2(\alpha+1)} \left(\frac{1}{3}\right)^{\frac{1}{\alpha}} + \frac{\alpha}{4(\alpha+2)} \left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{\alpha - 5}{4(\alpha+1)} \right) |f'(v_1)| \right. \\ &+ 2\left|f'\left(\frac{v_1 + v_2}{2}\right)\right| \left(\frac{\alpha+7}{4(\alpha+1)} + \frac{\alpha}{2(\alpha+1)} \left(\frac{1}{3}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{4(\alpha+2)} \left(\frac{1}{3}\right)^{\frac{2}{\alpha}} \right) \\ &\left. + \left(\frac{\alpha}{2(\alpha+1)} \left(\frac{1}{3}\right)^{\frac{1}{\alpha}} + \frac{\alpha}{4(\alpha+2)} \left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{\alpha - 5}{4(\alpha+1)} \right) |f'(v_2)| \right). \end{aligned} \tag{25}$$

Proof. Using Lemma 3.1, modulus, convexity of $|f'|$ and Lemma 2.1, we obtain

$$\begin{aligned} &|S(v_1, v_2, w, f)| \\ &\leq \frac{3(v_2 - v_1)^2}{8} \left(\int_0^1 |K_1(t)| \left|f'\left(tv_1 + (1-t)\frac{v_1 + v_2}{2}\right)\right| dt \right. \\ &\left. + \int_0^1 |K_2(t)| \left|f'\left(t\frac{v_1 + v_2}{2} + (1-t)v_2\right)\right| dt \right) \\ &\leq \frac{3(v_2 - v_1)^2}{8} \left(\int_0^1 |K_1(t)| \left(t|f'(v_1)| + (1-t)\left|f'\left(\frac{v_1 + v_2}{2}\right)\right| \right) dt \right. \\ &\left. + \int_0^1 |K_2(t)| \left(t\left|f'\left(\frac{v_1 + v_2}{2}\right)\right| + (1-t)|f'(v_2)| \right) dt \right) \\ &\leq \frac{3(v_2 - v_1)^2}{8} \|w\|_{[v_1,v_2],\infty} \\ &\times \left(\int_0^1 \left| \frac{2}{9} \int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1}) ds -; \frac{1}{3} \int_0^t ((1-s)^{\alpha-1} + s^{\alpha-1}) ds \right| \right. \\ &\times \left(t|f'(v_1)| + (1-t)\left|f'\left(\frac{v_1 + v_2}{2}\right)\right| \right) dt \\ &+ \int_0^1 \left| \frac{1}{9} \int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1}) ds -; \frac{1}{3} \int_0^t ((1-s)^{\alpha-1} + s^{\alpha-1}) ds \right| \\ &\left. \left(t\left|f'\left(\frac{v_1 + v_2}{2}\right)\right| + (1-t)|f'(v_2)| \right) dt \right) \\ &= \frac{3(v_2 - v_1)^2}{8} \|w\|_{[v_1,v_2],\infty} \\ &\times \left(\frac{1}{9\alpha} \int_0^1 |1 + 3((1-t)^\alpha - t^\alpha)| \left(t|f'(v_1)| + (1-t)\left|f'\left(\frac{v_1 + v_2}{2}\right)\right| \right) dt \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{9\alpha}\int_0^1 |3((1-t)^\alpha - t^\alpha) - 1| \left(t\left|f'\left(\frac{v_1+v_2}{2}\right)\right| + (1-t)|f'(v_2)|\right)dt\Bigg)\\
&= \frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\Bigg(|f'(v_1)|\int_0^1 |1+3((1-t)^\alpha - t^\alpha)|\, tdt\\
&+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|\int_0^1 |1+3((1-t)^\alpha - t^\alpha)|\,(1-t)dt\\
&+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|\int_0^1 |3((1-t)^\alpha - t^\alpha) - 1|\, tdt\\
&+|f'(v_2)|\int_0^1 |3((1-t)^\alpha - t^\alpha) - 1|\,(1-t)dt\Bigg)\\
&= \frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\\
&\times\Bigg(|f'(v_1)|\left(\int_0^{\frac{1}{2}} |3((1-t)^\alpha - t^\alpha) + 1|\, tdt + \int_{\frac{1}{2}}^1 |3(t^\alpha - (1-t)^\alpha) - 1|\, tdt\right)\\
&+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|\left(\int_0^{\frac{1}{2}} |3((1-t)^\alpha - t^\alpha) + 1|\,(1-t)dt + \int_{\frac{1}{2}}^1 |3(t^\alpha - (1-t)^\alpha) - 1|\,(1-t)dt\right)\\
&+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|\left(\int_0^{\frac{1}{2}} |3((1-t)^\alpha - t^\alpha) - 1|\, tdt + \int_{\frac{1}{2}}^1 |3(t^\alpha - (1-t)^\alpha) + 1|\, tdt\right)\\
&+|f'(v_2)|\left(\int_0^{\frac{1}{2}} |3((1-t)^\alpha - t^\alpha) - 1|\,(1-t)dt + \int_{\frac{1}{2}}^1 |3(t^\alpha - (1-t)^\alpha) + 1|\,(1-t)dt\right)\Bigg)\\
&\leq \frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\Bigg(|f'(v_1)|\left(\int_0^{\frac{1}{2}} |3(1-2t)^\alpha + 1|\, tdt + \int_{\frac{1}{2}}^1 |3(2t-1)^\alpha - 1|\, tdt\right)\\
&+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|\left(\int_0^{\frac{1}{2}} |3(1-2t)^\alpha + 1|\,(1-t)dt + \int_{\frac{1}{2}}^1 |3(2t-1)^\alpha - 1|\,(1-t)dt\right)\\
&+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|\left(\int_0^{\frac{1}{2}} |3(1-2t)^\alpha - 1|\, tdt + \int_{\frac{1}{2}}^1 |3(2t-1)^\alpha + 1|\, tdt\right)\\
&+|f'(v_2)|\left(\int_0^{\frac{1}{2}} |3(1-2t)^\alpha - 1|\,(1-t)dt + \int_{\frac{1}{2}}^1 |3(2t-1)^\alpha + 1|\,(1-t)dt\right)\Bigg)\\
&= \frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\Bigg(\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} + \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{\alpha-5}{4(\alpha+1)}\right)|f'(v_1)|\\
&+2\left|f'\left(\frac{v_1+v_2}{2}\right)\right|\left(\frac{\alpha+7}{4(\alpha+1)} + \frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}\right)\\
&+\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} + \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{\alpha-5}{4(\alpha+1)}\right)|f'(v_2)|\Bigg),
\end{aligned}
\tag{26}
$$

where we have used the facts that

$$\int_0^{\frac{1}{2}} |3(1-2t)^\alpha+1|\, t dt = \int_{\frac{1}{2}}^1 |3(2t-1)^\alpha+1|\,(1-t)dt = \frac{\alpha^2+3\alpha+8}{8(\alpha+1)(\alpha+2)}, \tag{27}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 |3(2t-1)^\alpha-1|\, t dt \\
&= \int_0^{\frac{1}{2}} |3(1-2t)^\alpha-1|\,(1-t)dt \\
&= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}} (1-3(2t-1)^\alpha) t dt + \int_{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1 (3(2t-1)^\alpha-1) t dt \\
&= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}} (t-3t(2t-1)^\alpha) dt + \int_{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1 (3t(2t-1)^\alpha-t) dt \\
&= \frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} + \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{3\alpha^2-3\alpha-12}{8(\alpha+1)(\alpha+2)},
\end{aligned} \tag{28}$$

$$\int_0^{\frac{1}{2}} |3(1-2t)^\alpha+1|\,(1-t)dt = \int_{\frac{1}{2}}^1 |3(2t-1)^\alpha+1|\, t dt = \frac{3\alpha^2+21\alpha+24}{8(\alpha+1)(\alpha+2)}, \tag{29}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} |3(1-2t)^\alpha-1|\, t dt \\
&= \int_{\frac{1}{2}}^1 |3(2t-1)^\alpha-1|\,(1-t)dt \\
&= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}} ((1-t)-3(2t-1)^\alpha+3t(2t-1)^\alpha) dt \\
&\quad + \int_{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1 (3(2t-1)^\alpha-3t(2t-1)^\alpha-(1-t)) dt \\
&= \frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{(\alpha-1)(\alpha+4)}{8(\alpha+1)(\alpha+2)}.
\end{aligned} \tag{30}$$

The proof is achieved. □

Corollary 3.1. In *Theorem 3.1,* if we take $w(u)=1/\upsilon_2-\upsilon_1$ we get

$$\begin{aligned}
&\left|\frac{1}{6}\left(f(\upsilon_1)+4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+f(\upsilon_2)\right)-\Omega(\upsilon_1,\upsilon_2,f)\right| \\
&\leq \frac{\upsilon_2-\upsilon_1}{24\alpha}\left(\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{\alpha-5}{4(\alpha+1)}\right)|f'(\upsilon_1)|\right. \\
&\quad +2\left|f'\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right|\left(\frac{\alpha+7}{4(\alpha+1)}+\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}\right) \\
&\quad \left.+\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{\alpha-5}{4(\alpha+1)}\right)|f'(\upsilon_2)|\right),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\Omega(\upsilon_1,\upsilon_2,f) &= \frac{2^{\alpha-1}\Gamma(\alpha)}{2(\upsilon_2-\upsilon_1)^{\alpha-2}}\left(J^\alpha_{\frac{\upsilon_1+\upsilon_2}{2}+}f(\upsilon_2)+J^\alpha_{\upsilon_2^-}f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right. \\
&\quad \left.+J^\alpha_{\upsilon_1^+}f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+J^\alpha_{\frac{\upsilon_1+\upsilon_2}{2}-}f(\upsilon_1)\right).
\end{aligned} \tag{32}$$

Remark 3.1. *In Theorem 3.1, if we take $\alpha=1$, we obtain the first inequality of Corollary 2.4 from [11]. Moreover, if we choose $w(u)=1/\upsilon_2-\upsilon_1$, we get the second inequality of Corollary 2.4 from [11].*

Corollary 3.2. In *Theorem 3.1,* if we use the convexity of $|f'|$ i.e. $|f'(\upsilon_1+\upsilon_2/2)|\leq|f'(\upsilon_1)|+|f'(\upsilon_2)|/2$, we obtain

$$\begin{aligned}
&|S(\upsilon_1,\upsilon_2,w,f)| \\
&\leq \frac{(\upsilon_2-\upsilon_1)^2}{8\alpha(\alpha+1)}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(1+\alpha\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}\right)(|f'(\upsilon_1)|+|f'(\upsilon_2)|).
\end{aligned} \tag{33}$$

Corollary 3.3. In *Corollary 3.2,* if we take $w(u) = 1/v_2 - v_1$ we get

$$\left|\frac{1}{6}\left(f(v_1) + 4f\left(\frac{v_1+v_2}{2}\right) + f(v_2)\right) - \Omega(v_1, v_2, f)\right| \leq \frac{v_2 - v_1}{8\alpha(\alpha+1)}\left(1 + \alpha\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}\right)(|f'(v_1)| + |f'(v_2)|), \tag{34}$$

where Ω is defined as in *Equation (32).*

Remark 3.2. *In Corollary 3.3, if we take $\alpha = 1$ we get Corollary 2.5 from [11]. Moreover, if we choose $w(u) = 1/v_2 - v_1$, we obtain Theorem 5 from [9].*

Theorem 3.2. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $v_1, v_2 \in I^{\circ}$ with $v_1 < v_2$, and let $w: [v_1, v_2] \to \mathbb{R}$ be symmetric with respect to $v_1 + v_2/2$ and $f', w \in L^1[v_1, v_2]$. If $|f'|^r$ is convex where $r, l > 1$ with $1/r + 1/l = 1$, then we have

$$\begin{aligned} &|S(v_1, v_2, w, f)| \\ &\leq \frac{(v_2 - v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\left(\frac{|f'(v_1)|^r + |f'(\frac{v_1+v_2}{2})|^r}{2}\right)^{\frac{1}{r}} + \left(\frac{|f'(\frac{v_1+v_2}{2})|^r + |f'(v_2)|^r}{2}\right)^{\frac{1}{r}}\right) \\ &\times\left(2^{2l-1} \cdot {}_2F_1\left(-l, 1, \frac{1+\alpha}{\alpha}; \frac{3}{4}\right) + \frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha}, l+1\right) + \frac{2^l \cdot {}_2F_1\left(\frac{\alpha-1}{\alpha}, 1, l+2; \frac{2}{3}\right)}{3\alpha(l+1)}\right)^{\frac{1}{l}}, \end{aligned} \tag{35}$$

where S is defined by *Equation (15),* B and ${}_2F_1$ are the beta and hypergeometric functions, respectively.

Proof. Using Lemma 3.1, modulus, Hölder inequality, convexity of $|f'|^r$, and Lemma 2.1, we obtain

$$\begin{aligned} &|S(v_1, v_2, w, f)| \\ &\leq \frac{3(v_2 - v_1)^2}{8}\left(\left(\int_0^1 |K_1(t)|^l dt\right)^{\frac{1}{l}}\left(\int_0^1 \left|f'\left(tv_1 + (1-t)\frac{v_1+v_2}{2}\right)\right|^r dt\right)^{\frac{1}{r}}\right. \\ &\left.+\left(\int_0^1 |K_2(t)|^l dt\right)^{\frac{1}{l}}\left(\int_0^1 \left|f'\left(t\frac{v_1+v_2}{2} + (1-t)v_2\right)\right|^r dt\right)^{\frac{1}{r}}\right) \\ &\leq \frac{(v_2 - v_1)^2}{24}\|w\|_{[v_1,v_2],\infty} \\ &\times\left(\left(\int_0^1 \left|\frac{2}{9}\int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1})ds -; \frac{1}{3}\int_0^t ((1-s)^{\alpha-1} + s^{\alpha-1})ds\right|^l dt\right)^{\frac{1}{l}}\right. \\ &\times\left(\int_0^1 \left(t|f'(v_1)|^r + (1-t)|f'\left(\frac{v_1+v_2}{2}\right)|^r\right)dt\right)^{\frac{1}{r}} \\ &+\left(\int_0^1 \left|\frac{1}{9}\int_0^1 ((1-s)^{\alpha-1} + s^{\alpha-1})ds -; \frac{1}{3}\int_0^t ((1-s)^{\alpha-1} + s^{\alpha-1})ds\right|^l dt\right)^{\frac{1}{l}} \\ &\left.\times\left(\int_0^1 \left(t|f'\left(\frac{v_1+v_2}{2}\right)|^r + (1-t)|f'(v_2)|^r\right)dt\right)^{\frac{1}{r}}\right) \\ &\leq \frac{(v_2 - v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty} \\ &\times\left(\left(\int_0^1 |1 + 3((1-t)^\alpha - t^\alpha)|^l dt\right)^{\frac{1}{l}}\left(\frac{|f'(v_1)|^r + |f'(\frac{v_1+v_2}{2})|^r}{2}\right)^{\frac{1}{r}}\right. \\ &\left.+\left(\int_0^1 |3((1-t)^\alpha - t^\alpha) - 1|^l dt\right)\left(\frac{|f'(\frac{v_1+v_2}{2})|^r + |f'(v_2)|^r}{2}\right)^{\frac{1}{r}}\right) \\ &= \frac{(v_2 - v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\int_0^1 |1 + 3((1-t)^\alpha - t^\alpha)|^l dt\right)^{\frac{1}{l}} \\ &\times\left(\left(\frac{|f'(v_1)|^r + |f'(\frac{v_1+v_2}{2})|^r}{2}\right)^{\frac{1}{r}} + \left(\frac{|f'(\frac{v_1+v_2}{2})|^r + |f'(v_2)|^r}{2}\right)^{\frac{1}{r}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{(\upsilon_2-\upsilon_1)^2}{24\alpha}\|w\|_{[\upsilon_1,\upsilon_2],\infty} \\
&\quad\times\left(\int_0^{\frac{1}{2}}|1+3((1-t)^\alpha-t^\alpha)|^l dt+\int_{\frac{1}{2}}^1|3(t^\alpha-(1-t)^\alpha)-1|^l dt\right)^{\frac{1}{l}} \\
&\quad\times\left(\left(\frac{|f'(\upsilon_1)|^r+|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r}{2}\right)^{\frac{1}{r}}+\left(\frac{|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}\right) \\
&\le \frac{(\upsilon_2-\upsilon_1)^2}{24\alpha}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(\int_0^{\frac{1}{2}}|1+3(1-2t)^\alpha|^l dt+\int_{\frac{1}{2}}^1|3(2t-1)^\alpha-1|^l dt\right)^{\frac{1}{l}} \\
&\quad\times\left(\left(\frac{|f'(\upsilon_1)|^r+|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r}{2}\right)^{\frac{1}{r}}+\left(\frac{|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}\right) \\
&= \frac{(\upsilon_2-\upsilon_1)^2}{24\alpha}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(\left(\frac{|f'(\upsilon_1)|^r+|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r}{2}\right)^{\frac{1}{r}}+\left(\frac{|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}\right) \\
&\quad\times\left(\int_0^{\frac{1}{2}}(1+3(1-2t)^\alpha)^l dt+\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2}(\frac{1}{3})^{\frac{1}{\alpha}}}(1-3(2t-1)^\alpha)^l dt\right. \\
&\quad\left.+\int_{\frac{1}{2}+\frac{1}{2}(\frac{1}{3})^{\frac{1}{\alpha}}}^1(3(2t-1)^\alpha-1)^l dt\right)^{\frac{1}{l}} \\
&= \frac{(\upsilon_2-\upsilon_1)^2}{24\alpha}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(\left(\frac{|f'(\upsilon_1)|^r+|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r}{2}\right)^{\frac{1}{r}}+\left(\frac{|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}\right) \\
&\quad\times\left(2^{2l-1}\cdot{}_2F_1\left(-l,1,\frac{1+\alpha}{\alpha};\frac{3}{4}\right)+\frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha},l+1\right)+\frac{2^l\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,l+2;\frac{2}{3}\right)}{3\alpha(l+1)}.\right)^{\frac{1}{l}},
\end{aligned}
\tag{36}
$$

where we have used the facts that

$$
\begin{aligned}
\int_0^{\frac{1}{2}}(1+3(1-2t)^\alpha)^l dt &= \frac{1}{6\alpha}\int_0^3(1+x)^l\left(\frac{x}{3}\right)^{\frac{1}{\alpha}-1}dx \\
&= \frac{1}{2\alpha}\int_0^1(1+3y)^l y^{\frac{1}{\alpha}-1}dy \\
&= \frac{2^{2l}}{2\alpha}\int_0^1(1-u)^{\frac{1}{\alpha}-1}\left(1-\frac{3}{4}u\right)^l dy \\
&= 2^{2l-1}\cdot{}_2F_1\left(-l,1,\frac{1+\alpha}{\alpha};\frac{3}{4}\right),
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2}(\frac{1}{3})^{\frac{1}{\alpha}}}(1-3(2t-1)^\alpha)^l dt &= \frac{3^{1-\frac{1}{\alpha}}}{6\alpha}\int_0^1 x^{\frac{1}{\alpha}-1}(1-x)^l dx \\
&= \frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha},l+1\right),
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
\int_{\frac{1}{2}+\frac{1}{2}(\frac{1}{3})^{\frac{1}{\alpha}}}^1(3(2t-1)^\alpha-1)^l dt &= \frac{3^{1-\frac{1}{\alpha}}}{6\alpha}\int_1^3(x-1)^l x^{\frac{1}{\alpha}-1}dx \\
&= \frac{2^{l+1}3^{1-\frac{1}{\alpha}}}{6\alpha}\int_0^1 u^l(2u+1)^{\frac{1}{\alpha}-1}du \\
&= \frac{2^l}{3\alpha}\int_0^1(1-y)^l\left(1-\frac{2}{3}y\right)^{\frac{1}{\alpha}-1}du \\
&= \frac{2^l}{3\alpha(l+1)}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,l+2;\frac{2}{3}\right).
\end{aligned}
\tag{39}
$$

the proof is achieved. □

Corollary 3.4. In *Theorem 3.2,* if we take $w(u)=1/\upsilon_2-\upsilon_1$ we get

$$
\begin{aligned}
&\left|\frac{1}{6}\left(f(\upsilon_1)+4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+f(\upsilon_2)\right)-\Omega(\upsilon_1,\upsilon_2,f)\right| \\
&\le \frac{(\upsilon_2-\upsilon_1)}{24\alpha}\left(\left(\frac{|f'(\upsilon_1)|^r+|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r}{2}\right)^{\frac{1}{r}}+\left(\frac{|f'(\frac{\upsilon_1+\upsilon_2}{2})|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}\right) \\
&\quad\times\left(2^{2l-1}\cdot{}_2F_1\left(-l,1,\frac{1+\alpha}{\alpha};\frac{3}{4}\right)+\frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha},l+1\right)+\frac{2^l\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,l+2;\frac{2}{3}\right)}{3\alpha(l+1)}\right)^{\frac{1}{l}},
\end{aligned}
\tag{40}
$$

where Ω is defined as in *Equation (32).*

Remark 3.3. *In Theorem 3.2, if we take $\alpha = 1$ we get Corollary 2.8 from [11]. Moreover if we take $w(u) = 1/v_2 - v_1$, we obtain Corollary 3.1 from [10].*

Corollary 3.5. In *Theorem 3.2,* if we use the convexity of $|f'|^r$ i.e. $|f'(v_1+v_2/2)|^r \le |f'(v_1)|^r + |f'(v_2)|^r/2$, we obtain

$$
\begin{aligned}
&|S(v_1,v_2,w,f)| \\
&\le \frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\left(\frac{3|f'(v_1)|^r+|f'(v_2)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{|f'(v_1)|^r+3|f'(v_2)|^r}{4}\right)^{\frac{1}{r}}\right) \\
&\times\left(2^{2l-1}\cdot{}_2F_1\left(-l,1,\frac{1+\alpha}{\alpha};\frac{3}{4}\right)+\frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha},l+1\right)+\frac{2^l\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,l+2;\frac{2}{3}\right)}{3\alpha(l+1)}\right)^{\frac{1}{l}},
\end{aligned}
\tag{41}
$$

Corollary 3.6. In *Corollary 3.5,* if we take $w(u) = 1/v_2 - v_1$.

$$
\begin{aligned}
&\left|\frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)-\Omega(v_1,v_2,f)\right| \\
&\le \frac{v_2-v_1}{24\alpha}\left(\left(\frac{3|f'(v_1)|^r+|f'(v_2)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{|f'(v_1)|^r+3|f'(v_2)|^r}{4}\right)^{\frac{1}{r}}\right) \\
&\times\left(2^{2l-1}\cdot{}_2F_1\left(-l,1,\frac{1+\alpha}{\alpha};\frac{3}{4}\right)+\frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha},l+1\right)+\frac{2^l\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,l+2;\frac{2}{3}\right)}{3\alpha(l+1)}\right)^{\frac{1}{l}},
\end{aligned}
\tag{42}
$$

where Ω is defined as in *Equation (32).*

Remark 3.4. *In Corollary 3.5, if we take $\alpha = 1$, we obtain the first inequality of Corollary 2.4 from [11]. Moreover if we choose $w(u) = 1/v_2 - v_1$, we get Theorem 6 from [9].*

Corollary 3.7. In *Corollary 3.5,* if we use the discrete power mean inequality, we get

$$
\begin{aligned}
&|S(v_1,v_2,w,f)| \\
&\le \frac{(v_2-v_1)^2}{12\alpha}\|w\|_{[v_1,v_2],\infty}\left(\frac{|f'(v_1)|^r+|f'(v_2)|^r}{2}\right)^{\frac{1}{r}} \\
&\times\left(2^{2l-1}\cdot{}_2F_1\left(-l,1,\frac{1+\alpha}{\alpha};\frac{3}{4}\right)+\frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha},l+1\right)+\frac{2^l\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,l+2;\frac{2}{3}\right)}{3\alpha(l+1)}\right)^{\frac{1}{l}}.
\end{aligned}
\tag{43}
$$

Corollary 3.8. In *Corollary 3.13,* if we take $w(u) = 1/v_2 - v_1$.

$$
\begin{aligned}
&\left|\frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)-\Omega(v_1,v_2,f)\right| \\
&\le \frac{v_2-v_1}{12\alpha}\left(\frac{|f'(v_1)|^r+|f'(v_2)|^r}{2}\right)^{\frac{1}{r}} \\
&\times\left(2^{2l-1}\cdot{}_2F_1\left(-l,1,\frac{1+\alpha}{\alpha};\frac{3}{4}\right)+\frac{3^{1-\frac{1}{\alpha}}}{6\alpha}B\left(\frac{1}{\alpha},l+1\right)+\frac{2^l\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,l+2;\frac{2}{3}\right)}{3\alpha(l+1)}\right)^{\frac{1}{l}},
\end{aligned}
\tag{44}
$$

where Ω is defined as in *Corollary 3.12.*

Corollary 3.9. In *Corollary 3.13,* if we take $\alpha = 1$

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right)\int_{\upsilon_1}^{\upsilon_2} w(u)du - \int_{\upsilon_1}^{\upsilon_2} w(u)f(u)du\right| \leq \frac{(\upsilon_2-\upsilon_1)^2}{6}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(\frac{1+2^{l+1}}{3(l+1)}\right)^{\frac{1}{l}}\left(\frac{|f'(\upsilon_1)|^r + |f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}. \tag{45}$$

Corollary 3.10. In *Corollary 3.13,* if we take $w(u) = 1/\upsilon_2 - \upsilon_1$ and $\alpha = 1$ we get

$$\left|\frac{1}{6}\left(f(\upsilon_1) + 4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + f(\upsilon_2)\right) - 7\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(u)du\right| \leq \frac{\upsilon_2-\upsilon_1}{6}\left(\frac{2^{l+1}+1}{3(l+1)}\right)^{\frac{1}{l}}\left(\frac{|f'(\upsilon_1)|^r + |f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}. \tag{46}$$

Theorem 3.3. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\upsilon_1, \upsilon_2 \in I^\circ$ with $\upsilon_1 < \upsilon_2$, and let $w: [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be symmetric with respect to $\upsilon_1 + \upsilon_2/2$ and $f', w \in L^1[\upsilon_1, \upsilon_2]$. If $|f'|^r$ is convex where $r \geq 1$, then we have

$$\begin{aligned}
&|S(\upsilon_1,\upsilon_2,w,f)| \\
&\leq \frac{(\upsilon_2-\upsilon_1)^2}{24\alpha}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(\frac{9}{2(\alpha+1)} - \frac{3}{\alpha+1}\left(\frac{1}{2}\right)^\alpha + \frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{r}} \\
&\times\left(\left(\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} + \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{2\alpha^2-6\alpha-20}{8(\alpha+1)(\alpha+2)}|f'(\upsilon_1)|^r\right)\right.\right. \\
&+\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} + \frac{2\alpha^2+18\alpha+28}{8(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right|^r\Bigg)^{\frac{1}{r}} \\
&+\left(\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} + \frac{2\alpha^2+18\alpha+28}{8(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right|^r\right. \\
&\times\left.\left.\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} + \frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}} - \frac{2\alpha^2-6\alpha-20}{8(\alpha+1)(\alpha+2)}\right)|f'(\upsilon_2)|^r\right)^{\frac{1}{r}}\right),
\end{aligned} \tag{47}$$

where S is defined by *Equation (15), B* and ${}_2F_1$ are the beta and hypergeometric functions, respectively.

Proof. Using Lemma 3.1, modulus, power mean inequality, convexity of $|f'|^r$ and Lemma 2.1, we obtain

$$\begin{aligned}
&|S(\upsilon_1,\upsilon_2,w,f)| \\
&\leq \frac{3(\upsilon_2-\upsilon_1)^2}{8}\left(\left(\int_0^1 |K_1(t)|dt\right)^{1-\frac{1}{r}}\left(\int_0^1 |K_1(t)|\left|f'\left(t\upsilon_1 + (1-t)\frac{\upsilon_1+\upsilon_2}{2}\right)\right|^r dt\right)^{\frac{1}{r}}\right. \\
&+\left.\left(\int_0^1 |K_2(t)|dt\right)^{1-\frac{1}{r}}\left(\int_0^1 |K_2(t)|\left|f'\left(t\frac{\upsilon_1+\upsilon_2}{2} + (1-t)\upsilon_2\right)\right|^r dt\right)^{\frac{1}{r}}\right) \\
&\leq \frac{3(\upsilon_2-\upsilon_1)^2}{8}\|w\|_{[\upsilon_1,\upsilon_2],\infty} \\
&\times\left(\left(\int_0^1\left|\frac{2}{9}\int_0^1((1-s)^{\alpha-1} + s^{\alpha-1})ds - ;\frac{1}{3}\int_0^t((1-s)^{\alpha-1} + s^{\alpha-1})ds\right|dt\right)^{1-\frac{1}{r}}\right. \\
&\times\left(\int_0^1\left|\frac{2}{9}\int_0^1((1-s)^{\alpha-1} + s^{\alpha-1})ds - ;\frac{1}{3}\int_0^t((1-s)^{\alpha-1} + s^{\alpha-1})ds\right|\right. \\
&\times\left.\left(t|f'(\upsilon_1)|^r + (1-t)\left|f'\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right|^r\right)dt\right)^{\frac{1}{r}} \\
&+\left(\int_0^1\left|\frac{1}{9}\int_0^1((1-s)^{\alpha-1} + s^{\alpha-1})ds - ;\frac{1}{3}\int_0^t((1-s)^{\alpha-1} + s^{\alpha-1})ds\right|dt\right)^{1-\frac{1}{r}}
\end{aligned}$$

$$\times\left(\int_0^1\left|\frac{1}{9}\int_0^1((1-s)^{\alpha-1}+s^{\alpha-1})ds-\frac{1}{3}\int_0^t((1-s)^{\alpha-1}+s^{\alpha-1})ds\right|\right.$$

$$\left.\times\left(t\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r+(1-t)|f'(v_2)|^r\right)dt\right)^{\frac{1}{r}}\Bigg)$$

$$\leq\frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\left(\int_0^1|1+3((1-t)^\alpha-t^\alpha)|\,dt\right)^{1-\frac{1}{r}}\right.$$

$$\times\left(\int_0^1|1+3((1-t)^\alpha-t^\alpha)|\left(t|f'(v_1)|^r+(1-t)\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r\right)dt\right)^{\frac{1}{r}}$$

$$+\left(\int_0^1|3((1-t)^\alpha-t^\alpha)-1|\,dt\right)^{1-\frac{1}{r}}$$

$$\left.\times\left(\int_0^1|3((1-t)^\alpha-t^\alpha)-1|\left(t\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r+(1-t)|f'(v_2)|^r\right)dt\right)^{\frac{1}{r}}\right)$$

$$\leq\frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\int_0^{\frac{1}{2}}|1+3((1-t)^\alpha-t^\alpha)|\,dt+\int_{\frac{1}{2}}^1|3(2t-1)^\alpha-1|\,dt\right)^{1-\frac{1}{r}}$$

$$\times\left(\left(|f'(v_1)|^r\int_0^1|1+3((1-t)^\alpha-t^\alpha)|\,tdt\right.\right.$$

$$\left.+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r\int_0^1|1+3((1-t)^\alpha-t^\alpha)|\,(1-t)dt\right)^{\frac{1}{r}}$$

$$\times\left(\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r\int_0^1|3((1-t)^\alpha-t^\alpha)-1|\,tdt\right.$$

$$\left.\left.+|f'(v_2)|^r\int_0^1|3((1-t)^\alpha-t^\alpha)-1|\,(1-t)dt\right)^{\frac{1}{r}}\right)$$

$$\leq\frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\int_0^{\frac{1}{2}}(1+3((1-t)^\alpha-t^\alpha))dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}(1-3(2t-1)^\alpha)dt+\int_{\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1(3(2t-1)^\alpha-1)dt\right)^{1-\frac{1}{r}}$$

$$\times\left(\left(|f'(v_1)|^r\left(\int_0^{\frac{1}{2}}|1+3(1-2t)^\alpha|\,tdt+\int_{\frac{1}{2}}^1|3(2t-1)^\alpha-1|\,tdt\right)\right.\right.$$

$$\left.+\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r\left(\int_0^{\frac{1}{2}}|1+3(1-2t)^\alpha|\,(1-t)dt+\int_{\frac{1}{2}}^1|3(2t-1)^\alpha-1|\,(1-t)dt\right)\right)^{\frac{1}{r}}$$

$$+\left(\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r\left(\int_0^{\frac{1}{2}}|3(1-2t)^\alpha-1|\,tdt+\int_{\frac{1}{2}}^1|3(2t-1)^\alpha+1|\,tdt\right)\right.$$

$$\left.\left.\times|f'(v_2)|^r\left(\int_0^{\frac{1}{2}}|3(1-2t)^\alpha-1|\,(1-t)dt+\int_{\frac{1}{2}}^1|3(2t-1)^\alpha+1|\,(1-t)dt\right)\right)^{\frac{1}{r}}\right)$$

$$=\frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\frac{9}{2(\alpha+1)}-\frac{3}{\alpha+1}\left(\frac{1}{2}\right)^\alpha+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{r}}$$

$$\times\Bigg(\Bigg(\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{2\alpha^2-6\alpha-20}{8(\alpha+1)(\alpha+2)}\Bigg)|f'(v_1)|^r$$
$$+\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{2\alpha^2+18\alpha+28}{8(\alpha+1)(\alpha+2)}\Bigg)\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r\Bigg)^{\frac{1}{r}}$$
$$+\Bigg(\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{2\alpha^2+18\alpha+28}{8(\alpha+1)(\alpha+2)}\Bigg)\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r$$
$$\times\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{2\alpha^2-6\alpha-20}{8(\alpha+1)(\alpha+2)}\Bigg)|f'(v_2)|^r\Bigg)^{\frac{1}{r}}\Bigg), \tag{48}$$

where we have used Equations (28)–(31). The proof is achieved. □

Corollary 3.11. In *Theorem 3.3*, if we take $w(u)=1/v_2-v_1$ we get

$$\left|\frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)-\Omega(v_1,v_2,f)\right|$$
$$\le\frac{v_2-v_1}{24\alpha}\left(\frac{9}{2(\alpha+1)}-\frac{3}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{r}}$$
$$\times\Bigg(\Bigg(\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{2\alpha^2-6\alpha-20}{8(\alpha+1)(\alpha+2)}\Bigg)|f'(v_1)|^r$$
$$+\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{2\alpha^2+18\alpha+28}{8(\alpha+1)(\alpha+2)}\Bigg)\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r\Bigg)^{\frac{1}{r}}$$
$$+\Bigg(\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{2\alpha^2+18\alpha+28}{8(\alpha+1)(\alpha+2)}\Bigg)\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r$$
$$\times\Bigg(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{2\alpha^2-6\alpha-20}{8(\alpha+1)(\alpha+2)}\Bigg)|f'(v_2)|^r\Bigg)^{\frac{1}{r}}\Bigg), \tag{49}$$

where Ω is defined as in *Equation (32)*.

Corollary 3.12. In *Theorem 3.3*, if we take $\alpha=1$ we get

$$\left|\frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)\int_{v_1}^{v_2}w(u)du-\int_{v_1}^{v_2}w(u)f(u)du\right|$$
$$\le\frac{5(v_2-v_1)^2}{72}\|w\|_{[v_1,v_2],\infty}$$
$$\times\left(\left(\frac{16|f'(v_1)|^r+29\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r}{45}\right)^{\frac{1}{r}}+\left(\frac{29\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r+16|f'(v_2)|^r}{45}\right)^{\frac{1}{r}}\right). \tag{50}$$

Corollary 3.13. In *Theorem 3.3*, if we take $w(u)=1/v_2-v_1$ and $\alpha=1$ we get

$$\left|\frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)-7\frac{1}{v_2-v_1}\int_{v_1}^{v_2}f(u)du\right|$$
$$\le\frac{5(v_2-v_1)}{72}\left(\left(\frac{16|f'(v_1)|^r+29\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r}{45}\right)^{\frac{1}{r}}+\left(\frac{29\left|f'\left(\frac{v_1+v_2}{2}\right)\right|^r+16|f'(v_2)|^r}{45}\right)^{\frac{1}{r}}\right). \tag{51}$$

Corollary 3.14. In *Theorem 3.3*, if we use the convexity of $|f'|^r$, we obtain

$$\begin{aligned}
&|S(v_1,v_2,w,f)| \\
&\leq \frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\frac{9}{2(\alpha+1)}-\frac{3}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{r}} \\
&\times\left(\left(\left(\frac{3\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{-\alpha^2+15\alpha+34}{8(\alpha+1)(\alpha+2)}\right)|f'(v_1)|^r\right.\right. \\
&\left.+\left(\frac{\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{\alpha^2+9\alpha+14}{8(\alpha+1)(\alpha+2)}\right)|f'(v_2)|^r\right)^{\frac{1}{r}} \\
&+\left(\left(\frac{\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{\alpha^2+9\alpha+14}{8(\alpha+1)(\alpha+2)}\right)|f'(v_1)|^r\right. \\
&\left.\left.\times\left(\frac{3\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{-\alpha^2+15\alpha+34}{8(\alpha+1)(\alpha+2)}\right)|f'(v_2)|^r\right)^{\frac{1}{r}}\right).
\end{aligned} \tag{52}$$

Corollary 3.15. In *Corollary 3.14*, if we take $w(u)=1/v_2-v_1$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left(f(v_1)+4f\left(\frac{v_1+v_2}{2}\right)+f(v_2)\right)-\Omega(v_1,v_2,f)\right| \\
&\leq \frac{v_2-v_1}{24\alpha}\left(\frac{9}{2(\alpha+1)}-\frac{3}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{r}} \\
&\times\left(\left(\left(\frac{3\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{-\alpha^2+15\alpha+34}{8(\alpha+1)(\alpha+2)}\right)|f'(v_1)|^r\right.\right. \\
&\left.+\left(\frac{\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{\alpha^2+9\alpha+14}{8(\alpha+1)(\alpha+2)}\right)|f'(v_2)|^r\right)^{\frac{1}{r}} \\
&+\left(\left(\frac{\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{\alpha^2+9\alpha+14}{8(\alpha+1)(\alpha+2)}\right)|f'(v_1)|^r\right. \\
&\left.\left.\times\left(\frac{3\alpha}{4(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{8(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{-\alpha^2+15\alpha+34}{8(\alpha+1)(\alpha+2)}\right)|f'(v_2)|^r\right)^{\frac{1}{r}}\right),
\end{aligned} \tag{53}$$

where Ω is defined as in *Equation (32)*.

Remark 3.5. *In Corollary 3.14, if we take $\alpha=1$, we obtain Corollary 2.12 from [11]. Moreover if we choose $w(u)=1/v_2-v_1$, we get Theorem 7 from [9].*

Corollary 3.16. In *Corollary 3.14*, if we use the discrete power mean, we obtain

$$\begin{aligned}
&|S(v_1,v_2,w,f)| \\
&\leq \frac{(v_2-v_1)^2}{24\alpha}\|w\|_{[v_1,v_2],\infty}\left(\frac{9}{2(\alpha+1)}-\frac{3}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{r}} \\
&\times\left(\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{3}{\alpha+1}\right)^{\frac{1}{r}}\left(\frac{|f'(v_1)|^r+|f'(v_2)|^r}{2}\right)^{\frac{1}{r}}.
\end{aligned} \tag{54}$$

Corollary 3.17. In *Corollary 3.16,* if we take $w(u) = 1/\upsilon_2 - \upsilon_1$ we get

$$\begin{aligned}&\left|\frac{1}{6}\left(f(\upsilon_1)+4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+f(\upsilon_2)\right)-\Omega(\upsilon_1,\upsilon_2,f)\right|\\ &\leq \frac{\upsilon_2-\upsilon_1}{24\alpha}\left(\frac{9}{2(\alpha+1)}-\frac{3}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{r}}\\ &\times\left(\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{3}{\alpha+1}\right)^{\frac{1}{r}}\left(\frac{|f'(\upsilon_1)|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}.\end{aligned} \tag{55}$$

where Ω is defined as in *Equation (32).*

Corollary 3.18. In *Corollary 3.14,* if we take $\alpha = 1$ we get

$$\begin{aligned}&\left|\frac{1}{6}\left(f(\upsilon_1)+4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+f(\upsilon_2)\right)\int_{\upsilon_1}^{\upsilon_2}w(u)du-\int_{\upsilon_1}^{\upsilon_2}w(u)f(u)du\right|\\ &\leq \frac{5(\upsilon_2-\upsilon_1)^2}{72}\|w\|_{[\upsilon_1,\upsilon_2],\infty}\left(\frac{|f'(\upsilon_1)|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}.\end{aligned} \tag{56}$$

Corollary 3.19. In *Corollary 3.18,* if we take $w(u) = 1/\upsilon_2 - \upsilon_1$ and $\alpha = 1$ we get

$$\begin{aligned}&\left|\frac{1}{6}\left(f(\upsilon_1)+4f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+f(\upsilon_2)\right)-7\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2}f(u)du\right|\\ &\leq \frac{5(\upsilon_2-\upsilon_1)}{72}\left(\frac{|f'(\upsilon_1)|^r+|f'(\upsilon_2)|^r}{2}\right)^{\frac{1}{r}}.\end{aligned} \tag{57}$$

4. Example

In this section, we utilize Matlab software to provide a graphical representation for analyzing the behavior of the estimates. The right-hand side is depicted in red color, while the left-hand side is depicted in blue color.

Example 4.1. Assume that $\upsilon_1 = 0, \upsilon_2 = 1, f(t) = \left(\frac{1}{2} - t\right)^2$ *and* $w(t) = \left(\frac{1}{2} - t\right)^2$. *Clearly we have*

$$\begin{aligned}J_{0^+}^{\alpha}w\left(\frac{1}{2}\right) &= \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^{\alpha+1}dt = \frac{1}{2^{\alpha+2}(\alpha+2)\Gamma(\alpha)},\\ J_{\frac{1}{2}^-}^{\alpha}w(0) &= \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}}t^{\alpha-1}\left(\frac{1}{2}-t\right)^2dt = \frac{1}{2^{\alpha+1}\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha)},\\ J_{\frac{1}{2}^+}^{\alpha}wf(1) &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{\alpha-1}\left(\frac{1}{2}-t\right)^4dt = \frac{3}{2^{\alpha+1}\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\Gamma(\alpha)},\\ J_{1^-}^{\alpha}wf\left(\frac{1}{2}\right) &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right)^{\alpha+3}dt = \frac{1}{(\alpha+4)2^{\alpha+4}\Gamma(\alpha)},\\ J_{0^+}^{\alpha}wf\left(\frac{1}{2}\right)\Bigg) &= \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^{\alpha+3}dt = \frac{1}{(\alpha+4)2^{\alpha+4}\Gamma(\alpha)},\\ J_{\frac{1}{2}^-}^{\alpha}wf(0) &= \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}}t^{\alpha-1}\left(\frac{1}{2}-t\right)^4dt = \frac{3}{2^{\alpha+1}\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\Gamma(\alpha)}.\end{aligned} \tag{58}$$

and $|f'(t)| = |2t - 1|$.

From Theorem 3.1, *we derive the following inequality, which is illustrated in* Figure 1.

$$\begin{aligned}&\left|\frac{\alpha^2+\alpha+2}{192\alpha(\alpha+1)(\alpha+2)}-7\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)+24}{32\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\right|\\ &\leq \frac{1}{96\alpha}\left(\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{\alpha-5}{4(\alpha+1)}\right)\right.\\ &\left.+\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}-\frac{\alpha-5}{4(\alpha+1)}\right)\right).\end{aligned} \tag{59}$$

5. Applications

This section will be devoted to the applications of the results obtained, knowing that in general in this type of problem the applications relate to the special averages, the random variable or in the numerical integration of which it is the center of our interest.

Let Σ be the partition of the points $\upsilon_1 = \tau_0 < \tau_1 < \ldots < \tau_n = \upsilon_2$ of the interval $[\upsilon_1, \upsilon_2]$, and consider the quadrature formula [11]

$$\int_{\upsilon_1}^{\upsilon_2}w(u)f(u)du = \Theta_w(f,\Sigma) + R_w(f,\Sigma), \tag{60}$$

where

$$\Theta_w(f,\Sigma) = \sum_{i=0}^{n-1}\frac{f(\tau_i)+4f\left(\frac{\tau_i+\tau_{i+1}}{2}\right)+f(\tau_{i+1})}{6}\int_{\upsilon_1}^{\upsilon_2}w(u)du. \tag{61}$$

and $R_w(f,\Sigma)$ denotes the associated approximation error.

Proposition 5.1. Let $n \in \mathbb{N}, r > 1$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\upsilon_1, \upsilon_2 \in I^\circ$ with $\upsilon_1 < \upsilon_2$, and let w:

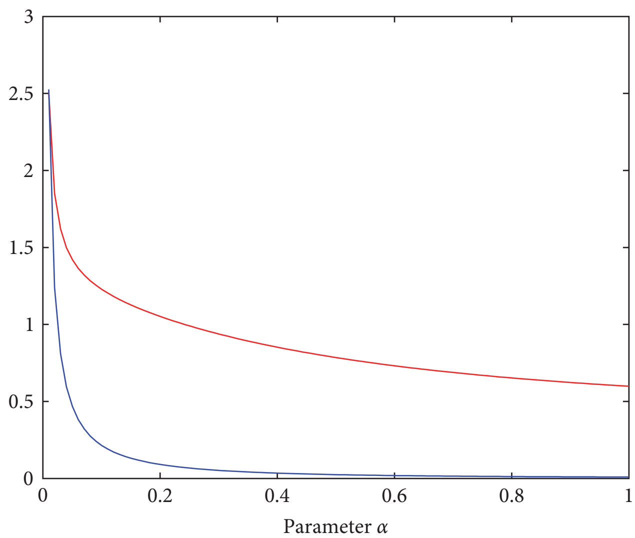

FIGURE 1: The function curve for $\alpha \in (0,1)$.

$[v_1, v_2] \to \mathbb{R}$ be symmetric with respect to $v_1 + v_2/2$ and f', $w \in L^1[v_1, v_2]$. If $|f'|^r$ is convex, then we have

$$|R_w(f,\Sigma)| \le \frac{(v_2 - v_1)^2}{6} \|w\|_{[v_1,v_2],} \\ \infty \left(\frac{1+2^{l+1}}{3(l+1)}\right)^{\frac{1}{l}} \left(\frac{|f'(v_1)|^r + |f'(v_2)|^r}{2}\right)^{\frac{1}{r}}, \tag{62}$$

where $1/r + 1/l = 1$.

Proof. Applying Corollary 3.9 on the subintervals $[\tau_i, \tau_{i+1}]$ for $(i = 0, 1, \ldots, n-1)$ of the partition Σ, we get

$$\left|\frac{f(\tau_i) + 4f\left(\frac{\tau_i+\tau_{i+1}}{2}\right) + f(\tau_{i+1})}{6} \int_{\tau_i}^{\tau_{i+1}} w(u)du \right. \\ \left. - \int_{\tau_i}^{\tau_{i+1}} w(u)f(u)du \right| \\ \le \frac{(\tau_{i+1} - \tau_i)^2}{6} \|w\|_{[\tau_i,\tau_{i+1}],} \\ \infty \left(\frac{1+2^{l+1}}{3(l+1)}\right)^{\frac{1}{l}} \left(\frac{|f'(\tau_i)|^r + |f'(\tau_{i+1})|^r}{2}\right)^{\frac{1}{r}}. \tag{63}$$

Summing above inequality for all $i = 0, \ldots, n-1$ and using property of modulus, we obtain the desired result. □

Proposition 5.2. Let $n \in \mathbb{N}$, $r > 1$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $v_1, v_2 \in I^\circ$ with $v_1 < v_2$, and let $w : [v_1, v_2] \to \mathbb{R}$ be symmetric with respect to $v_1 + v_2/2$ and f', $w \in L^1[v_1, v_2]$. If $|f'|^r$ is convex, then we have

$$|R_w(f,\Sigma)| \le \frac{(v_2 - v_1)^2}{6} \|w\|_{[v_1,v_2],\infty} \left(\frac{1+2^{l+1}}{3(l+1)}\right)^{\frac{1}{l}} \left(\frac{|f'(v_1)|^r + |f'(v_2)|^r}{2}\right)^{\frac{1}{r}}, \tag{64}$$

where $1/r + 1/l = 1$.

Proof. Applying Corollary 3.12 on the subintervals $[\tau_i, \tau_{i+1}]$ for $(i = 0, 1, \ldots, n-1)$ of the partition Σ, we get

$$\left|\frac{f(\tau_i) + 4f\left(\frac{\tau_i+\tau_{i+1}}{2}\right) + f(\tau_{i+1})}{6} \int_{\tau_i}^{\tau_{i+1}} w(u)du - \int_{\tau_i}^{\tau_{i+1}} w(u)f(u)du \right| \\ \le \frac{5(\tau_{i+1} - \tau_i)^2}{72} \|w\|_{[\tau_i,\tau_{i+1}],\infty} \\ \times \left(\left(\frac{16|f'(\tau_i)|^r + 29\left|f'\left(\frac{\tau_i+\tau_{i+1}}{2}\right)\right|^r}{45}\right)^{\frac{1}{r}} + \left(\frac{29\left|f'\left(\frac{\tau_i+\tau_{i+1}}{2}\right)\right|^r + 16|f'(\tau_{i+1})|^r}{45}\right)^{\frac{1}{r}} \right). \tag{65}$$

Summing above inequality for all $i = 0, \ldots, n-1$ and using property of modulus, we obtain the desired result. □

Proposition 5.3. Let $n \in \mathbb{N}$, $r > 1$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $v_1, v_2 \in I^\circ$ with $v_1 < v_2$, and let $w : [v_1, v_2] \to \mathbb{R}$ be symmetric with respect to $v_1 + v_2/2$ and f', $w \in L^1[v_1, v_2]$. If $|f'|^r$ is convex, then we have

$$|R_w(f,\Sigma)| \le \frac{(v_2 - v_1)^2}{6} \|w\|_{[v_1,v_2],\infty} \left(\frac{1+2^{l+1}}{3(l+1)}\right)^{\frac{1}{l}} \left(\frac{|f'(v_1)|^r + |f'(v_2)|^r}{2}\right)^{\frac{1}{r}}, \tag{66}$$

where $1/r + 1/l = 1$.

Proof. Applying Corollary 3.18 on the subintervals $[\tau_i, \tau_{i+1}]$ for $(i = 0, 1, \ldots, n-1)$ of the partition Σ, we get

$$\begin{aligned} &\left| \frac{f(\tau_i) + 4f\left(\frac{\tau_i+\tau_{i+1}}{2}\right) + f(\tau_{i+1})}{6} \int_{\tau_i}^{\tau_{i+1}} w(u)du - \int_{\tau_i}^{\tau_{i+1}} w(u)f(u)du \right| \\ &\leq \frac{5(\tau_{i+1} - \tau_i)^2}{72} \|w\|_{[\tau_i, v_2],\infty} \left(\frac{|f'(\tau_i)|^r + |f'(\tau_{i+1})|^r}{2} \right)^{\frac{1}{r}}. \end{aligned} \tag{67}$$

Summing above inequality for all $i = 0, \ldots, n-1$ and using property of modulus, we obtain the desired result. □

Now, recalling some particularly means.
The Arithmetic mean: $A(a, b) = a + b/2$.
The harmonic mean: $H(a, b) = 2ab/a + b$ with $a, b > 0$.
The logarithmic mean: $L(a, b = \begin{cases} a \text{ if } a = b \\ \dfrac{b-a}{\ln b - \ln a} \text{ if } a = b \end{cases} a, b > 0.$

Proposition 5.4. Let $a, b \in \mathbb{R}$ with $0 < a < b$ and let $r > 2$, then we have

$$|H^{-}(a, b) + 2A^{-1}(a, b) - 3L^{-1}(a, b)| \leq \frac{5(b-a)}{24} \left(\frac{1}{2a^{2r}} + \frac{1}{2b^{2r}} \right)^{\frac{1}{r}}. \tag{68}$$

Proof. The assertion Corollary 3.19, applied to function $f(x) = 1/x$. □

6. Conclusion

In this study, we have considered the fractional weighted Simpson type integral inequalities for functions whose first derivatives are convex. We have established a novel weighted identity that incorporates the Riemann–Liouville integral operator. We have established some new fractional weighted Simpson type inequalities. We have discussed according to the values of the parameter α or the weight function w some special cases. Several known results have been derived. Applications of our findings are provided. We hope that the ideas of this paper will inspire researchers working in field of inequalities to generalize our results for different kinds of classical and generalized convexity. In future work, we can expand upon our research to encompass various types of calculus, including quantum calculus as well as non-Newtonian calculus.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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