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Research Article

Stability of a Fractional Opinion Formation Model with and without Leadership Using the New Generalized Hattaf Fractional Derivative

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In this paper, we propose and analyze the dynamical behaviors of two opinion formation models, one with leadership and the other without leadership. The two proposed models are formulated by fractional differential equations (FDEs) with the frame of the new generalized Hattaf fractional (GHF) derivative. The stability in the sense of Mittag–Leffler is rigorously established for both models. The convergence of agents' opinions to the consensus opinion is fully investigated. Numerical simulations are given to illustrate the analytical results.

1. Introduction

Opinion formation is a complex process that involves the development or change of beliefs, opinions, or viewpoints on a specific subject. It is influenced by various factors such as education, personal experience, culture, media, social interactions, and external influences. In the field of education, formal education acquired in schools and universities can have a significant impact on people's opinions on various issues. The media also plays a major role in opinion formation, as the information and messages disseminated by the media can influence an individual's perception of a particular issue. Social interactions, including those with family, friends, work colleagues, and the community, can also influence opinion formation. Additionally, the way opinions are formed can be influenced by the culture in which an individual resides. Furthermore, external influences such as opinion leaders, celebrities, interest groups, political groups, educational campaigns, and advertising can also play a role in shaping opinions [1–7].

Opinion formation models have been established in the discipline of mathematical modeling to describe the dynamics of how opinions arise and change over time within a population, taking into account factors, such as personal experiences, social interactions, and external sources of information. Such models are highly relevant in various fields such as politics [8], consumerism [9], and social media [10–12], as they provide organizations with valuable insights into public opinion and enable them to adapt their strategies accordingly. For instance, Degroot [9] presented the initial agent-based opinion formation model to discuss the importance of consensus in group decision-making processes, he suggested different approaches to achieving it and discussed methods of voting, compromise, negotiation, and the use of mathematical and statistical models. The author also pointed out that the appropriate method depends on the specific context of the group decision-making situation. Chen et al. [13] studied a fractional-order memristor neural network model that describes the effects of memristor memory and nonlinearity. They proved that the system is Mittag–Leffler stable. Furthermore, they proposed synchronization conditions for neural memory networks. Almeida et al. [14] used the theory of fractional differential equations (FDEs) to describe the opinion formation model and the proposed leader–follower control method. They showed that the proposed method can minimize the gap between the opinions of group members and the leader while limiting the influence of the leader on the opinions of other members. In [15], the authors proposed a leader–follower control approach to a fractional opinion formation model, in which the leader is responsible for directing the opinion formation of group members. The goal of the proposed approach was to minimize the discrepancy between the opinions of group members and the leader while limiting the influence of the leader on the opinions of other members. Somjaiwang and Ngiamsunthorn [16] investigated the exponential stability of a leader–follower opinion formation model based on a nonlinear system of FDEs.

It is important to note that the FDEs used in [13–16] have been formulated by the classical Caputo [17] fractional derivative. To avoid the singularity of such derivative, Caputo and Fabrizio [18] replaced the singular kernel with an exponential kernel. The version of Caputo–Fabrizio (FC) derivative was extended by Atangana and Baleanu [19] to Mittag–Leffler kernel. In 2020, Hattaf [20] developed a new generalized Hattaf fractional (GHF) derivative that includes the CF derivative [18] and Atangana–Baleanu (AB) derivative [19]. This study aims to extend the exponential stability defined in [21] and used in [16], as well as the Mittag–Leffler stability introduced in [13] to FDEs with GHF derivative.

The second objective of this work is to propose two mathematical models incorporating the new GHF derivative for opinion formation with and without leadership. In order to accomplish this, the fundamental ideas and the development of the two fractional models are covered in Section 2. The Mittag–Leffler stability of the fractional model in the absence of leadership is investigated in Section 3. The stability and consensus of the second fractional model with leadership are established in Section 4. Three examples are provided in Section 5 to illustrate the analytical results. Finally, we end our current paper with a conclusion and some future works.

2. Basic Concepts and Models Formulation

In this section, we introduce the essential concepts and formulate models both with and without leadership.

Definition 1 (Hattaf [20]). The GHF derivative of order η in the Caputo sense of the function $u(t)$ with respect to the weight function $w(t)$ is given by the following equation:

$$^{C}D_{a,t,w}^{\eta,\theta,\sigma}u(t)=\frac{\mathrm{GHF}(\eta)}{1-\eta}\frac{1}{w(t)}\int_a^t E_\theta\left[-\mu_\eta(t-s)^\sigma\right]\frac{d}{ds}(wu)(s)ds, \tag{1}$$

where $\eta\in[0,1)$, θ, and σ are both positive, while u belongs to the space $H^1(a,b)$, $w\in C^1(a,b)$ with $w>0$ on [a, b], $\mathrm{GHF}(\eta)$ is a normalization function such that $\mathrm{GHF}(0)=\mathrm{GHF}(1)=1$, $\mu_\eta=\frac{\eta}{1-\eta}$ and $E_\theta(t)=\sum_{k=0}^{+\infty}\frac{t^k}{\Gamma(\theta k+1)}$ is the Mittag–Leffler function of parameter θ.

Here, $H^1(a,b)=\{f\in L^2(a,b):f'\in L^2(a,b)\}$ be the Sobolev space of order one.

Denote $^{C}D_{a,t,w}^{\eta,\theta,\theta}$ by $\mathscr{D}_{a,w}^{\eta,\theta}$. It follows from Hattaf's [20] study that the GHF integral associated with $\mathscr{D}_{a,w}^{\eta,\theta}$ is given by the following definition:

Definition 2 (Hattaf [20]). The GHF integral associated with $\mathscr{D}_{a,w}^{\eta,\theta}$ is given by the following equation:

$$I_{a,w}^{\eta,\theta}u(t)=\frac{1-\eta}{\mathrm{GHF}(\eta)}u(t)+\frac{\eta}{\mathrm{GHF}(\eta)}\,{}^{\mathrm{RL}}I_{a,w}^{\theta}u(t), \tag{2}$$

where ${}^{\mathrm{RL}}I_{a,w}^{\theta}$ is the weighted Riemann–Liouville fractional integral of order θ defined by the following formula:

$${}^{\mathrm{RL}}I_{a,w}^{\theta}u(t)=\frac{1}{\Gamma(\theta)}\frac{1}{w(t)}\int_a^t(t-s)^{\theta-1}w(s)u(s)\mathrm{d}s. \tag{3}$$

Now, we consider the following FDE:

$$\mathscr{D}_{0,w}^{\eta,\theta}O(t)=g(t,O(t)), \tag{4}$$

where $O(t)\in\mathbb{R}^n$ and $g:[0,+\infty)\times\Omega\rightarrow\mathbb{R}^n$ is a continuous locally Lipschitz function.

Definition 3. System (4) is Mittag–Leffler stable if:

$$\|O_1(t)-O_2(t)\|\leq[m(O_1(t_0)-O_2(t_0))E_\theta(-\lambda(t-t_0)^\sigma)]^\nu, \tag{5}$$

for any solutions $O_1(t)$ and $O_2(t)$ of System (4), where t_0 is the initial time, $\lambda\geq 0$, $\nu>0$, $m(0)=0$, $m(O)\geq 0$, and $m(O)$ is locally Lipschitz on $O\in\mathbb{R}^n$.

Definition 3 extends the Mittag–Leffler stability introduced in [13] when $\sigma=\theta$, $\nu=1$, and $m(O)=M\|O\|$ with M being a positive constant. Additionally, the exponential stability employed in [21] is a particular case of Equation (5), it suffices to take $m(O)=M\|O\|$ and $\theta=\nu=1$.

Lemma 1 (Hattaf [22]). *Let $O_1(t)$ and $O_2(t)$ be two functions defined on $[t_0,+\infty)$ with $\mathscr{D}_{t_0,w}^{\eta,\theta}O_1(t)\geq\mathscr{D}_{t_0,w}^{\eta,\theta}O_2(t)$ and $O_1(t_0)\geq O_2(t_0)$. Then, $O_1(t)\geq O_2(t)$, for all $t\geq t_0$.*

Lemma 2 (Hattaf [23]). *Let $\lambda>0$ and $O(t)$ be a function satisfying the following inequality:*

$$\mathscr{D}_{0,w}^{\eta,\theta}O(t)\leq-\lambda O(t). \tag{6}$$

Then:

$$O(t) \leq O(0) E_\theta \left(\frac{-\eta\lambda t^\theta}{\mathrm{GHF}(\eta) + \lambda(1-\eta)} \right). \quad (7)$$

Next, we propose two fractional opinion formation models. The first one without leadership, which is formulated by the following system:

$$\mathscr{D}_{0,w}^{\eta,\theta} O_i(t) = \sum_{1\leq j\leq N} a_{ij}\left(f_j\left(O_j(t)\right)\right),\ i = 1, 2, \ldots, N, \quad (8)$$

with the given initial conditions $O_i(0) = \xi_i \in \mathbb{R}$, for $i = 1, 2, \ldots, N$. The state variable $O_i(t)$ represents the opinion of the ith follower at time t. The matrix $A = [a_{ij}]_{N\times N}$ denotes the adjacency matrix of the social network connecting the followers, and $f_j(\cdot)$ is the opinion function of the jth follower.

The second opinion formation fractional model with leadership is given by the following equation:

$$\begin{cases} \mathscr{D}_{0,w}^{\eta,\theta} O_0(t) &= 0, \\ \mathscr{D}_{0,w}^{\eta,\theta} O_i(t) &= \sum\limits_{1\leq j\leq N} a_{ij}\left(f_j\left(t, O_j(t)\right)\right) + c_i(O_0(t) - O_i(t)) + I_i(t), \quad i = 1, 2, \ldots, N, \end{cases} \quad (9)$$

where $I_i(t)$ represents external inputs, c_i is a constant that indicates the leader's influence on the ith follower and $O_0(t)$ is the leader's opinion at time t. As in Equation (8), we consider System (9) with initial conditions $O_i(0) = \xi_i \in \mathbb{R}$, for $i = 0, 1, \ldots, N$.

3. Stability of the Model without Leadership

This section establishes the Mittag–Leffler stability of System (8) without leadership.

Theorem 1. *Assume that there exists a symmetric positive definite matrix $P \in IR^{N\times N}$ such that $Q = -(A^TP + PA)$ is a positive definite matrix and let $f_i(O_i) = O_i$ for every $i = 1, 2, \ldots, N$. Then, System (8) is Mittag–Leffler stable.*

Proof. Let $O(t) = (O_1(t), O_2(t), \ldots, O_N(t))^T$ and $\bar{O}(t) = (\bar{O}_1(t), \bar{O}_2(t), \ldots, \bar{O}_N(t))^T$ be the solutions of System (8) subject to the different initial conditions $O_i(0) = \xi_i$ and $\bar{O}_i(0) = \zeta_i$, for $i = 1, 2, \ldots, N$.

Consider the following Lyapunov function:

$$L(t) = X(t)^T P X(t), \quad (10)$$

where $X(t) = O(t) - \bar{O}(t)$. So, we have the following equation:

$$\lambda_1 \|X(t)\|^2 \leq L(t) \leq \lambda_2 \|X(t)\|^2, \quad (11)$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are the minimum and the maximum eigenvalues of the matrix P, respectively. It follows from Lemma 1 of [24] that:

$$\begin{aligned} \mathscr{D}_{0,w}^{\eta,\theta} L(t) &\leq 2X(t)^T P \mathscr{D}_{0,w}^{\eta,\theta} X(t) \\ &= -X(t)^T Q X(t). \end{aligned} \quad (12)$$

Let us denote by δ the minimum eigenvalue of the matrix Q. Since Q is a positive definite matrix, we have $\delta > 0$. We therefore have using Equation (11):

$$\mathscr{D}_{0,w}^{\eta,\theta} L(t) \leq -\delta \|X(t)\|^2 \leq -\frac{\delta}{\lambda_2} L(t). \quad (13)$$

By applying Lemma 2, we deduce the following equation:

$$L(t) \leq L(0) E_\theta \left(\frac{-\eta\delta t^\theta}{\lambda_2 \mathrm{GHF}(\eta) + \delta(1-\eta)} \right). \quad (14)$$

By Equation (11), we have the following equation:

$$\|X(t)\|^2 \leq \frac{1}{\lambda_1} L(0) E_\theta \left(\frac{-\eta\delta t^\theta}{\lambda_2 \mathrm{GHF}(\eta) + \delta(1-\eta)} \right). \quad (15)$$

Therefore, it follows from Definition 3 that System (8) is Mittag–Leffler stable. This completes the proof. □

4. Stability and Consensus of the Model with Leadership

In this section, we discuss the stability and consensus of the second opinion formation fractional model with leadership presented by System (9).

In the following, we consider the following assumptions:

(A1) The functions f_i, $i = 1, 2, \ldots, N$, are continuous and satisfy the following Lipschitz condition on $\mathbb{R}$:

$$|f_i(t, O_1) - f_i(t, O_2)| \leq L_i |O_1 - O_2|, \quad (16)$$

where $L_i > 0$ are the Lipschitz constants.

(A2) The constants c_i are positive for $i = 1, 2, \ldots, N$ and satisfy the following equation:

$$\Lambda_i = c_i - \sum_{1\le j\le N} |a_{ji}| L_i > 0. \tag{17}$$

(A3) The functions I_i are bounded, i.e., there exists a $M_i > 0$ such that:

$$|I_i(t)| \le M_i, \tag{18}$$

for every $i = 1, 2, \ldots, N$.

Theorem 2. *Assume that (A1), (A2), and (A3) are satisfied. Then, System (9) is Mittag–Leffler stable.*

Proof. Let $O(t) = (O_1(t), O_2(t), \ldots, O_N(t))^T$ and $\bar{O}(t) = (\bar{O}_1(t), \bar{O}_2(t), \ldots, \bar{O}_N(t))^T$ be the solutions of System (9) subject to the different initial conditions $O_i(0) = \xi_i$ and $\bar{O}_i(0) = \zeta_i$, $i = 1, 2, \ldots, N$. Let $u_i^*(t) = \bar{O}_i(t) - O_i(t)$, for $i = 1, 2, \ldots, N$. Then, $u_i^*(0) \ne 0$. From Equation (9), we have the following equation:

$$\mathscr{D}_{0,w}^{\eta,\theta} u_i^*(t) = -c_i u_i^*(t) + \sum_{1\le j\le N} a_{ij}\left(f_j\left(t, \bar{O}_j(t)\right) - f_j\left(t, O_j(t)\right)\right). \tag{19}$$

If $u_i^*(t)$ is positive, then:

$$\begin{aligned}\mathscr{D}_{0,w}^{\eta,\theta} |u_i^*(t)| &= \frac{\mathrm{GHF}(\eta)}{1-\eta}\frac{1}{w(t)}\int_0^t E_\theta\left[-\mu_\eta (t-s)^\theta\right](wu)'(s)ds \\ &= \mathscr{D}_{0,w}^{\eta,\theta} u_i^*(t).\end{aligned} \tag{20}$$

If $u_i^*(t)$ is negative, then:

$$\begin{aligned}\mathscr{D}_{0,w}^{\eta,\theta} |u_i^*(t)| &= -\frac{\mathrm{GHF}(\eta)}{1-\eta}\frac{1}{w(t)}\int_0^t E_\theta\left[-\mu_\eta (t-s)^\theta\right](wu)'(s)ds \\ &= -\mathscr{D}_{0,w}^{\eta,\theta} u_i^*(t).\end{aligned} \tag{21}$$

Hence:

$$\mathscr{D}_{0,w}^{\eta,\theta} |u_i^*(t)| = \mathrm{sign}(u_i^*(t)) \mathscr{D}_{0,w}^{\eta,\theta} u_i^*(t). \tag{22}$$

Consider the following function:

$$U(t) = \sum_{1\le i\le N} |u_i^*(t)|. \tag{23}$$

Then:

$$\begin{aligned}\mathscr{D}_{0,w}^{\eta,\theta} U(t) &= \sum_{1\le i\le N} \mathscr{D}_{0,w}^{\eta,\theta} |u_i^*(t)| \\ &= \sum_{1\le i\le N} \mathrm{sign}(u_i^*(t))\left(-c_i u_i^*(t) + \sum_{1\le j\le N} a_{ij}\left(f_j\left(t, \bar{O}_j(t)\right) - f_j\left(t, O_j(t)\right)\right)\right) \\ &\le \sum_{1\le i\le N}\left(-c_i|u_i^*(t)| + \sum_{1\le j\le N}|a_{ij}|\left|f_j\left(t, \bar{O}_j(t)\right) - f_j\left(t, O_j(t)\right)\right|\right) \\ &\le \sum_{1\le i\le N}\left(-c_i|u_i^*(t)| + \sum_{1\le j\le N}|a_{ij}|L_j\left|u_j^*(t)\right|\right) \\ &= -\sum_{1\le i\le N}\left(c_i - \sum_{1\le j\le N}|a_{ji}|L_i\right)|u_i^*(t)| \\ &\le -\Lambda U(t),\end{aligned} \tag{24}$$

where $\Lambda = \min_{1\le i\le N} \Lambda_i$. It follows from Lemma 2 that:

$$U(t) \le U(0) E_\theta\left(\frac{-\eta\Lambda t^\theta}{\mathrm{GHF}(\eta) + \Lambda(1-\eta)}\right). \tag{25}$$

By using Definition 3, we conclude that System (9) is Mittag–Leffler stable. □

Theorem 3. *Assume that (A1), (A2), and (A3) are satisfied. Then, for any solution $O(t)$ of System (9) with $w(t) = 1$, there exists a $t_0 > 0$ such that:*

$$\|O(t)\| \le \frac{u}{\Lambda} + \varepsilon, \text{ for all } t \ge t_0, \tag{26}$$

where $\varepsilon > 0$ is an arbitrary small constant and $u = \sum_{1\le i\le N}\left(\sum_{1\le j\le N}|a_{ij}| \sup_{t\in[0,\infty)} |f_j(t,0)| + c_i|\xi_0| + M_i\right)$.

Proof. Let $O(t) = (O_1(t), O_2(t), \ldots, O_N(t))^T$ be a solution $O(t)$ of System (9). So, consider the the following equation:

$$V(t) = \|O(t)\| = \sum_{1\le i\le N} |O_i(t)|. \tag{27}$$

Hence:

$$\begin{aligned}
\mathscr{D}_{0,1}^{\eta,\theta}V(t) &= \sum_{1\le i\le N} \mathscr{D}_{0,w}^{\eta,\theta}|O_i(t)| \\
&= \sum_{1\le i\le N} \operatorname{sign}(O_i(t))\left(-c_iO_i(t) + \sum_{1\le j\le N} a_{ij}\left(f_j\left(t, O_j(t)\right) - f_j(t,0)\right)\right) \\
&\quad + \sum_{1\le i\le N} \operatorname{sign}(O_i(t))\left(\sum_{1\le j\le N} a_{ij}\, f_j(t,0)\right) + \sum_{1\le i\le N} \operatorname{sign}(O_i(t))(c_iO_0(t) + I_i(t)) \\
&\le \sum_{1\le i\le N}\left(-c_i|O_i(t)| + \sum_{1\le j\le N}\left|a_{ij}\right|L_j\left|O_j(t)\right| + \sum_{1\le j\le N}\left|a_{ij}\right|\left|f_j(t,0)\right| + c_i|\xi_0| + M_i\right) \\
&= -\sum_{1\le i\le N}\left(c_i - \sum_{1\le j\le N}\left|a_{ji}\right|L_i\right)|O_i(t)| + \sum_{1\le i\le N}\left(\sum_{1\le j\le N}\left|a_{ij}\right|\left|f_j(t,0)\right| + c_i|\xi_0| + M_i\right) \\
&\le -\Lambda V(t) + u.
\end{aligned} \tag{28}$$

Now, consider the following fractional system:

$$\mathscr{D}_{0,1}^{\eta,\theta}G(t) = -\Lambda G(t) + u. \tag{29}$$

According to Remark 2 of Hattaf's [20] study, we get the following equation:

$$G(t) = \frac{u}{\Lambda} + \frac{\mathrm{GHF}(\eta)}{\mathrm{GHF}(\eta) + \Lambda(1-\eta)}\left(G(0) - \frac{u}{\Lambda}\right)E_\theta\left(\frac{-\eta\Lambda t^\theta}{\mathrm{GHF}(\eta) + \Lambda(1-\eta)}\right). \tag{30}$$

Hence:

$$\lim_{t\longrightarrow+\infty} G(t) = \frac{u}{\Lambda}. \tag{31}$$

Therefore, for any $\varepsilon > 0$, there exists a $t_0 > 0$ such that:

$$G(t) \le \frac{u}{\Lambda} + \varepsilon,\ t \ge t_0. \tag{32}$$

From Lemma 1, we deduce that $V(t) \le G(t)$ with $V(0) = G(0)$. Since $V(t) = \|O(t)\|$, we deduce the following equation:

$$\|O(t)\| \le \frac{u}{\Lambda} + \varepsilon,\ t \ge t_0. \tag{33}$$

This ends the proof. □

Theorem 4. *Let $f_i(t,\xi_0) = 0$ for $t \in [0,\infty)$ and $\lim_{t\to\infty} I_i(t) = 0$ for all $i = 1, 2, \ldots, N$. If (A1), (A2), and (A3) are satisfied, then the consensus opinion ξ_0 is reached by all agent opinions $O_i(t)$ of System (9) $w(t) = 1$, i.e., $\lim_{t\to\infty} O_i(t) = \xi_0$ for every $i = 1, 2, \ldots, N$.*

Proof. Let $(O_0(t), O_1(t), \ldots, O_N(t))$ be a solution of System (9). As $\mathscr{D}_{0,w}^{\eta,\theta}O_0(t) = 0$, we get $O_0(t) = \xi_0$. For each $i = 1, \ldots, N$, let $\bar{O}_i = O_i - \xi_0$ and $h_i(t,\bar{O}_i) := f_i(t,\bar{O}_i + \xi_0)$. Then, System (9) becomes:

$$\mathscr{D}_{0,1}^{\eta,\theta}\bar{O}_i(t) = \sum_{1\le j\le N} a_{ij}\left(h_j\left(t,\bar{O}_j(t)\right)\right) - c_i\bar{O}_i(t) + I_i(t). \tag{34}$$

For any $x, y \in \mathrm{I\,R}$, we have the following equation:

$$|h_i(t,x) - h_i(t,y)| = |f_i(t, x+\xi_0) - f_i(t, y+\xi_0)| \le L_i|x-y|. \tag{35}$$

Then, the condition (A1) holds for h_i. In addition, the other conditions (A2) and (A3) are satisfied by System (34) with initial conditions $\bar{O}_i(0) = 0$, for $i = 1, 2, \ldots, N$. By applying Theorem 3, we deduce that there exists a $t_0 > 0$ such that:

$$\|\bar{O}(t)\| \le \frac{u}{\Lambda} + \varepsilon,\ \text{for all}\ \ t \ge t_0, \tag{36}$$

where $u = \sum_{1\le i\le N}\left(\sum_{1\le j\le N}|a_{ij}|\sup_{t\in[0,\infty)}|h_j(t,0)| + c_i(0) + M_i\right)$.

Since $h_j(t,0) = f_j(t,\xi) = 0$, we get $u = \sum_{1\le i\le N} M_i$.

As $\lim_{t\to\infty} I_i(t) = 0$, then there exists a $t_1 > 0$ such that $|I_i(t)| \le \varepsilon$ for all $t \ge t_1$. In the same way used in the proof of Theorem 3, we can easily obtain the following equation:

$$\|\bar{O}_i(t)\| \le \frac{N\varepsilon}{\Lambda} + \varepsilon,\ t \ge t_1. \tag{37}$$

Hence, $\lim_{t\to\infty}\bar{O}_i(t) = 0$, which leads to $\lim_{t\longrightarrow\infty} O_i(t) = \xi_0$. This completes the proof. □

Remark 5. Obviously, the vector $(\xi_0, \ldots, \xi_0) \in IR^{N+1}$ is the equilibrium of System (9) when $f_i(t,\xi_0) = 0$ and $I_i(t) = 0$ for all $i = 1, 2, \ldots, N$. This equilibrium is asymptotically stable according to Theorems 3 and 4.

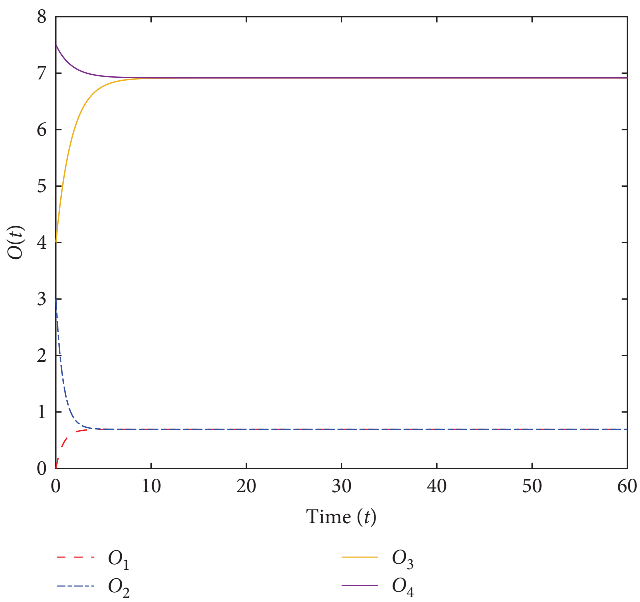

FIGURE 1: Behavior of Model (38) with $\eta = 1$ and $\theta = 1$.

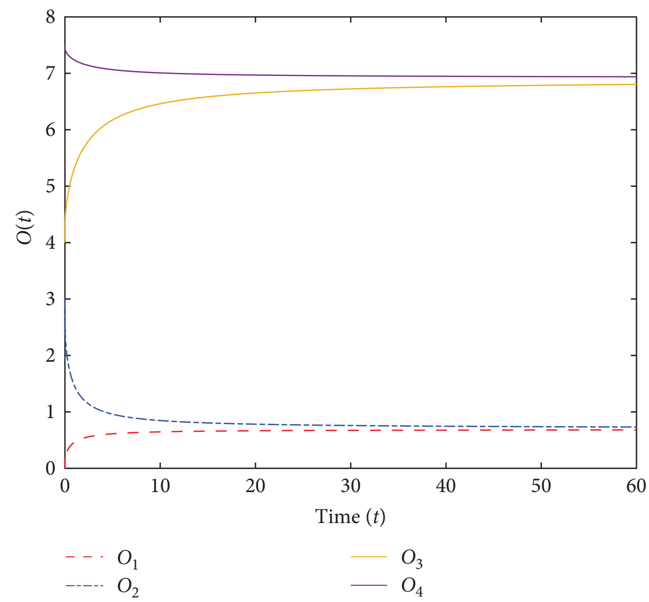

FIGURE 2: Behavior of Model (38) with $\eta = 0.8$ and $\theta = 0.7$.

5. Numerical Simulations

This section presents three examples as in [16] and [14] to demonstrate our analytical results using numerical simulations based on the numerical method presented in [25].

Example 6. Consider the following system:

$$\begin{cases} \mathscr{D}_{0,1}^{\eta,\theta} O_1(t) = 0.3O_2(t) - 0.3O_1(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_2(t) = O_1(t) - O_2(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_3(t) = 0.5(O_4(t) - O_3(t)) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_4(t) = 0.1O_3(t) - 0.1O_4(t), \end{cases} \tag{38}$$

with the initial conditions $O_1(0) = 0$, $O_2(0) = 3$, $O_3(0) = 4$, and $O_4(0) = 7.5$.

Figures 1–3 show the dynamical behaviors of Model (38) for different fractional orders η and θ.

We notice that each agent is influenced by the opinions of its neighbors. Also, in a leaderless environment where each agent acts independently, the agents' opinions do not converge to a consensus state.

Example 7. Consider the following system:

$$\begin{cases} \mathscr{D}_{0,1}^{\eta,\theta} O_0(t) = 0 \\ \mathscr{D}_{0,1}^{\eta,\theta} O_1(t) = 0.03 \sin(O_1(t)) + 9e^{-t} - 0.3O_1(t) + 0.3O_0(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_2(t) = 0.015 \sin(O_1(t)) - 0.0006 \sin(O_4(t)) + 0.09e^{-t} - 0.3O_2(t) + 0.3O_0(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_3(t) = -0.003 \sin(O_3(t)) + 0.003 \sin(O_4(t)) + 0.1e^{-t} - 0.2O_3(t) + 0.2O_0(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_4(t) = 0.009 \sin(O_4(t)) - 0.006 \sin(O_5(t)) - 0.4O_4(t) + 0.4O_0(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_5(t) = -0.003(O_3(t)) + 0.02e^{-t} - 0.1O_5(t) + 0.1O_0(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_6(t) = 0.015 \sin(O_1(t)) + 0.027 \sin(O_3(t)) - 0.4e^{-t} - 0.4O_6(t) + 0.4O_0(t), \end{cases} \tag{39}$$

with the initial conditions $O_0(0) = 0$, $O_1(0) = 25$, $O_2(0) = 10$, $O_3(0) = 30$, $O_4(0) = 1.5$, $O_5(0) = -20$, and $O_6(0) = -10$.

Figures 4–6 show the dynamical behaviors of Model (39) for different fractional orders η and θ.

In this case, the leader $O_0(t) = \xi_0 = 0$, $f_i(t, \xi_0) = 0$, $\lim_{t\to\infty} I_i(t) = 0$, and $L_i = 1$ for $i = 1, 2, \ldots, 6$. As a result, the fractional opinion formation Model (39) is Mittag–Leffler stable. The states of O_0, O_1, O_2, O_3, O_4, O_5, and O_6 converge to a consensus state of $(0, 0, 0, 0, 0, 0, 0)$.

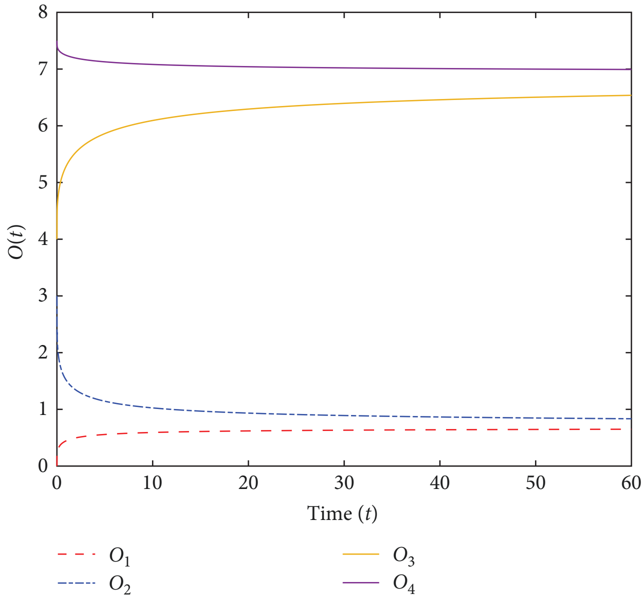

Figure 3: Behavior of Model (38) with $\eta = 0.8$ and $\theta = 0.5$.

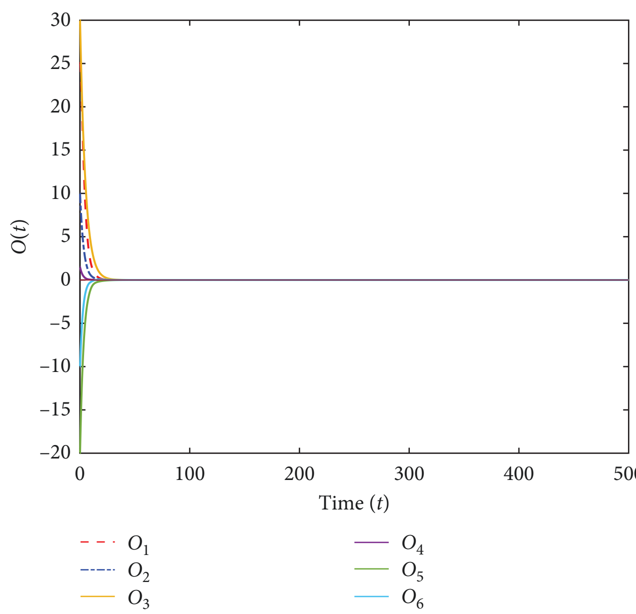

Figure 4: Behavior of Model (39) with $\eta = 1$ and $\theta = 1$.

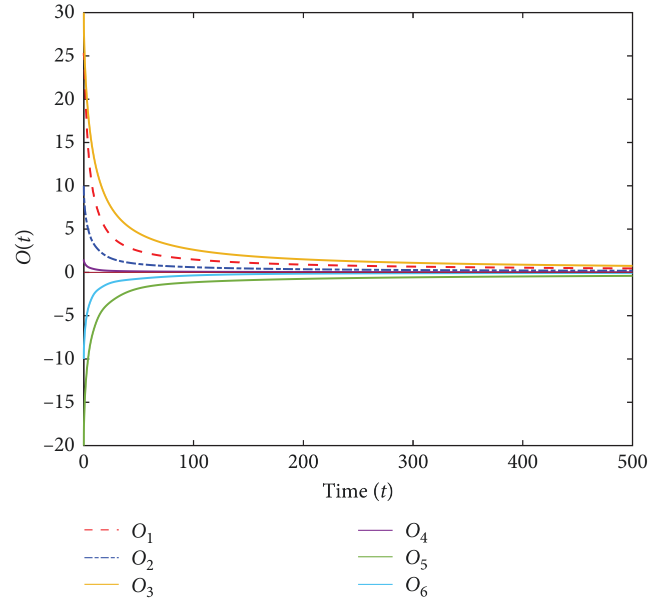

Figure 5: Behavior of Model (39) with $\eta = 0.8$ and $\theta = 0.7$.

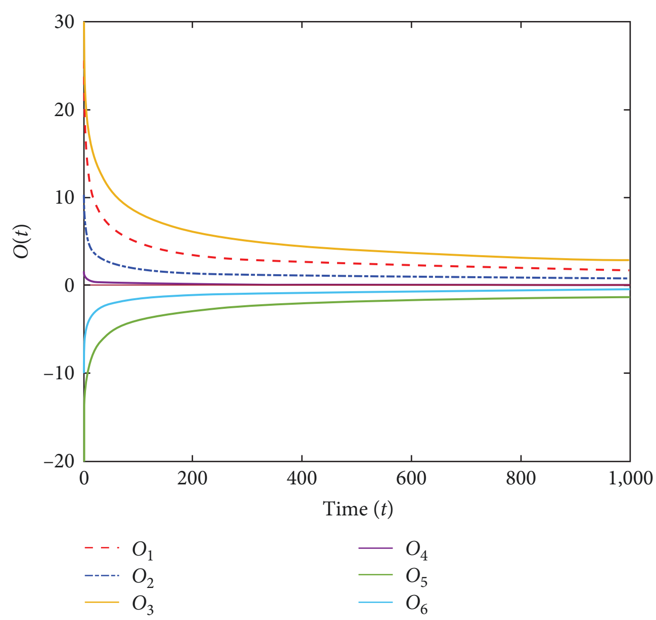

Figure 6: Behavior of Model (39) with $\eta = 0.8$ and $\theta = 0.5$.

Example 8. Consider the following system:

$$\begin{cases} \mathscr{D}_{0,1}^{\eta,\theta} O_0(t) = 0 \\ \mathscr{D}_{0,1}^{\eta,\theta} O_1(t) = -0.36 O_1 + 0.01 O_2(t) + \dfrac{\sin(t)}{t} + 0.2 O_0(t) \\ \mathscr{D}_{0,1}^{\eta,\theta} O_2(t) = -0.03 O_1(t) - 0.22 O_2(t) - 2e^{-t} + 0.3 O_0(t), \end{cases} \tag{40}$$

with the initial conditions $O_0(0) = 1, O_1(0) = -10$, and $O_2(0) = 15$.

In this example, we have $f_i(t, \xi_0) = 0$, $\lim_{t\to\infty} I_i(t) = 0$, and $L_i = 1$ for $i = 1, 2, 3$. System (40) is Mittag–Leffler stable, as well as the states of O_0, O_1, and O_2 converge to a consensus state $(1, 1, 1)$, in which all agents share the same opinion. Figures 7–9 demonstrate these results. In addition, we observe that the speed of consensus depends on the values of the fractional orders η and θ. More precisely, the time for arriving at the consensus with large fractional derivative order value is faster than that with small fractional derivative order value.

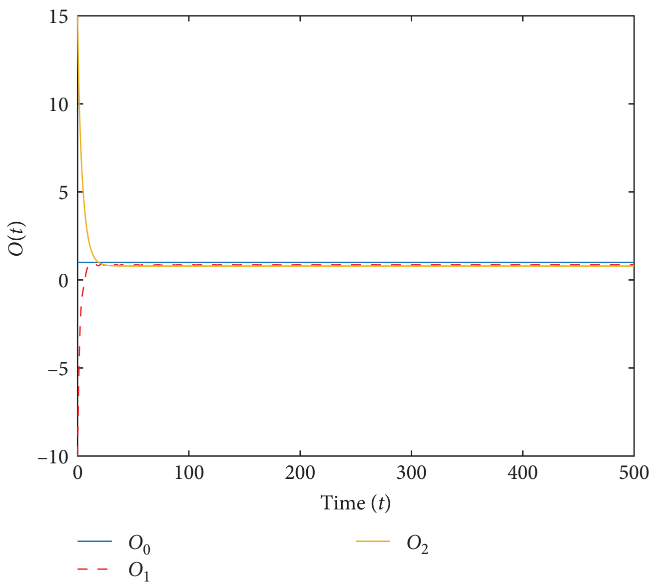

FIGURE 7: Behavior of Model (40) with $\eta = 1$ and $\theta = 1$.

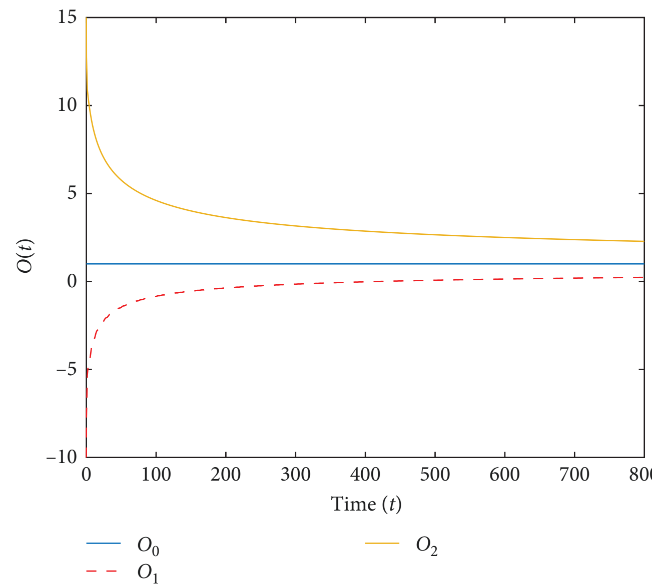

FIGURE 9: Behavior of Model (40) with $\eta = 0.8$ and $\theta = 0.5$.

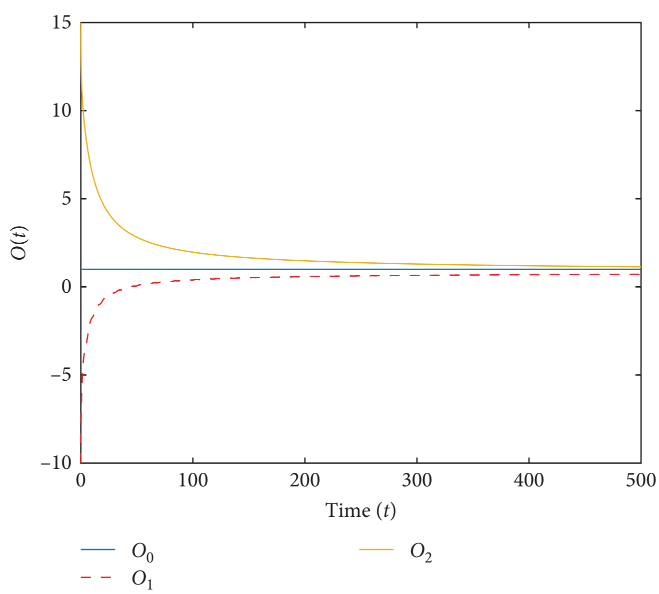

FIGURE 8: Behavior of Model (40) with $\eta = 0.8$ and $\theta = 0.7$.

6. Conclusion

In this work, we have presented two fractional opinion formation models involving the new GHF derivative. The first model is without leadership, while the second one is with leadership and time-dependent external inputs. At the beginning, we have proposed a definition for Mittag–Leffler stability in order to investigate the dynamical behaviors of such fractional models. The proposed definition extends that introduced in [13] and generalizes the exponential stability given in [21] and used in [16]. Furthermore, the convergence of agents' opinions toward the consensus opinion was carefully examined. At the last, three numerical examples and corresponding numerical simulations have been given to illustrate our main analytical results.

On the other hand, it would be very interesting to extend our work to the case of delay systems and to model the dynamics of opinion formation using the novel mixed fractional derivative [26] that covers the GHF and Caputo fractional derivatives. These issues will be the topics of future work.

Data Availability

All data used during the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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