# Total Face Irregularity Strength of Certain Graphs 

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#### Abstract

The edge $k$-labeling $\psi$ of $G$ is defined by a mapping from $E(G)$ to a set of integers $\{1,2, \ldots, k\}$, where the integer weight assigned to the vertex $x \in V(G)$ is given as $w_{\psi}(x)=\Sigma \psi(x y)$, such that the sum is taken over every vertex of $y \in V(G)$ that is adjacent to $x$ and the integer weights of adjacent vertices must be distinct for all vertices with $x \neq y$. An irregular assignment of $G$ using atmost $k$ labels which is considered to be a minimum $k$ is defined as irregularity strength of a graph $G$ and can be denoted as $s(G)$. There are also further works on familiar irregular assignments, such as edge irregular labelings, vertex irregular total labelings, edge irregular total labelings, and face irregular entire $k$-labelings of plane graphs. A plane graph can be defined as a graph that is embedded in the plane in which no two lines will be intersected. In a plane graph the number of regions present are called faces and we denote it as $F$. The concept of total face irregularity strength is defined by the motivation of irregular networks and entire irregular face $k$-labeling. In our paper, we have obtained a minimum bound for the total face irregularity strength of two-connected plane graphs like cycle-of-ladder, $C$-necklace graph, $P$-necklace graph, sibling tree, and triangular graph.


## 1. Introduction

Graph theory is an interesting topic of research in combinatorics. Graph-based models are a powerful and indispensable tool for solving practical issues. In general, new algorithms that are appropriate for graph structures have been explored as a result of the development of contemporary theoretical issues such as labeling in graph theory. At present the utilization of computers is increasing in human life and our future is entirely becoming computer oriented. Most of the applications use graphs as their underlying structures, say for example, Google search engines are primarily based on the concept of shortest paths in the graph theory. As a result, graph theory has grown to be an enormous field of study in mathematics.

The concept of graph labeling was established in the middle of 1960's. Graph labeling can be defined as the assignment of integer labels (values) to the links and/or nodes of a graph. One important motive in graph labeling is to fulfill some conditions that are imposed on the graph. In general, a lot of graph labeling problems started off in connection with
some real-time applications such as the exam scheduling and tournament scheduling problems in sports. In recent times, graphs with a defined set of labels (integers) allocated to the vertices, edges, or both based on certain given conditions are being explored. These graphs are termed as labeled graphs. Labeled graphs are studied as an important concept in graph theory because of their innumerable applications. These labeled graphs are also of interest in their own way because of their theoretical mathematical properties of the underlying graphs. Research works on the labeling of graphs are being encouraged in several domains such as human inquiry, including resolution of conflicts in psychology, cybernetic systems, and energy shortages. Mathematical labeling of various graphs has resulted in rather complex domains of application, such as the theory of coding problems, such as the development of excellent radar position codes, missile navigation codes, and convolution signals. There are numerous theoretical applications for labeled graphs in combinatorial number theory, group theory, and in linear algebra [1]. In general, the graph labeling problem can be described as follows: for a given graph, find the optimal way of labeling the
vertices/edges or both with distinct integers or $k$-tuples of integers subject to certain objectives. Many fascinating applications for graph labeling are found in Bloom and Golomb's [2, 3] papers.

We consider the vertex set and edge set of a finite, simple, and undirected graph $G$. An irregular assignment of $G$ maybe defined as the labeling of edges of $G$ with a set of positive integer such that the total (sum) of the labels that are incident with some vertex is distinct for every vertex. Chartrand et al. [4] introduced this concept. They discovered that for any graph, the irregularity strength $s(G)$, which is the lowest possible value $k$ for which $G$ constitutes an irregular assignment with label at most $k$, is extremely difficult to find. In recent times, motivated by this particular concept, many researchers are having specific interest for these types of irregular labeling and have found the irregularity strength for some graphs [5-10]. Shabbir et al. [11] have proved their exact values of the strength of total vertex (edge) irregularities of a randomly convex unions of $(3,6)$-fullerene graphs. In 2022, Bača et al. [12] investigated the irregular labelings with respect to face of plane graphs and have found a new graph characteristic which can be termed as face irregularity strength of a few types $(\alpha, \beta, \gamma)$. Also, Tilukay et al. [13] have estimated the bounds of total face irregularity strength $t f_{s}(G)$ and have proved that the lower bound is sharp for $G$ isomorphic to a cycle, a book with $m$ polygonal pages, or a wheel. Further, Jamil and Mughal [14] have studied $t f s$ of generalized plane grid graphs $G_{m}^{n}$ and wheel graphs $W_{n}$ under a graph $k$-labeling of type $(\alpha, \beta, \gamma)$ where $\alpha, \beta \in 0,1$. Since a total labeling is defined for both edges and vertices, it is difficult to find the lower bounds for certain higher dimensions of graphs.

In our paper, we have obtained a lower bound for the $t f s(G)$ of two-connected plane graphs like cycle-of-ladder, $C$ necklace graph $\mathrm{CN}\left[C_{m} ; C_{m}^{i}\right], P$-necklace graph $\mathrm{PN}\left[\mathrm{P}_{\mathrm{m}} ; \mathrm{C}_{\mathrm{m}}^{\mathrm{i}}\right]$, sibling tree, as well as a lower bound for $t f s(G)$ is obtained for triangular graph.

## 2. Preliminaries

The concept of irregularity strengths and a recent publication on entire coloring of plane graphs [15] served as the inspiration for Bača's et al. [16] study of face irregular entire $k$-labeling of plane graphs in 2015 . In the sections below, for a graph $G$ we denote the number of cycles of length $d$ as $n_{d}$.

In the year 2016, Packiam [17] has defined the concept of total face irregularity strength. The following theorems give the lower bound for total face irregularity strength of a plane graph.

Theorem 1 [17]. Let $G=(V, E, F)$ be a plane graph with $m_{i} r_{i}$-sided faces where $r_{i}<r_{i+1}$ and $1 \leq i \leq s$. Then $t f s(G) \geq$ $\left\lceil\frac{2 r_{1}+|F|-1}{2 r_{s}}\right\rceil$.


Figure 1: Total face irregularity strength of cycle-of-ladder.

Theorem 2 [17]. Let $G=(V, E, F)$ be a two-connected plane graph with $m_{i} r_{i}$-sided faces where $r_{i} \leq r_{i+1}$ and $1 \leq i \leq s$. Then $t f s(G) \geq \max _{i}\left\lceil\frac{2 r_{i}+m_{i}-1}{2 r_{i}}\right\rceil$.

We now propose the following theorem:

Theorem 3. Let $G$ be a two-connected plane graph and $d$ be the girth in $G$. Then $t f s(G) \geq\left\lceil\frac{2 d+n_{d}-1}{2 d}\right\rceil$.

Proof. Let $t f s(G)=k$. Clearly, $2 d k \geq 2 d+n_{d}-1$. Therefore, $k \geq\left\lceil\frac{2 d+n_{d}-1}{2 d}\right\rceil$.

Remark 1. Theorem 3 implies that it is not always necessary to consider all the cycles induced by the plane graph faces.

## 3. Main Results

3.1. Cycle-of-Ladder. In this section, the $t f s(G)$ is obtained for cycle-of-ladder (Figure 1) where its graph theoretical definition given in [18, 19].

Lemma 1. Let $G$ be the $C L(2 k, r)$, where $r_{1}=r_{2}=\ldots=r_{k}=r$ are considered as the rungs of equal length and $d$ be a girth in G. Then $t f(G) \geq\left\lceil\frac{n_{d}+7}{8}\right\rceil$.

Proof. In $G$, the shortest cycle length is 4 . Hence, $d=4$. By Theorem 3, $t f s(G) \geq\left\lceil\frac{2 d+n_{d}-1}{2 d}\right\rceil$. Therefore, $t f_{s}(G) \geq\left\lceil\frac{n_{d}+7}{8}\right\rceil$.

Input: Cycle-of-ladder, $\mathrm{CL}(2 k, r), k \geq 3$.
Algorithm: Begin labeling the 4 -cycles in the ladders $L_{1}, L_{2}, \ldots, L_{k}$ as follows.
Step 1: The labels of the bottom rung $R_{1}^{1}$, the two parallel sides perpendicular to the bottom rung and the top rung $R_{2}^{1}$ constitute a 8tuple divided as 3 -tuple representing the labels of the bottom rung followed by 2 -tuple representing the labels of the parallel edges followed by 3-tuple representing the labels of the top rung of the respective 4 -cycle.
Step 2: We sequentially label the 4-cycles in each ladder, beginning from the bottom rung of $L_{1}$, go up the ladder till all vertices and edges in $L_{1}$ are labeled, and repeat the same with $L_{2}, \ldots, L_{k}$ in the same order.
Step 3: We list the labels of the first 4-cycles as

$$
s_{1}=(\overbrace{111} \overbrace{11} \overbrace{111}),
$$

and the next eight 4-cycles as follows

$$
\left.\begin{array}{l}
s_{2}=\left(\begin{array}{llllll}
1 & 1 & 1 & 1 & 2 & 1
\end{array} 1\right.
\end{array}\right)
$$

Step 4: If the number of 4-cycles in $L_{1} \cup L_{2} \cup \ldots \cup L_{k}$ is more than 9 , continue labeling the subsequent cycles as $s_{2}+1, s_{3}+1, \ldots, s_{9}+$ 1 , where $s_{i}+1$ is the 8 -tuple obtained from $s_{i}$ by adding 1 to each bit, $2 \leq i \leq 9$.
Step 5: Repeat this procedure by adding $2,3, \ldots$ to each $s_{i}, 2 \leq i \leq 9$, till all the cycles are labeled.
Step 6: By our labeling, we get consecutive labels for each of the faces enclosed by these 4-cycles. Let $q$ denote the greatest of these face labels. Finally, we label the edge of the $C_{s}$ which are not yet labeled using the already used labels to arrive at a face label greater than $q$.
Output: $t f_{s}\left(\mathrm{CL}(2 k, r)=\left\lceil\frac{n_{d}+7}{8}\right\rceil\right.$.
Proof of correctness: $\mathrm{CL}(2 k, r)$ has $k(r-1)$ number of 4-cycles. Then for every set of eight 4 -cycles beginning from the second, considered sequentially from bottom to top and in the clockwise direction, the label is incremented by 1 , beginning from 2 . Hence the number of labels used is $\left\lceil\frac{\mathrm{k}(r-1)-1}{8}\right\rceil+1=\left\lceil\frac{n_{d}+7}{8}\right\rceil$.

Algorithm 1: Total face irregularity strength of cycle-of-ladder.

Input: $C$-necklace graph, $\mathrm{CN}\left[\mathrm{C}_{\mathrm{m}} ; \mathrm{C}_{\mathrm{m}}^{\mathrm{i}}\right], 1 \leq i \leq m$.

## Algorithm:

Step 1: Assign labels to the vertices and the edges of the inner cycle $C_{m}$ with label 1 such that $2 m$ is the weight of the inner face.
Step 2: Begin labeling the unlabeled vertices and edges of $C_{m}^{1}, C_{m}^{2}, \ldots, C_{m}^{i}, 1 \leq i \leq m$, as follows:
(i) Since one vertex of $C_{m}^{i}$ is identified with the $i^{\text {th }}$ vertex of $C_{m}$, it is already labeled with label 1 .
(ii) The remaining unlabeled $m-1$ vertices and $m$ edges in each outer cycles $C_{m}^{i}, 1 \leq i \leq m$ receive labels $\left(d_{1}^{1}, d_{2}^{1}, \ldots, d_{2 m-1}^{1}\right)$ in such a way that the hamming distance $\sum_{j=1}^{2 m-1}\left|d_{j}^{i+1}-d_{j}^{i}\right|=1$. Thus the weight of each of the $m$ outer faces varies from $2 m+1$ to $3 m$ as shown in Figure 2.
Output: $t f s\left(\operatorname{CN}\left[C_{m} ; C_{m}^{i}\right]\right)=1+\left\lceil\frac{m-1}{2 m}\right\rceil$.

Algorithm 2: Total face irregularity strength of $C$-necklace graph.

Remark 2. Algorithm 1 holds good for $\mathrm{CL}\left(2 k, r_{1}, r_{2}, \ldots, r_{k}\right)$, $k \geq 3$.
3.2. C-Necklace Graph, $C N\left[C_{m} ; C_{m}^{i}\right]$. In this section, we obtain the $t f s(G)$ of $C$-necklace graph, $\mathrm{CN}\left[C_{m} ; C_{m}^{i}\right], 1 \leq$ $i \leq m$.

Definition 1 [20]. Let us consider $C_{m}$ to be a cycle with $m$ vertices which is called the inner cycle and for $1 \leq i \leq m$; let $C_{m}^{i}$ be a cycle on $m$ vertices called the outer cycle. The resulting graph is called $C$-necklace graph that is obtained by attaching any one vertex of $C_{m}^{i}$ to the corresponding $i^{t^{m}}$ vertex of the cycle $C_{m}, 1 \leq i \leq m$ and we denote it by $\mathrm{CN}\left[C_{m} ; C_{m}^{i}\right]$. It has $m^{2}$ vertices and $m(m+1)$ edges.


Figure 2: Total face irregularity strength of $C$-necklace graph $\mathrm{CN}\left[C_{5} ; C_{5}^{5}\right]$.

Input: $P$-necklace graph, $\mathrm{PN}\left[P_{m} ; C_{m}^{i}\right], 1 \leq i \leq m$.

## Algorithm:

Step 1: Label the vertices and edges of the cycles $C_{m}^{1}, C_{m}^{2}, \ldots, C_{m}^{i}, 1 \leq i \leq m$, beginning with the minimum label 1 such that $2 m$ is the minimum weight of the $C_{m}^{1}$ and the weights of each of these faces varies from $2 m$ to $3 m-1$.
Step 2: Label the unlabeled edges of the path with 1 as shown in Figure 3.
Output: $t f s\left(\operatorname{PN}\left[P_{m} ; C_{m}^{i}\right]\right)=1+\left\lceil\frac{m-1}{2 m}\right\rceil$.

Algorithm 3: Total face irregularity strength of $P$-necklace graph.

Proof of correctness: Let $G$ be a $C$-necklace graph, $\mathrm{CN}\left[C_{m} ; C_{m}^{i}\right], 1 \leq i \leq m$. Let $\left(u_{1}, u_{2}, \ldots, u_{m}\right)$ and $\left(e_{1}, e_{2}, \ldots\right.$, $e_{m}$ ) be the vertices and edges of the inner cycle. Then by our labeling Algorithm 2, the weight of that face is $2 m$. Further, if $\left(u_{1}, u_{2}, \ldots, u_{m-1}\right)$ and $\left(e_{1}, e_{2}, \ldots, e_{m}\right)$ are the vertices and edges of the outer cycles, respectively, then the weights of $C_{m}^{1}, C_{m}^{2}, \ldots, C_{m}^{i}, 1 \leq i \leq m$, varies from $2 m+1$ to $3 m$, thereby $w_{\phi}(f) \neq w_{\phi}(g)$, for every two distinct faces $f$ and $g$ of $G$. Further, $t f s(G)=1+\left\lceil\frac{m-1}{2 m}\right\rceil$.

### 3.3. P-Necklace Graph, $\operatorname{PN}\left[P_{m} ; C_{m}^{i}\right]$.

In this section, a lower bound for the total face irregularity strength is obtained for $P$-necklace graph, $\operatorname{PN}\left[P_{m} ; C_{m}^{i}\right], 1 \leq$ $i \leq m$. The definition of $\mathrm{PN}\left[P_{m} ; C_{m}^{i}\right]$ is given in [20].

Definition 2 [20]. Let us consider $P_{m}$ to be a path with $m$ vertices and $C_{m}^{i}$ is a cycle with $m$ vertices, $1 \leq i \leq m$. The resulting graph is called $P$-necklace graph that is obtained by attaching any one vertex of $C_{m}^{i}$ to the corresponding $i^{\text {th }}$ vertex of $P_{m}, 1 \leq i \leq m$ and we denote it by $\operatorname{PN}\left[P_{m} ; C_{m}^{i}\right]$.

Proof of correctness: Let $G$ be a $P$-necklace graph, $\mathrm{PN}\left[\mathrm{P}_{\mathrm{m}} ; \mathrm{C}_{\mathrm{m}}^{\mathrm{i}}\right], 1 \leq i \leq m$. Let $\left(u_{1}^{i}, u_{2}^{i}, \ldots, u_{m}^{i}\right)$ and $\left(e_{1}^{i}, e_{2}^{i}, \ldots\right.$, $\left.e_{m}^{i}\right)$ be the vertices and edges in $C_{m}^{i}$, respectively. Then by our labeling Algorithm 3, the weight of this face is $2 m+$ $(i-1)$. The weights of all the faces vary between $2 m$ and $3 m-1$. Therefore, for every two distinct faces $f$ and $g$ of $G, w_{\phi}(f) \neq w_{\phi}(g)$.
3.4. Sibling Tree. In this section, the total face irregularity strength is obtained for sibling tree. The formal definition of sibling tree can be seen in [21,22].

Lemma 2. Let $\operatorname{ST}(n)$ be a sibling tree. The total face irregularity strength of $\mathrm{ST}(n)$ is given by $t f(\mathrm{ST}(n)) \geq\left\lceil\frac{n}{2}(n-1)+\right.$ 1) for $n \geq 1$.

Proof of correctness: Let $G$ be a sibling tree, $\operatorname{ST}(n)$, for $n \geq 1$. Let $\left(u_{1}^{i}, u_{2}^{i}, \ldots, u_{m}^{i}\right)$ and $\left(e_{1}^{i}, e_{2}^{i}, \ldots, e_{m}^{i}\right)$ be the vertices and edges in each cycle $C_{m}^{i}$, respectively. Then by our labeling Algorithm 4, the weight of the face of the first cycle of length 3 is $2 m$. The weights of all the faces of the cycles of

Input: Sibling tree $\mathrm{ST}(n+2), n \geq 1$.

## Algorithm:

Step 1: Let $C_{3}(x, y)$ denote a cycle of length three which is formed in Level $x$, for $x \geq 1$ and the number of copies, represented by $y_{i}$ for $1 \leq i \leq 2^{x}$. The root vertices of each level of $C_{3}\left(x, y_{i}\right)$ is called as the pivot vertices and the sum of the weights of $C_{3}\left(x, y_{i}\right)$ is called as the weight of $C_{3}\left(x, y_{i}\right)$. Notice that there will be $2^{x-1}$ copies of $C_{3}\left(x, y_{i}\right)$ in each Level $x$.
Step 2: Let us begin by defining a rule for the labeling process. In order to label ST( $n+1$ ), we retain the labeling of $\operatorname{ST}(n)$ and after retaining the labeling of $\mathrm{ST}(n)$, we start labeling the remaining copies of $C_{3}\left(n+1, y_{i}\right)$. In $\mathrm{ST}(2)$, there will be three copies of $C_{3}\left(x, y_{i}\right)$ and we label the vertices and edges of $C_{3}\left(0, y_{1}\right)$ by 1 . Thus, the weight of $C_{3}\left(1, y_{1}\right)$ is 6 . Similarly, the vertices and edges of $C_{3}\left(2, y_{1}\right)$ and $C_{3}\left(2, y_{2}\right)$ will be labeled as follows. The vertices and edges of copies of $C_{3}\left(1, y_{i}\right)$ are labeled temporarily as they were for $C_{3}\left(0, y_{1}\right)$, except for the pivot vertices. Observe that the two copies of $C_{3}\left(2, y_{i}\right)$ will have the same weight as $C_{3}\left(1, y_{1}\right)$. In order to make the weights distinct, we change the label of the left non-pivot vertex of $C_{3}\left(2, y_{1}\right)$ to 2 ; and in the rightmost copy of $C_{3}\left(2, y_{2}\right)$, we change the label of two non-pivot vertices to 2 . Thus, the weight of the $C_{3}\left(2, y_{1}\right)$ and $C_{3}\left(2, y_{2}\right)$ are 7 and 8 , respectively. Now we can say that copies of $C_{3}\left(1, y_{i}\right)$ are permanently labeled. We can now begin labeling the vertices and edges of $\operatorname{ST}(n)$, for $n \geq 3$ from Step 3 .
Step 3: When $n \geq 3$, the vertices and edges in $C_{3}\left(n, y_{1}\right)$ of $S T(n)$ are labeled temporarily as they were for $C_{3}\left(n-1, y_{2^{n-2}}\right)$, except for the pivot vertices. Observe that, there will arise two cases.
Case (i): When the labels of the pivot vertices of $C_{3}\left(n, y_{1}\right)=C_{3}\left(n-1, y_{2^{n-2}}\right)$, then the weights, $w\left(C_{3}\left(n, y_{1}\right)\right)=w\left(C_{3}\left(n-1, y_{2^{n-2}}\right)\right)$.
Case (ii): When the labels of the pivot vertices of $C_{3}\left(n, y_{1}\right) \neq C_{3}\left(n-1, y_{2^{n-2}}\right)$ or $C_{3}\left(n, y_{2}\right) \neq C_{3}\left(n-1, y_{2^{n-2}}\right)$, then the weights, $w\left(C_{3}\left(n-1, y_{2^{n-2}}\right)\right)<w\left(C_{3}\left(n, y_{1}\right)\right)$.
Hence $C_{3}\left(n, y_{1}\right)$ satisfies Case (ii) and the weights of $C_{3}\left(n, y_{1}\right)$ and $C_{3}\left(n-1, y_{2^{n-2}}\right)$ are all distinct.
Now the vertices and edges of $C_{3}\left(n, y_{i+1}\right)$ are labeled temporarily as they were for $C_{3}\left(n, y_{i}\right)$, except for the pivot vertices. Observe that, there will arise another case.
Case (iii): When the labels of the pivot vertices of $C_{3}\left(n, y_{i}\right)>C_{3}\left(n, y_{i+1}\right)$ or $C_{3}\left(n, y_{i}\right)>C_{3}\left(n, y_{i+2}\right)$, then the weights, $w\left(C_{3}(n-1\right.$, $\left.\left.y_{2^{n-2}}\right)\right)=w\left(C_{3}\left(n, y_{i+1}\right)\right)$.
Thus all the weights of $C_{3}\left(n, y_{i}\right)$ are all distinct.
Output: $t f s(\mathrm{ST}(\mathrm{n}+2))=\frac{\mathrm{n}^{2}+\mathrm{n}+2}{2}$, for $n \geq 1$.

Algorithm 4: Total face irregularity strength of sibling tree.


Figure 3: Total face irregularity strength of $P$-necklace graph $\operatorname{PN}\left[P_{4} ; C_{4}^{4}\right]$.
length 3 vary between $2 m$ and $\left(6+\sum_{i=1}^{m-2} 2^{i}\right)$. Therefore, for every two different faces $f$ and $g$ of $G, w_{\phi}(f) \neq w_{\phi}(g)$ as illustrated in Figure 4.
3.5. Triangular Graph. In this section, a lower bound for the $t f s(G)$ is obtained for triangular graph. Figure 5 shows the diagrammatic representation of triangular graph.

Lemma 3. Let $n_{d}$ be the number of cycles of length 6 in $\mathrm{TG}_{n}$. The total face irregularity strength of triangular graph is given by:

$$
\begin{equation*}
t f_{s}\left(\mathrm{TG}_{n}\right) \geq\left\lceil\frac{n_{d}+11}{12}\right\rceil \tag{1}
\end{equation*}
$$

Proof. With minimum label 1, the weight of a face is 12 . If $n_{d}$ is the number of cycles of length 6 , then the weights vary between 12 and $12+n_{d}-1=n_{d}+11$. Hence $t f s\left(\mathrm{TG}_{n}\right) \geq$ $\left\lceil\frac{n_{d}+11}{12}\right\rceil$.

Remark 3. Label the vertices and edges in Levels 1, 2, and 3 of $\mathrm{TG}_{3}$ as shown in Figure 5(b) as 1 and the vertices and edges in Level 4 as $\{1,1,1,1,2,1,1,1,2,1,1,2,2\}$ and the perpendicular edges between Levels 1 and 2 as 1, between Levels 2 and 3 as $\{1,2,2\}$ and between Levels 3 and 4 as 2 . The graph has six faces. It is easy to see that the weights of the faces are all distinct and sequential. Hence $t f\left(\mathrm{TG}_{3}\right)=\left\lceil\frac{6+11}{12}\right\rceil=2$.


Figure 4: Total face irregularity strength of sibling tree ST(4).


Figure 5: (a) Triangular graph $\mathrm{TG}_{3}$ and (b) brick representation of $\mathrm{TG}_{3}$.

Open problem: Let $n_{d}$ be the number of cycles of length 6 in $\mathrm{TG}_{n}, n \geq 1$. Then $t f\left(\mathrm{TG}_{n}\right)=\left\lceil\frac{n_{d}+11}{12}\right\rceil$.

## 4. Conclusion

We have found the total face irregularity strength of plane graphs like cycle-of-ladder, necklace graph, necklace graph, and sibling tree. For triangular graphs, we have established a lower bound on their total face irregularity strength. Our labeling approach on the vertices and edges of graphs is focused on estimating face weights of graphs in order to prove the sharpness of $k$-labeling. Excitingly, we are now exploring the total face irregularity strength of plane graphs like Sierpinski-like graphs, Schreier graphs, and WK-recursive networks, all of which find numerous practical applications in network theory. In general, it would be really challenging to improve the lower bound of total face irregularity strength for both plane graphs and their dual counterparts.

## Data Availability

No underlying data were collected or produced in this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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