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Research Article

Solving Partial Differential Equations of Fractional Order by Using a Novel Double Integral Transform

Shams A. Ahmed,[1,2] **Tarig M. Elzaki**,[3] **and Anis Mohamed**[1,4]

[1]*Department of Mathematics, Faculty of Sciences and Arts, Jouf University, Tubarjal, Saudi Arabia*
[2]*Department of Mathematics, University of Gezira, Sudan*
[3]*Department of Mathematics, Faculty of Sciences and Arts, Alkamil, University of Jeddah, Jeddah, Saudi Arabia*
[4]*Laboratory for Mathematical and Numerical Modeling in Engineering Science, University of Tunis El Manar, Tunisia*

Correspondence should be addressed to Shams A. Ahmed; shamsalden20@hotmail.com

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In this work, the double Sumudu–Elzaki transform was used for solving fractional-partial differential equations (FPDEs) with starting and boundary conditions. We will use the fractional-order derivative (Caputo's derivatives) idea. Theorems and facts that are crucial to the newly introduced transform are also discussed and illustrated. By using this newly designed integral transform and its properties, FPDEs can be reduced into algebraic equations. This strategy has the precise answer since it does not need any discrimination, transformation, or limited assumptions. Five further instances were given to support our conclusions. The results showed that the recommended strategy is superb, reliable, and efficient. It is also a simple method for solving specific problems in a number of applied scientific and technical fields.

1. Introduction

Integral transformations are seen as the most efficient method of resolving fractional-partial differential equations (FPDEs). FPDEs can mathematically describe a wide variety of phenomena in mathematical physics and in many other scientific fields, making them valuable [1–5]. With integral transformations [6–9], these equations can also be modified to identify precise FPDE solutions. The direct power of transformation techniques has been the inspiration for ongoing research to understand and improve them. Many integral transforms were developed and implemented to solve FPDEs. These transformations allow us to get the exact solutions of the target equations without having to linearize or discretize. They are used to convert FPDEs to ordinary equations when using only one transformation and to algebraic equations when using a double integral transformation. Some examples of these transformations are: the Sumudu transform [10], the natural transform [11], the Elzaki transform [12], the novel transform [13], the Aboodh transform [14], the double Sumudu transform [15, 16], the double Elzaki transform [17], the double Shehu transform [18], and the double Laplace–Sumudu transform [19, 20].

Diverse partial differential equations have recently been effectively solved using the double Sumudu–Elzaki transform (DSET), a novel double integral transform technique [21]. Unfortunately, unlike other integral transforms, this transformation is unable to handle complex mathematical models or nonlinear problems. In order to handle a variety of nonlinear differential equations, some researchers have combined these integral transforms with additional techniques, such as the homotopy perturbation method, the variational iteration method, the differential transform method, and the Adomian decomposition method [22, 23].

The primary goal of this research is to broaden the application of DSET by using it to solve FPDEs. We show the effectiveness of the proposed method by applying DSET to a number of interesting applications to get the exact solutions.

The following subjects will be covered in this essay's succeeding sections. We provide some basic definitions and theorems of CFDs in Section 2. Section 3 presents the fundamental DSET definitions, features, and theorems. Section 4 describes the model and process for using the DSET to provide accurate analytical answers to the specified FPDEs. Five exemplary scenarios are utilized in Section 5 to illustrate the

recommended approach's liability, convergence, and efficacy. In Section 6, we explain the numerical results and show how the DSET is accurate and efficient. Section 7 also has conclusions.

2. Preliminaries

In this section, we present basic definitions and notions that will be used in the present work.

Definition 1 (see [24]). Suppose that $\xi(u,t)$ is a continuous function. Then, the RLPFIs (Riemann–Liouville partial fractional integrals) are given by:

$$ {}_tI^{\beta}\xi(u,t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\xi(u,\tau)d\tau = \frac{1}{\Gamma(\beta)}t^{\beta-1}\times\xi(u,t), \tag{1}$$

$$ {}_uI^{\alpha}\xi(u,t) = \frac{1}{\Gamma(\alpha)}\int_0^u (u-\varsigma)^{\alpha-1}\xi(\varsigma,t)d\varsigma = \frac{1}{\Gamma(\alpha)}u^{\alpha-1}\times\xi(u,t). \tag{2}$$

Definition 2. The CPFDs (Caputo partial fractional derivatives) of order $\varsigma>0$, and $\tau>0$, *of* $\xi(u,t)$, are given by:

$$ \frac{\partial^{\beta}\xi(u,t)}{\partial t^{\beta}} = \frac{1}{\Gamma(m-\beta)}\int_0^t (t-\tau)^{m-\beta-1}\frac{\partial^m\xi(u,\tau)}{\partial\tau^m}d\tau, \quad m-1<\beta<m, m\in N, \tag{3}$$

$$ \frac{\partial^{\alpha}\xi(u,t)}{\partial u^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^u (u-\varsigma)^{n-\alpha-1}\frac{\partial^n\xi(\varsigma,t)}{\partial\varsigma^n}d\varsigma, \quad n-1<\alpha<n, n\in N. \tag{4}$$

Definition 3. Assume that the function $E_{\beta,\alpha}(u)$ denoted to Mittag–Leffler [25], then

$$ E_{\beta,\alpha}(u) = \sum_{k=0}^{\infty}\frac{u^k}{\Gamma(\beta k+\alpha)}, \quad u,\in\mathbb{C}, \Re e(\beta)>0, \Re e(\alpha)>0. \tag{5}$$

3. Double Sumudu–Elzaki Transform (DSET)

In this section, a new integral transform called the DSET is introduced that combines the Sumudu transform and the Elzaki transform. We present the fundamental DSET definitions, features, and theorems.

Definition 4. Let $\xi(u,t)$ is a real-valued function of two variables u, and v, then

(i) The SST (single Sumudu transform) of $\xi(u,t)$ w.r.t u denoted by $S_u[\xi(u,t):w] = \Psi(w,t)$ and defined as follows:

$$ S_u[\xi(u,t):w] = \Psi(w,t) = \frac{1}{w}\int_0^{\infty} e^{-\frac{u}{w}}\xi(u,t)du, u>0. \tag{6}$$

(ii) The SET (single Elzaki transform) of $\xi(u,t)$ w.r.t t, denoted by $E_t[\xi(u,t):q] = \Psi(u,q)$, and defined as follows:

$$ E_t[\xi(u,t):q] = \Psi(u,q), = q\int_0^{\infty} e^{-\frac{t}{q}}\xi(u,t)dt, t>0. \tag{7}$$

Proposition 1 (see [26]). *Assume that* $\Psi(w,t)$, *and* $\Omega(w,t)$ *be the ST of* $\xi(u,t)$ *and* $\psi(u,t)$, *respectively, then the ST of the convolution theorem is given by as follows:*

$$ S_u[(\xi\times\psi)(u,t):(w,t)] = w\,\Psi(w,t)\Omega(w,t). \tag{8}$$

Proposition 2 (see [12]). *Assume that* $\Psi(u,q)$, *and* $\Omega(u,q)$ *be the ET of* $\xi(u,t)$ *and* $\psi(u,t)$, *respectively, then the ET of the convolution theorem is given by*

$$ E_t[(\xi\times\psi)(u,t):(u,q)] = \frac{1}{q}\Psi(u,q)\Omega(u,q). \tag{9}$$

Lemma 1 (see [26]). *Assume that* $\alpha>0$, *and* $\xi(u,t)$ *is the exponential order. Then, the SST of* ${}_uI^{\alpha}\xi(u,t)$ *is given by:*

$$ S_u[{}_uI^{\alpha}\xi(u,t)] = w^{\alpha}\,S_u[\xi(u,t)]. \tag{10}$$

Proof. From Equation (2) above,

$$ {}_uI^{\alpha}\xi(u,t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}\times\xi(u,t), \tag{11}$$

by applying ST to Equation (11), and using Proposition 1, we get

$$\begin{aligned} S_u[\,{}_uI^{\alpha}\xi(u,t)] &= S_u\left(\frac{1}{\Gamma(\alpha)}u^{\alpha-1}\times\xi(u,t)\right)\\ &= \frac{w}{\Gamma(\alpha)}S_u(u^{\alpha-1})S_u(\xi(u,t))\\ &= \frac{w}{\Gamma(\alpha)}(\alpha-1)!w^{\alpha-1}S_u(\xi(u,t)) = w^{\alpha}S_u(\xi(u,t)). \end{aligned} \tag{12}$$

□

Lemma 2 (see [27]). *Assume that $\beta>0$, $\xi(u,t)$ is the exponential order. Then, the SET of ${}_tI^{\beta}\xi(u,t)$ is given by:*

$$E_t[\,{}_tI^{\beta}\xi(u,t)] = q^{\beta}\,E_t[\xi(u,t)]. \tag{13}$$

Proof. From Equation (1) above,

$${}_tI^{\beta}\xi(u,t) = \frac{1}{\Gamma(\beta)}t^{\beta-1}\times\xi(u,t), \tag{14}$$

by applying ET to Equation (11), and using Proposition 2, we get

$$\begin{aligned} E_t[\,{}_tI^{\beta}\xi(u,t)] &= E_t\left(\frac{1}{\Gamma(\beta)}t^{\beta-1}\times\xi(u,t)\right)\\ &= \frac{1}{q\Gamma(\beta)}E_t(t^{\beta-1})E_t(\xi(u,t))\\ &= \frac{1}{q\Gamma(\beta)}(\beta-1)!q^{\beta+1}E_t(\xi(u,t)) = q^{\beta}E_t(\xi(u,t)). \end{aligned} \tag{15}$$

□

Definition 5. The DSET) of $\xi(u,t)$, w.r.t u and t denoted by $S_uE_t[\xi(u,t):(w,q)] = \Psi(w,q)$ and defined as follows:

$$\begin{aligned} S_uE_t[\xi(u,t):(w,q)] &= \Psi(w,q)\\ &= \frac{q}{w}\int_0^{\infty}\int_0^{\infty}e^{-\left(\frac{u}{w}+\frac{t}{q}\right)}\ \xi(u,t)du\,dt, \end{aligned} \tag{16}$$

provided the integral exists.

Or

$$\begin{aligned} S_uE_t[\xi(u,t):(w,q)] &= \Psi(w,q)\\ &= q^2\int_0^{\infty}\int_0^{\infty}e^{-(u+t)}\ \ \xi(wu,qt)du\,dt. \end{aligned} \tag{17}$$

Recall that: $S_uE_t[\xi(u,t)] = E_tS_u[\xi(u,t)]$, when $\xi(u,t)$ satisfies the necessary conditions [28].

The inverse $(S_w)^{-1}(E_q)^{-1}[\Psi(w,q)] = \xi(u,t)$ is defined by:

$$\begin{aligned} &(S_w)^{-1}(E_q)^{-1}[\Psi(w,q)] = \xi(u,t)\\ &= \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{1}{w}e^{\frac{u}{w}}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}q\,e^{\frac{t}{q}}\Psi(w,q)\,dq\right]dw. \end{aligned} \tag{18}$$

Theorem 1 (see [21]) (existence condition). *If a function $\xi(u,t)$ in all finite interval $(0,U)$ and $(0,T)$ is a continuous function as well as on an exponential scale $e^{c_1u+c_2t}$, then DSET of $\xi(u,t)$ exists for all $\frac{1}{w}$ and $\frac{1}{q}$ supplied $\mathrm{Re}[\frac{1}{w}]>c_1$ and $\mathrm{Re}[\frac{1}{q}]>c_2$.*

Lemma 3 (see [21]). *If $S_uE_t[\xi(u,t)] = \Psi(w,q)$, then the DSET of the FPDs $\frac{\partial\xi}{\partial u}, \frac{\partial\xi}{\partial t}, \frac{\partial^2\xi}{\partial u^2}$, and $\frac{\partial^2\xi}{\partial t^2}$, can be represented as follows:*

(I) $S_uE_t[\frac{\partial\xi}{\partial u}] = \frac{1}{w}\Psi(w,q) - \frac{1}{w}\Psi(0,q)$.
(II) $S_uE_t[\frac{\partial\xi}{\partial t}] = \frac{1}{q}\Psi(w,q) - q\,\Psi(w,0)$.
(III) $S_uE_t[\frac{\partial^2\xi}{\partial u^2}] = \frac{1}{w^2}\Psi(w,q) - \frac{1}{w^2}E(\xi(0,t)) - \frac{1}{w}E(\xi_u(0,t))$.
(IV) $S_uE_t[\frac{\partial^2\xi}{\partial t^2}] = \frac{1}{q^2}\Psi(w,q) - S(\xi(u,0)) - q\,S(\xi_t(u,0))$.

The results mentioned above can be generally expanded as follows:

$$S_uE_t\left[\frac{\partial^n\xi}{\partial u^n}\right] = w^{-n}\,\Psi(w,q) - \sum_{k=0}^{n-1}w^{-n+k}E_t\left[\frac{\partial^k}{\partial u^k}\xi(0,t)\right], \tag{19}$$

$$S_uE_t\left[\frac{\partial^m\xi}{\partial t^m}\right] = q^{-m}\,\Psi(w,q) - \sum_{j=0}^{m-1}q^{2-m+j}\,S_u\left[\frac{\partial^j}{\partial t^j}\xi(u,0)\right]. \tag{20}$$

Theorem 2. *The DSET for some functions is given below*

(I) $S_uE_t[c:(w,q)] = c\,q^2, c\in\mathbb{R}$.
(II) $S_uE_t[u^mt^n:(w,q)] = m!n!w^mq^{n+2}, m,n\in\mathbb{Z}^+$.
(III) $S_uE_t[e^{c_1u+c_2t}:(w,q)] = \frac{q^2}{(1-c_1w)(1-c_2q)}$.
(IV) $S_uE_t[\sin(c_1u)\sin(c_2t):(w,q)] = \frac{c_1w}{(1+c_1{}^2w^2)}\frac{c_2q^3}{1+c_2{}^2q^2}$.
(V) $S_uE_t[1-e^{c_2t}:(w,q)] = \frac{-c_2q^3}{(1-c_2q)}$.
(VI) $S_uE_t[(1-e^{c_2t})\sin(c_1u):(w,q)] = \frac{-c_1c_2wq^3}{(1-c_2q)(1+c_1{}^2w^2)}$.

Proof. Here, we will provide evidence for results (I), (III), and (VI).

Table 1: DSET for some functions.

Sr. no.	$\xi(u,t)$	$S_uE_t[\xi(u,t)]=\Psi(w,q)$
1	c	$c\,q^2,\ c\in\mathbb{R}$
2	$u^m t^n,\ m,n\in\mathbb{Z}^+$	$m!n!w^m q^{n+2}$
3	$e^{c_1u+c_2t}$	$\frac{q^2}{(1-c_1w)(1-c_2q)}$
4	$\sin(c_1u+c_2t)$	$\frac{q^2(c_1w+c_2q)}{(1+c_1^2w^2)(1+c_2^2q^2)}$
5	$\cos(c_1u+c_2t)$	$\frac{q^2(1-c_1c_2wq)}{(1+c_1^2w^2)(1+c_2^2q^2)}$
6	$\sinh(c_1u+c_2t)$	$\frac{q^2(c_1w+c_2q)}{(1-c_1^2w^2)(1-c_2^2q^2)}$
7	$\cosh(c_1u+c_2t)$	$\frac{q^2(1+c_1c_2wq)}{(1-c_1^2w^2)(1-c_2^2q^2)}$
8	$J_0(b\ \sqrt{u\,t})$	$\frac{4q^2}{4+b^2wq}$

(I) Gives us

$$S_uE_t[c:(w,q)]=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u}{w}+\frac{t}{q}\right)}\ \xi(u,t)du\,dt$$
$$=\left(\frac{1}{w}\int_0^\infty e^{-\frac{u}{w}}c\,du\right)\left(q\int_0^\infty e^{-\frac{t}{q}}(1)dt\right)=c\,q^2. \tag{21}$$

(III) Gives us

$$S_uE_t[e^{c_1u+c_2t}:(w,q)]=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u}{w}+\frac{t}{q}\right)}\ e^{c_1u+c_2t}\,du\,dt$$
$$=\left(\frac{1}{w}\int_0^\infty e^{-\left(\frac{1}{w}-c_1\right)u}\,du\right)\left(q\int_0^\infty e^{-\left(\frac{1}{q}-c_2\right)t}\,dt\right)$$
$$=\frac{1}{(1-c_1w)}\frac{q^2}{(1-c_2q)}=\frac{q^2}{(1-c_1w)(1-c_2q)}. \tag{22}$$

(VI) Gives us

$$S_uE_t[(1-e^{c_2t})\sin(c_1u):(w,q)]$$
$$=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u}{w}+\frac{t}{q}\right)}\ (1-e^{c_2t})\sin(c_1u)\,du\,dt$$
$$=\left(\frac{1}{w}\int_0^\infty e^{-\frac{u}{w}}\sin(c_1u)du\right)\left(q\int_0^\infty e^{-\frac{t}{q}}(1-e^{c_2t})dt\right)$$
$$=S_u[\sin(c_1u):w\,]E_t[1-e^{c_2t}:q]=\frac{c_1w}{(1+{c_1}^2w^2)}\frac{-c_2q^3}{(1-c_2q)}$$
$$=\frac{-c_1c_2wq^3}{(1+{c_1}^2w^2)(1-c_2q)}. \tag{23}$$

The same method can be used to demonstrate the remaining results.

The DSET for some fundamental functions is summed up in Table 1 below. □

Lemma 4 (see [20, 27]). *The single ST of $u^{-1+\alpha}E_{\beta,\alpha}(\mu u^\beta)$, takes the form:*

$$S_u\left[u^{\alpha-1}E_{\beta,\alpha}(\mu u^\beta)\right]=w^{\alpha-1}(1-\mu w^\beta)^{-1},\quad |\mu|<|w^\beta|, \tag{24}$$

and the single ET of $t^{-1+\alpha}E_{\beta,\alpha}(\mu t^\beta)$ takes the form:

$$E_t\left[t^{\alpha-1}E_{\beta,\alpha}(\mu t^\beta)\right]=q^{\alpha+1}(1-\mu q^\beta)^{-1},\quad |\mu|<|q^\beta|. \tag{25}$$

Lemma 5 (see [12, 29]) (DL-DSE duality). *If the DSET of $\xi(u,t)$ exist, then*

$$S_uE_t[\xi(u,t):(w,q)]=\frac{q}{w}L_uL_t\left[\xi(u,t):\left(\frac{1}{w},\frac{1}{q}\right)\right], \tag{26}$$

where $L_uL_t[\xi(u,t):(w,q)]=\Psi(w,q)=\int_0^\infty\int_0^\infty e^{-(w\,u+q\,t)}\xi(u,t)du\,dt$.

Theorem 3. *Assume $\xi(u,t)$ and $\psi(u,t)$ are two functions with the DSET, then*

(I) $S_uE_t[c_1\xi(u,t)+c_2\psi(u,t)]=c_1S_uE_t[\xi(u,t):(w,q)]+c_2S_uE_t[\psi(u,t):(w,q)]$.
(II) $S_uE_t[e^{-c_1u-c_2t}\xi(u,t):(w,q)]=\frac{(1+c_2q)}{(1+c_1w)}\Psi(\frac{w}{1+c_1w},\frac{q}{1+c_2q})$.
(III) $S_uE_t[\xi(\lambda\,u,\mu\,t):(w,q)]=\frac{1}{r}\Psi(\frac{w}{\lambda},\frac{q}{\mu})$; $r=\lambda\mu$.
(IV) $(-1)^{m+n}S_uE_t[u^mt^n\,\xi(u,t):(w,q)]=\frac{q}{w}\frac{\partial^{m+n}}{\partial w^m\partial q^n}[\frac{w}{q}S_uE_t[\xi(u,\ t):(w,\ q)\ ]]$.

Proof.

(I) The use of the DSET specification makes the proof of (I) simple to demonstrate.

(II)

$$S_uE_t[e^{-c_1u-c_2t}\xi(u,t):(w,q)]$$
$$=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u}{w}+\frac{t}{q}\right)}\ e^{-c_1u-c_2t}\xi(u,t)du\,dt$$
$$=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{w}+c_1\right)u-\left(\frac{1}{q}+c_2\right)t}\ \xi(u,t)du\,dt$$
$$=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{1+c_1w}{w}\right)u-\left(\frac{1+c_2q}{q}\right)t}\ \xi(u,t)du\,dt. \tag{27}$$

Put $p=\frac{w}{1+c_1 w}$, $s=\frac{q}{1+c_2 q}$, then

$$\begin{aligned}
&S_u E_t[e^{-c_1 u-c_2 t}\xi(u,t):(w,q)]\\
&=\frac{1+c_2 q}{1+c_1 w}\left(\frac{s}{p}\int_0^\infty\int_0^\infty e^{-\left(\frac{u}{p}+\frac{t}{s}\right)}\xi(u,t)du\,dt\right)\\
&=\frac{1+c_2 q}{1+c_1 w}\Psi(p,s)=\frac{1+c_2 q}{1+c_1 w}\Psi\left(\frac{w}{1+c_1 w},\frac{q}{1+c_2 q}\right).
\end{aligned}\tag{28}$$

(III) Suppose $\gamma=\lambda\, u$ and $\eta=\mu\, t$, then

$$\begin{aligned}
&S_u E_t[\zeta(\lambda\, u,\mu\, t):(w,q)]\\
&=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u}{w}+\frac{t}{q}\right)}\xi(\lambda\, u,\mu\, t)du\,dt\\
&=\frac{1}{w}\int_0^\infty e^{-\frac{u}{w}}\left(q\int_0^\infty e^{-\frac{t}{q}}\xi(\lambda\, u,\mu\, t)\,dt\right)du\\
&=\frac{1}{\mu w}\int_0^\infty e^{-\frac{u}{w}}\left(q\int_0^\infty e^{-\frac{\eta}{q\mu}}\xi(\lambda u,\eta)\,d\eta\right)du\\
&=\frac{1}{\mu w}\int_0^\infty e^{-\frac{u}{w}}\Psi\left(\lambda u,\frac{q}{\mu}\right)du\\
&=\frac{1}{\mu\lambda}\int_0^\infty \frac{1}{w}e^{-\frac{\gamma}{w\lambda}}\Psi\left(\gamma,\frac{q}{\mu}\right)d\gamma\\
&=\frac{1}{\mu\lambda}\Psi\left(\frac{w}{\lambda},\frac{q}{\mu}\right).
\end{aligned}\tag{29}$$

(IV) Here, by combining Lemma 5 with the properties of DLT in [23], we obtain,

$$\begin{aligned}
&(-1)^{m+n}S_u E_t[u^m\, t^n\,\xi(u,t):(w,q)]\\
&=\frac{q}{w}(-1)^{m+n}L_u L_t\left[u^m\, t^n\,\xi(u,t):\left(\frac{1}{w},\frac{1}{q}\right)\right]\\
&=\frac{q}{w}\frac{\partial^{m+n}}{\partial w^m\partial q^n}\left[L_u L_t\left[\xi(u,t):\left(\frac{1}{w},\frac{1}{q}\right)\right]\right]\\
&=\frac{q}{w}\frac{\partial^{m+n}}{\partial w^m\partial q^n}\left[\frac{w}{q}S_u E_t[\xi(u,t):(w,q)\,]\right].
\end{aligned}\tag{30}$$

□

Theorem 4 (see [12, 29]) (convolution theorem). *Assume that $\xi(u,t)$ and $\psi(u,t)$ are two functions with the DSET, then,*

$$S_u E_t[(\xi\times\psi)(u,t):(w,q)]=\frac{w}{q}\Psi(w,q)\Omega(w,q),\tag{31}$$

where $(\xi\times\psi)(u,t)=\int_0^u\int_0^t\xi(u-\tau,t-\varsigma)\,\psi(\tau,\varsigma)\,d\tau\,d\varsigma$.

Proof. Using Lemma 5, we obtain,

$$\begin{aligned}
&S_u E_t[(\xi\times\psi)(u,t):(w,q)]=\frac{q}{w}L_u L_t\left[(\xi\times\psi)(u,t):\left(\frac{1}{w},\frac{1}{q}\right)\right]\\
&=\frac{q}{w}\left(L_u L_t\left[\xi(u,t):\left(\frac{1}{w},\frac{1}{q}\right)\right]L_u L_t\left[\psi(u,t):\left(\frac{1}{w},\frac{1}{q}\right)\right]\right)\\
&=\frac{q}{w}\left(\left(\frac{w}{q}S_u E_t[\xi(u,t):(w,q)]\right)\left(\frac{w}{q}S_u E_t[\psi(u,t):(w,q)]\right)\right)\\
&=\frac{w}{q}(S_u E_t[\xi(u,t):(w,q)])(S_u E_t[\psi(u,t):(w,q)])\\
&=\frac{w}{q}\Psi(w,q)\Omega(w,q).
\end{aligned}\tag{32}$$

□

Lemma 6 (see [24]). *Assume that $\beta,\alpha>0$, and $\xi(u,t)$ are exponential orders. Then, the DSET of ${}_tI^\beta\xi(u,t)$, and ${}_uI^\alpha\xi(u,t)$, respectively, are given by as follows:*

$$S_u E_t[\,{}_tI^\beta\xi(u,t)]=q^\beta\, S_u E_t[\xi(u,t)],\tag{33}$$

$$S_u E_t[\,{}_uI^\alpha\xi(u,t)]=w^\alpha\, S_u E_t[\xi(u,t)].\tag{34}$$

Lemma 7 (see [24]). *Assume that $\beta,\alpha>0$, and $\xi(u,t)$ are exponential orders. Then, the DSET of ${}_tI^\beta\,{}_uI^\alpha\xi(u,t)$ is given by:*

$$S_u E_t[\,{}_tI^\beta\,{}_uI^\alpha\xi(u,t)]=w^\alpha q^\beta S_u E_t[\xi(u,t)].\tag{35}$$

Theorem 5. *The DSET for CFDs can be expressed as follows:*

(I) $S_u E_t[\frac{\partial^\alpha\xi}{\partial u^\alpha}]=\frac{\Psi(w,q)}{w^\alpha}-\sum_{k=0}^{n-1}w^{-\alpha+k}E_t[\frac{\partial^k}{\partial u^k}\xi(0,t)]$, $n-1<\alpha<n$.

(II) $S_u E_t[\frac{\partial^\beta\xi}{\partial t^\beta}]=\frac{\Psi(q,w)}{q^\beta}-\sum_{j=0}^{m-1}q^{2-\beta+j}S_u[\frac{\partial^j}{\partial t^j}\xi(u,0)]$, $m-1<\beta<m$.

Proof. Here, we will provide evidence for result (I).

The CFD w.r.t u, for the function $\xi(u,t)$ can be rewritten as follows [30]:

$$\frac{\partial^\alpha\xi}{\partial u^\alpha}=\frac{1}{\Gamma(n-\alpha)}u^{n-\alpha-1}\times\frac{\partial^n\xi(u,t)}{\partial u^n},\tag{36}$$

by applying DSET to Equation (36), we get

$$\begin{aligned} S_uE_t\left[\frac{\partial^\alpha \xi}{\partial u^\alpha}\right] &= S_uE_t\left(\frac{1}{\Gamma(n-\alpha)}u^{n-\alpha-1}\times\frac{\partial^n\xi(u,t)}{\partial u^n}\right)\\ &=\frac{1}{\Gamma(n-\alpha)}S_uE_t(u^{n-\alpha-1})S_uE_t\left(\frac{\partial^n\xi(u,t)}{\partial u^n}\right)\\ &=w^{n-\alpha}\left[\frac{\Psi(w,q)}{w^n}-\sum_{k=0}^{n-1}w^{-n+k}E_t\left[\frac{\partial^k}{\partial u^k}\xi(0,t)\right]\right]\\ &=\frac{\Psi(w,q)}{w^\alpha}-\sum_{k=0}^{n-1}w^{-\alpha+k}E_t\left[\frac{\partial^k}{\partial u^k}\xi(0,t)\right]. \end{aligned} \tag{37}$$

The same method can be used to demonstrate the remaining result. □

4. Applications of DSET

In this section, we will apply the DSET to a family of FPDEs and get a simple formula for the general solution.

We consider a general nonhomogeneous FPDE of the form:

$$\begin{aligned} &a\frac{\partial^\beta\xi(u,t)}{\partial t^\beta}+b\frac{\partial^\alpha\xi(u,t)}{\partial u^\alpha}+c\,R(\xi(u,t))=h(u,t),\ u,t>0,\\ &m-1<\beta\le m\ ,\ \ n-1<\alpha\le n,\ \ \ m,n\in\mathbb{N}, \end{aligned} \tag{38}$$

on the ICs:

$$\frac{\partial^j\xi(u,0)}{\partial t^j}=g_j(u),\ j=0,1,\ldots,m-1, \tag{39}$$

and the BCs:

$$\frac{\partial^k\xi(0,t)}{\partial u^k}=f_k(t),\ k=0,1,\ldots,n-1, \tag{40}$$

where $a, b,$ and c are constants, $R(\xi(u,t))$ is a linear operator, and $h(u,t)$ is the source term.

Applying DSET to Equation (38), we get

$$\begin{aligned} &a\left(q^{-\beta}\,\Psi(w,q)-\sum_{j=0}^{m-1}q^{2-\beta+j}\,S_u\left[\frac{\partial^j}{\partial t^j}\xi(u,0)\right]\right)\\ &+b\left(w^{-\alpha}\Psi(w,q)-\sum_{k=0}^{n-1}w^{-\alpha+k}E_t\left[\frac{\partial^k}{\partial u^k}\xi(0,t)\right]\right)\\ &+c\,S_uE_t\,[R(\xi(u,t))]=H(w,q)\ . \end{aligned} \tag{41}$$

Using the SST for the conditions Equation (39) and the SET for the conditions Equation (40), to get

$$\begin{aligned} &S_u\left[\frac{\partial^j\xi(u,0)}{\partial t^j}\right]=G_j(w),\ j=0,1,\ldots,m-1, E_t\left[\frac{\partial^k\xi(0,t)}{\partial u^k}\right]\\ &=F_k(q), k=0,1,\ldots,n-1. \end{aligned} \tag{42}$$

By substituting Equation (42) into Equation (41), we have

$$\begin{aligned} &a\left(q^{-\beta}\Psi(w,q)-\sum_{j=0}^{m-1}q^{2-\beta+j}\,G_j(w)\right)\\ &+b\left(w^{-\alpha}\Psi(q,w)-\sum_{k=0}^{n-1}w^{-\alpha+k}F_k(q)\right)\\ &=H(w,q)-c\,S_uE_t\,[R(\xi(u,t))]\ . \end{aligned} \tag{43}$$

Simplifying Equation (43), we obtain

$$\Psi(w,q)=[a\,q^{-\beta}+b\,w^{-\alpha}]^{-1}\left\{\begin{array}{l} a\left(\sum_{j=0}^{m-1}q^{2-\beta+j}\,G_j(w)\right)+b\left(\sum_{k=0}^{n-1}w^{-\alpha+k}\,F_k(q)\right)+H(w,q)\\ -c\,S_uE_t\,[R(\xi(u,t))] \end{array}\right\}. \tag{44}$$

Taking $(S_w)^{-1}(E_q)^{-1}$ of Equation (44), we get

$$\begin{aligned} &\xi(u,t)=\\ &(S_w)^{-1}(E_q)^{-1}\left[[a\,q^{-\beta}+b\,w^{-\alpha}]^{-1}\left\{\begin{array}{l} a\left(\sum_{j=0}^{m-1}q^{2-\beta+j}\,G_j(w)\right)+b\left(\sum_{k=0}^{n-1}w^{-\alpha+k}F_k(q)\right)+H(q,w)\\ -c\,S_uE_t\,[R(\xi(u,t))] \end{array}\right\}\right]. \end{aligned} \tag{45}$$

5. Illustrative Examples

In this section, we will construct a few different examples to show how the DSET can be used and how effective it is.

Example 1. Consider the linear fractional heat equation:

$$\frac{\partial^{\beta}\xi}{\partial t^{\beta}} - \frac{\partial^{2}\xi}{\partial u^{2}} = 0, 0 < \beta \leq 1, \tag{46}$$

with the ICs:

$$\xi(u, 0) = \sin u, \tag{47}$$

and the BCs:

$$\xi(0, t) = 0, \ \xi_u(0, t) = E_{\beta}(-t^{\beta}). \tag{48}$$

Solution. Operating the DSET on Equation (46) and SST on Equation (47) and the SET on Equation (48), we get

$$\begin{aligned} &q^{-\beta}\,\Psi(w, q) - q^{2-\beta}S_u[\xi(u, 0)] \\ &-\left(w^{-2}\Psi(w, q) - w^{-2}E_t[\xi(0, t)] - w^{-1}E_t\left[\frac{\partial}{\partial u}\xi(0, t)\right]\right) = 0, \end{aligned} \tag{49}$$

substituting the SST and SET of initial and boundary conditions

$$\begin{aligned} S_u[\xi(u, 0)] &= \frac{w}{1 + w^2}, \ E_t[\xi(0, t)] = 0, E_t\left[\frac{\partial}{\partial u}\xi(0, t)\right] \\ &= q^2(1 + q^{\beta})^{-1}, \end{aligned} \tag{50}$$

in Equation (49), and simplifying, we get

$$\Psi(w, q) = \frac{wq^2}{(w^2 + 1)(1 + q^{\beta})}, \tag{51}$$

taking $(S_w)^{-1}(E_q)^{-1}$ of Equation (51), we get

$$\xi(u, t) = (S_w)^{-1}(E_q)^{-1}\left[\frac{wq^2}{(w^2 + 1)(1 + q^{\beta})}\right] = E_{\beta}(-t^{\beta})\sin u. \tag{52}$$

In Figure 1, we sketch the approximate solution of Equation (52) with different values of the fractional order β when $t = 0.03$ and $u \in (0, 6)$.

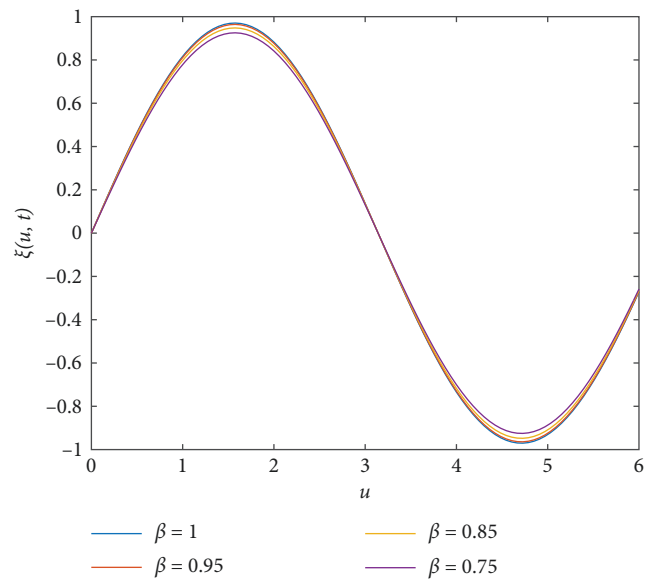

FIGURE 1: 2D approximate solution of Equation (52).

Example 2. Consider the linear fractional Klein–Gordon equation:

$$\frac{\partial^{\beta}\xi}{\partial t^{\beta}} - \frac{\partial^{2}\xi}{\partial u^{2}} - \xi = 0, 1 < \beta \leq 2, \tag{53}$$

on the ICs:

$$\xi(u, 0) = \sin u + 1, \xi_t(u, 0) = 0, \tag{54}$$

and the BCs:

$$\xi(0, t) = E_{\beta}(t^{\beta}), \ \ \xi_u(0, t) = 1. \tag{55}$$

Solution. Operating the DSET on Equation (53) and SST on Equation (54) and the SET on Equation (55), we get

$$\begin{aligned} &q^{-\beta}\,\Psi(w, q) - q^{2-\beta}S_u[\xi(u, 0)] - q^{3-\beta}S_u\left[\frac{\partial}{\partial t}\xi(u, 0)\right] \\ &-\left(w^{-2}\Psi(w, q) - w^{-2}E_t[\xi(0, t)] - w^{-1}E_t\left[\frac{\partial}{\partial u}\xi(0, t)\right]\right) \\ &-\Psi(w, q) = 0, \end{aligned} \tag{56}$$

substituting the SST and SET of initial and boundary conditions

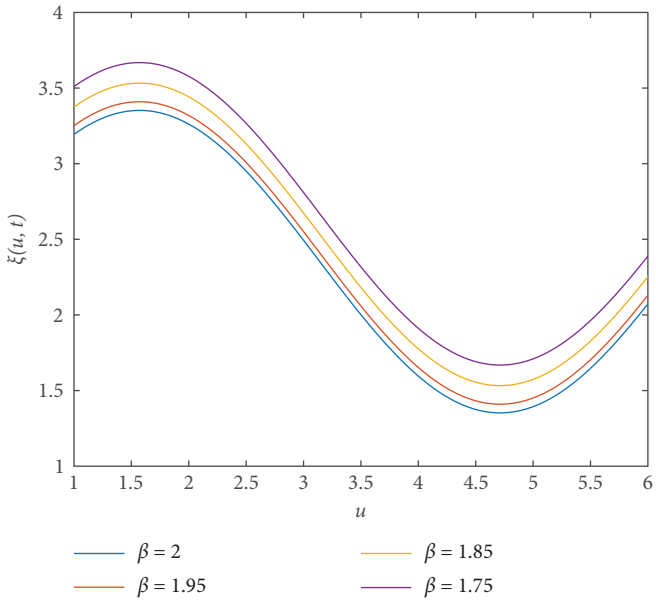

FIGURE 2: 2D approximate solution of Equation (60).

$$S_u[\xi(u,0)] = \frac{w}{1+w^2}+1, S_u\left[\frac{\partial}{\partial t}\xi(u,0)\right]=0, E_t[\xi(0,t)] = q^2(1-q^\beta)^{-1}, \quad E_t\left[\frac{\partial}{\partial u}\xi(0,t)\right]=q^2, \tag{57}$$

in Equation (56), we get

$$\Psi(w,q)= \frac{1}{(q^{-\beta}-w^{-2}-1)}\left[q^{2-\beta}\left(\frac{w}{1+w^2}+1\right) - w^{-2}(q^2(1-q^\beta)^{-1}) - w^{-1}q^2\right], \tag{58}$$

simplifying Equation (58), we obtain

$$\Psi(w,q)=\frac{wq^2}{w^2+1}+\frac{q^2}{(1-q^\beta)}, \tag{59}$$

taking $(S_w)^{-1}(E_q)^{-1}$ of Equation (59), we get

$$\xi(u,t)=(S_w)^{-1}(E_q)^{-1}\left[\frac{wq^2}{w^2+1}+\frac{q^2}{(1-q^\beta)}\right]=\sin u+E_\beta(t^\beta). \tag{60}$$

In Figure 2, we sketch the approximate solution of Equation (60) with different values of the fractional order β when $t=1.5$ and $u\in(1,6)$.

Example 3. Consider the linear one-dimensional time fractional Burgers equation:

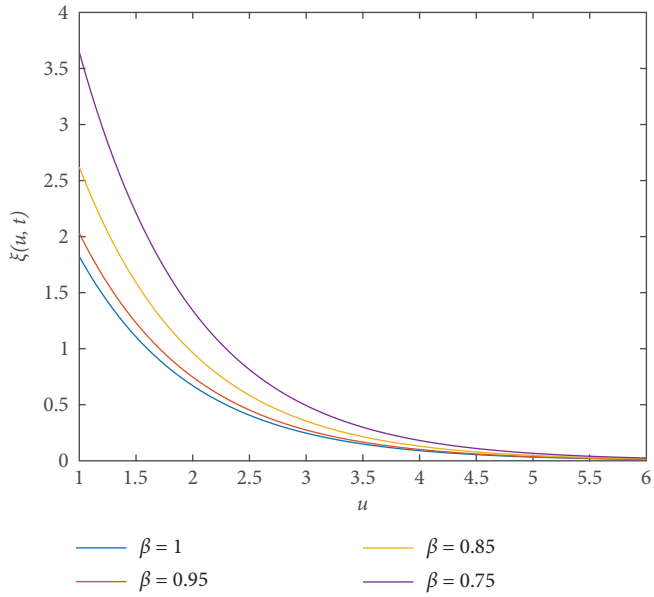

FIGURE 3: 2D approximate solution of Equation (68).

$$\frac{\partial^\beta \xi}{\partial t^\beta}-\frac{\partial^2\xi}{\partial u^2}+\frac{\partial \xi}{\partial u}=0, 0<\beta\le 1, \tag{61}$$

on the ICs:

$$\xi(u,0)=e^{-u}, \tag{62}$$

and the BCs:

$$\xi(0,t)=E_\beta(2t^\beta), \quad \xi_u(0,t)=-E_\beta(2t^\beta). \tag{63}$$

Solution. Operating the DSET on Equation (61) and SST on Equation (62) and the SET on Equation (63), we get

$$q^{-\beta}\,\Psi(w,q)-q^{2-\beta}S_u[\xi(u,0)] -\left(w^{-2}\Psi(w,q)-w^{-2}E_t[\xi(0,t)]-w^{-1}E_t\left[\frac{\partial}{\partial u}\xi(0,t)\right]\right) + w^{-1}\Psi(w,q)-w^{-1}E_t[\xi(0,t)]=0, \tag{64}$$

substituting

$$S_u[\xi(u,0)]=\frac{1}{1+w}, \; E_t[\xi(0,t)]=\frac{q^2}{(1-2q^\beta)}, \; E_t\left[\frac{\partial}{\partial u}\xi(0,t)\right] = -\frac{q^2}{(1-2q^\beta)}, \tag{65}$$

in Equation (64), we get

$$\Psi(w,q)=\frac{1}{(q^{-\beta}-w^{-2}+w^{-1})}\left[q^{2-\beta}\frac{1}{1+w}-w^{-2}\frac{q^2}{(1-2q^{\beta})}+w^{-1}\frac{q^2}{(1-2q^{\beta})}+w^{-1}\frac{q^2}{(1-2q^{\beta})}\right], \tag{66}$$

simplifying Equation (66), we obtain

$$\Psi(w,q)=\frac{q^2}{(w+1)(1-2q^{\beta})}, \tag{67}$$

taking $(S_w)^{-1}(E_q)^{-1}$ of Equation (67), we get

$$\xi(u,t)=(S_w)^{-1}(E_q)^{-1}\left[\frac{q^2}{(w+1)(1-2q^{\beta})}\right]=e^{-u}E_{\beta}(2t^{\beta}). \tag{68}$$

In Figure 3, we sketch the approximate solution of Equation (68) with different values of the fractional order β when $t=0.8$ and $u\in(1,6)$.

Example 4. Consider the linear fractional Fokker–Planck equation:

$$\frac{\partial^{\beta}\xi}{\partial t^{\beta}}-\frac{\partial^2\xi}{\partial u^2}-\frac{\partial\xi}{\partial u}=0, 0<\beta\leq 1, \tag{69}$$

on the ICs:

$$\xi(u,0)=u, \tag{70}$$

and the BCs:

$$\xi(0,t)=\frac{t^{\beta}}{\Gamma(1+\beta)},\quad \xi_u(0,t)=1. \tag{71}$$

Solution. Operating the DSET on Equation (69) and SST on Equation (70) and the SET on Equation (71), we get

$$\begin{aligned}&q^{-\beta}\,\Psi(w,q)-q^{2-\beta}S_u[\xi(u,0)]\\&-\left(w^{-2}\Psi(w,q)-w^{-2}E_t[\xi(0,t)]-w^{-1}E_t\left[\frac{\partial}{\partial u}\xi(0,t)\right]\right)\\&-(w^{-1}\Psi(w,q)-w^{-1}E_t[\xi(0,t)])=0,\end{aligned} \tag{72}$$

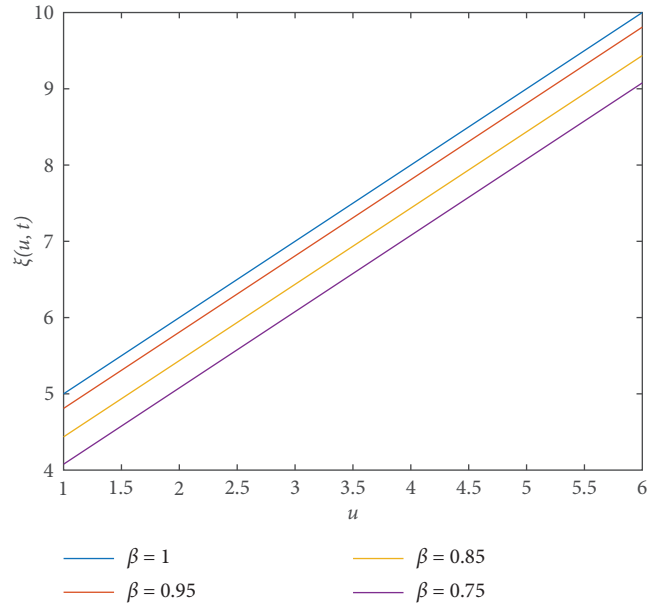

FIGURE 4: 2D approximate solution of Equation (76).

substituting

$$S_u[\xi(u,0)]=w,\quad E_t[\xi(0,t)]=q^{\beta+2},\quad E_t\left[\frac{\partial}{\partial u}\xi(0,t)\right]=q^2, \tag{73}$$

in Equation (72), we get

$$\Psi(w,q)=\frac{1}{(q^{-\beta}-w^{-2}-w^{-1})}[q^{2-\beta}\,w-w^{-2}q^{\beta+2}+w^{-1}q^2-w^{-1}q^{\beta+2}], \tag{74}$$

simplifying Equation (74), we obtain

$$\Psi(w,q)=q^2w+q^{\beta+2}, \tag{75}$$

taking $(S_w)^{-1}(E_q)^{-1}$ of Equation (75), we get

$$\xi(u,t)=(S_w)^{-1}(E_q)^{-1}[q^2w+q^{\beta+2}]=u+\frac{t^{\beta}}{\Gamma(1+\beta)}. \tag{76}$$

In Figure 4, we sketch the approximate solution of Equation (76) with different values of the fractional order β when $t=4$ and $u\in(1,6)$.

Example 5. Consider the linear fractional telegraph equation:

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \xi}{\partial t} + \xi = \frac{\partial^\alpha \xi}{\partial u^\alpha}, 1 < \alpha \leq 2, \tag{77}$$

on the ICs:

$$\begin{aligned} \xi(u,0) &= E_\alpha(u^\alpha) + uE_{\alpha,2}(u^\alpha), \quad \xi_t(u,0) \\ &= -\left[E_\alpha(u^\alpha) + uE_{\alpha,2}(u^\alpha)\right], \end{aligned} \tag{78}$$

and the BCs:

$$\xi(0,t) = e^{-t}, \quad \xi_u(0,t) = e^{-t}. \tag{79}$$

Solution. Operating the DSET on Equation (77) and SST on Equation (78) and the SET on Equation (79), we get

$$\begin{aligned} &q^{-2}\Psi(w,q) - S_u[\xi(u,0)] - q\, S_u\left[\frac{\partial}{\partial t}\xi(u,0)\right] \\ &\quad + q^{-1}\Psi(w,q) - q\, S_u[\xi(u,0)] + \Psi(w,q) \\ &= \left(w^{-\alpha}\Psi(w,q) - w^{-\alpha}E_t[\xi(0,t)] - w^{-\alpha+1}E_t\left[\frac{\partial}{\partial u}\xi(0,t)\right]\right), \end{aligned} \tag{80}$$

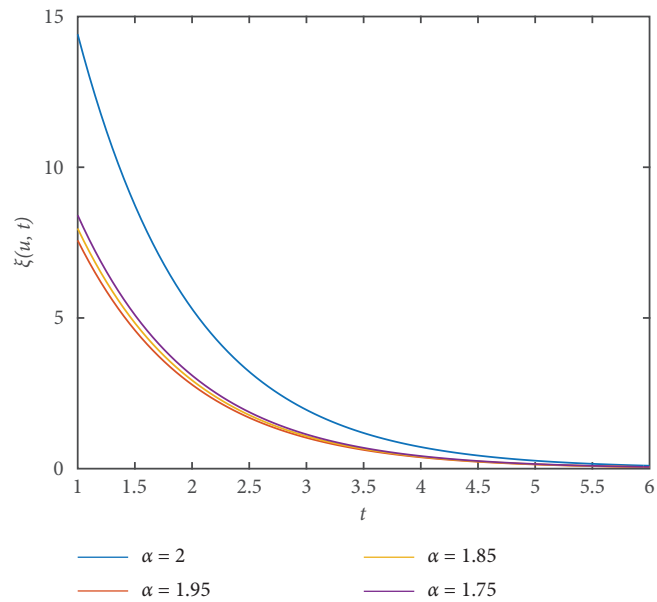

FIGURE 5: 2D approximate solution of Equation (84).

substituting

$$\begin{aligned} S_u[\xi(u,0)] &= \frac{(1+w)}{w(1-w^\alpha)}, \quad S_u\left[\frac{\partial}{\partial t}\xi(u,0)\right] \\ &= -\frac{(1+w)}{w(1-w^\alpha)}, \quad E_t[\xi(0,t)] = E_t\left[\frac{\partial}{\partial u}\xi(0,t)\right] = \frac{q^2}{1+q}, \end{aligned} \tag{81}$$

in Equation (81), we get

$$\Psi(w,q) = \frac{1}{(q^{-2} + q^{-1} + 1 - w^{-\alpha})}\left[\begin{array}{c} \dfrac{(1+w)}{w(1-w^\alpha)} - q\dfrac{(1+w)}{w(1-w^\alpha)} + q\dfrac{(1+w)}{w(1-w^\alpha)} \\ -w^{-\alpha}\dfrac{q^2}{1+q} - w^{-\alpha+1}\dfrac{q^2}{1+q} \end{array}\right], \tag{82}$$

simplifying Equation (82), we obtain

$$\Psi(w,q) = \frac{q^2(1+w)}{w(1+q)(1-w^\alpha)}, \tag{83}$$

taking $(S_w)^{-1}(E_q)^{-1}$ of Equation (83), we get

$$\begin{aligned} \xi(u,t) &= (S_w)^{-1}(E_q)^{-1}\left[\frac{q^2}{(1+q)}\frac{(1+w)}{w(1-w^\alpha)}\right] \\ &= e^{-t}\left[E_\alpha(u^\alpha) + uE_{\alpha,2}(u^\alpha)\right]. \end{aligned} \tag{84}$$

In Figure 5, we sketch the approximate solution of Equation (84) with different values of the fractional order α when $u = 3$ and $t \in (1, 6)$.

6. Results and Discussion

In order to show the accuracy and usefulness of the recommended approach, in this section we will look at the numerical evaluation of the results of fractional equations that have been proposed to be solved. Furthermore, we will compare the numerical behavior of the solutions to FPDEs with that of equations with integer derivatives. When $\beta = 1$ and $\alpha, \beta = 2$, the closed-form solutions for Examples 1–5 is simply calculated. We have chosen to look at the numerical results for different values of fractional-order values α and β. We noticed that the solutions obtained for $\beta = 1, 0.95, 0.85, 0.75$, and $\alpha, \beta = 2, 1.95, 1.85, 1.75$, are in coordination with the solutions of the closed forms for $\beta = 1$ and $\alpha, \beta = 2$, as shown in Figures 1–5. It is sufficient to note that when $\beta \to 1$ and $\alpha, \beta \to 2$, the solutions resulting from the fractional equations approach these exact solutions.

7. Conclusion

This article discusses a new double transformation called DSET. First, we applied the DSET to a few particular functions; following that, some theorems and properties connected to the DSET were presented and proved. To demonstrate the applicability and efficacy of the proposed transform, we used DSET to solve a wide range of FPDEs in mathematical physics. Based on the obtained findings, we conclude that the provided transform is efficient, suitable, reliable, and adequate to acquire the accurate solutions of FPDEs according to the taken-into-account starting and boundary conditions. Therefore, we may state that a broad class of linear FPDE schemes can be solved using this approach.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

S.A.A, T.M.E., and A.M. contributed in the data curation, formal analysis, investigation, supervision, writing—original draft, and writing—review and editing. S.A.A. and T.M.E contributed in the methodology. All authors have read and agreed to the published version of the manuscript.

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