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Research Article

Few More Series of Reciprocals with Binomial Coefficients and Their Evaluations

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In the present work, utilizing the known series, new series involving reciprocals of binomial coefficients, alternating positive, and negative binomial coefficients are constructed. Further, several new series of reciprocals of binomial coefficients with two odd terms in the denominator are obtained. In the end, we use these to establish the closed form evaluations of hypergeometric functions for the argument 1/16.

1. Introduction

The generalized hypergeometric function ${}_{s+1}F_s(z)$ is defined by Rainville [1] as follows:

$$ {}_{s+1}F_s\left[\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, b_2, \ldots, b_s \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \ldots (a_s)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} \frac{z^n}{n!}, \quad |z|<1, \tag{1} $$

where $(x)_m$ is given by:

$$ (x)_m = \begin{cases} x(x+1)\ldots(x+m-1) & ; m \in \mathbb{N} \\ 1 & ; m = 0, \end{cases} \tag{2} $$

which is known as Pochhammer's symbol and in terms of gamma function it is seen that:

$$ (x)_m = \frac{\Gamma(x+m)}{\Gamma(x)}. \tag{3} $$

In Equation (1), $a_0, a_1, \ldots, a_s, b_1, b_2, \ldots, b_s$ are any complex numbers, except that, of course, none of $\{b_j\}_{j=1}^{s}$ is a nonpositive integer. By D'Alembert ratio test, the above series ${}_{s+1}F_s$ converges absolutely for $|z|<1$. It is evident that ${}_{s+1}F_s(z)$ appears in a wide range of theoretical and real-world context, like statistics, engineering, theoretical physics, and mathematics. For more information, the following articles can be referred [2–5]. Further, it is widely acknowledged that splitting even and odd component of a generalized hypergeometric function can result in unique results. The following identities are used to facilitate this composition:

$$ (x)_{2m} = 2^{2m} \left(\frac{x}{2}\right)_m \left(\frac{x}{2} + \frac{1}{2}\right)_m, \tag{4} $$

and

$$ (x)_{2m+1} = x 2^{2m} \left(\frac{x}{2} + \frac{1}{2}\right)_m \left(\frac{x}{2} + 1\right)_m. \tag{5} $$

The well-known binomial coefficients and central binomial coefficients are given by:

$$ \binom{m}{n} = \begin{cases} \dfrac{m!}{n!(m-n)!} & ; m \geqslant n \\ 0 & ; m<n. \end{cases} \tag{6} $$

for $m, n \in \mathbb{N} \cup \{0\}$ and for $n \geqslant 0$:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}, \tag{7}$$

respectively. In several branches of mathematics, including number theory, graph theory, probability, and many others, the binomial coefficients and their reciprocals play a crucial role. For a long time, the sums comprising central binomial coefficients and its reciprocals have been examined, extensive work on this area can be found in [6–13]. For the recent work on closed-form evaluations of the hypergeometric functions, one may refer to the studies by Srivatsa Kumar et al. [14, 15].

Motivated by the above work, we aim at obtaining some closed form evaluations of ${}_{s+1}F_s(z)$ of $\frac{1}{16}$. Accordingly, we state two lemmas in Section 2 of the present paper that will be useful for our main results. In Section 3 of this article, we establish the reciprocal series of binomial coefficients with one odd term in the denominator. In Section 4, we establish some closed form evaluations of hypergeometric function ${}_4F_3$ of argument $\frac{1}{16}$. In the last section, we provide several reciprocal series of binomial coefficients with the combination of two odd terms in the denominator.

2. Preliminaries

Lemma 1. *The following results hold good for the reciprocal series of positive and negative binomial coefficients with one odd factor in the denominator:*

$$\sum_{m=0}^{\infty} \frac{(m!)^2 (2x)^{2m}}{(2m)!(2m+1)} = \frac{\arcsin(x)}{x\sqrt{1-x^2}} =: K, \tag{8}$$

$$\sum_{m=0}^{\infty} \frac{(2x)^{2m}}{\binom{2m}{m}(2m+3)} = \left(\frac{2}{x^2} - 1\right)K - \frac{2}{x^2}, \tag{9}$$

$$\sum_{m=0}^{\infty} \frac{(2x)^{2m}}{\binom{2m}{m}(2m+5)} = \left(\frac{8}{3x^4} - \frac{4}{3x^2} - \frac{1}{3}\right)K - \frac{8}{3x^4} - \frac{4}{9x^2}, \tag{10}$$

$$\sum_{m=0}^{\infty} \frac{(2x)^{2m}}{\binom{2m}{m}(2m+7)} = \left(\frac{16}{5x^6} - \frac{8}{5x^4} - \frac{2}{5x^2} - \frac{1}{5}\right)K - \frac{16}{5x^6} - \frac{8}{15x^4} - \frac{6}{25x^2}, \tag{11}$$

$$\sum_{m=0}^{\infty} \frac{(2x)^{2m}}{\binom{2m}{m}(2m+9)} = \left(\frac{128}{35x^8} - \frac{64}{35x^6} - \frac{16}{35x^4} - \frac{8}{35x^2} - \frac{1}{7}\right)K - \frac{128}{35x^8} - \frac{64}{105x^6} - \frac{48}{175x^4} - \frac{8}{49x^2}. \tag{12}$$

Proof. For the proof of the above identities, one may refer to the study by Ji and Hei [16]. □

Lemma 2. *We have:*

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2x)^{2m}}{\binom{2m}{m}(2m+1)} = \frac{\ln\left(x + \sqrt{x^2+1}\right)}{x\sqrt{x^2+1}} =: L, \tag{13}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2x)^{2m}}{\binom{2m}{m}(2m+3)} = -\left(\frac{2}{x^2} + 1\right)L + \frac{2}{x^2}, \tag{14}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2x)^{2m}}{\binom{2m}{m}(2m+5)} = \left(\frac{8}{3x^4} + \frac{4}{3x^2} - \frac{1}{3}\right)L - \frac{8}{3x^4} + \frac{4}{9x^2}, \tag{15}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2x)^{2m}}{\binom{2m}{m}(2m+7)} = \left(-\frac{16}{5x^6} - \frac{8}{5x^4} + \frac{2}{5x^2} - \frac{1}{5}\right)L + \frac{16}{5x^6} - \frac{8}{15x^4} + \frac{6}{25x^2}, \tag{16}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2x)^{2m}}{\binom{2m}{m}(2m+9)} = \left(\frac{128}{35x^8} + \frac{64}{35x^6} - \frac{16}{35x^4} + \frac{8}{35x^2} - \frac{1}{7}\right)L - \frac{128}{35x^8} + \frac{64}{105x^6} - \frac{48}{175x^4} + \frac{8}{49x^2}. \tag{17}$$

Proof. For the proof of the above identities, one may refer to the study by Ji and Zhang [17]. □

3. Main Results

In this section, the results containing reciprocal series of binomial coefficients with one odd factor in the denominator are established.

Theorem 1. *We have the following reciprocal series of binomial coefficients with one odd factor in the denominator:*

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+11)} = \left(\frac{256}{63x^{10}}-\frac{128}{63x^{8}}-\frac{32}{63x^{6}}-\frac{16}{63x^{4}}-\frac{10}{63x^{2}}-\frac{1}{9}\right)K - \frac{256}{63x^{10}}-\frac{128}{189x^{8}}-\frac{32}{105x^{6}}-\frac{80}{441x^{4}}-\frac{10}{81x^{2}}, \tag{18}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+11)} = -\left(\frac{256}{63x^{10}}+\frac{128}{63x^{8}}-\frac{32}{63x^{6}}+\frac{16}{63x^{4}}-\frac{10}{63x^{2}}+\frac{1}{9}\right)L + \frac{256}{63x^{10}}-\frac{128}{189x^{8}}+\frac{32}{105x^{6}}-\frac{80}{441x^{4}}+\frac{10}{81x^{2}}, \tag{19}$$

where K and L are defined as in Equations (8) and (13), respectively.

Proof of Equation (18). From the study by Gradshteyn and Ryzhik [18], we have:

$$\sum_{n=0}^{\infty}\frac{4^n(n!)^2x^{2n+2}}{(2n+1)!(n+1)} = (\arcsin x)^2, \quad (x^2 \geqslant 1). \tag{20}$$

On taking the derivative of both ends with respect to x and on multiplying by $\frac{1}{x}$, we get Equation (8). On splitting the terms in Equation (8), we have:

$$1+\frac{2x^2}{3}+\frac{8x^4}{15}+\frac{16x^6}{35}+\frac{128x^8}{315} + \sum_{n=5}^{\infty}\frac{n(n-1)(n-2)(n-3)(n-4)((n-5)!)^2(2x)^{2n}}{2^5(2n-1)(2n-3)(2n-5)(2n-7)(2n-9)(2n-10)!(2n+1)} = K. \tag{21}$$

On letting $n-5=m$ in above, we get:

$$1+\frac{2x^2}{3}+\frac{8x^4}{15}+\frac{16x^6}{35}+\frac{128x^8}{315} + \sum_{m=0}^{\infty}\frac{(m!)^2(m+1)^2(m+2)^2\ldots(m+5)^2(2x)^{2m+10}}{(2m+11)!} = K. \tag{22}$$

On multiplying by $\frac{1}{x^{10}}$, we get:

$$\frac{1}{x^{10}}+\frac{2}{3x^8}+\frac{8}{15x^6}+\frac{16}{35x^4}+\frac{128}{315x^2} + \sum_{m=0}^{\infty}\frac{(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{(2m+2)(2m+4)\ldots(2m+10)}{(2m+1)(2m+3)\ldots(2m+11)}\right\} = \frac{K}{x^{10}}, \tag{23}$$

which is equivalent to:

$$\frac{1}{x^{10}}+\frac{2}{3x^8}+\frac{8}{15x^6}+\frac{16}{35x^4}+\frac{128}{315x^2} + \sum_{m=0}^{\infty}\frac{(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{63}{256(2m+1)}+\frac{35}{256(2m+3)} + \frac{15}{128(2m+5)}+\frac{15}{128(2m+7)}+\frac{35}{256(2m+9)} + \frac{63}{256(2m+11)}\right\} = \frac{K}{x^{10}}. \tag{24}$$

On employing Equations (8)–(12) and on simplifying, we obtain the result. □

Proof of Equation (19). We have:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2 4^n x^{2n+1}}{(2n+1)!} = \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}. \tag{25}$$

On multiplying the above by $\frac{1}{x}$ and rewriting, we obtain:

$$1-\frac{2x^2}{3}+\frac{8x^4}{15}-\frac{16x^6}{35}+\frac{128x^8}{315}+\sum_{n=5}^{\infty}\frac{(-1)^n(n!)^2(2x)^{2n}}{(2n+1)!} = L. \tag{26}$$

On letting $n-5=m$, we obtain:

$$1-\frac{2x^2}{3}+\frac{8x^4}{15}-\frac{16x^6}{35}+\frac{128x^8}{315} - \sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(m+1)^2(m+2)^2\ldots(m+5)^2(2x)^{2m+10}}{(2m+11)!} = L. \tag{27}$$

On multiplying by $\frac{1}{x^{10}}$, we obtain:

$$\frac{1}{x^{10}}-\frac{2}{3x^8}+\frac{8}{15x^6}-\frac{16}{35x^4}+\frac{128}{315x^2} - \sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(2x)^{2m}}{(2m)!} \left\{\frac{(2m+2)(2m+4)\ldots(2m+10)}{(2m+1)(2m+3)\ldots(2m+11)}\right\}=\frac{L}{x^{10}}, \tag{28}$$

which is equivalent to:

$$\frac{1}{x^{10}}-\frac{2}{3x^8}+\frac{8}{15x^6}-\frac{16}{35x^4}+\frac{128}{315x^2} - \sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{63}{256(2m+1)} + \frac{35}{256(2m+3)}+\frac{15}{128(2m+5)}+\frac{15}{128(2m+7)} + \frac{35}{256(2m+9)}+\frac{63}{256(2m+11)}\right\}=\frac{L}{x^{10}}. \tag{29}$$

On employing Equations (13)–(17) in the above and then simplifying, we obtain the desired result. □

Theorem 2. *We have the following reciprocal series of binomial coefficients with one odd factor in the denominator.*

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+13)} =\left(\frac{1024}{231x^{12}}-\frac{512}{231x^{10}}-\frac{128}{231x^8}-\frac{64}{231x^6}-\frac{40}{231x^4}-\frac{4}{33x^2}-\frac{1}{11}\right)K -\frac{1024}{231x^{12}}-\frac{512}{693x^{10}}-\frac{128}{385x^8}-\frac{320}{1617x^6}-\frac{40}{297x^4}-\frac{12}{121x^2}, \tag{30}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+13)} =\left(\frac{1024}{231x^{12}}+\frac{512}{231x^{10}}-\frac{128}{231x^8}+\frac{64}{231x^6}-\frac{40}{231x^4}+\frac{4}{33x^2}-\frac{1}{11}\right)L -\frac{1024}{231x^{12}}+\frac{512}{693x^{10}}-\frac{128}{385x^8}+\frac{320}{1617x^6}-\frac{40}{297x^4}+\frac{12}{121x^2}, \tag{31}$$

where K and L are defined as in Equations (8) and (13), respectively.

Proof of Equation (30). On splitting the term Equation (8), up to the term containing x^{10}, we have:

$$1+\frac{2x^2}{3}+\frac{8x^4}{15}+\frac{16x^6}{35}+\frac{128x^8}{315}+\frac{256x^{10}}{693} +\sum_{n=6}^{\infty}\frac{n(n-1)\ldots(n-5)((n-6)!)^2(2x)^{2n}}{2^6(2n-1)(2n-3)\ldots(2n-11)(2n-12)!(2n+1)}=K. \tag{32}$$

On letting $n-6=m$, we obtain:

$$1+\frac{2x^2}{3}+\frac{8x^4}{15}+\frac{16x^6}{35}+\frac{128x^8}{315}+\frac{256x^{10}}{693} +\sum_{m=0}^{\infty}\frac{(m!)^2(m+1)^2(m+2)^2\ldots(m+6)^2(2x)^{2m+12}}{(2m+13)!}=K. \tag{33}$$

On multiplying by $\frac{1}{x^{12}}$, we obtain:

$$\frac{1}{x^{12}}+\frac{2}{3x^{10}}+\frac{8}{15x^8}+\frac{16}{35x^6}+\frac{128}{315x^4}+\frac{256}{693x^2} +\sum_{m=0}^{\infty}\frac{(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{(2m+2)(2m+4)\ldots(2m+12)}{(2m+1)(2m+3)\ldots(2m+13)}\right\}=\frac{K}{x^{12}}, \tag{34}$$

which is equivalent to:

$$\frac{1}{x^{12}}+\frac{2}{3x^{10}}+\frac{8}{15x^8}+\frac{16}{35x^6}+\frac{128}{315x^4}+\frac{256}{693x^2} +\sum_{m=0}^{\infty}\frac{(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{231}{4(2m+1)} +\frac{63}{26(2m+3)}+\frac{105}{4(2m+5)}+\frac{25}{(2m+7)}+\frac{105}{4(2m+9)} +\frac{63}{2(2m+11)}+\frac{231}{4(2m+13)}\right\}=\frac{K}{x^{12}}. \tag{35}$$

By employing Equations (8)–(12), and (18) and simplifying further, we obtain the result in Equation (30). □

Proof of Equation (31). On multiplying Equation (25) by $\frac{1}{x}$ and rewriting, we have

$$1-\frac{2x^2}{3}+\frac{8x^4}{15}-\frac{16x^6}{35}+\frac{128x^8}{315}-\frac{256x^{10}}{693} +\sum_{n=6}^{\infty}\frac{(-1)^n(n!)^2(2x)^{2n}}{(2n+1)!}=L. \tag{36}$$

On letting $n-6=m$, we obtain:

$$1-\frac{2x^2}{3}+\frac{8x^4}{15}-\frac{16x^6}{35}+\frac{128x^8}{315}-\frac{256x^{10}}{693} +\sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(m+1)^2(m+2)^2\ldots(m+6)^2(2x)^{2m+12}}{(2m+13)!}=L. \tag{37}$$

On multiplying by $\frac{1}{x^{12}}$, we obtain:

$$\frac{1}{x^{12}}-\frac{2}{3x^{10}}+\frac{8}{15x^8}-\frac{16}{35x^6}+\frac{128}{315x^4}-\frac{256}{693x^2}+\sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{(2m+2)(2m+4)\ldots(2m+12)}{(2m+1)(2m+3)\ldots(2m+13)}\right\}=\frac{L}{x^{12}}, \tag{38}$$

which is equivalent to:

$$\frac{1}{x^{12}}-\frac{2}{3x^{10}}+\frac{8}{15x^8}-\frac{16}{35x^6}+\frac{128}{315x^4}-\frac{256}{693x^2}+\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}(m!)^2}{(2m)!}\left\{\frac{231}{4(2m+1)}+\frac{63}{26(2m+3)}+\frac{105}{4(2m+5)}+\frac{25}{(2m+7)}+\frac{105}{4(2m+9)}+\frac{63}{2(2m+11)}+\frac{231}{4(2m+13)}\right\}=\frac{L}{x^{12}}. \tag{39}$$

On employing Equations (13)–(17), and (19) and simplifying, we obtain the desired result. □

Theorem 3. *We have the following reciprocal series of binomial coefficients with one odd factor in the denominator:*

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+15)}=\left(\frac{2048}{429x^{14}}-\frac{1024}{429x^{12}}-\frac{256}{429x^{10}}-\frac{128}{429x^8}-\frac{80}{429x^6}-\frac{56}{429x^4}-\frac{14}{143x^2}-\frac{1}{13}\right)K-\frac{2048}{429x^{14}}-\frac{1024}{1287x^{12}}-\frac{256}{715x^{10}}-\frac{640}{3003x^8}-\frac{560}{3861x^6}-\frac{168}{1573x^4}-\frac{14}{169x^2}, \tag{40}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+15)}=-\left(\frac{2048}{429x^{14}}+\frac{1024}{429x^{12}}-\frac{256}{429x^{10}}+\frac{128}{429x^8}-\frac{80}{429x^6}+\frac{56}{429x^4}-\frac{14}{143x^2}+\frac{1}{13}\right)L+\frac{2048}{429x^{14}}-\frac{1024}{1287x^{12}}+\frac{256}{715x^{10}}-\frac{640}{3003x^8}+\frac{560}{3861x^6}-\frac{168}{1573x^4}+\frac{14}{169x^2}, \tag{41}$$

where K and L are defined as in Equations (8) and (13), respectively.

Proof of Equation (40). On splitting the term Equation (8), up to the term containing x^{12}, we have:

$$1+\frac{2x^2}{3}+\frac{8x^4}{15}+\frac{16x^6}{35}+\frac{128x^8}{315}+\frac{256x^{10}}{693}+\frac{1024x^{12}}{3003}+\sum_{n=7}^{\infty}\frac{n(n-1)\ldots(n-6)((n-7)!)^2(2x)^{2n}}{2^7(2n-1)(2n-3)\ldots(2n-13)(2n-14)!(2n+1)}=K. \tag{42}$$

On letting $n-7=m$ in above, we get:

$$1+\frac{2x^2}{3}+\frac{8x^4}{15}+\frac{16x^6}{35}+\frac{128x^8}{315}+\frac{256x^{10}}{693}+\frac{1024x^{12}}{3003}+\sum_{m=0}^{\infty}\frac{(m!)^2(m+1)^2(m+2)^2\ldots(m+7)^2(2x)^{2m+14}}{(2m+15)!}=K. \tag{43}$$

On multiplying by $\frac{1}{x^{14}}$, we obtain:

$$\frac{1}{x^{14}}+\frac{2}{3x^{12}}+\frac{8}{15x^{10}8}+\frac{16}{35x^8}+\frac{128}{315x^6}+\frac{256}{693x^4}+\frac{1024}{3003x^2}+\sum_{m=0}^{\infty}\frac{(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{(2m+2)(2m+4)\ldots(2m+14)}{(2m+1)(2m+3)\ldots(2m+15)}\right\}=\frac{K}{x^{14}}, \tag{44}$$

which is equivalent to:

$$\frac{1}{x^{14}}+\frac{2}{3x^{12}}+\frac{8}{15x^{10}}+\frac{16}{35x^8}+\frac{128}{315x^6}+\frac{256}{693x^4}+\frac{1024}{3003x^2}+\frac{1}{2048}\sum_{m=0}^{\infty}\frac{(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{429}{(2m+1)}+\frac{231}{(2m+3)}+\frac{189}{(2m+5)}+\frac{175}{(2m+7)}+\frac{175}{(2m+9)}+\frac{189}{(2m+11)}+\frac{231}{(2m+13)}+\frac{429}{(2m+15)}\right\}=\frac{K}{x^{14}}. \tag{45}$$

By employing Equations (8)–(12), (18), and (30) and simplifying further, we obtain the result in Equation (40). □

Proof of Equation (41). On multiplying Equation (25) by $\frac{1}{x}$ and rewriting, we deduce:

$$1-\frac{2x^2}{3}+\frac{8x^4}{15}-\frac{16x^6}{35}+\frac{128x^8}{315}-\frac{256x^{10}}{693}+\frac{1024x^{12}}{3003}+\sum_{n=7}^{\infty}\frac{(-1)^n(n!)^2(2x)^{2n}}{(2n+1)!}=L. \tag{46}$$

On letting $n-7=m$, we obtain:

$$1-\frac{2x^2}{3}+\frac{8x^4}{15}-\frac{16x^6}{35}+\frac{128x^8}{315}-\frac{256x^{10}}{693}+\frac{1024x^{12}}{3003}-\sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(m+1)^2(m+2)^2\ldots(m+7)^2(2x)^{2m+14}}{(2m+15)!}=L. \tag{47}$$

On multiplying by $\frac{1}{x^{14}}$, we obtain:

$$\frac{1}{x^{14}}-\frac{2}{3x^{12}}+\frac{8}{15x^{10}8}-\frac{16}{35x^8}+\frac{128}{315x^6}-\frac{256}{693x^4}+\frac{1024}{3003x^2}-\sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{(2m+2)(2m+4)\ldots(2m+14)}{(2m+1)(2m+3)\ldots(2m+15)}\right\}=\frac{L}{x^{14}}, \tag{48}$$

which is equivalent to:

$$\frac{1}{x^{14}}-\frac{2}{3x^{12}}+\frac{8}{15x^{10}}-\frac{16}{35x^8}+\frac{128}{315x^6}-\frac{256}{693x^4}+\frac{1024}{3003x^2}-\frac{1}{2048}\sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2(2x)^{2m}}{(2m)!}\left\{\frac{429}{(2m+1)}+\frac{231}{(2m+3)}+\frac{189}{(2m+5)}+\frac{175}{(2m+7)}+\frac{175}{(2m+9)}+\frac{189}{(2m+11)}+\frac{231}{(2m+13)}+\frac{429}{(2m+15)}\right\}=\frac{L}{x^{14}}. \tag{49}$$

On employing Equations (13)–(17), (19), and (31) and simplifying, we obtain the desired result. □

Corollary 1. *The following results hold for the reciprocal series of positive and negative binomial coefficients with odd factors in the denominator:*

$$\sum_{m=0}^{\infty}\frac{1}{\binom{2m}{m}(2m+11)}=\frac{50450}{63}\sqrt{3}\pi-\frac{86470472}{19845}, \tag{50}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{\binom{2m}{m}(2m+11)}=-\frac{1172348\sqrt{5}\ln\phi}{315}+\frac{79473992}{19845}, \tag{51}$$

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2m}{m}(2m+13)}=\frac{2421586}{693}\sqrt{3}\pi-\frac{15219120592}{800415}, \tag{52}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{\binom{2m}{m}(2m+13)}=\frac{18757484\sqrt{5}\ln\phi}{1155}-\frac{13987105072}{800415}, \tag{53}$$

$$\sum_{m=0}^{\infty}\frac{1}{\binom{2m}{m}(2m+15)}=\frac{19372666}{1287}\sqrt{3}\pi-\frac{1582794944888}{19324305}, \tag{54}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{\binom{2m}{m}(2m+15)}=-\frac{150060004\sqrt{5}\ln\phi}{2145}+\frac{1454665330808}{19324305}. \tag{55}$$

Proof. The corollary is evident from the above Theorems 1–3 by letting $x=\frac{1}{2}$ and $\frac{1+\sqrt{5}}{2}=\phi$. □

4. Main Results: Closed Form Evaluation

Theorem 4. *These closed form evaluations for the generalized hypergeometric functions hold good:*

$${}_4F_3\left(\begin{matrix}1,1,\frac{1}{2},\frac{11}{4}\\ \frac{1}{4},\frac{3}{4},\frac{15}{4}\end{matrix};\frac{1}{16}\right)=\frac{277475}{63}\sqrt{3}\pi-\frac{1172348}{315}\sqrt{5}\ln\phi-\frac{132037868}{6615}, \tag{56}$$

$${}_4F_3\left(\begin{matrix}1,1,\frac{3}{2},\frac{13}{4}\\ \frac{3}{4},\frac{5}{4},\frac{17}{4}\end{matrix};\frac{1}{16}\right)=\frac{655850}{63}\sqrt{3}\pi+\frac{15240524}{315}\sqrt{5}\ln\phi-\frac{308182576}{2835}, \tag{57}$$

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix}1,1,\frac{1}{2},\frac{13}{4}\\ \frac{1}{4},\frac{3}{4},\frac{17}{4}\end{matrix};\frac{1}{16}\right) &= \frac{31480618}{693}\sqrt{3}\pi + \frac{243847292}{1155}\sqrt{5}\ln\phi - \frac{4930921216}{10395},\\
{}_4F_3\left(\begin{matrix}1,1,\frac{3}{2},\frac{15}{4}\\ \frac{3}{4},\frac{5}{4},\frac{19}{4}\end{matrix};\frac{1}{16}\right) &= \frac{12107930}{231}\sqrt{3}\pi - \frac{18757484}{77}\sqrt{5}\ln\phi - \frac{410671840}{17787},\\
{}_4F_3\left(\begin{matrix}1,1,\frac{1}{2},\frac{15}{4}\\ \frac{1}{4},\frac{3}{4},\frac{19}{4}\end{matrix};\frac{1}{16}\right) &= \frac{96863330}{429}\sqrt{3}\pi - - \frac{15006004}{143}\sqrt{5}\ln\phi \frac{3285374720}{33033},\\
{}_4F_3\left(\begin{matrix}1,1,\frac{3}{2},\frac{17}{4}\\ \frac{3}{4},\frac{5}{4},\frac{21}{4}\end{matrix};\frac{1}{16}\right) &= \frac{329335322}{1287}\sqrt{3}\pi + \frac{2551020068}{2145}\sqrt{5}\ln\phi - \frac{670608112816}{250965}.
\end{aligned}
\tag{58}
$$

Proof of Equation (56). On employing Legendre's duplication formula by Rainville [1] in the left hand side of Equations (50) and (51), on simplification yields

$$
\begin{aligned}
\sum_{m=0}^{\infty}\frac{1}{\binom{2m}{m}(2m+11)} &= {}_3F_2\left(\begin{matrix}1,1,\frac{11}{2}\\ \frac{1}{2},\frac{13}{2}\end{matrix};\frac{1}{4}\right)\\
&= 11\left(\frac{50450}{63}\sqrt{3}\pi - \frac{86470472}{19845}\right)
\end{aligned}
\tag{59}
$$

and

$$
\begin{aligned}
\sum_{m=0}^{\infty}\frac{(-1)^m}{\binom{2m}{m}(2m+11)} &= {}_3F_2\left(\begin{matrix}1,1,\frac{11}{2}\\ \frac{1}{2},\frac{13}{2}\end{matrix};-\frac{1}{4}\right)\\
&= 11\left(-\frac{1172348}{315}\sqrt{5}\ln\phi + \frac{79473992}{19845}\right),
\end{aligned}
\tag{60}
$$

respectively. From the study by Prudnikov et al. [19], we have:

$$
\begin{aligned}
&{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};z\right] + {}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};-z\right]\\
&= 2\,{}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2},\frac{a_1}{2}+\frac{1}{2},\ldots,\frac{a_{q+1}}{2},\frac{a_{q+1}}{2}+\frac{1}{2}\\ \frac{1}{2},\frac{b_1}{2},\frac{b_1}{2}+\frac{1}{2},\ldots,\frac{b_q}{2},\frac{b_q}{2}+\frac{1}{2}\end{matrix};z^2\right],
\end{aligned}
\tag{61}
$$

and

$$
\begin{aligned}
&{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};z\right] - {}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};-z\right]\\
&= \frac{2za_1a_2\ldots a_{q+1}}{b_1b_2\ldots b_q}\,{}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+1,\ldots,\frac{a_{q+1}}{2}+\frac{1}{2},\frac{a_{q+1}}{2}+1\\ \frac{3}{2},\frac{b_1}{2}+\frac{1}{2},\frac{b_1}{2}+1,\ldots,\frac{b_{q+1}}{2}+\frac{1}{2},\frac{b_q}{2}+1\end{matrix};z^2\right].
\end{aligned}
\tag{62}
$$

By letting $q=2$ and substituting $a_1=1$, $a_2=1$, $a_3=11/2$, $b_1=1/2$, $b_2=13/2$, and $z=1/4$ in Equations (61) and (62), respectively and making use of Equations (59) and (60) and after some simplification, we obtain the results Equations (56) and (57). Similarly, other results can be established by choosing the appropriate parameters and making use of other results mentioned above. We however omit the details. □

5. Several Series of Binomial Coefficients

In the present section, many more results containing reciprocal series of binomial coefficients with two odd terms in the denominator are listed, proofs of which are left to the interest of the reader.

Theorem 5. *The following results hold good:*

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+11)}=\left(-\frac{128}{315x^{10}}+\frac{64}{315x^{8}}+\frac{16}{315x^{6}}+\frac{8}{315x^{4}}+\frac{1}{63x^{2}}+\frac{1}{9}\right)K+\frac{128}{315x^{10}}+\frac{64}{945x^{8}}+\frac{16}{525x^{6}}+\frac{8}{441x^{4}}+\frac{1}{81x^{2}}, \quad (63)$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+11)}=\left(\frac{128}{315x^{10}}+\frac{64}{315x^{8}}-\frac{16}{315x^{6}}+\frac{8}{315x^{4}}-\frac{1}{63x^{2}}+\frac{1}{9}\right)L-\frac{128}{315x^{10}}+\frac{64}{945x^{8}}-\frac{16}{525x^{6}}+\frac{8}{441x^{4}}-\frac{1}{81x^{2}}, \quad (64)$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+3)(2m+11)}=\left(-\frac{32}{63x^{10}}+\frac{16}{63x^{8}}+\frac{4}{63x^{6}}+\frac{2}{63x^{4}}+\frac{17}{63x^{2}}-\frac{1}{9}\right)K-\frac{32}{63x^{10}}+\frac{16}{189x^{8}}+\frac{4}{105x^{6}}+\frac{10}{441x^{4}}-\frac{19}{81x^{2}}, \quad (65)$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+3)(2m+11)}=\left(\frac{32}{63x^{10}}+\frac{16}{63x^{8}}-\frac{4}{63x^{6}}+\frac{2}{63x^{4}}-\frac{17}{63x^{2}}-\frac{1}{9}\right)L-\frac{32}{63x^{10}}+\frac{16}{189x^{8}}-\frac{4}{105x^{6}}+\frac{10}{441x^{4}}-\frac{19}{81x^{2}}, \quad (66)$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+5)(2m+11)}=\left(-\frac{128}{189x^{10}}+\frac{64}{189x^{8}}+\frac{16}{189x^{6}}+\frac{92}{189x^{4}}-\frac{37}{189x^{2}}-\frac{1}{27}\right)K+\frac{128}{189x^{10}}+\frac{64}{567x^{8}}+\frac{16}{315x^{6}}-\frac{548}{1323x^{4}}-\frac{13}{243x^{2}}, \quad (67)$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+5)(2m+11)}=\left(\frac{128}{189x^{10}}+\frac{64}{189x^{8}}-\frac{16}{189x^{6}}+\frac{92}{189x^{4}}+\frac{37}{189x^{2}}-\frac{1}{27}\right)L-\frac{128}{189x^{10}}+\frac{64}{567x^{8}}-\frac{16}{315x^{6}}-\frac{548}{1323x^{4}}+\frac{13}{243x^{2}}, \quad (68)$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+7)(2m+11)}=\left(-\frac{64}{63x^{10}}+\frac{32}{63x^{8}}+\frac{292}{315x^{6}}-\frac{106}{315x^{4}}-\frac{19}{315x^{2}}-\frac{1}{45}\right)K+\frac{64}{63x^{10}}+\frac{32}{189x^{8}}-\frac{76}{105x^{6}}-\frac{194}{2205x^{4}}-\frac{59}{2025x^{2}}, \quad (69)$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+7)(2m+11)}=\left(\frac{64}{63x^{10}}+\frac{32}{63x^{8}}-\frac{292}{315x^{6}}-\frac{106}{315x^{4}}+\frac{19}{315x^{2}}-\frac{1}{45}\right)L-\frac{64}{63x^{10}}+\frac{32}{189x^{8}}+\frac{76}{105x^{6}}-\frac{194}{2205x^{4}}+\frac{59}{2025x^{2}}, \tag{70}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+9)(2m+11)}=\left(-\frac{128}{63x^{10}}+\frac{128}{45x^{8}}-\frac{208}{315x^{6}}-\frac{32}{315x^{4}}-\frac{11}{315x^{2}}-\frac{1}{63}\right)K+\frac{128}{63x^{10}}-\frac{1408}{945x^{8}}-\frac{16}{105x^{6}}-\frac{512}{11025x^{4}}-\frac{79}{3969x^{2}}, \tag{71}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+9)(2m+11)}=\left(\frac{128}{63x^{10}}+\frac{128}{45x^{8}}+\frac{208}{315x^{6}}-\frac{32}{315x^{4}}+\frac{11}{315x^{2}}-\frac{1}{63}\right)L-\frac{128}{63x^{10}}-\frac{1408}{945x^{8}}+\frac{16}{105x^{6}}-\frac{512}{11025x^{4}}+\frac{79}{3969x^{2}}, \tag{72}$$

where K and L are as defined as in Equations (8) and (13), respectively.

Proof of Equation (63). Consider

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+11)}. \tag{73}$$

By the method of partial fraction, one can easily deduce:

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+11)}=\frac{1}{10}\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}}\left[\frac{1}{(2m+1)}-\frac{1}{(2m+11)}\right]. \tag{74}$$

On employing Equations (8) and (18) and simplifying further, we obtain the result in Equation (63). Similarly, the remaining results can be deduced using appropriate results listed in the previous sections. □

Theorem 6. *The following results hold good:*

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+13)}=\left(-\frac{256}{693x^{12}}+\frac{128}{693x^{10}}+\frac{32}{693x^{8}}+\frac{16}{693x^{6}}+\frac{10}{693x^{4}}+\frac{1}{99x^{2}}+\frac{1}{11}\right)K+\frac{256}{693x^{12}}+\frac{128}{2079x^{10}}+\frac{32}{1155x^{8}}+\frac{80}{4851x^{6}}+\frac{10}{891x^{4}}+\frac{1}{121x^{2}}, \tag{75}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+13)}=-\left(\frac{256}{693x^{12}}+\frac{128}{693x^{10}}-\frac{32}{693x^{8}}+\frac{16}{693x^{6}}-\frac{10}{693x^{4}}+\frac{1}{99x^{2}}-\frac{1}{11}L\right)+\frac{256}{693x^{12}}-\frac{128}{2079x^{10}}+\frac{32}{1155x^{8}}-\frac{80}{4851x^{6}}+\frac{10}{891x^{4}}-\frac{1}{121x^{2}}, \tag{76}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+3)(2m+13)}=\left(-\frac{512}{1155x^{12}}+\frac{256}{1155x^{10}}+\frac{64}{1155x^{8}}+\frac{32}{1155x^{6}}+\frac{4}{231x^{4}}\right.$$
$$\left.+\frac{7}{33x^{2}}-\frac{1}{11}\right)K+\frac{512}{1155x^{12}}+\frac{256}{3465x^{10}}+\frac{64}{1925x^{8}}+\frac{32}{1617x^{6}}+\frac{4}{297x^{4}}-\frac{23}{121x^{2}}, \tag{77}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^{m}(2x)^{2m}}{\binom{2m}{m}(2m+3)(2m+13)}=-\left(\frac{512}{1155x^{12}}+\frac{256}{1155x^{10}}-\frac{64}{1155x^{8}}+\frac{32}{1155x^{6}}-\frac{4}{231x^{4}}\right.$$
$$\left.+\frac{7}{33x^{2}}+\frac{1}{11}\right)L+\frac{512}{1155x^{12}}-\frac{256}{3465x^{10}}+\frac{64}{1925x^{8}}-\frac{32}{1617x^{6}}+\frac{4}{297x^{4}}+\frac{23}{121x^{2}}, \tag{78}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+5)(2m+13)}=\left(-\frac{128}{231x^{12}}+\frac{64}{231x^{10}}+\frac{16}{231x^{8}}+\frac{8}{231x^{6}}+\frac{82}{231x^{4}}\right.$$
$$\left.-\frac{5}{33x^{2}}-\frac{1}{33}\right)K+\frac{128}{231x^{12}}+\frac{64}{693x^{10}}+\frac{16}{385x^{8}}+\frac{40}{1617x^{6}}-\frac{94}{297x^{4}}-\frac{47}{1089x^{2}}, \tag{79}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^{m}(2x)^{2m}}{\binom{2m}{m}(2m+5)(2m+13)}=-\left(\frac{128}{231x^{12}}+\frac{64}{231x^{10}}-\frac{16}{231x^{8}}+\frac{8}{231x^{6}}-\frac{82}{231x^{4}}\right.$$
$$\left.-\frac{5}{33x^{2}}+\frac{1}{33}\right)L+\frac{128}{231x^{12}}-\frac{64}{693x^{10}}+\frac{16}{385x^{8}}-\frac{40}{1617x^{6}}-\frac{94}{297x^{4}}+\frac{47}{1089x^{2}}, \tag{80}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+7)(2m+13)}=(-\frac{512}{693x^{12}}+\frac{256}{693x^{10}}+\frac{64}{693x^{8}}+\frac{2008}{3465x^{6}}-\frac{824}{3465x^{4}}$$
$$-\frac{23}{495x^{2}}-\frac{1}{55})K+\frac{512}{693x^{12}}+\frac{256}{2079x^{10}}+\frac{64}{1155x^{8}}-\frac{12136}{24255x^{8}}-\frac{296}{4455x^{4}}-\frac{71}{3025x^{2}}, \tag{81}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^{m}(2x)^{2m}}{\binom{2m}{m}(2m+7)(2m+13)}=-\left(\frac{512}{693x^{12}}+\frac{256}{693x^{10}}-\frac{64}{693x^{8}}+\frac{2008}{3465x^{6}}+\frac{824}{3465x^{4}}\right.$$
$$\left.-\frac{23}{495x^{2}}+\frac{1}{55}\right)L+\frac{512}{693x^{12}}-\frac{256}{2079x^{10}}+\frac{64}{1155x^{8}}+\frac{12136}{24255x^{6}}-\frac{296}{4455x^{4}}+\frac{71}{3025x^{2}}, \tag{82}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+9)(2m+13)}=\left(-\frac{256}{231x^{12}}+\frac{128}{231x^{10}}+\frac{1216}{1155x^{8}}-\frac{64}{165x^{6}}-\frac{82}{1155x^{4}}\right.$$
$$\left.-\frac{31}{1155x^{2}}-\frac{1}{77}\right)K+\frac{256}{231x^{12}}+\frac{128}{693x^{10}}-\frac{64}{77x^{8}}-\frac{832}{8085x^{6}}-\frac{1814}{51975x^{4}}-\frac{95}{5929x^{2}}, \tag{83}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+9)(2m+13)}=-\left(\frac{256}{231x^{12}}+\frac{128}{231x^{10}}-\frac{1216}{1155x^8}-\frac{64}{165x^6}+\frac{82}{1155x^4}-\frac{31}{1155x^2}+\frac{1}{77}\right)L+\frac{256}{231x^{12}}-\frac{128}{693x^{10}}-\frac{64}{77x^8}+\frac{832}{8085x^6}-\frac{1814}{51975x^4}+\frac{95}{5929x^2}, \quad (84)$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+11)(2m+13)}=\left(-\frac{512}{231x^{12}}+\frac{2176}{693x^{10}}-\frac{512}{693x^8}-\frac{80}{693x^6}-\frac{4}{99x^4}-\frac{13}{693x^2}-\frac{1}{99}\right)K+\frac{512}{231x^{12}}-\frac{128}{77x^{10}}-\frac{256}{1485x^8}-\frac{144}{2695x^6}-\frac{340}{14553x^4}-\frac{119}{9801x^2}, \quad (85)$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+11)(2m+13)}=-\left(\frac{512}{231x^{12}}+\frac{2176}{693x^{10}}+\frac{512}{693x^8}-\frac{80}{693x^6}+\frac{4}{99x^4}-\frac{13}{693x^2}+\frac{1}{99}\right)L+\frac{512}{231x^{12}}+\frac{128}{77x^{10}}-\frac{256}{1485x^8}+\frac{144}{2695x^6}-\frac{340}{14553x^4}+\frac{119}{9801x^2}, \quad (86)$$

where K and L are as defined as in Equations (8) and (13), respectively.

Proof. The proof of the above theorem is similar to that of Theorem 5. □

Theorem 7. *The following results hold good:*

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+15)}=\left(-\frac{1024}{3003x^{14}}+\frac{512}{3003x^{12}}+\frac{128}{3003x^{10}}+\frac{64}{3003x^8}+\frac{40}{3003x^6}+\frac{4}{429x^4}+\frac{1}{143x^2}+\frac{1}{13}\right)K+\frac{1024}{3003x^{14}}+\frac{512}{9009x^{12}}+\frac{128}{5005x^{10}}+\frac{320}{21021x^8}+\frac{40}{3861x^6}+\frac{12}{1573x^4}+\frac{1}{169x^2}, \quad (87)$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+1)(2m+15)}=\left(\frac{1024}{3003x^{14}}+\frac{512}{3003x^{12}}-\frac{128}{3003x^{10}}+\frac{64}{3003x^8}-\frac{40}{3003x^6}+\frac{4}{429x^4}-\frac{1}{143x^2}+\frac{1}{13}\right)L-\frac{1024}{3003x^{14}}+\frac{512}{9009x^{12}}-\frac{128}{5005x^{10}}+\frac{320}{21021x^8}-\frac{40}{3861x^6}+\frac{12}{1573x^4}-\frac{1}{169x^2}, \quad (88)$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+3)(2m+15)}=\left(-\frac{512}{1287x^{14}}+\frac{256}{1287x^{12}}+\frac{64}{1287x^{10}}+\frac{32}{1287x^8}+\frac{20}{1287x^6}+\frac{14}{1287x^4}+\frac{25}{143x^2}-\frac{1}{13}\right)K+\frac{512}{1287x^{14}}+\frac{256}{3861x^{12}}+\frac{64}{2145x^{10}}+\frac{160}{9009x^8}+\frac{140}{11583x^6}+\frac{14}{1573x^4}-\frac{27}{169x^2}, \quad (89)$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+3)(2m+15)}=\left(\frac{512}{1287x^{14}}+\frac{256}{1287x^{12}}-\frac{64}{1287x^{10}}+\frac{32}{1287x^{8}}-\frac{20}{1287x^{6}}+\frac{14}{1287x^{4}}-\frac{25}{143x^{2}}-\frac{1}{13}\right)L-\frac{512}{1287x^{14}}+\frac{256}{3861x^{12}}-\frac{64}{2145x^{10}}+\frac{160}{9009x^{8}}-\frac{140}{11583x^{6}}+\frac{14}{1573x^{4}}+\frac{27}{169x^{2}}, \tag{90}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+5)(2m+15)}=\left(-\frac{1024}{2145x^{14}}+\frac{512}{2145x^{12}}+\frac{128}{2145x^{10}}+\frac{64}{2145x^{8}}+\frac{8}{429x^{6}}+\frac{40}{143x^{4}}-\frac{53}{429x^{2}}-\frac{1}{39}\right)K+\frac{1024}{2145x^{14}}+\frac{512}{6435x^{12}}+\frac{128}{3575x^{10}}+\frac{64}{3003x^{8}}+\frac{56}{3861x^{6}}-\frac{1208}{4719x^{4}}-\frac{55}{1521x^{2}}, \tag{91}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+5)(2m+15)}=\left(\frac{1024}{2145x^{14}}+\frac{512}{2145x^{12}}-\frac{128}{2145x^{10}}+\frac{64}{2145x^{8}}-\frac{8}{429x^{6}}+\frac{40}{143x^{4}}+\frac{53}{429x^{2}}-\frac{1}{39}\right)L-\frac{1024}{2145x^{14}}+\frac{512}{6435x^{12}}-\frac{128}{3575x^{10}}+\frac{64}{3003x^{8}}-\frac{56}{3861x^{6}}-\frac{1208}{4719x^{4}}+\frac{55}{1521x^{2}}, \tag{92}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+7)(2m+15)}=\left(-\frac{256}{429x^{14}}+\frac{128}{429x^{12}}+\frac{32}{429x^{10}}+\frac{16}{429x^{8}}+\frac{908}{2145x^{6}}-\frac{394}{2145x^{4}}-\frac{27}{715x^{2}}-\frac{1}{65}\right)K+\frac{256}{429x^{14}}+\frac{128}{1287x^{12}}+\frac{32}{715x^{10}}+\frac{80}{3003x^{8}}-\frac{7372}{19305x^{6}}-\frac{1258}{23595x^{4}}-\frac{83}{4225x^{2}}, \tag{93}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+7)(2m+15)}=\left(\frac{256}{429x^{14}}+\frac{128}{429x^{12}}-\frac{32}{429x^{10}}+\frac{16}{429x^{8}}-\frac{908}{2145x^{6}}-\frac{394}{2145x^{4}}+\frac{27}{715x^{2}}-\frac{1}{65}\right)L-\frac{256}{429x^{14}}+\frac{128}{1287x^{12}}-\frac{32}{715x^{10}}+\frac{80}{3003x^{8}}+\frac{7372}{19305x^{6}}-\frac{1258}{23595x^{4}}+\frac{83}{4225x^{2}}, \tag{94}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+9)(2m+15)}=\left(-\frac{1024}{1287x^{14}}+\frac{512}{1287x^{12}}+\frac{128}{1287x^{10}}+\frac{29696}{45045x^{8}}-\frac{12328}{45045x^{6}}-\frac{2452}{45045x^{4}}-\frac{109}{5005x^{2}}-\frac{1}{91}\right)K+\frac{1024}{1287x^{14}}+\frac{512}{3861x^{12}}+\frac{128}{2145x^{10}}-\frac{25856}{45045x^{8}}-\frac{31384}{405405x^{6}}-\frac{7684}{275275x^{4}}-\frac{111}{8281x^{2}}, \tag{95}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+9)(2m+15)}=\left(\frac{1024}{1287x^{14}}+\frac{512}{1287x^{12}}-\frac{128}{1287x^{10}}+\frac{29696}{45045x^{8}}+\frac{12328}{45045x^{6}}-\frac{2452}{45045x^{4}}+\frac{109}{5005x^{2}}-\frac{1}{91}\right)L-\frac{1024}{1287x^{14}}+\frac{512}{3861x^{12}}-\frac{128}{2145x^{10}}-\frac{25856}{45045x^{8}}+\frac{31384}{405405x^{6}}-\frac{7684}{275275x^{4}}+\frac{111}{8281x^{2}}, \tag{96}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+11)(2m+15)}=\left(-\frac{512}{429x^{14}}+\frac{256}{429x^{12}}+\frac{10496}{9009x^{10}}-\frac{3904}{9009x^{8}}-\frac{724}{9009x^{6}}-\frac{278}{9009x^{4}}-\frac{137}{9009x^{2}}-\frac{1}{117}\right)K+\frac{512}{429x^{14}}+\frac{256}{1287x^{12}}-\frac{41728}{45045x^{10}}-\frac{448}{3861x^{8}}-\frac{5396}{135135x^{6}}-\frac{12938}{693693x^{4}}-\frac{139}{13689x^{2}}, \tag{97}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}}{\binom{2m}{m}(2m+11)(2m+15)}=\left(\frac{512}{429x^{14}}+\frac{256}{429x^{12}}-\frac{10496}{9009x^{10}}-\frac{3904}{9009x^{8}}+\frac{724}{9009x^{6}}-\frac{278}{9009x^{4}}+\frac{137}{9009x^{2}}-\frac{1}{117}\right)L-\frac{512}{429x^{14}}\frac{256}{1287x^{12}}+\frac{41728}{45045x^{10}}-\frac{448}{3861x^{8}}+\frac{5396}{135135x^{6}}-\frac{12938}{693693x^{4}}+\frac{139}{13689x^{2}}, \tag{98}$$

$$\sum_{m=0}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}(2m+13)(2m+15)}=\left(-\frac{1024}{429x^{14}}+\frac{10240}{3003x^{12}}-\frac{2432}{3003x^{10}}-\frac{128}{1001x^{8}}-\frac{136}{3003x^{6}}-\frac{64}{3003x^{4}}-\frac{5}{429x^{2}}-\frac{1}{143}\right)K+\frac{1024}{429x^{14}}-\frac{16384}{9009x^{12}}-\frac{8576}{45045x^{10}}-\frac{128}{2145x^{8}}-\frac{5000}{189189x^{6}}-\frac{592}{42471x^{4}}-\frac{167}{20449x^{2}}, \tag{99}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2x)^{2m}}{\binom{2m}{m}(2m+13)(2m+15)} = \left(\frac{1024}{429x^{14}} + \frac{10240}{3003x^{12}} + \frac{2432}{3003x^{10}} - \frac{128}{1001x^8} + \frac{136}{3003x^6} - \frac{64}{3003x^4} + \frac{5}{429x^2} - \frac{1}{143}\right) L - \frac{1024}{429x^{14}} - \frac{16384}{9009x^{12}} + \frac{8576}{45045x^{10}} - \frac{128}{2145x^8} + \frac{5000}{189189x^6} - \frac{592}{42471x^4} + \frac{167}{20449x^2}, \tag{100}$$

where K and L are as defined as in Equations (8) and (13), respectively.

Proof. The proof of the above theorem is similar to that of Theorem 5. □

The reciprocals of binomial coefficient are focused in the present work and the study on harmonic numbers is left to the interest of the reader.

6. Conclusion

By employing a known series, we have constructed many new infinite series of reciprocals of the binomial coefficients. Further, several new closed form evaluations of ${}_{s+1}F_s(z)$ for $s = 3$ with argument $\frac{1}{16}$ are obtained. Also, several new series of the similar type are obtained by splitting the terms and the evaluation of these is left to the interest of the reader. Furthermore, we hope that these evaluations would be useful in the areas of mathematical physics and statistics.

Data Availability

All the data used to support the findings of this study are included within the article.

Conflicts of Interest

Authors declare that there is no conflict of interest regarding the publication of this article.

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