Hindawi
Mathematical Problems in Engineering
Volume 2024, Article ID 6844686, 7 pages
https://doi.org/10.1155/2024/6844686

Research Article

Existence and Uniqueness of Solutions for Fractional-Differential Equation with Boundary Condition Using Nonlinear Multi-Fractional Derivatives

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Received 14 September 2023; Revised 8 November 2023; Accepted 8 January 2024; Published 12 February 2024

Academic Editor: Xinan Hao

In this article the existence as well as the uniqueness (EU) of the solutions for nonlinear multiorder fractional-differential equations (FDE) with local boundary conditions and fractional derivatives of different orders (Caputo and Riemann–Liouville) are covered. The existence result is derived from Krasnoselskii's fixed point theorem and its uniqueness is shown using the Banach contraction mapping principle. To illustrate the reliability of the results, two examples are given.

1. Introduction

Fractional-differential equations (FDEs) have gained significant attention recently due to their wide-ranging applications in the diverse scientific and engineering fields. These equations find relevance in areas such as fractal theory, potential theory, biology, chemistry, and diffusion, among others [1, 2]. Some specific physical phenomena associated with the fractional-oscillator equations and fractional Euler–Lagrange equations containing mixed fractional derivatives can be found in [1, 3]. Once a FDE model is established to represent a real-world problem, the subsequent challenge lies in solving the model. Finding the exact solution to a FDE is often a difficult task. Therefore, researchers strive to identify as many aspects of the solution as possible, addressing questions such as its existence and uniqueness. These inquiries explore whether a solution exists for the problem and if so, whether it is the only possible solution.

Consequently, the investigation of existence and uniqueness solutions for the FDEs with initial and boundary conditions has attracted the interest of numerous scientists and mathematicians [4–6]. This line of study aims to develop theoretical frameworks, techniques, and methodologies for analyzing and solving the FDEs. By understanding the behavior and properties of solutions, researchers can devise effective solution approaches and gain insights into the practical applications of FDEs. Many authors have studied the existence and uniqueness theorem for the FDEs involving mixed fractional derivatives in recent years [7–14]. Sarwar [15] has shown the variable order Caputo type FDEs of the form as follows:

$$\begin{cases} {}^{C}\mathscr{D}_{0,\phi}^{\omega(\phi)}\varphi(\phi) = \mathfrak{f}(\phi,\varphi), & 0<\omega(\phi)<1 \\ \varphi(\phi)|_{\phi=0} = \varphi_0, \quad \varphi\in\mathbb{R}, & \phi\in(0,+\infty) \end{cases}, \tag{1}$$

where ${}^{C}D^{\omega(\phi)}(.)$ denotes Caputo derivative with the variable order. Chai [16] investigated the existence of solutions to the boundary-value problem as follows:

$$\begin{cases} {}^{C}\mathscr{D}^{\omega}\mathfrak{x}(\phi) + r\,{}^{C}\mathscr{D}^{\omega-1}\mathfrak{x}(\phi) = \mathfrak{f}(\phi,\mathfrak{x}(\phi)), & \phi\in(0,1) \\ \mathfrak{x}(0) = \mathfrak{x}(1), \quad \mathfrak{x}(\xi) = \nu, \quad \xi\in(0,1), \end{cases} \tag{2}$$

where ${}^{C}\mathscr{D}^{\omega}$ and ${}^{C}\mathscr{D}^{\omega-1}$ denote the standard Caputo derivatives of order ω and $\omega-1$, respectively, in this case with $1<\omega\le 2$, and $r\ne 0$. Additionally, more recently, Xu et al. [17] considered the existence of solutions and Ulam–Hyers stability for the fractional boundary value problem:

$$\begin{cases} \xi\mathscr{D}^{\omega}\mathfrak{x}(\phi) + \mathscr{D}^{\upsilon}\mathfrak{x}(\phi) = \mathfrak{f}(\phi,\mathfrak{x}(\phi)), & \phi\in(0,T) \\ \mathfrak{x}(0) = 0, \quad \mu\mathscr{D}^{\gamma_1}\mathfrak{x}(T) + I^{\gamma_2}\mathfrak{x}(\nu) = \gamma_3, \end{cases} \tag{3}$$

where $\mathscr{D}^{\omega}$ denotes the Riemann–Liouville fractional derivative operator of order ω, $1<\omega\le 2$, $1\le\upsilon<\omega$, $0<\xi\le 1$, $0<\mu\le 1$, $0\le\gamma_1\le\omega-\upsilon$, $\gamma_2\ge 0$, and I^{γ_2} denotes the Riemann–Liouville fractional integral operator of order γ_2, and $0<\nu<T$. Motivated by the analysis and outcomes obtained for the aforementioned challenges. In this study, we will look at the existence and uniqueness of solutions to the following nonlinear FDE boundary value issue using multi-fractional derivatives.

$$\begin{cases} \xi\,{}^{C}\mathscr{D}^{\omega}\mathfrak{x}(\phi) - {}^{RL}\mathscr{D}^{\upsilon}\mathfrak{x}(\phi) = \mathfrak{f}(\phi,\mathfrak{x}(\phi)), & 1<\omega<2, 0<\upsilon<1, \\ \mathfrak{x}(0) = 0, \quad \mathfrak{x}'(1) = 0, & \xi>0, \phi\in[0,1], \end{cases} \tag{4}$$

where ${}^{C}\mathscr{D}^{\omega}$ denotes the Caputo's derivative of fractional order ω with $1<\omega<2$ and ${}^{\mathrm{RL}}\mathscr{D}^{\upsilon}$ denotes the Riemann–Liouville derivative of fractional order υ with $0<\upsilon<1$. It should be noted that our work makes some fundamental assumptions for the order of multi-fractional derivatives; prospective relaxations of these restrictions may be taken into account in future research. This article's key contribution is as follows:

(1) A generalization of the results obtained in [15].
(2) A generalization of the results obtained in [16].
(3) A generalization of the results obtained in [17].

The rest of the work is structured as follows: Section 2 covers the required definitions as well as the fundamental tools that will be utilized in the next sections; Section 3 derives and solves various requirements for the existence and uniqueness of solutions for multi-fractional derivatives. Finally, some specific examples are provided to describe the achieved findings.

2. Preliminaries

Here, we introduce certain definitions, desired lemmas, and theorems, which are essential to find the main result.

Definition 1 (see [1, 18]). The Riemann–Liouville fractional integral of order $\omega>0$ of a function $\mathfrak{f}:[a,b]\longrightarrow\mathbf{R}$ at the point ϕ is defined by

$$I_a^{\omega}\mathfrak{f}(\phi) = \int_a^{\phi}\frac{(\phi-\varphi)^{\omega-1}}{\Gamma(\omega)}\mathfrak{f}(\varphi)d\varphi, \tag{5}$$

provided the right side is point-wisely defined, where Γ is the Gamma function.

Definition 2 (see [1, 18]). The Riemann–Liouville fractional derivative of order of $\omega>0$ a function $\mathfrak{f}:[a,b]\longrightarrow\mathbf{R}$ at the point ϕ is defined by

$${}^{\mathrm{RL}}\mathscr{D}_a^{\omega}\mathfrak{f}(\phi) = \frac{1}{\Gamma(n-\omega)}\int_a^{\phi}(\phi-\varphi)^{n-\omega-1}\mathfrak{f}(\varphi)d\varphi, \tag{6}$$

provided the right side is point-wisely defined, where $n=[\omega]+1$, $[\omega]$ denotes the integer part of ω.

Definition 3 (see [1, 18]). The Caputo derivative of fractional order ω for an n-times differentiable function $\mathfrak{f}:[a,b]\longrightarrow\mathbf{R}$ is defined as follows:

$${}^{C}\mathscr{D}_a^{\omega}\mathfrak{f}(\phi) = \frac{1}{\Gamma(n-\omega)}\int_a^{\phi}(\phi-\varphi)^{n-\omega-1}\left(\frac{d}{ds}\right)^{n}\mathfrak{f}(\varphi)d\varphi, \tag{7}$$

where $n=[\omega]+1$ and $\omega>0$.

Property 1 (see [1]). *Assume* $0<\omega\le 1$ *and* $\mathfrak{m}=[\omega]+1$. *If* $\mathfrak{x}(\phi)\in(C^{\mathfrak{m}}[0,1])$

$${}^{C}I_a^{\omega}\,{}^{C}\mathscr{D}_a^{\omega}\mathfrak{x}(\phi) = \mathfrak{x}(\phi) - \mathfrak{x}(0). \tag{8}$$

Theorem 1 (Krasnosel'skii fixed point theorem [19]). *Suppose* M *is a closed, convex, and bounded nonempty subset of a Banach space* X. *Let* P *and* Q *be two operators satisfying the following conditions:*

(1) $Px+Qy\in M$, *whenever* $\mathfrak{x},\mathfrak{y}\in M$;
(2) Q *is a contraction mapping;*
(3) P *is both compact and continuous, then there exists an element* $\omega\in M$ *such that the equation* $\omega=P\omega+Q\omega$ *holds true.*

Theorem 2 (contraction mapping principle [20]). *Let* M *be a Banach space. If* $T:M\longrightarrow M$ *is a contraction, then* T *has a unique fixed point in* M.

3. Main Result

Lemma 1. *The solution to the boundary value problem (4) satisfies the integral equation as follows:*

$$\mathfrak{x}(\phi) = \frac{1}{\xi} I^{\omega-\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) - \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}(1) - \frac{1}{\xi} I^{\omega-1}\mathfrak{f}(1, \mathfrak{x}(1)). \tag{9}$$

Proof. From Equation (4), we have

$$^{C}\mathscr{D}^{\omega}\mathfrak{x}(\phi) = \frac{1}{\xi}\, {}^{RL}D^{\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi}\mathfrak{f}(\phi, \mathfrak{x}(\phi)). \tag{10}$$

Taking the Riemann–Liouville fractional integral of order ω on both sides, we get

$$\mathfrak{x}(\phi) - \mathfrak{x}(0) = \frac{1}{\xi} I^{\omega-\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) + c_1 + c_2\mathfrak{x}, \tag{11}$$

$$\mathfrak{x}(\phi) = \frac{1}{\xi} I^{\omega-\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) + c_1 + c_2\mathfrak{x}. \tag{12}$$

Then, $\mathfrak{x}(0) = 0$ implies $c_1 = 0$. Hence,

$$\mathfrak{x}(\phi) = \frac{1}{\xi} I^{\omega-\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) + c_2\mathfrak{x}. \tag{13}$$

On differentiating both side we get:

$$\mathfrak{x}'(\phi) = \frac{1}{\xi}\frac{d}{d\phi} I^{\omega-\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi}\frac{d}{d\phi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) + c_2. \tag{14}$$

Applying the boundary condition $\mathfrak{x}'(1) = 0$ in above equation, we get:

$$c_2 = -\frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}(1) - \frac{1}{\xi} I^{\omega-1}\mathfrak{f}(1, \mathfrak{x}(1)). \tag{15}$$

Now, putting the value of c_1 and c_2 in Equation (10), we get:

$$\mathfrak{x}(\phi) = \frac{1}{\xi} I^{\omega-\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) - \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}(1) - \frac{1}{\xi} I^{\omega-1}\mathfrak{f}(1, \mathfrak{x}(1)). \tag{16}$$

□

4. Existence and Uniqueness

In this section, we will utilize the Krasnoselskii fixed point theorem and Banach contraction principle to establish both the existence and uniqueness of a solution to problem (4) within the Banach space C. Now, let's examine the following assumptions that are necessary for the forthcoming analysis:

(A1) Let $\mathfrak{f}(C[0, 1]\mathbb{R})$ denotes the Banach space as the set of all continuous functions from the interval $[0, 1]$ into $\mathbb{R}$ equipped with the norm determined by

$$\|h(\mathfrak{x})\| = \|\,\mathfrak{f}(\phi, \mathfrak{x}(\phi))\| = \sup\|\mathfrak{x}(\phi)\|, \phi \in [0, 1]. \tag{17}$$

(A2) There exists a constant $\mathscr{L} > 0$ such that $|\,\mathfrak{f}(\phi, \mathfrak{x}_1) - \mathfrak{f}(\phi, \mathfrak{x}_2)| \le \mathscr{L}|\mathfrak{x}_1 - \mathfrak{x}_2|$, $\forall \mathfrak{x}_1, \mathfrak{x}_2 \in \mathbb{R}$, and $\phi \in [0, 1]$.

(A3) For each $\eta_\circ > 0$, $B_{\eta_\circ} \in \{\mathfrak{x} \in C([0, 1], \mathbb{R}), \|\mathfrak{x}(\phi)\| \le \eta_\circ\}$, then $B_{\eta_\circ}$ is evidently a bounded, closed, and convex subset within $C([0, 1], \mathbb{R})$.

Now, we will demonstrate the existence of a solution for the problem using the Krasnoselskii fixed point theorem.

Theorem 3. *Assume that A1–A3 are satisfied, and if* $M < 1$ *where* $M = (\frac{\mu_\circ}{\Gamma(\omega-\upsilon+1)} + \frac{\|h\|}{\Gamma(\omega+1)} + \frac{\|\mathfrak{x}(1)\|}{\Gamma(\omega-\upsilon)} + \frac{\|h(1)\|}{\Gamma(\omega+1)}$ *then the boundary value problem (4) possesses at least one solution in* $C([0, 1], \mathbb{R})$.

Proof. In order to demonstrate the existence of a solution for problem (4), we will proceed with the proof by considering the following steps:

Step 1: For any constant $\xi > 0$, we will define two operator T_1 and T_2 as follows:

$$T_1 = \tfrac{1}{\xi_0} I_{\phi}^{\omega-\upsilon}\mathfrak{x}(\phi),$$
$$T_2 = \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) - \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}(1) - \frac{1}{\xi} I^{\omega-1}\mathfrak{f}(1, \mathfrak{x}(1)). \tag{18}$$

Now, we will show that the operator $T_1 + T_2 = T$ is bounded as follows:

$$
\begin{aligned}
\|T(\mathfrak{x}(\phi))\| &\le \left|\frac{1}{\xi}I^{\omega-\upsilon}\mathfrak{x}(\phi)+\frac{1}{\xi}I^{\omega}\mathfrak{f}(\phi,\mathfrak{x}(\phi))-\frac{1}{\xi}I^{\omega-\upsilon-1}\mathfrak{x}(1)-\frac{1}{\xi}I^{\omega}\mathfrak{f}(1,\mathfrak{x}(1))\right| \\
&\le \frac{1}{\xi\Gamma(\omega-\upsilon)}\left|\int_0^{\phi}(\phi-\varphi)^{\omega-\upsilon-1}\mathfrak{x}(\varphi)\,d\varphi\right|+\frac{1}{\xi\Gamma\omega}\left|\int_0^{\phi}(\phi-\varphi)^{\omega-1}\mathfrak{f}(\varphi,\mathfrak{x}(\varphi))\,d\varphi\right| \\
&\quad+\frac{1}{\xi\Gamma(\omega-\upsilon-1)}\left|\int_0^{\phi}(\phi-\varphi)^{\omega-\upsilon-2}\mathfrak{x}(1)\,d\varphi\right|+\frac{1}{\xi\Gamma\omega}\left|\int_0^{\phi}(\phi-\varphi)^{\omega-1}\mathfrak{f}(1,\mathfrak{x}(1))\,d\varphi\right|, \\
&\le \frac{1}{\xi\Gamma(\omega-\upsilon)}\|\mathfrak{x}(\phi)\|\left|\int_0^{\phi}(\phi-\varphi)^{\omega-\upsilon-1}d\varphi\right|+\frac{1}{\xi\Gamma\omega}\|h(\phi)\|\left|\int_0^{\phi}(\phi-\varphi)^{\omega-1}d\varphi\right| \\
&\quad+\frac{1}{\xi\Gamma(\omega-\upsilon-1)}\|\mathfrak{x}(1)\|\left|\int_0^{\phi}(\phi-\varphi)^{\omega-\upsilon-2}d\varphi\right|+\frac{1}{\xi\Gamma\omega}\|h(1)\|\left|\int_0^{\phi}(\phi-\varphi)^{\omega-1}d\varphi\right|, \\
&\le \frac{\mu_\circ}{\xi\Gamma(\omega-\upsilon+1)}+\frac{\|h\|}{\xi\Gamma(\omega+1)}+\frac{\|\mathfrak{x}(1)\|}{\xi\Gamma(\omega-\upsilon)}+\frac{\|h(1)\|}{\xi\Gamma(\omega+1)}, \\
&\le \frac{1}{\xi}\left(\frac{\mu_\circ}{\Gamma(\omega-\upsilon+1)}+\frac{\|h\|}{\Gamma(\omega+1)}-\frac{\|\mathfrak{x}(1)\|}{\Gamma(\omega-\upsilon)}+\frac{\|h(1)\|}{\Gamma(\omega+1)}\right), \\
&\le \frac{M}{\xi}, \\
&\textit{where}\ M=\left(\frac{\mu_\circ}{\Gamma(\omega-\upsilon+1)}+\frac{\|h\|}{\Gamma(\omega+1)}+\frac{\|\mathfrak{x}(1)\|}{\Gamma(\omega-\upsilon)}+\frac{\|h(1)\|}{\Gamma(\omega+1)}\right.
\end{aligned}
\tag{19}
$$

Hence, operator T is bounded.

Step 2: Contraction

$$
\begin{aligned}
&\|T_1(\mathfrak{x}(\phi))-T_2(\mathfrak{y}(\phi))\| \\
&\quad=\left\|\frac{1}{\xi}{}_0I_{\phi}^{\omega-\upsilon}\mathfrak{x}(\phi)-\frac{1}{\xi}{}_0I_{\phi}^{\omega-\upsilon}\mathfrak{y}(\phi)\right\|, \\
&\quad=\left\|\frac{1}{\xi\Gamma(\omega-\upsilon)}\int_0^{\phi}(\phi-\varphi)^{\omega-\upsilon-1}(\mathfrak{x}(\varphi))d\varphi-\frac{1}{\xi\Gamma(\omega-\upsilon)}\int_0^{\phi}(\phi-\varphi)^{\omega-\upsilon-1}(\mathfrak{y}(\varphi))d\varphi\right\|, \\
&\quad\le\frac{1}{\xi}\frac{1}{\Gamma(\omega-\upsilon)}\|\mathfrak{x}-\mathfrak{y}\|\left\|\int_0^{\phi}(\phi-\varphi)^{\omega-\upsilon-1}(\mathfrak{x}(\varphi))d\varphi\right\|, \\
&\quad\le\frac{1}{\xi}\frac{1}{\Gamma(\omega-\upsilon)}\|\mathfrak{x}-\mathfrak{y}\|\left\{\frac{1}{\omega-\upsilon}\right\}, \\
&\quad\le\frac{1}{\xi}\frac{1}{\Gamma(\omega-\upsilon+1)}\|\mathfrak{x}-\mathfrak{y}\|, \\
&\quad\le\Updownarrow\|\mathfrak{x}-\mathfrak{y}\|.
\end{aligned}
\tag{20}
$$

where $\Updownarrow=\frac{1}{\xi}\frac{1}{\Gamma(\omega-\upsilon+1)}$.

Step 3: In order to establish the complete continuity of the operator T_2, it is necessary to demonstrate both its continuity and equicontinuity. By proving these two properties, we can establish the complete continuity of T_2.

$$
\begin{aligned}
&\|T_2(\mathfrak{x}_n(\phi)) - T_2(\mathfrak{x}(\phi))|\\
&\quad= \left|\frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}_n(\phi)) - \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}_n(1) - \frac{1}{\xi} I^{\omega}\mathfrak{f}(1, \mathfrak{x}_n(1))\right.\\
&\qquad\left. - \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi, \mathfrak{x}(\phi)) - \frac{1}{\xi} I^{\omega-\upsilon 1)}\mathfrak{x}(1) - \frac{1}{\xi} I^{\omega}\mathfrak{f}(1, \mathfrak{x}(1))\right|,\\
&\quad\leq \frac{1}{\xi\Gamma\omega}\left\|\int_0^{\phi}(\phi-\varphi)^{\omega-1}\left(\mathfrak{f}(1, \mathfrak{x}_n(1))d\varphi - \int_0^{\phi}(\phi-\varphi)^{\omega-1}\mathfrak{f}(1, \mathfrak{x}(1))\right)d\varphi\right\|\\
&\qquad+ \frac{1}{\xi\Gamma\omega+\upsilon}\left\|\int_0^{1}(\phi-\varphi)^{\omega-\upsilon-2}\mathfrak{x}_n(\varphi)d\varphi - \int_0^{1}(\phi-\varphi)^{\omega+\upsilon-2}\mathfrak{x}(\varphi)d\varphi\right\|\\
&\qquad+ \frac{1}{\xi\Gamma\omega-1}\left\|\int_0^{1}(\phi-\varphi)^{\omega-2}\left(\mathfrak{f}(\varphi, \mathfrak{x}_n(\varphi)) - \int_0^{1}(\phi-\varphi)^{\omega-2}\mathfrak{f}(\varphi, \mathfrak{x}(\varphi))\right)d\varphi\right\|,\\
&\quad\leq \frac{1}{\xi\Gamma\omega}\mathscr{L}\|\mathfrak{x}_n - \mathfrak{x}\|\int_0^{1}(\phi-\varphi)^{\omega-1}d\varphi + \frac{1}{\xi\Gamma\omega+\upsilon}|\mathfrak{x}_n(\phi) - \mathfrak{x}(\phi)|\left\|\int_0^{1}(\phi-\varphi)^{\omega+\upsilon-2}d\varphi\right.\\
&\qquad+ \frac{1}{\xi\Gamma\omega-1}\mathscr{L}\left\|\mathfrak{x}_n(\phi) - \mathfrak{x}(\phi)\right\|\left\|\int_0^{1}(\phi-\varphi)^{\omega-2}d\varphi\right.,\\
&\quad\leq \frac{1}{\xi\Gamma\omega+1}\mathscr{L}\left\|\mathfrak{x}_n(\phi) - \mathfrak{x}(\phi)\right\| + \frac{1}{\xi\Gamma(\omega+\upsilon+1)}\left\|\mathfrak{x}_n(\phi) - \mathfrak{x}(\phi)\right\| + \frac{1}{\xi\Gamma\omega}\mathscr{L}\left\|\mathfrak{x}_n(\phi) - \mathfrak{x}(\phi)\right\|,\\
&\quad\leq \frac{1}{\xi}\left(\frac{\mathscr{L}}{\Gamma\omega+1} + \frac{1}{\Gamma(\omega+\upsilon+1)} + \frac{\mathscr{L}}{\Gamma\omega}\right)\|\mathfrak{x}_n(\phi) - \mathfrak{x}(\phi)\|.
\end{aligned}
\tag{21}
$$

As $\mathfrak{x}_n(\phi)\longrightarrow\mathfrak{x}(\phi)$ as $n\longrightarrow\infty$, so $\|T_2(\mathfrak{x}_n(\phi)) - T_2(\mathfrak{x}(\phi))\|\longrightarrow 0$ as $n\longrightarrow\infty$.

Hence, T_2 is continuous.

Next, we will show that T_2 is equicontinuous operator.

$$
\begin{aligned}
&\|T_2(\mathfrak{x}(\phi_2)) - T_2(\mathfrak{x}(\phi_1))\|\\
&\quad= \left|\frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi_2, \mathfrak{x}(\phi_2)) - \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}(1) - \frac{1}{\xi} I^{\omega}\mathfrak{f}(1, \mathfrak{x}(1))\right.\\
&\qquad\left. - \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi_1, \mathfrak{x}(\phi_1)) + \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}(1) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(1, \mathfrak{x}(1))\right|,\\
&\quad\leq \frac{1}{\xi\Gamma\omega}\left|\int_0^{\phi_2}(\phi_2-\varphi)^{\omega-1}\left(\mathfrak{f}(\varphi, \mathfrak{x}(\varphi)) - \int_0^{\phi_1}(\phi_1-\varphi)^{\omega-1}\mathfrak{f}(\varphi, \mathfrak{x}(\varphi))\right)d\varphi\right|,\\
&\quad\leq \frac{1}{\xi\Gamma\omega}\left|\int_0^{\phi_1}(\phi_2-\varphi)^{\omega-1}\mathfrak{f}(\varphi, \mathfrak{x}(\varphi))ds + \int_{\phi_1}^{\phi_2}(\phi_2-\varphi)^{\omega-1}\mathfrak{f}(\varphi, \mathfrak{x}(\varphi))d\varphi\right.\\
&\qquad\left. - \int_0^{\phi_1}(\phi_2-\varphi)^{\omega-1}\mathfrak{f}(\varphi, \mathfrak{x}(\varphi))ds\right|,\\
&\quad\leq \frac{1}{\xi\Gamma\omega}\left|\int_{\phi_1}^{\phi_2}(\phi_2-\varphi)^{\omega-1}\mathfrak{f}(\varphi, \mathfrak{x}(\varphi))d\varphi\right|.
\end{aligned}
\tag{22}
$$

As $\phi_2\longrightarrow\phi_1$, and $\mathfrak{x}(\phi_2)\longrightarrow\mathfrak{x}(\phi_1)$.

Hence, ϕ_2 is equicontinuous operator.

Given that all the conditions of Krasnosel'skii fixed point theorem are met, namely the existence of a fixed point, this concludes the proof. □

Theorem 4. *Let* $u(\mathfrak{x})\in C[0,1]$ *such that* $\mathfrak{f}\in C([a,b]\mathbb{R})$ *and* $\Lambda<1$ *where,*

$$
\Lambda = \frac{1}{\omega}\left\{\frac{1}{\Gamma((\omega-\upsilon))+1} + \frac{\mathscr{L}(\omega+1)}{\Gamma\omega+1}\right\}.
\tag{23}
$$

Then the problem (4) has a unique solution.

Proof. First, we will establish the boundedness of the operator T.

Step 1: In the previous result confirms that T is indeed bounded. Moving forward, we will utilize the Banach contraction mapping theorem to demonstrate the uniqueness of T.

$$
\begin{aligned}
&\|T(\mathfrak{x}(\phi)) - T(\mathfrak{y}(\phi))\| \\
&\quad = \left| \frac{1}{\xi} I^{\omega-\upsilon}\mathfrak{x}(\phi) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi,\mathfrak{x}(\phi)) - \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{x}(1) - \frac{1}{\xi} I^{\omega}\mathfrak{f}(1,\mathfrak{x}(1)) \right. \\
&\qquad \left. - \frac{1}{\xi} I^{\omega-\upsilon}\mathfrak{y}(\phi) - \frac{1}{\xi} I^{\omega}\mathfrak{f}(\phi,\mathfrak{y}(\phi)) + \frac{1}{\xi} I^{\omega-\upsilon-1}\mathfrak{y}(1) + \frac{1}{\xi} I^{\omega}\mathfrak{f}(1,\mathfrak{y}(1)) \right|,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
&\le \frac{1}{\xi\Gamma(\omega-\upsilon)} \left| \int_0^{\phi} (\phi-\varphi)^{\omega-\upsilon-1} (\mathfrak{x}(\phi) - \mathfrak{y}(\phi)) d\varphi \right| \\
&\quad + \frac{1}{\xi\Gamma\omega} \left| \int_0^{\phi} (\phi-\varphi)^{\omega-\upsilon-1} (\mathfrak{f}(\phi,\mathfrak{x}(\phi)) - \mathfrak{f}(\phi,\mathfrak{y}(\phi))) d\varphi \right| \\
&\quad + \frac{1}{\xi\Gamma(\omega-\upsilon)} \left| \int_0^{\phi} (\phi-\varphi)^{\omega-\upsilon-1} (\mathfrak{y}(1) - \mathfrak{x}(1)) d\varphi \right| \\
&\quad + \frac{1}{\xi\Gamma\omega} \left| \int_0^{\phi} (\phi-\varphi)^{\omega-\upsilon-1} (\mathfrak{f}(1,\mathfrak{y}(1)) - \mathfrak{f}(1,\mathfrak{y}(1))) \right| d\varphi, \\
&\le \frac{1}{\xi}\frac{1}{\Gamma(\omega-\upsilon+1)} \|\mathfrak{x}-\mathfrak{y}\| + \frac{1}{\xi}\frac{1}{\Gamma\omega+1}\mathscr{L}\|\mathfrak{x}-\mathfrak{y}\| \\
&\quad + \frac{1}{\xi}\frac{1}{\Gamma\omega-\upsilon+1} \|\mathfrak{x}-\mathfrak{y}\| + \frac{1}{\xi}\frac{1}{\Gamma\omega}\mathscr{L}\|\mathfrak{x}-\mathfrak{y}\|, \\
&\le \left(\frac{2}{\xi}\frac{1}{\Gamma(\omega-\upsilon+1)} + \frac{\mathscr{L}}{\xi\Gamma\omega+1} + \frac{\mathscr{L}}{\xi\Gamma\omega} \right) \|\mathfrak{x}-\mathfrak{y}\|, \\
&\le \frac{1}{\xi} \left\{ \frac{2}{\Gamma(\omega-\upsilon+1)} + \frac{\mathscr{L}}{\Gamma\omega}\left(\frac{1}{\omega}+1\right) \right\} \|\mathfrak{x}-\mathfrak{y}\|, \\
&\le \Lambda\|\mathfrak{x}-\mathfrak{y}\|; \Lambda < 1,
\end{aligned} \tag{25}
$$

where $\Lambda = \frac{1}{\xi}\{\frac{2}{\Gamma(\omega-\upsilon+1)} + \frac{\mathscr{L}}{\Gamma\omega}(\frac{1}{\omega}+1)\}$.

Upon examining the situation, it becomes evident that the criteria of the Banach contraction principle is fulfilled, allowing us to deduce that a unique fixed point exists for the given problem. □

5. Examples

Example 1. Let us consider the multi-FDE

$$
\begin{cases}
\dfrac{10}{11}\mathscr{D}^{\frac{5}{4}}\mathfrak{x}(\phi) - D^{\frac{3}{4}}\mathfrak{x}(\phi) = \dfrac{e^t}{1+e^t}\cos\mathfrak{x}(\phi), & t \in [0,1], \\
\mathfrak{x}(0) = 0, \quad \mathfrak{x}'(1) = 0.
\end{cases} \tag{26}
$$

Here, $\omega = \frac{5}{4}$, $\upsilon = \frac{3}{4}$, $\xi = \frac{10}{11}$, and $\mathfrak{f}(\phi,\mathfrak{x}(\phi)) = \frac{e^t}{1+e^t}\cos\mathfrak{x}(\phi)$.

It is clear that $|\mathfrak{f}(\phi,\mathfrak{x}_1(t) - \mathfrak{f}(\phi,\mathfrak{x}_2(t)| \le \frac{e^t}{1+e^t}|x_1(t) - x_2(t)|$, which fulfills condition (A2).

Here $\mathscr{L} = \frac{e^t}{1+e^t} < 1$.

Hence, by employing the concept of uniqueness and utilizing the Lipschitz condition, $\Lambda.\mathscr{L} < 1$ where $\Lambda = \frac{1}{\omega}\{\frac{1}{\Gamma((\omega-\upsilon))+1} + \frac{\mathscr{L}(\omega+1)}{\Gamma\omega+1}\}$. Based on our deductions, we can assert that the boundary value problem (4) possesses a solitary solution, which is unique.

Example 2. Let us consider the multi-FDE

$$
\begin{cases}
\dfrac{6}{5}\mathscr{D}^{\frac{3}{2}}\mathfrak{x}(\phi) - D^{\frac{1}{2}}\mathfrak{x}(\phi) = 100\pi + \dfrac{\sin\mathfrak{x}(\phi)}{100\pi} + \dfrac{1}{100\pi}\mathfrak{x}(\phi), & t \in [0,1], \\
\mathfrak{x}(0) = 0, \quad \mathfrak{x}'(1) = 0.
\end{cases} \tag{27}
$$

Here, $\omega = \frac{3}{2}$, $\upsilon = \frac{1}{2}$, $\xi = \frac{6}{5}$, and $\mathfrak{f}(\phi,\mathfrak{x}(\phi)) = 100\pi + \frac{\sin\mathfrak{x}(\phi)}{100\pi} + \frac{1}{100\pi}\mathfrak{x}(\phi)$.

It is clear that $|\mathfrak{f}(\phi,\mathfrak{x}_1(t) - \mathfrak{f}(\phi,\mathfrak{x}_2(t)| \le \frac{1}{50\pi}|x_1(t) - x_2(t)|$, which fulfills condition (A2).

Here $\mathscr{L} = \frac{1}{50\pi} < 1$.

Hence, by employing the concept of uniqueness and utilizing the Lipschitz condition, $\Lambda.\mathscr{L} < 1$ where $\Lambda = \frac{1}{\omega}\{\frac{1}{\Gamma((\omega-\upsilon))+1} + \frac{\mathscr{L}(\omega+1)}{\Gamma\omega+1}\}$. Based on our deductions, we can assert that the boundary value problem (4) possesses a solitary solution, which is unique.

6. Conclusion

This research paper delves into the analysis of nonlinear multifractional differential equations, specifically focusing on the boundary-value problems involving mixed FDEs. The investigation involves the utilization of Krasnoselskii's fixed point theorems to establish existence results, while the

Banach contraction mapping principle is employed to obtain a uniqueness theorem. Furthermore, the validity of the obtained results is verified through the inclusion of two illustrative examples.

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Chanon Promsakon, Intesham Ansari, Mecieu Wetsah, Anoop Kumar, Kulandhaivel Karthikeyan, and Thanin Sitthiwirattham contributed to the study conception and design, material preparation, data collection, and analysis and drafted the first manuscript. All authors commented on the previous versions of the manuscript. All authors read and approved the final manuscript.

Acknowledgments

This research was funded by the National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-65-49.

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