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Research Article

Modified One-Parameter Liu Estimator for the Linear Regression Model

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Motivated by the ridge regression (Hoerl and Kennard, 1970) and Liu (1993) estimators, this paper proposes a modified Liu estimator to solve the multicollinearity problem for the linear regression model. This modification places this estimator in the class of the ridge and Liu estimators with a single biasing parameter. Theoretical comparisons, real-life application, and simulation results show that it consistently dominates the usual Liu estimator. Under some conditions, it performs better than the ridge regression estimators in the smaller MSE sense. Two real-life data are analyzed to illustrate the findings of the paper and the performances of the estimators assessed by MSE and the mean squared prediction error. The application result agrees with the theoretical and simulation results.

1. Introduction

The linear regression model (LRM) is

$$y = X\theta + \epsilon, \tag{1}$$

where $y_{n\times 1}$ is a vector of the predictand, $X_{n\times p}$ is a known matrix of predictor variables, $\theta_{p\times 1}$ is a vector of unknown regression parameters, $\epsilon_{n\times 1}$ is a vector of errors such that $E(\epsilon) = 0$ and $V(\varepsilon) = \sigma^2 I_n$, and I_n is an $n \times n$ identity matrix. The parameters in (1) are mostly estimated by the ordinary least square (OLS) estimator defined in (2)

$$\hat{\theta} = \left(X' X\right)^{-1} X' y. \tag{2}$$

The performance of the estimator is conditional on the non-violation of the assumption of model (1) that the predictor variables are independent. However, in most real-life applications, we observed that the predictor variables grow together, which result in the problem termed multicollinearity. The consequence of this on the OLS estimator is that it reduces its efficiency and it became unstable (for examples, [1, 2]). Many methods exist in literature to combat the multicollinearity problem. Biased estimators with one biasing parameter include the ridge regression estimator by Hoerl and Kennard [1] and the Liu estimator by Kejian [3], among others.

The objective of this paper is to propose a new one-parameter Liu-type estimator for the regression parameter when the predictor variables of the model are linearly related. Since we want to compare the performance of the proposed estimator with the usual Liu and ridge regression estimators, we will give a brief description of each of them as follows.

1.1. Ridge Regression Estimator (RRE). Hoerl and Kennard [1] proposed $\hat{\theta}_k$ by augmenting $0 = k^{1/2}\theta + \epsilon'$ to the linear regression model (1). The ridge regression estimator is defined as

$$\hat{\theta}_k = \left(X'X + kI_p\right)^{-1} X'y. \quad (3)$$

1.2. Liu Estimator. Since the ridge regression estimator is a complicated function of k, Liu [3] derived $\hat{\theta}_d$ by augmenting $d\hat{\theta}_d = \theta + \epsilon'$ to the linear regression model (1) [4]. The Liu estimator of θ is

$$\hat{\theta}_d = \left(X'X + I_p\right)^{-1}\left(X'X + dI_p\right)\hat{\theta}. \quad (4)$$

This estimator is a linear function of the shrinkage parameter d.

1.3. Proposed One-Parameter Liu Estimator. One of the limitations of the shrinkage parameter by Kejian [3] is that it returns a negative value most of the time which affects the performance of the estimator [4, 5]. In this study, we augment $-d\hat{\theta}_d = \theta + \epsilon'$ to the LRM. This is done by minimizing $(\theta + d\theta^*)'(\theta + d\theta^*)$ subject to $(y - X\theta^*)'(y - X\theta^*) = c$ where c is a constant. The modified Liu estimator of θ is

$$\hat{\theta}_{\mathrm{ML}} = \left(X'X + I_p\right)^{-1}\left(X'X - d_{\mathrm{ML}}I_p\right)\hat{\theta}. \quad (5)$$

This modification provides a substantial improvement in the performance of the modified Liu estimator and will give a positive value of the shrinkage parameter d. The estimator will always produce a smaller mean square error compared to the OLS estimator.

The bias, variance, and MSE of the proposed estimator are, respectively, given as follows:

$$\mathrm{Bias}\left(\hat{\theta}_{\mathrm{ML}}\right) = -(d_{\mathrm{ML}} - 1)\left(X'X + I_p\right)^{-1}\theta, \quad (6)$$

$$\mathrm{Var}\left(\hat{\theta}_{\mathrm{ML}}\right) = \left(X'X + I_p\right)^{-1}\left(X'X - d_{\mathrm{ML}}I_p\right)\sigma^2\left(X'X\right)^{-1} \cdot \left(X'X - d_{\mathrm{ML}}I_p\right)\left(X'X + I_p\right)^{-1}, \quad (7)$$

$$\mathrm{MSEM}\left(\hat{\theta}_{\mathrm{ML}}\right) = \left(X'X + I_p\right)^{-1}\left(X'X - d_{\mathrm{ML}}I_p\right)\sigma^2 \cdot \left(X'X\right)^{-1}\left(X'X - d_{\mathrm{ML}}I_p\right)\left(X'X + I_p\right)^{-1} + (d_{\mathrm{ML}} - 1)^2\left(X'X + I_p\right)^{-1}\theta\theta'\left(X'X + I_p\right)^{-1}. \quad (8)$$

To compare the performance of the estimators, we will consider the linear regression model in canonical form, which is given as follows:

$$y = Z\alpha + \epsilon, \quad (9)$$

where $Z = XV$, $\alpha = V'\theta$, $Z'Z = V'X'XV = A$. A and V' are the eigenvalues and eigenvectors of $X'X$. The ordinary, ridge, Liu, and modified Liu estimators of α are

$$\hat{\alpha} = A^{-1}Z'y, \quad (10)$$

$$\hat{\alpha}_k = \left(A + kI_p\right)^{-1}Z'y, \quad (11)$$

$$\hat{\alpha}_d = \left(A + I_p\right)^{-1}\left(A + dI_p\right)\hat{\alpha}, \quad (12)$$

$$\hat{\alpha}_{\mathrm{ML}} = \left(A + I_p\right)^{-1}\left(A - d_{\mathrm{ML}}I_p\right)\hat{\alpha}. \quad (13)$$

The following notations and lemmas will be used to prove the statistical property of $\hat{\alpha}_{\mathrm{ML}}$.

Lemma 1. *Given that matrix $G > 0$ and α is a vector, $G - \alpha\alpha' \geq 0$ if and only if $\alpha' M^{-1}\alpha \leq 1$ [6].*

Lemma 2. *Given two estimators of θ, $\hat{\theta}_1 = B_1 y$ and $\hat{\theta}_2 = B_2 y$. Suppose that $D = Cov(\hat{\theta}_1) - Cov(\hat{\theta}_2) > 0$, where $Cov(\hat{\theta}_1)$ and $Cov(\hat{\theta}_2)$ represent the covariance matrix of $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively. Therefore, $MSEM(\hat{\theta}_1) - MSEM(\hat{\theta}_2) > 0$ if and only if $c_2'[\sigma^2 D + c_1 c_1']^{-1} c_2 < 1$ where $MSEM(\hat{\theta}_j) = Cov(\hat{\theta}_j) + c_j c_j'$ where $MSEM(\hat{\theta}_j)$ and c_j represent the mean squared error matrix and bias vector of $\hat{\theta}_i$ [7].*

According to Özkale and Kaçiranlar [4], if $\mathrm{MSEM}(\hat{\theta}_1) - \mathrm{MSEM}(\hat{\theta}_2) > 0$, then $\mathrm{SMSE}(\hat{\theta}_1) - \mathrm{SMSE}(\hat{\theta}_2) > 0$ where SMSE is the scalar mean square error.

The rest of the paper is as follows. The theoretical comparison among the estimators and the estimation of the biasing parameter of the proposed estimator are given in Section 2. A simulation study and numerical examples are conducted in Sections 3 and 4, respectively. This paper ends up with some concluding remarks in Section 5.

2. Comparison among Estimators

In this section, we will show theoretical comparisons among the estimators. First, we will compare between the proposed estimator and OLSE.

2.1. The Proposed Estimator and OLSE

Theorem 3. *Given two estimators of α, $\hat{\alpha}$, and $\hat{\alpha}_{ML}$, if $d_{ML} > 0$, then the estimator $\hat{\alpha}_{ML}$ is better than $\hat{\alpha}$, that is, $MSEM(\hat{\alpha}) - MSEM(\hat{\alpha}_{ML}) > 0$ if and only if, $c_2'[\sigma^2 \left(A^{-1} - (A + I_p)^{-1}(A - d_{ML}I_p)A^{-1}(A - d_{ML}I_p)(A + I_p)^{-1}\right)]^{-1}c_2 < 1$ where $c_2 = -(d_{ML} - 1)(A + I_p)^{-1}\alpha$.*

Proof. Recall that

$$\mathrm{Var}[\hat{\alpha}] = \sigma^2 A^{-1}, \quad (14)$$

$$\mathrm{Var}(\hat{\alpha}_{\mathrm{ML}}) = \sigma^2\left(A + I_p\right)^{-1}\left(A - d_{\mathrm{ML}}I_p\right)A^{-1} \cdot \left(A - d_{\mathrm{ML}}I_p\right)\left(A + I_p\right)^{-1}. \quad (15)$$

Table 1: Estimated MSE when $n = 30$ and $p = 3$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
	0.1	0.3621	0.3517	0.2909	0.2782	0.5473	0.5186	0.3750	0.3461	1.1540	1.0117	0.5323	0.4458	12.774	4.1554	0.8567	0.3894
	0.2	0.3621	0.3420	0.2977	0.2722	0.5473	0.4929	0.3908	0.3329	1.1540	0.8995	0.5826	0.4096	12.774	2.3395	1.3877	0.4531
	0.3	0.3621	0.3329	0.3047	0.2666	0.5473	0.4699	0.4073	0.3206	1.1540	0.8094	0.6376	0.3781	12.774	1.6033	2.1170	0.7151
	0.4	0.3621	0.3245	0.3121	0.2612	0.5473	0.4493	0.4248	0.3092	1.1540	0.736	0.6973	0.3513	12.774	1.2143	3.0446	1.1753
	0.5	0.3621	0.3167	0.3197	0.2561	0.5473	0.4306	0.4431	0.2985	1.1540	0.6754	0.7617	0.3292	12.774	0.9779	4.1705	1.8339
1	0.6	0.3621	0.3093	0.3276	0.2513	0.5473	0.4137	0.4622	0.2888	1.1540	0.6247	0.8307	0.3117	12.774	0.8214	5.4946	2.6906
	0.7	0.3621	0.3025	0.3358	0.2468	0.5473	0.3991	0.4822	0.2799	1.1540	0.5819	0.9045	0.2990	12.774	0.7117	7.0169	3.7457
	0.8	0.3621	0.2961	0.3443	0.2425	0.5473	0.3845	0.5031	0.2718	1.1540	0.5454	0.9830	0.2910	12.774	0.6314	8.7375	4.9990
	0.9	0.3621	0.29	0.353	0.2385	0.5473	0.3718	0.5248	0.2646	1.1540	0.5139	1.0661	0.2876	12.774	0.5708	10.656	6.4505
	1.0	0.3621	0.2844	0.3621	0.2348	0.5473	0.3601	0.5473	0.2583	1.1540	0.4867	1.1540	0.2890	12.774	0.5239	12.774	8.1003
	0.1	8.0209	7.7585	6.1372	5.7600	12.967	12.232	8.3642	7.5029	28.462	24.840	12.067	9.5146	319.35	102.39	17.451	4.1230
	0.2	8.0209	7.5109	6.3314	5.5771	12.967	11.567	8.8169	7.0943	28.462	21.945	13.492	8.3869	319.35	56.008	31.445	4.7897
	0.3	8.0209	7.2770	6.5294	5.3980	12.967	10.962	9.2843	6.7003	28.462	19.588	15.017	7.3584	319.35	36.978	50.327	10.344
	0.4	8.0209	7.0557	6.7311	5.2226	12.967	10.411	9.7664	6.321	28.462	17.641	16.640	6.4290	319.35	26.816	74.095	20.784
	0.5	8.0209	6.8461	6.9367	5.0510	12.967	9.9071	10.263	5.9564	28.462	16.010	18.362	5.5988	319.35	20.580	102.75	36.112
5	0.6	8.0209	6.6474	7.146	4.8832	12.967	9.4445	10.775	5.6065	28.462	14.627	20.184	4.8677	319.35	16.415	136.29	56.327
	0.7	8.0209	6.4588	7.3591	4.7192	12.967	9.0189	11.301	5.2713	28.462	13.442	22.105	4.2357	319.35	13.467	174.72	81.429
	0.8	8.0209	6.2795	7.5759	4.5589	12.967	8.6262	11.842	4.9508	28.462	12.418	24.124	3.7028	319.35	11.293	218.04	111.42
	0.9	8.0209	6.1090	7.7953	4.4024	12.967	8.2630	12.397	4.6450	28.462	11.526	26.243	3.2691	319.35	9.6369	266.24	146.30
	1.0	8.0209	5.9467	8.0209	4.2497	12.967	7.9262	12.967	4.3538	28.462	10.741	28.462	2.9345	319.35	8.3433	319.33	186.06
	0.1	31.993	30.939	24.421	22.901	51.819	48.871	33.333	29.865	113.84	99.331	48.088	37.822	1277.4	409.25	69.149	15.713
	0.2	31.993	29.945	25.203	22.163	51.819	46.201	35.155	28.218	113.84	87.726	53.814	33.282	1277.4	223.57	125.20	18.323
	0.3	31.993	29.005	26.000	21.440	51.819	43.775	37.034	26.629	113.84	78.278	59.935	29.137	1277.4	147.37	200.79	40.484
	0.4	31.993	28.116	26.812	20.731	51.819	41.561	38.972	25.098	113.84	70.466	66.450	25.387	1277.4	106.67	295.94	82.197
	0.5	31.993	27.274	27.639	20.038	51.819	39.536	40.968	23.626	113.84	63.919	73.361	22.031	1277.4	81.687	410.64	143.46
10	0.6	31.993	26.474	28.480	19.359	51.819	37.677	43.022	22.211	113.84	58.368	80.667	19.071	1277.4	64.998	544.90	224.28
	0.7	31.993	25.715	29.336	18.695	51.819	35.966	45.134	20.855	113.84	53.612	88.368	16.506	1277.4	53.189	698.70	324.65
	0.8	31.993	24.994	30.207	18.045	51.819	34.387	47.304	19.557	113.84	49.498	96.464	14.336	1277.4	44.476	872.06	444.57
	0.9	31.993	24.307	31.092	17.411	51.819	32.926	49.533	18.316	113.84	45.910	104.96	12.562	1277.4	37.839	1065.0	584.04
	1.0	31.993	23.654	31.993	16.791	51.819	31.570	51.819	17.134	113.84	42.758	113.84	11.182	1277.4	32.655	1277.4	743.07

Table 2: Estimated MSE when $n = 50$ and $p = 3$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
1	0.1	0.1336	0.1325	0.1249	0.1232	0.1806	0.1785	0.1646	0.1614	0.3253	0.3172	0.2685	0.2577	2.8206	2.1349	0.7453	0.5194
	0.2	0.1336	0.1314	0.1258	0.1224	0.1806	0.1765	0.1662	0.1599	0.3253	0.3097	0.2741	0.2526	2.8206	1.6977	0.8903	0.4386
	0.3	0.1336	0.1304	0.1267	0.1216	0.1806	0.1746	0.1679	0.1584	0.3253	0.3025	0.2799	0.2477	2.8206	1.3996	1.0568	0.3791
	0.4	0.1336	0.1294	0.1277	0.1208	0.1806	0.1727	0.1696	0.1570	0.3253	0.2958	0.2859	0.2429	2.8206	1.1861	1.2446	0.3410
	0.5	0.1336	0.1285	0.1286	0.1200	0.1806	0.1709	0.1713	0.1556	0.3253	0.2895	0.2920	0.2383	2.8206	1.0274	1.4538	0.3243
	0.6	0.1336	0.1275	0.1296	0.1192	0.1806	0.1692	0.1731	0.1542	0.3253	0.2836	0.2983	0.2339	2.8206	0.9058	1.6844	0.3289
	0.7	0.1336	0.1266	0.1305	0.1185	0.1806	0.1676	0.1749	0.1529	0.3253	0.2780	0.3048	0.2297	2.8206	0.8105	1.9363	0.3550
	0.8	0.1336	0.1257	0.1315	0.1177	0.1806	0.1660	0.1767	0.1516	0.3253	0.2727	0.3115	0.2256	2.8206	0.7344	2.2097	0.4025
	0.9	0.1336	0.1249	0.1325	0.1170	0.1806	0.1644	0.1786	0.1504	0.3253	0.2677	0.3183	0.2217	2.8206	0.6726	2.5044	0.4713
	1	0.1336	0.1241	0.1336	0.1163	0.1806	0.1630	0.1806	0.1491	0.3253	0.2630	0.3253	0.2180	2.8206	0.6217	2.8206	0.5615
5	0.1	3.0492	3.0174	2.7831	2.7259	4.1707	4.1067	3.6537	3.5443	7.5964	7.3662	5.9057	5.5633	69.766	52.391	15.944	9.5391
	0.2	3.0492	2.9860	2.8119	2.6976	4.1707	4.0443	3.7092	3.4904	7.5964	7.1477	6.0815	5.3966	69.766	41.181	19.904	7.0945
	0.3	3.0492	2.9553	2.8410	2.6695	4.1707	3.9836	3.7651	3.4370	7.5964	6.9399	6.2603	5.2329	69.766	33.446	24.369	5.1550
	0.4	3.0492	2.9251	2.8702	2.6415	4.1707	3.9244	3.8216	3.3840	7.5964	6.7421	6.4421	5.0723	69.766	27.840	29.339	3.7205
	0.5	3.0492	2.8954	2.8996	2.6138	4.1707	3.8667	3.8785	3.3315	7.5964	6.5538	6.6269	4.9146	69.766	23.620	34.814	2.7912
	0.6	3.0492	2.8662	2.9291	2.5862	4.1707	3.8104	3.9360	3.2796	7.5964	6.3742	6.8148	4.7601	69.766	20.350	40.794	2.3669
	0.7	3.0492	2.8375	2.9589	2.5588	4.1707	3.7555	3.9939	3.2281	7.5964	6.2028	7.0056	4.6085	69.766	17.755	47.280	2.4478
	0.8	3.0492	2.8094	2.9888	2.5315	4.1707	3.7020	4.0523	3.1771	7.5964	6.0391	7.1995	4.4599	69.766	15.655	54.270	3.0337
	0.9	3.0492	2.7817	3.0189	2.5045	4.1707	3.6498	4.1113	3.1267	7.5964	5.8826	7.3965	4.3144	69.766	13.929	61.766	4.1247
	1	3.0492	2.7544	3.0492	2.4776	4.1707	3.5988	4.1707	3.0767	7.5964	5.7330	7.5964	4.1719	69.766	12.489	69.767	5.7209
10	0.1	12.177	12.049	11.108	10.877	16.661	16.404	14.582	14.141	30.351	29.427	23.561	22.182	279.07	209.56	63.558	37.824
	0.2	12.177	11.923	11.224	10.763	16.661	16.154	14.805	13.923	30.351	28.550	24.267	21.511	279.07	164.70	79.447	27.978
	0.3	12.177	11.800	11.340	10.650	16.661	15.909	15.030	13.707	30.351	27.716	24.986	20.852	279.07	133.72	97.350	20.146
	0.4	12.177	11.678	11.458	10.537	16.661	15.671	15.257	13.494	30.351	26.922	25.717	20.205	279.07	111.26	117.27	14.328
	0.5	12.177	11.559	11.576	10.425	16.661	15.439	15.487	13.282	30.351	26.166	26.459	19.569	279.07	94.352	139.20	10.525
	0.6	12.177	11.442	11.695	10.314	16.661	15.213	15.718	13.072	30.351	25.444	27.214	18.945	279.07	81.240	163.14	8.7356
	0.7	12.177	11.327	11.814	10.204	16.661	14.992	15.951	12.864	30.351	24.755	27.980	18.334	279.07	70.829	189.10	8.9606
	0.8	12.177	11.213	11.934	10.094	16.661	14.776	16.186	12.658	30.351	24.097	28.758	17.734	279.07	62.401	217.08	11.200
	0.9	12.177	11.102	12.055	9.9840	16.661	14.566	16.423	12.454	30.351	23.468	29.548	17.146	279.07	55.468	247.06	15.453
	1	12.177	10.992	12.177	9.8760	16.661	14.360	16.661	12.252	30.351	22.866	30.351	16.570	279.07	49.684	279.07	21.721

Table 3: Estimated MSE when $n = 100$ and $p = 3$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
1	0.1	0.1124	0.1121	0.1105	0.1103	0.1492	0.1478	0.1396	0.1380	0.2867	0.2781	0.2304	0.2209	3.0721	2.1409	0.6883	0.4808
	0.2	0.1124	0.1118	0.1107	0.1102	0.1492	0.1465	0.1404	0.1372	0.2867	0.2702	0.2356	0.2165	3.0721	1.6206	0.8360	0.4210
	0.3	0.1124	0.1116	0.1108	0.1101	0.1492	0.1453	0.1414	0.1365	0.2867	0.2629	0.2410	0.2124	3.0721	1.2979	1.0130	0.3905
	0.4	0.1124	0.1114	0.1110	0.1100	0.1492	0.1442	0.1423	0.1358	0.2867	0.2562	0.2468	0.2086	3.0721	1.0825	1.2192	0.3893
	0.5	0.1124	0.1112	0.1112	0.1100	0.1492	0.1432	0.1434	0.1352	0.2867	0.2501	0.2527	0.2050	3.0721	0.9305	1.4548	0.4174
	0.6	0.1124	0.111	0.1114	0.1100	0.1492	0.1422	0.1444	0.1346	0.2867	0.2444	0.2590	0.2017	3.0721	0.8186	1.7197	0.4748
	0.7	0.1124	0.1108	0.1116	0.1099	0.1492	0.1412	0.1455	0.1341	0.2867	0.2391	0.2655	0.1987	3.0721	0.7333	2.0138	0.5615
	0.8	0.1124	0.1106	0.1119	0.1099	0.1492	0.1403	0.1467	0.1336	0.2867	0.2342	0.2723	0.1959	3.0721	0.6667	2.3373	0.6775
	0.9	0.1124	0.1105	0.1121	0.1099	0.1492	0.1395	0.1479	0.1332	0.2867	0.2297	0.2793	0.1934	3.0721	0.6134	2.6900	0.8228
	1.0	0.1124	0.1104	0.1124	0.1099	0.1492	0.1387	0.1492	0.1328	0.2867	0.2255	0.2867	0.1912	3.0721	0.5699	3.0721	0.9973
5	0.1	2.0631	2.0452	1.9126	1.8802	3.2440	3.1954	2.8523	2.7697	6.9576	6.7191	5.2564	4.9172	76.772	53.314	14.747	8.2979
	0.2	2.0631	2.0276	1.9289	1.8642	3.2440	3.1480	2.8942	2.7290	6.9576	6.4945	5.4313	4.7529	76.772	39.906	18.971	6.0738
	0.3	2.0631	2.0102	1.9454	1.8483	3.2440	3.1019	2.9365	2.6887	6.9576	6.2827	5.6097	4.5921	76.772	31.412	23.862	4.5166
	0.4	2.0631	1.9932	1.9619	1.8325	3.2440	3.0570	2.9793	2.6487	6.9576	6.0826	5.7917	4.4349	76.772	25.626	29.420	3.6261
	0.5	2.0631	1.9764	1.9785	1.8167	3.2440	3.0133	3.0224	2.6092	6.9576	5.8934	5.9772	4.2811	76.772	21.466	35.645	3.4025
	0.6	2.0631	1.9599	1.9952	1.8011	3.2440	2.9707	3.0659	2.5701	6.9576	5.7143	6.1662	4.1310	76.772	18.350	42.537	3.8457
	0.7	2.0631	1.9436	2.0121	1.7856	3.2440	2.9291	3.1098	2.5314	6.9576	5.5445	6.3587	3.9843	76.772	15.939	50.096	4.9557
	0.8	2.0631	1.9276	2.029	1.7702	3.2440	2.8887	3.1542	2.4931	6.9576	5.3834	6.5548	3.8412	76.772	14.024	58.321	6.7325
	0.9	2.0631	1.9119	2.0460	1.7548	3.2440	2.8492	3.1989	2.4553	6.9576	5.2305	6.7544	3.7016	76.772	12.471	67.213	9.1762
	1.0	2.0631	1.8964	2.0631	1.7396	3.2440	2.8108	3.2440	2.4178	6.9576	5.0850	6.9576	3.5656	76.772	11.189	76.772	12.287
10	0.1	8.1632	8.0901	7.5481	7.4154	12.920	12.723	11.334	10.999	27.809	26.853	20.970	19.601	307.09	213.26	58.717	32.755
	0.2	8.1632	8.0182	7.615	7.3496	12.920	12.532	11.505	10.833	27.809	25.951	21.675	18.937	307.09	159.58	75.684	23.760
	0.3	8.1632	7.9474	7.6822	7.2842	12.920	12.346	11.676	10.669	27.809	25.100	22.394	18.287	307.09	125.56	95.308	17.423
	0.4	8.1632	7.8777	7.7498	7.2191	12.920	12.164	11.849	10.507	27.809	24.296	23.126	17.650	307.09	102.37	117.59	13.743
	0.5	8.1632	7.8091	7.8178	7.1544	12.920	11.987	12.024	10.346	27.809	23.535	23.872	17.028	307.09	85.681	142.53	12.721
	0.6	8.1632	7.7415	7.8862	7.0901	12.920	11.814	12.200	10.186	27.809	22.815	24.632	16.419	307.09	73.175	170.13	14.356
	0.7	8.1632	7.675	7.9549	7.0261	12.920	11.646	12.378	10.028	27.809	22.131	25.406	15.823	307.09	63.493	200.38	18.648
	0.8	8.1632	7.6096	8.0240	6.9625	12.920	11.4821	12.557	9.8717	27.809	21.482	26.193	15.242	307.09	55.802	233.29	25.598
	0.9	8.1632	7.5451	8.0934	6.8992	12.920	11.322	12.738	9.7168	27.809	20.865	26.994	14.674	307.09	49.561	268.86	35.206
	1.0	8.1632	7.4816	8.1632	6.8364	12.920	11.166	12.920	9.5634	27.809	20.279	27.809	14.120	307.09	44.407	307.09	47.470

Table 4: Estimated MSE when $n = 30$ and $p = 7$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
1	0.1	0.8375	0.8019	0.6073	0.5645	1.3038	1.2137	0.8030	0.7166	2.7626	2.3689	1.1868	0.9622	30.134	10.720	2.2263	0.9170
	0.2	0.8375	0.7694	0.6299	0.5442	1.3038	1.1356	0.8496	0.6767	2.7626	2.0758	1.3162	0.8670	30.134	6.4830	3.5481	0.9295
	0.3	0.8375	0.7395	0.6532	0.5246	1.3038	1.0673	0.8985	0.6392	2.7626	1.8492	1..4571	0.7833	30.134	4.5630	5.3147	1.3867
	0.4	0.8375	0.7121	0.6773	0.5058	1.3038	1.0072	0.9496	0.6039	2.7626	1.6691	1.6093	0.7109	30.134	3.4720	7.5260	2.2886
	0.5	0.8375	0.6868	0.7021	0.4877	1.3038	0.9539	1.0030	0.5708	2.7626	1.5224	1.7730	0.6500	30.134	2.7762	10.182	3.6354
	0.6	0.8375	0.6635	0.7277	0.4705	1.3038	0.9063	1.0586	0.5400	2.7626	1.4006	1.9481	0.6005	30.134	2.2996	13.283	5.4268
	0.7	0.8375	0.6418	0.7540	0.4539	1.3038	0.8636	1.1166	0.5115	2.7626	1.2980	2.1346	0.5624	30.134	1.9565	16.828	7.6631
	0.8	0.8375	0.6217	0.7811	0.4381	1.3038	0.8252	1.1767	0.4853	2.7626	1.2104	2.3325	0.5357	30.134	1.7004	20.819	10.344
	0.9	0.8375	0.6030	0.8089	0.4231	1.3038	0.7903	1.2391	0.4613	2.7626	1.1348	2.5418	0.5204	30.134	1.5038	25.254	13.470
	1.0	0.8375	0.5855	0.8375	0.4088	1.3038	0.7587	1.3038	0.4395	2.7626	1.0688	2.7626	0.5166	30.134	1.3493	30.134	17.040
5	0.1	20.936	20.040	15.024	13.885	32.596	30.329	19.803	17.525	69.065	59.182	29.022	23.128	753.34	266.98	50.691	15.905
	0.2	20.936	19.216	15.618	13.339	32.596	28.357	21.019	16.463	69.065	51.794	32.378	20.591	753.34	160.14	84.633	15.061
	0.3	20.936	18.457	16.227	12.809	32.596	26.624	22.286	15.452	69.065	46.060	36.008	18.328	753.34	111.36	129.61	25.249
	0.4	20.936	17.755	16.852	12.295	32.596	25.091	23.605	14.493	69.065	41.478	39.911	16.337	753.34	83.415	185.61	46.470
	0.5	20..936	17.104	17.493	11.797	32.596	23.724	24.975	13.586	69.065	37.727	44.087	14.620	753.34	65.445	252.65	78.724
	0.6	20.936	16.498	18.150	11.315	32.596	22.498	26.396	12.729	69.065	34.597	48.536	14.176	753.34	53.028	330.73	122.01
	0.7	20.936	15.933	18.823	10.848	32.596	21.392	27.869	11.924	69.065	31.943	53.258	12.005	753.34	44.012	419.83	176.33
	0.8	20.936	15.405	19.512	10.398	32.596	20.389	29.393	11.171	69.065	29.662	58.254	11.107	753.34	37.220	519.97	241.68
	0.9	20.936	14.911	20.216	9.9630	32.596	19.474	30.969	10.469	69.065	27.679	63.523	10.482	753.34	31.956	631.14	318.06
	1.0	20.936	14.447	20.936	9.5441	32.596	18.638	32.596	9.8179	69.065	25.938	69.065	10.131	753.34	27.782	753.34	405.48
10	0.1	83.746	80.157	60.064	55.497	130.38	121.31	79.156	70.028	276.26	236.71	115.97	92.358	3013.4	1067.6	201.97	62.631
	0.2	83.746	76.859	62.443	53.307	130.38	113.42	84.028	65.771	276.26	207.15	129.41	82.190	3013.4	640.15	337.83	59.156
	0.3	83.746	73.819	64.885	51.181	130.38	106.48	89.105	61.719	276.26	184.20	143.95	73.113	3013.4	444.94	517.82	99.809
	0.4	83.746	71.006	67.390	49.118	130.38	100.34	94.387	57.872	276.26	165.85	159.57	65.129	3013.4	333.07	741.94	184.59
	0.5	83.746	68.397	69.958	47.119	130.38	94.868	99.874	54.230	276.26	150.84	176.29	58.236	3013.4	261.12	1010.2	313.50
	0.6	83.746	65.970	72.589	45.182	130.38	89.957	105.57	50.793	276.26	138.30	194.10	52.434	3013.4	211.39	1322.6	486.54
	0.7	83.746	63.707	75.284	43.309	130.38	85.526	111.46	47.561	276.26	127.67	213.00	47.725	3013.4	175.28	1679.1	703.71
	0.8	83.746	61.592	78.041	41.498	130.38	81.506	117.56	44.534	276.26	118.54	233.00	44.107	3013.4	148.06	2079.7	965.01
	0.9	83.746	59.610	80.862	39.751	130.38	77.842	123.87	41.712	276.26	110.59	254.08	41.581	3013.4	126.97	2524.5	1270.4
	1.0	83.746	57.749	83.746	38.067	130.38	74.490	130.38	39.095	276.26	103.62	276.26	40.147	3013.4	110.24	3013.4	1620.0

Table 5: Estimated MSE when $n = 50$ and $p = 7$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
1	0.1	0.4876	0.4746	0.3942	0.3756	0.7704	0.7350	0.5473	0.5057	1.6417	1.4749	0.8662	0.7433	17.673	7.9846	1.9910	1.0357
	0.2	0.4876	0.4624	0.4038	0.3666	0.7704	0.7029	0.5693	0.4859	1.6417	1.3411	0.9343	0.6886	17.673	5.3441	2.8135	0.9031
	0.3	0.4876	0.4508	0.4136	0.3578	0.7704	0.6738	0.5919	0.4669	1.6417	1.2314	1.0070	0.6384	17.673	4.0252	3.8661	1.0004
	0.4	0.4876	0.4399	0.4236	0.3492	0.7704	0.6471	0.6152	0.4485	1.6417	1.1399	1.0842	0.5927	17.673	3.2147	5.1486	1.3276
	0.5	0.4876	0.4295	0.4338	0.3408	0.7704	0.6227	0.6393	0.4309	1.6417	1.0623	1.1659	0.5515	17.673	2.6636	6.6610	1.8849
	0.6	0.4876	0.4196	0.4441	0.3326	0.7704	0.6003	0.6641	0.4140	1.6417	0.9957	1.2520	0.5148	17.673	2.2653	8.4035	2.6720
	0.7	0.4876	0.4103	0.4547	0.3246	0.7704	0.5796	0.6896	0.3979	1.6417	0.9379	1.3427	0.4826	17.673	1.9654	10.376	3.6892
	0.8	0.4876	0.4014	0.4655	0.3167	0.7704	0.5604	0.7158	0.3824	1.6417	0.8873	1.4379	0.4549	17.673	1.7327	12.578	4.9363
	0.9	0.4876	0.3929	0.4764	0.3091	0.7704	0.5427	0.7428	0.3677	1.6417	0.8424	1.5375	0.4317	17.673	1.5479	15.011	6.4134
	1.0	0.4876	0.3849	0.4876	0.3016	0.7704	0.5261	0.7704	0.3537	1.6417	0.8025	1.6417	0.4130	17.673	1.3984	17.673	8.1205
5	0.1	12.190	11.866	9.8393	9.3648	19.261	18.376	13.650	12.589	41.042	36.874	21.539	18.403	441.82	199.51	47.225	22.001
	0.2	12.190	11.560	10.083	9.1341	19.261	17.573	14.206	12.084	41.042	33.525	23.270	16.999	441.82	133.20	68.355	17.906
	0.3	12.190	11.270	10.331	8.9077	19.261	16.842	14.779	11.595	41.042	30.775	25.110	15.704	441.82	99.857	95.164	19.490
	0.4	12.190	10.994	10.584	8.6856	19.261	16.174	15.369	11.124	41.042	28.475	27.059	14.517	441.82	79.213	127.65	26.753
	0.5	12.190	10.733	10.841	8.4680	19.261	15.560	15.975	10.669	41.042	26.522	29.117	13.440	441.82	65.056	165.82	39.694
	0.6	12.190	10.733	10.841	8.4679	19.261	14.994	16.598	10.232	41.042	24.840	31.284	12.471	441.82	54.734	209.66	58.313
	0.7	12.190	10.733	10.841	8.4680	19.261	14.470	17.239	9.8108	41.042	23.375	33.560	11.612	441.82	46.888	259.18	82.611
	0.8	12.190	10.484	11.102	8.2545	19.261	13.985	17.896	9.4068	41.042	22.086	35.945	10.861	441.82	40.740	314.38	112.59
	0.9	12.190	10.248	11.367	8.0455	19.261	13.533	18.570	9.0197	41.042	20.941	38.439	10.220	441.82	35.807	375.26	148.24
	1.0	12.190	9.5999	12.190	7.4446	19.261	13.111	19.261	8.6495	41.042	19.917	41.042	9.6868	441.82	31.774	441.82	189.58
10	0.1	48.759	47.464	39.357	37.458	77.044	73.503	54.600	50.354	164.17	147.50	86.148	73.599	1767.3	798.06	188.64	87.571
	0.2	48.759	46.239	40.333	36.535	77.044	70.294	56.824	48.331	164.17	134.10	93.076	67.978	1767.3	532.77	273.23	71.092
	0.3	48.759	45.079	41.325	35.629	77.044	67.370	59.116	46.377	164.17	123.10	100.44	62.791	1767.3	399.37	380.53	77.317
	0.4	48.759	43.978	42.335	34.740	77.044	64.696	61.475	44.489	164.17	113.90	108.24	58.041	1767.3	316.75	510.52	106.245
	0.5	48.759	42.933	43.363	33.868	77.044	62.240	63.901	42.669	164.17	106.09	116.47	53.725	1767.3	260.08	663.22	157.88
	0.6	48.759	41.938	44.407	33.014	77.044	59.976	66.395	40.917	164.17	99.359	125.14	49.845	1767.3	218.75	838.63	232.21
	0.7	48.759	40.991	45.469	32.177	77.044	57.882	68.956	39.231	164.17	93.497	134.24	46.400	1767.3	187.32	1036.7	329.25
	0.8	48.759	40.087	46.548	31.357	77.044	55.939	71.584	37.613	164.17	88.338	143.78	43.390	1767.3	162.69	1257.6	449.00
	0.9	48.759	39.224	47.645	30.555	77.044	54.131	74.280	36.063	164.17	83.757	153.76	40.816	1767.3	142.92	1501.1	591.44
	1.0	48.759	38.399	48.759	29.770	77.044	52.443	77.044	34.580	164.17	79.656	164.17	38.677	1767.3	126.75	1767.3	756.59

Table 6: Estimated MSE when $n = 100$ and $p = 7$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
1	0.1	0.2225	0.2195	0.1989	0.1942	0.3446	0.3366	0.2865	0.2751	0.7153	0.6783	0.4938	0.4543	7.4001	4.6647	1.6335	1.1431
	0.2	0.2225	0.2165	0.2014	0.1919	0.3446	0.3290	0.2924	0.2697	0.7153	0.6453	0.5149	0.4358	7.4001	3.4961	1.9866	1.0057
	0.3	0.2225	0.2137	0.2038	0.1896	0.3446	0.3218	0.2985	0.2643	0.7153	0.6158	0.5369	0.4182	7.4001	2.8469	2.4116	0.9403
	0.4	0.2225	0.2110	0.2064	0.1874	0.3446	0.3150	0.3046	0.2591	0.7153	0.5892	0.5597	0.4015	7.4001	2.4268	2.9085	0.9468
	0.5	0.2225	0.2084	0.2090	0.1852	0.3446	0.3085	0.3110	0.2540	0.7153	0.5652	0.5835	0.3857	7.4001	2.1278	3.4773	1.0252
	0.6	0.2225	0.2058	0.2116	0.1831	0.3446	0.3024	0.3174	0.2491	0.7153	0.5434	0.6081	0.3707	7.4001	1.9014	4.118	1.1755
	0.7	0.2225	0.2034	0.2142	0.1810	0.3446	0.2966	0.3240	0.2443	0.7153	0.5236	0.6336	0.3566	7.4001	1.7224	4.8307	1.3977
	0.8	0.2225	0.2010	0.2169	0.1789	0.3446	0.2910	0.3308	0.2397	0.7153	0.5055	0.6599	0.3434	7.4001	1.5767	5.6152	1.6918
	0.9	0.2225	0.1987	0.2197	0.1769	0.3446	0.2858	0.3376	0.2352	0.7153	0.4889	0.6872	0.3311	7.4001	1.4552	6.4717	2.0578
	1.0	0.2225	0.1965	0.2225	0.1750	0.3446	0.2808	0.3446	0.2308	0.7153	0.4736	0.7153	0.3197	7.4001	1.3523	7.4001	2.4958
5	0.1	5.5620	5.4869	4.9618	4.8354	8.6161	8.4163	7.1380	6.8372	17.882	16.960	12.260	11.220	185.00	116.57	39.235	26.109
	0.2	5.5620	5.4139	5.0260	4.7731	8.6161	8.2263	7.2921	6.6906	17.882	16.134	12.809	10.728	185.00	87.159	48.425	22.173
	0.3	5.5620	5.3430	5.0908	4.7115	8.6161	8.0453	7.4488	6.5466	17.882	15.390	13.377	10.256	185.00	70.707	59.367	19.988
	0.4	5.5620	5.2741	5.1562	4.6505	8.6161	7.8728	7.6980	6.4050	17.882	14.717	13.963	9.8020	185.00	59.977	72.060	19.556
	0.5	5.5620	5.2071	5.2223	4.5901	8.6161	7.7081	7.7697	6.2660	17.882	14.105	15.687	9.3674	185.00	52.275	86.505	20.874
	0.6	5.5620	5.1419	5.2890	4.5304	8.6161	7.5508	7.9339	6.1294	17.882	13.546	15.193	8.9517	185.00	46.388	102.70	23.944
	0.7	5.5620	5.0785	5.3563	4.4712	8.6161	7.4004	8.1007	5.9954	17.882	13.035	16.000	8.5551	185.00	41.691	120.65	28.766
	0.8	5.5620	5.0169	5.4242	4.4128	8.6161	7.2565	8.2700	5.8640	17.882	12.565	16.500	8.1774	185.00	37.827	140.35	35.339
	0.9	5.5620	4.9568	5.4928	4.3549	8.6161	7.1186	8.4418	5.7350	17.882	12.131	17.181	7.8188	185.00	34.577	161.80	43.664
	1.0	5.5620	4.8983	5.5620	4.2977	8.6161	6.9863	8.6161	5.6086	17.882	11.731	17.882	7.4791	185.00	31.796	185.00	53.740
10	0.1	22.248	21.948	19.849	19.343	34.464	33.666	28.553	27.349	71.528	67.843	49.040	44.873	740.01	466.29	156.77	104.14
	0.2	22.248	21.656	20.106	19.093	34.464	32.907	29.170	26.762	71.528	64.540	51.236	42.903	740.01	348.64	193.58	88.328
	0.3	22.248	21.373	20.365	18.846	34.464	32.183	29.797	26.185	71.528	61.564	53.508	41.009	740.01	282.80	237.39	79.512
	0.4	22.248	21.098	20.626	18.602	34.464	31493	30.434	25.617	71.528	58.870	55.856	39.190	740.01	239.86	288.20	77.693
	0.5	22.248	20.830	20.891	18.360	34.464	30.835	31.081	25.060	71.528	56.422	58.279	37.447	740.01	209.02	346.00	82.874
	0.6	22.248	20.569	21.157	18.120	34.464	30.206	31.738	24.513	71.528	54.187	60.777	35.779	740.01	185.44	410.81	95.052
	0.7	22.248	20.316	21.426	17.883	34.464	29.604	32.405	23.975	71.528	52.141	63.352	34.187	740.01	166.62	482.61	114.23
	0.8	22.248	20.069	21.698	17.649	34.464	29.028	33.081	23.448	71.528	50.259	66.001	32.671	740.01	151.14	561.41	140.41
	0.9	22.248	19.829	21.972	17.416	34.464	28.476	33.768	22.930	71.528	48.524	68.727	31.230	740.01	138.11	647.21	173.58
	1.0	22.248	19.594	22.248	17.187	34.464	27.946	34.464	22.422	71.528	46.919	71.528	29.864	740.01	126.96	740.01	213.75

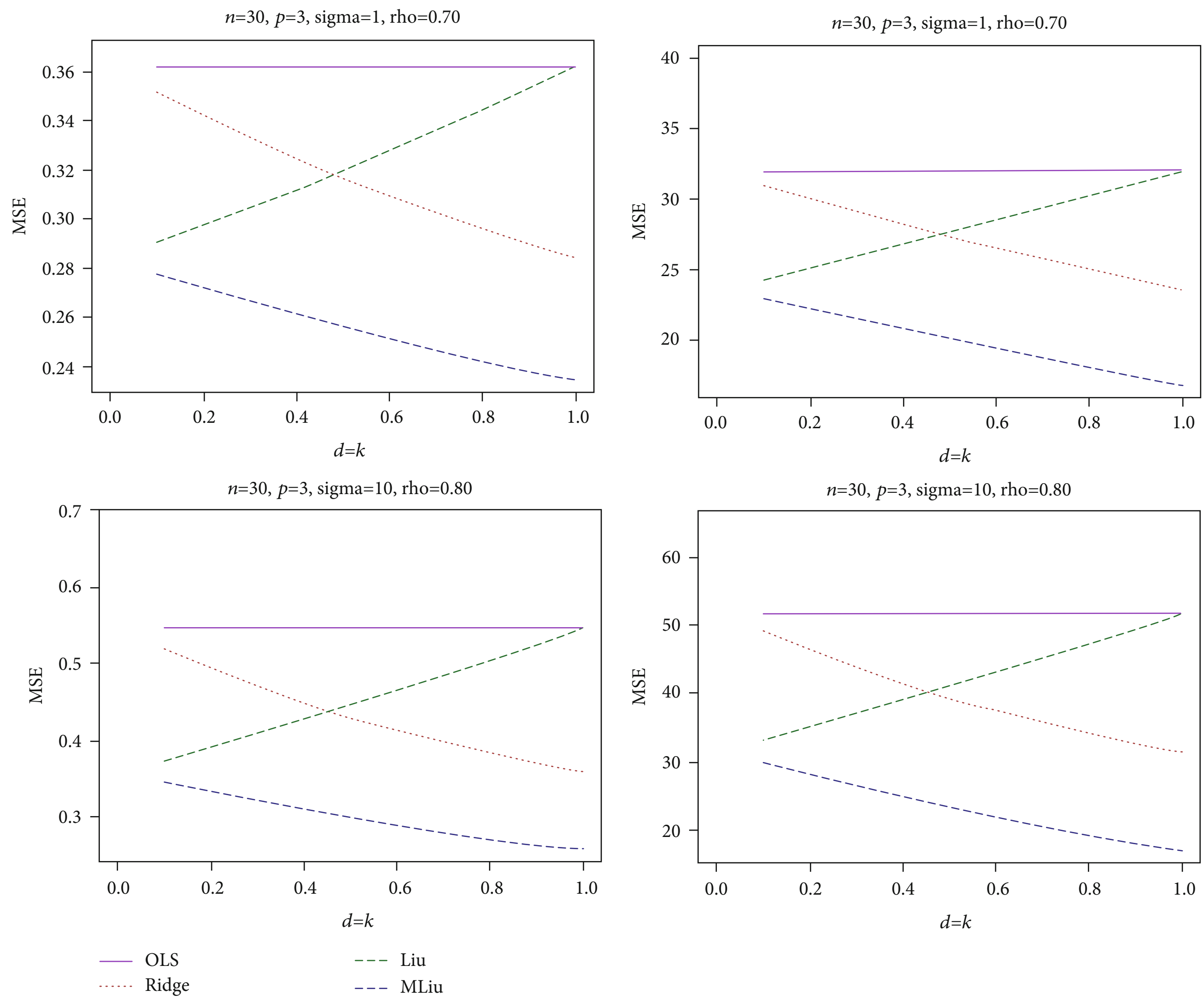

Figure 1: Estimated MSEs for $n = 30$. Sigma = 1, 10; rho = 0.70, 0.80; and different values of $k = d$.

Then,

$$\begin{aligned}\operatorname{Var}[\hat{\alpha}] - \operatorname{Var}(\hat{\alpha}_{\mathrm{ML}}) &= \sigma^2 \Lambda^{-1} - \sigma^2 (\Lambda + I_p)^{-1} (\Lambda - d_{\mathrm{ML}} I_p) \Lambda^{-1} \\ &\quad \cdot (\Lambda - d_{\mathrm{ML}} I_p)(\Lambda + I_p)^{-1} \\ &= \sigma^2 dg \left[\frac{1}{\lambda_j} - \frac{(\lambda_j - d_{\mathrm{ML}})^2}{\lambda_j (\lambda_j + 1)^2} \right]_{j=1}^{p}.\end{aligned} \tag{16}$$

We observed from equation (16) that $(\lambda_j + 1)^2 > (\lambda_j - d_{\mathrm{ML}})^2$ which shows that $\operatorname{Var}[\hat{\alpha}] - \operatorname{Var}(\hat{\alpha}_{\mathrm{ML}}) > 0$.

2.2. The Proposed Estimator and RRE

Theorem 4. *Given two estimators of α, $\hat{\alpha}_k$, and $\hat{\alpha}_{ML}$, if k, $d_{ML} > 0$, then the estimator $\hat{\alpha}_{ML}$ is better than $\hat{\alpha}_k$, that is, $MSEM(\hat{\alpha}_k) - MSEM(\hat{\alpha}_{ML}) > 0$ if and only if, $c_2'[\sigma^2 ((\Lambda + kI_p)^{-1} \Lambda (\Lambda + kI_p)^{-1} - (\Lambda + I_p)^{-1} (\Lambda - d_{ML} I_p) \Lambda^{-1} (\Lambda - d_{ML} I_p)(\Lambda + I_p)^{-1}) + c_1 c_1']^{-1} c_2 < 1$ where $c_1 = -k (\Lambda + kI_p)^{-1} \alpha$ and $c_2 = -(d_{ML} - 1)(\Lambda + I_p)^{-1} \alpha$.*

Proof. Recall that

$$\operatorname{Var}[\hat{\alpha}_k] = (\Lambda + kI_p)^{-1} \Lambda (\Lambda + kI_p)^{-1}. \tag{17}$$

Therefore,

$$\begin{aligned}\operatorname{Var}[\hat{\alpha}_k] - \operatorname{Var}(\hat{\alpha}_{\mathrm{ML}}) &= \sigma^2 (\Lambda + kI_p)^{-1} \Lambda (\Lambda + kI_p)^{-1} - \sigma^2 (\Lambda + I_p)^{-1} \\ &\quad \cdot (\Lambda - d_{\mathrm{ML}} I_p) \Lambda^{-1} (\Lambda - d_{\mathrm{ML}} I_p)(\Lambda + I_p)^{-1} \\ &= \sigma^2 dg \left[\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{(\lambda_j - d_{\mathrm{ML}})^2}{\lambda_j (\lambda_j + 1)^2} \right]_{j=1}^{p}.\end{aligned} \tag{18}$$

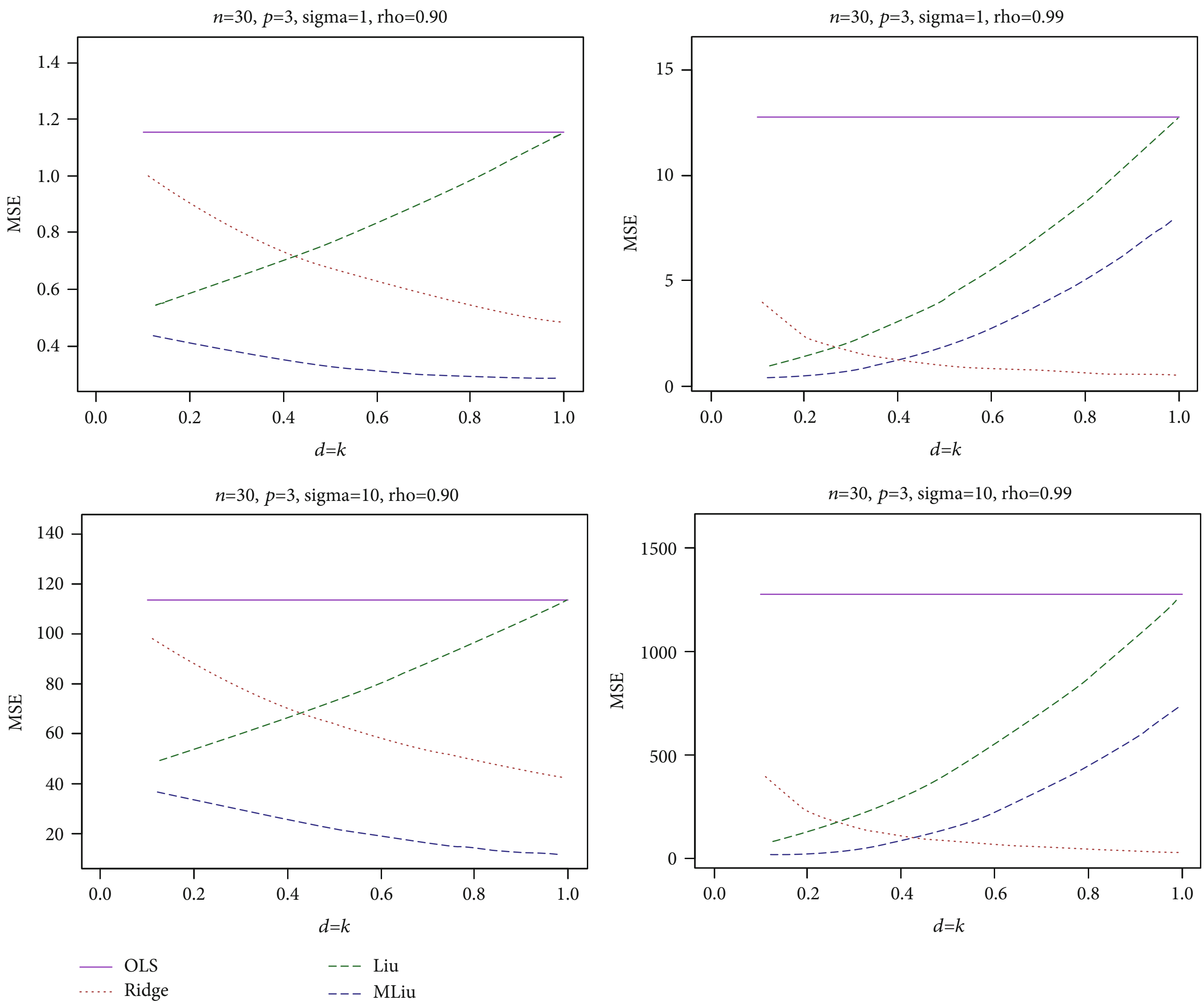

Figure 2: Estimated MSEs for $n = 30$. Sigma $(\sigma) = 1$, 10; rho $(\rho) = 0.90$, 0.99; and different values of $k = d$.

From equation (18), we observed that $\lambda_j^2(\lambda_j + 1)^2 > (\lambda_j - d_{\mathrm{ML}})^2(\lambda_j + k)^2 = \lambda_j(1 + d_{\mathrm{ML}} - k) + kd_{\mathrm{ML}}$.. Hence, this shows that $\mathrm{Var}[\hat{\alpha}_k] - \mathrm{Var}(\hat{\alpha}_{\mathrm{ML}}) > 0$..

2.3. The Proposed Estimator and Liu Estimator

Theorem 5. *Given two estimators of* α, $\hat{\alpha}_d$, *and* $\hat{\alpha}_{ML}$, *if* $0 < d < 1, d_{ML} > 0$, *then the estimator* $\hat{\alpha}_{ML}$ *is better than* $\hat{\alpha}_d$, *that is,* $MSEM(\hat{\alpha}_d) - MSEM(\hat{\alpha}_{ML}) > 0$ *if and only if,* $c_2' [\sigma^2 (A + I_p)^{-1}(A + dI_p)A^{-1}(A + dI_p)(A + I_p)^{-1} - (A + I_p)^{-1}(A - d_{ML}I_p)A^{-1}(A - d_{ML}I_p)(A + I_p)^{-1} + c_1c_1']^{-1}c_2 < 1$ *where* $c_1 = (d - 1)(A + I_p)^{-1}\alpha$ *and* $c_2 = -(d_{ML} - 1)(A + I_p)^{-1}\alpha$.

Proof. Note that

$$\mathrm{Var}[\hat{\alpha}_d] = \left(A + I_p\right)^{-1}\left(A + dI_p\right)A^{-1}\left(A + dI_p\right)\left(A + I_p\right)^{-1}. \tag{19}$$

Therefore,

$$\begin{aligned}\mathrm{Var}[\hat{\alpha}_d] - \mathrm{Var}(\hat{\alpha}_{\mathrm{ML}}) &= \sigma^2\left(A + I_p\right)^{-1}\left(A + dI_p\right)A^{-1}\left(A + dI_p\right)\left(A + I_p\right)^{-1} \\ &\quad - \sigma^2\left(A + I_p\right)^{-1}\left(A - d_{\mathrm{ML}}I_p\right)A^{-1}\left(A - d_{\mathrm{ML}}I_p\right)\left(A + I_p\right)^{-1} \\ &= \sigma^2 dg\left[\frac{\left(\lambda_j + d\right)^2}{\lambda_j\left(\lambda_j + 1\right)^2} - \frac{\left(\lambda_j - d_{\mathrm{ML}}\right)^2}{\lambda_j\left(\lambda_j + 1\right)^2}\right]_{j=1}^{p}.\end{aligned} \tag{20}$$

From equation (20), we observed that $(\lambda_j + d)^2 > (\lambda_j - d_{\mathrm{ML}})^2$. Hence, this shows that$\mathrm{Var}[\hat{\alpha}_d] - \mathrm{Var}(\hat{\alpha}_{\mathrm{ML}}) > 0$.

2.4. Determination of d_{ML}. The shrinkage parameter d_{ML} is selected according to Kejian [3], Kibria [8], Kibria and Banik [9], Lukman and Ayinde [10], and Qasim et al. [11].

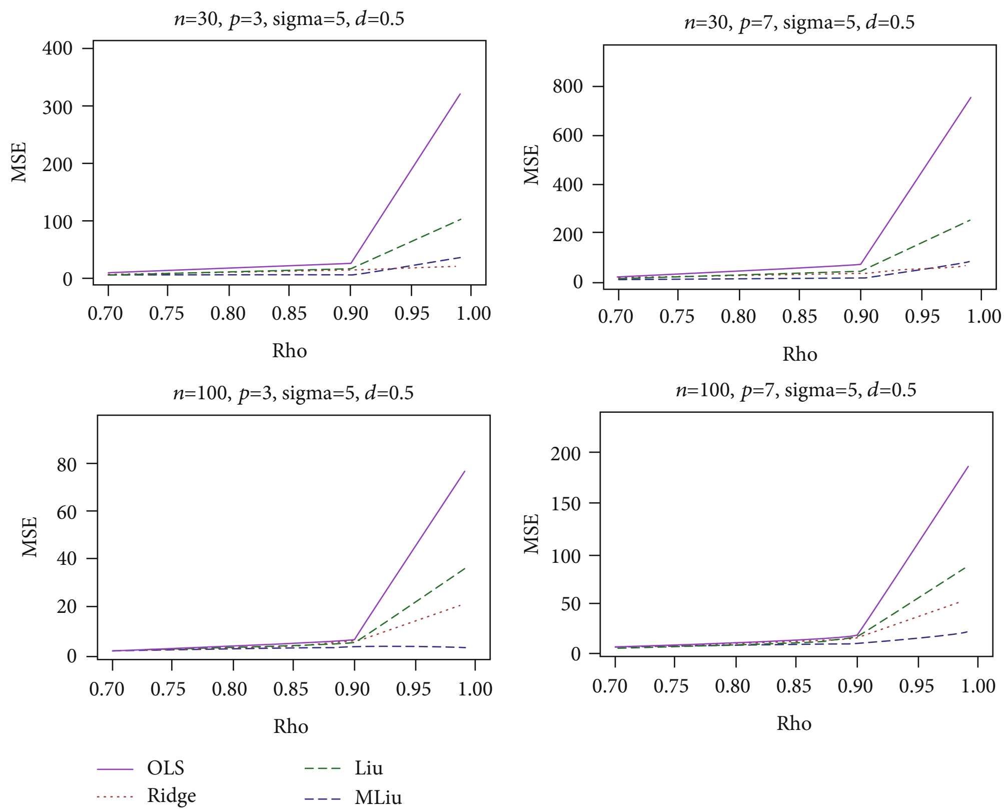

Figure 3: Estimated MSEs for sigma = 5, $p = 3$, $d = 0.5$, and different values of ρ and n.

$$\mathrm{SMSE}(\hat{\alpha}_{ML}) = \sigma^2 \sum_{j=1}^{p} \frac{(\lambda_j - d_{ML} I_p)^2}{\lambda_j(\lambda_j + I_p)^2} + (d_{ML} - 1)^2 \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j + I_p)^2}. \quad (21)$$

The partial derivative of (21) with respect to d_{ML} and setting it to zero, we obtained

$$d_{ML} = \frac{\lambda_j(\sigma^2 + \alpha_j^2)}{\sigma^2 + \lambda_i \alpha_j^2}. \quad (22)$$

In eqn. (22), we replace σ^2 and with its unbiased estimate and obtained:

$$\hat{d}_{ML} = \frac{\lambda_j(\sigma\wedge^2 + \hat{\alpha}_j^2)}{\sigma\wedge^2 + \lambda_j \hat{\alpha}_j^2} \quad (23)$$

Taking a critical look at equation (23), the estimate of the shrinkage parameter will often return a positive value since $\sigma\wedge^2$ and $\alpha\wedge^2$ will always be greater than zero. For practical purposes, we obtained the minimum value of (24) as

$$\hat{d}_{ML}^{\min} = \min\left(\frac{\lambda_j(\sigma\wedge^2 + \hat{\alpha}_j^2)}{\sigma\wedge^2 + \lambda_j \hat{\alpha}_j^2}\right). \quad (24)$$

3. Simulation Study

As theoretical comparison among the estimators in Section 2 gives the conditional dominance among the estimators, a simulation study conducted using the R 3.4.1 programming languages is considered in this section to grasp the better picture about the performance of the estimators.

3.1. Simulation Technique. We generated the explanatory variables by the following references of Gibbons [12] and Qasim et al. [11]:

$$X_{ij} = (1 - \gamma^2)^{1/2} z_{ij} + \gamma z_{ip+1}, \quad i = 1, 2, \cdots, n, j = 1, 2, \cdots, p, \quad (25)$$

where z_{ij} are independent standard normal pseudorandom numbers, and γ^2 represents the correlation between any two predictor variables. The number of predictor variables is three and seven. The predictands for the regression models are generated as follows:

$$y_i = \theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3 + \epsilon_i, \quad (26)$$

$$y_i = \theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3 + \theta_4 X_4 + \theta_5 X_5 + \theta_6 X_6 + \theta_7 X_7 + \epsilon_i, \quad (27)$$

where $\epsilon_i \sim (0, \sigma^2)$. $\theta'\theta$ is constrained to unity, according to Newhouse and Oman [13], Lukman et al. [14], and Lukman et al. [15]. We examined the performances of the estimators,

Table 7: Estimated MSE when $n = 200$ and $p = 3$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
1	0.1	0.0260	0.0259	0.0253	0.0252	0.0355	0.0354	0.0343	0.0341	0.0650	0.0645	0.0612	0.0604	0.5874	0.5532	0.3747	0.3374
	0.2	0.0260	0.0258	0.0254	0.0251	0.0355	0.0352	0.0344	0.0339	0.0650	0.0641	0.0616	0.0600	0.5874	0.5221	0.3947	0.3201
	0.3	0.0260	0.0257	0.0254	0.0250	0.0355	0.0351	0.0346	0.0338	0.0650	0.0636	0.0620	0.0597	0.5874	0.4940	0.4156	0.3038
	0.4	0.0260	0.0257	0.0255	0.0250	0.0355	0.0350	0.0347	0.0337	0.0650	0.0632	0.0624	0.0593	0.5874	0.4685	0.4374	0.2883
	0.5	0.0260	0.0256	0.0256	0.0249	0.0355	0.0348	0.0348	0.0336	0.0650	0.0628	0.0628	0.0589	0.5874	0.4452	0.4602	0.2737
	0.6	0.0260	0.0255	0.0257	0.0248	0.0355	0.0347	0.0350	0.0335	0.0650	0.0624	0.0632	0.0586	0.5874	0.4240	0.4838	0.2601
	0.7	0.0260	0.0254	0.0257	0.0248	0.0355	0.0346	0.0351	0.0334	0.0650	0.0620	0.0636	0.0582	0.5874	0.4046	0.5084	0.2473
	0.8	0.0260	0.0254	0.0258	0.0247	0.0355	0.0344	0.0352	0.0333	0.0650	0.0616	0.0641	0.0579	0.5874	0.3868	0.5338	0.2355
	0.9	0.0260	0.0253	0.0259	0.0247	0.0355	0.0343	0.0354	0.0331	0.0650	0.0612	0.0645	0.0576	0.5874	0.3705	0.5602	0.2246
	1.0	0.0260	0.0252	0.0260	0.0246	0.0355	0.0342	0.0355	0.0330	0.0650	0.0608	0.0650	0.0572	0.5874	0.3556	0.5874	0.2145
5	0.1	0.6208	0.6195	0.6089	0.6063	0.8560	0.8533	0.8322	0.8270	1.5796	1.5699	1.4957	1.4774	14.6151	13.7762	9.1902	8.1593
	0.2	0.6208	0.6181	0.6102	0.6050	0.8560	0.8506	0.8348	0.8244	1.5796	1.5603	1.5049	1.4684	14.6151	13.0082	9.7294	7.6677
	0.3	0.6208	0.6168	0.6115	0.6037	0.8560	0.8479	0.8374	0.8218	1.5796	1.5507	1.5141	1.4593	14.6151	12.3033	10.2845	7.1920
	0.4	0.6208	0.6155	0.6128	0.6024	0.8560	0.8453	0.8401	0.8192	1.5796	1.5413	1.5234	1.4503	14.6151	11.6549	10.8556	6.7321
	0.5	0.6208	0.6141	0.6142	0.6011	0.8560	0.8426	0.8427	0.8166	1.5796	1.5319	1.5327	1.4413	14.6151	11.0570	11.4424	6.2882
	0.6	0.6208	0.6128	0.6155	0.5998	0.8560	0.8400	0.8454	0.8140	1.5796	1.5227	1.5420	1.4324	14.6151	10.5045	12.0452	5.8601
	0.7	0.6208	0.6115	0.6168	0.5985	0.8560	0.8373	0.8480	0.8114	1.5796	1.5135	1.5513	1.4234	14.6151	9.9929	12.6639	5.4479
	0.8	0.6208	0.6102	0.6181	0.5972	0.8560	0.8347	0.8507	0.8088	1.5796	1.5045	1.5607	1.4146	14.6151	9.5184	13.2984	5.0515
	0.9	0.6208	0.6089	0.6195	0.5959	0.8560	0.8321	0.8533	0.8063	1.5796	1.4955	1.5702	1.4057	14.6151	9.0774	13.9488	4.6711
	1.0	0.6208	0.6076	0.6208	0.5946	0.8560	0.8296	0.8560	0.8037	1.5796	1.4866	1.5796	1.3969	14.6151	8.6668	14.6151	4.3065
10	0.1	2.4784	2.4731	2.4315	2.4211	3.4193	3.4086	3.3251	3.3044	6.3127	6.2741	5.9791	5.9063	58.4499	55.0988	36.7499	32.6162
	0.2	2.4784	2.4678	2.4367	2.4159	3.4193	3.3980	3.3355	3.2940	6.3127	6.2359	6.0157	5.8701	58.4499	52.0297	38.9107	30.6432
	0.3	2.4784	2.4626	2.4418	2.4108	3.4193	3.3874	3.3459	3.2837	6.3127	6.1980	6.0524	5.8340	58.4499	49.2118	41.1340	28.7328
	0.4	2.4784	2.4573	2.4470	2.4056	3.4193	3.3768	3.3563	3.2734	6.3127	6.1605	6.0893	5.7980	58.4499	46.6183	43.4199	26.8850
	0.5	2.4784	2.4521	2.4523	2.4005	3.4193	3.3664	3.3668	3.2631	6.3127	6.1233	6.1262	5.7621	58.4499	44.2260	45.7685	25.0998
	0.6	2.4784	2.4469	2.4575	2.3953	3.4193	3.3559	3.3773	3.2529	6.3127	6.0865	6.1633	5.7263	58.4499	42.0146	48.1796	23.3772
	0.7	2.4784	2.4417	2.4627	2.3902	3.4193	3.3456	3.3877	3.2426	6.3127	6.0500	6.2005	5.6907	58.4499	39.9662	50.6533	21.7172
	0.8	2.4784	2.4366	2.4679	2.3851	3.4193	3.3352	3.3982	3.2324	6.3127	6.0139	6.2378	5.6552	58.4499	38.0652	53.1896	20.1197
	0.9	2.4784	2.4314	2.4732	2.3800	3.4193	3.3250	3.4088	3.2222	6.3127	5.9781	6.2752	5.6198	58.4499	36.2978	55.7884	18.5849
	1.0	2.4784	2.4263	2.4784	2.3748	3.4193	3.3147	3.4193	3.2120	6.3127	5.9427	6.3127	5.5845	58.4499	34.6518	58.4499	17.1126

Table 8: Estimated MSE when $n = 200$ and $p = 7$.

σ	ρ	0.7				0.8				0.9				0.99			
	$k = d$	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu	OLS	Ridge	Liu	MLiu
1	0.1	0.0608	0.0607	0.0598	0.0595	0.0856	0.0854	0.0835	0.0830	0.1620	0.1611	0.1543	0.1527	1.5604	1.4782	1.0328	0.9358
	0.2	0.0608	0.0606	0.0599	0.0594	0.0856	0.0851	0.0837	0.0828	0.1620	0.1602	0.1552	0.1519	1.5604	1.4031	1.0840	0.8901
	0.3	0.0608	0.0605	0.0600	0.0593	0.0856	0.0849	0.0839	0.0825	0.1620	0.1593	0.1560	0.1511	1.5604	1.3343	1.1371	0.8463
	0.4	0.0608	0.0603	0.0601	0.0592	0.0856	0.0846	0.0842	0.0823	0.1620	0.1585	0.1568	0.1504	1.5604	1.2712	1.1921	0.8042
	0.5	0.0608	0.0602	0.0602	0.0591	0.0856	0.0844	0.0844	0.0821	0.1620	0.1576	0.1577	0.1496	1.5604	1.2132	1.2488	0.7641
	0.6	0.0608	0.0601	0.0603	0.0590	0.0856	0.0841	0.0846	0.0819	0.1620	0.1568	0.1585	0.1488	1.5604	1.1598	1.3075	0.7257
	0.7	0.0608	0.0600	0.0605	0.0589	0.0856	0.0839	0.0849	0.0817	0.1620	0.1559	0.1594	0.1481	1.5604	1.1105	1.3679	0.6892
	0.8	0.0608	0.0599	0.0606	0.0588	0.0856	0.0837	0.0851	0.0815	0.1620	0.1551	0.1602	0.1473	1.5604	1.0649	1.4302	0.6546
	0.9	0.0608	0.0598	0.0607	0.0587	0.0856	0.0834	0.0854	0.0812	0.1620	0.1543	0.1611	0.1466	1.5604	1.0226	1.4944	0.6218
	1.0	0.0608	0.0596	0.0608	0.0586	0.0856	0.0832	0.0856	0.0810	0.1620	0.1535	0.1620	0.1459	1.5604	0.9834	1.5604	0.5908
5	0.1	1.5209	1.5177	1.4931	1.4869	2.1402	2.1338	2.0844	2.0722	4.0504	4.0274	3.8518	3.8085	39.0100	36.9446	25.4493	22.8472
	0.2	1.5209	1.5146	1.4961	1.4839	2.1402	2.1275	2.0906	2.0660	4.0504	4.0047	3.8736	3.7869	39.0100	35.0465	26.8065	21.6023
	0.3	1.5209	1.5115	1.4992	1.4808	2.1402	2.1213	2.0967	2.0599	4.0504	3.9822	3.8954	3.7654	39.0100	33.2972	28.2010	20.3948
	0.4	1.5209	1.5084	1.5023	1.4778	2.1402	2.1150	2.1029	2.0539	4.0504	3.9598	3.9174	3.7440	39.0100	31.6813	29.6329	19.2247
	0.5	1.5209	1.5053	1.5054	1.4747	2.1402	2.1088	2.1091	2.0478	4.0504	3.9377	3.9393	3.7226	39.0100	30.1850	31.1023	18.0919
	0.6	1.5209	1.5022	1.5085	1.4717	2.1402	2.1027	2.1153	2.0417	4.0504	3.9158	3.9614	3.7013	39.0100	28.7964	32.6090	16.9966
	0.7	1.5209	1.4991	1.5116	1.4686	2.1402	2.0965	2.1215	2.0357	4.0504	3.8941	3.9835	3.6801	39.0100	27.5052	34.1531	15.9387
	0.8	1.5209	1.4961	1.5147	1.4656	2.1402	2.0904	2.1277	2.0296	4.0504	3.8726	4.0057	3.6590	39.0100	26.3022	35.7347	14.9181
	0.9	1.5209	1.4930	1.5178	1.4626	2.1402	2.0843	2.1339	2.0236	4.0504	3.8512	4.0280	3.6379	39.0100	25.1794	37.3536	13.9350
	1.0	1.5209	1.4900	1.5209	1.4595	2.1402	2.0783	2.1402	2.0175	4.0504	3.8301	4.0504	3.6169	39.0100	24.1296	39.0100	12.9893
10	0.1	6.0835	6.0709	5.9721	5.9475	8.5606	8.5353	8.3373	8.2882	16.2015	16.1096	15.4061	15.2323	156.0399	147.7764	101.7462	91.3151
	0.2	6.0835	6.0584	5.9844	5.9352	8.5606	8.5101	8.3620	8.2636	16.2015	16.0186	15.4934	15.1457	156.0399	140.1802	107.1845	86.3224
	0.3	6.0835	6.0459	5.9967	5.9230	8.5606	8.4850	8.3867	8.2392	16.2015	15.9284	15.5810	15.0595	156.0399	133.1789	112.7715	81.4783
	0.4	6.0835	6.0335	6.0091	5.9107	8.5606	8.4600	8.4114	8.2147	16.2015	15.8390	15.6688	14.9735	156.0399	126.7096	118.5070	76.7827
	0.5	6.0835	6.0211	6.0214	5.8985	8.5606	8.4352	8.4362	8.1903	16.2015	15.7504	15.7569	14.8878	156.0399	120.7180	124.3910	72.2358
	0.6	6.0835	6.0088	6.0338	5.8863	8.5606	8.4105	8.4610	8.1659	16.2015	15.6626	15.8453	14.8024	156.0399	115.1568	130.4237	67.8373
	0.7	6.0835	5.9965	6.0462	5.8741	8.5606	8.3859	8.4858	8.1416	16.2015	15.5756	15.9339	14.7172	156.0399	109.9844	136.6049	63.5875
	0.8	6.0835	5.9842	6.0586	5.8619	8.5606	8.3614	8.5107	8.1173	16.2015	15.4893	16.0229	14.6323	156.0399	105.1643	142.9346	59.4862
	0.9	6.0835	5.9720	6.0710	5.8497	8.5606	8.3370	8.5356	8.0931	16.2015	15.4038	16.1120	14.5476	156.0399	100.6645	149.4130	55.5334
	1.0	6.0835	5.9598	6.0835	5.8375	8.5606	8.3127	8.5606	8.0689	16.2015	15.3190	16.2015	14.4633	156.0399	96.4563	156.0399	51.7293

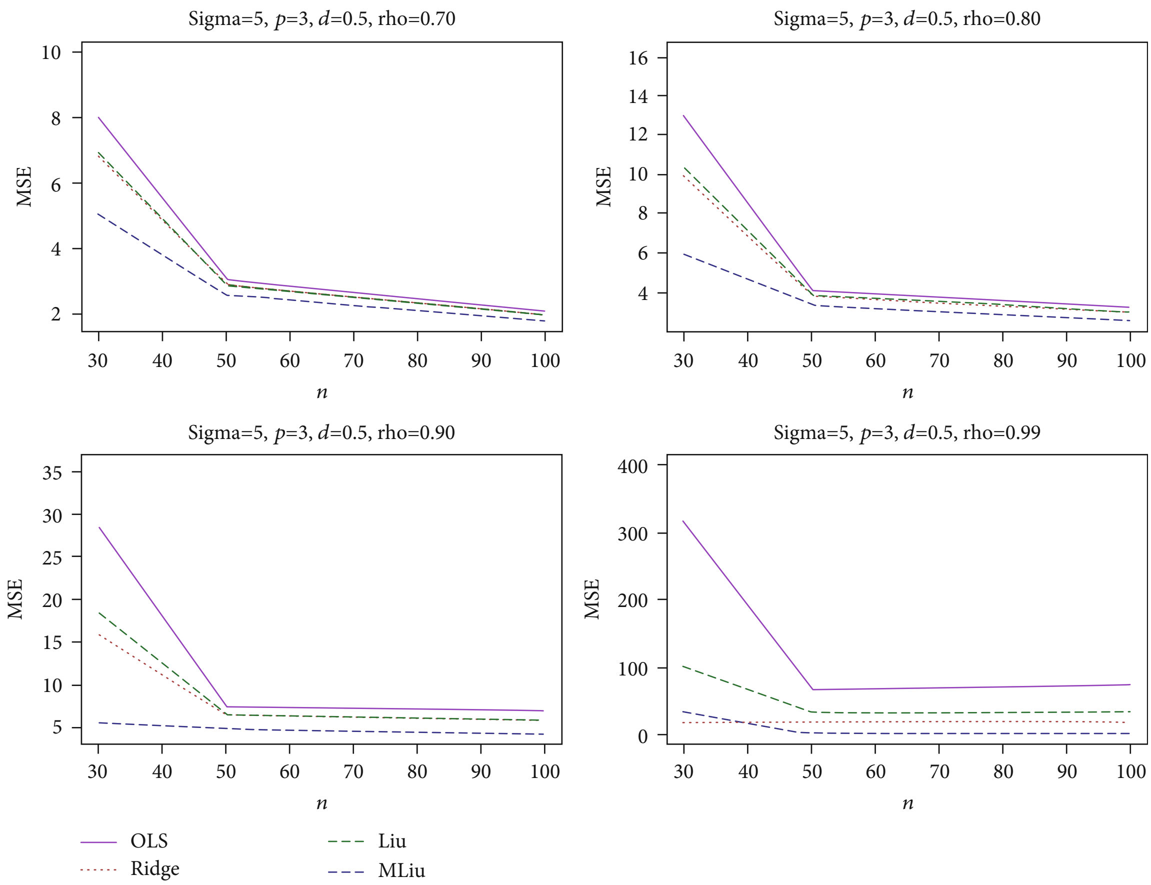

Figure 4: Estimated MSEs for $n = 30$, 100; sigma = 5; $p = 3$; $d = 0.5$; and different values of ρ.

using mean square error criteria. The simulation is performed using the following condition:

(1) Sample sizes: $n = 30$, 50, 100, and 200

(2) Number of replications: 1000

(3) The variances: σ^2 are 1, 25, and 100

(4) The multicollinearity levels: rho $(\rho) = 0.7$, 0.8, 0.9, and 0.99

(5) $k = d = 0.1, 0.2, \ldots, 0.9$, and 1

The simulated results for $p = 3$ and $\rho = 0.70$, 0.80, 0.90, and 0.99 are presented in Tables 1–3 for $n = 30$, 50, and 100, respectively. $p = 7$ and $\rho = 0.70$, 0.80, 0.90, and 0.99 are presented in Tables 4–6 for n = 30, 50, and 100, respectively. For a better picture, we have plotted MSE vs. d for $n = 30$, $\sigma = 1$, and $\rho = 0.70$, 0.90, and 0.99 in Figures 1 and 2, respectively. We also plotted MSE vs. ρ and MSE vs. n in Figure 3.

3.2. Simulation Result Discussion. From Tables 1–8 and Figures 1–4, we observed that the modified Liu estimator consistently performs better than the Liu estimator and other existing estimators in this study.

Results from Tables 1–8 show that increasing the sample size results in a decrease in the MSE values for each of the estimator. It is evident that MSE values of the estimators increase as the degree of correlation and the number of explanatory variables increase. The simulation results show that the proposed estimator performed best at most levels of the multicollinearity, sample size (n), and number of explanatory variables with few exceptions. The only exceptions to its performance are when$\rho = 0.99$, and they are defined as follows:

(i) $n = 30$, $p = 3$, $\rho = 0.99$, and $k = d > 0.4$ for $\sigma = 1$, 5, and 10

(ii) $n = 100$, $p = 3$, and $k = d = 1.0$ for $\sigma = 5$ and 10

(iii) $n = 100$, $p = 3$, and $k = d = 0.8$ and 0.9 when $\sigma = 1$

(iv) $n = 30$, $p = 7$, $\rho = 0.99$, and $k = d > 0.4$ for $\sigma = 1$, 5, and 10

(v) $n = 50$, $p = 7$, $\rho = 0.99$, and $k = d > 0.5$ for $\sigma = 1$, 5, and 10

(vi) $n = 100$ and $\rho = 0.99$ for $k = d > 0.8$ at $\sigma = 1$, 5, and 10 including $k = d = 0.8$ and 0.9 at $\sigma = 1$

The instances mentioned above are the only times that ridge regression dominates the proposed estimator. The new estimator consistently dominates the OLS and the Liu estimator. Also, from Tables 1–8, we observed that the values of OLS and Liu are the same when $d = 1$.

Table 9: The regression output.

Coef.	$\hat{\alpha}$	$\hat{\alpha}_k$	$\hat{\alpha}_{k\ \min}$	$\hat{\alpha}_d\left(\hat{d}_{\min}\right)$	$\hat{\alpha}_{ML}\left(\hat{d}_{ML}^{\min}\right)$
$\hat{\alpha}_1$	2.1930	2.1478	2.1695	2.1930	2.1425
$\hat{\alpha}_2$	1.1533	1.1638	1.1587	1.1533	1.1650
$\hat{\alpha}_3$	0.7585	0.7261	0.7416	0.7583	0.7221
$\hat{\alpha}_4$	0.4863	0.4931	0.4899	0.4864	0.4939
MSE	0.0638	0.0636	0.0629	0.0617	0.0608
MSPE	2.8966	2.2665	2.8805	2.1342	1.2789
k/d	—	3.3636	1.7256	0.9782	2.6817

Consistently when $\rho = 0.7, 0.8$, and 0.9, the proposed estimator performs better than other estimators at the different sample sizes irrespective of the values of the biasing parameter k and d. The fact that the ridge estimator dominates the proposed estimator in some of the exceptions mentioned earlier does not show that it performs better. It only shows that at those intervals, the performance of the new estimator drops. Thus, this necessitates the use of real-life data in the next session because the values of k and d will be estimated rather than choosing it arbitrarily.

4. Applications

We adopt two datasets to illustrate the theoretical findings of the paper. These include the Portland cement data and the French economy data.

4.1. Portland Dataset. The first user of this dataset was Woods et al. [16] and later adopted by Kaciranlar et al. [17] and Li and Yang [18]. It consists of one predictand, y_i, which is the heat evolved after 180 days of curing measured in calories per gram of cement, and four predictors: X_1 is the tricalcium aluminate, X_2 is the tricalcium silicate, X_3 is the tetracalcium aluminoferrite, and X_4 is the β-dicalcium silicate. Variance inflation factors (VIFs) and a condition number are adopted to diagnose multicollinearity in the model [19]. The VIFs are 38.50, 254.42, 46.87, and 282.51 while the condition number is approximately 424. Both tests are evidence that the model possesses severe multicollinearity. The regression output is available in Table 9. According to Kejian [3], the optimum biasing parameter is expressed as

$$\hat{d}_{\text{opt}} = \frac{\sum_{j=1}^{p}\left(\hat{\alpha}_j^2 - \sigma\wedge^2\right)/\left(\lambda_j + 1\right)^2}{\sum_{j=1}^{p}\left(\sigma^2 + \lambda_j\hat{\alpha}_j^2\right)/\lambda_j\left(\lambda_j + 1\right)^2}. \quad (28)$$

Following Özkale and Kaçiranlar [4], we replaced $\hat{d}_{\text{opt}}$ with $\hat{d}_{\min}$ if $\hat{d}_{\text{opt}} < 0$.

$$\hat{d}_{\min} = \min\left[\frac{\sigma\wedge^2}{\hat{\alpha}_j^2 + \sigma\wedge^2/\lambda_j}\right]. \quad (29)$$

The ridge biasing parameters are computed by

$$\hat{k} = \left[\frac{p\sigma\wedge^2}{\sum_{j=1}^{p}\hat{\alpha}_j^2}\right], \quad (30)$$

$$\hat{k}_{\min} = \min\left[\frac{\sigma\wedge^2}{\hat{\alpha}_j^2}\right]. \quad (31)$$

We also adopt the leave-one-out crossvalidation to validate the performance of the estimators (see [20]). The performance of the estimator is assessed through the mean squared prediction error (MSPE). The result is presented in Table 9.

The theoretical results are computed as follows:

$$\begin{aligned}
&\operatorname{tr}\left(c_2'\left[\sigma^2\left(A^{-1} - \left(A + I_p\right)^{-1}\left(A - d_{\text{ML}}I_p\right)A^{-1}\right.\right.\right.\\
&\quad\left.\left.\left.\cdot\left(A - d_{\text{ML}}I_p\right)\left(A + I_p\right)^{-1}\right)\right]^{-1}c_2\right) = 0.4586,\\
&\operatorname{tr}\left(c_2'\left[\sigma^2\left(\left(A + kI_p\right)^{-1}A\left(A + kI_p\right)^{-1}\right.\right.\right.\\
&\quad - \left(A + I_p\right)^{-1}\left(A - d_{\text{ML}}I_p\right)A^{-1}\left(A - d_{\text{ML}}I_p\right)\\
&\quad\left.\left.\left.\cdot\left(A + I_p\right)^{-1}\right) + c_1c_1'\right]^{-1}c_2\right) = 0.0424,\\
&\operatorname{tr}\left(c_2'\left[\sigma^2\left(A + I_p\right)^{-1}\left(A + dI_p\right)A^{-1}\left(A + dI_p\right)\left(A + I_p\right)^{-1}\right.\right.\\
&\quad - \left(A + I_p\right)^{-1}\left(A - d_{\text{ML}}I_p\right)A^{-1}\left(A - d_{\text{ML}}I_p\right)\\
&\quad\left.\left.\cdot\left(A + I_p\right)^{-1} + c_1c_1'\right]^{-1}c_2\right) = 0.3134.
\end{aligned} \quad (32)$$

We observed that the theoretical comparisons stated in Sections 2.1, 2.2, and 2.3 are valid since each of the estimates are less than 1. From Table 9, the regression coefficients and MSE of the OLS and Liu estimators are approximately the same because d is close to unity. Recall that the Liu estimator becomes OLS when $d = 1$. The proposed estimator possesses the smallest mean square error and average MSE of the validation error. Also, the performances of the estimators largely depend on their biasing parameters.

4.2. French Economy Dataset. The detail about this dataset is initially described in Chatterjee and Price [21] and is later available in the following references Malinvard [22] and Kejian [3]. It comprises one predictand, imports, and three predictor variables (domestic production, stock information, and domestic consumption) with eighteen observations. The variance inflation factors are $\text{VIF}_1 = 469.688$, $\text{VIF}_2 = 1.047$, and $\text{VIF}_3 = 469.338$ and the condition number 32612. Both results indicate the presence of severe multicollinearity. We analyzed the data using the biasing parameters for each of the estimators and present the results in Table 10. The proposed estimator performed the best in the sense of smaller MSE and MSPE. As mentioned earlier, the estimators'

Table 10: The regression output.

Coef.	$\hat{\alpha}$	$\hat{\alpha}_k$	$\hat{\alpha}_{k\ \min}$	$\hat{\alpha}_d\left(\hat{d}_{\text{opt}}\right)$	$\hat{\alpha}_{ML}\left(\hat{d}_{ML}^{\min}\right)$
$\hat{\alpha}_0$	-19.7127	-16.7613	-18.8782	-18.8410	-19.2613
$\hat{\alpha}_1$	0.0327	0.1419	0.0636	0.0648	0.0493
$\hat{\alpha}_2$	0.4059	0.3576	0.3922	0.3914	0.3984
$\hat{\alpha}_3$	0.2421	0.0709	0.1937	0.1918	0.2161
MSE	17.3326	21.30519	16.6017	16.6029	4.9448
MSPE	0.8267	0.4371	0.1411	0.1302	0.0871
k/d	—	0.0527	0.0132	0.9424	0.9701

performance is a function of the biasing parameter. The proposed estimator with the biasing parameter $\hat{d}_{\text{ML}}^{\min}$ performs best. Also, the theoretical results agree with Section 2 and support the simulation and real-life findings.

The theoretical results are computed as follows:

$$\begin{aligned}
&\operatorname{tr}\left(c_2'\left[\sigma^2\left(\Lambda^{-1}-\left(\Lambda+I_p\right)^{-1}\left(\Lambda-d_{\text{ML}}I_p\right)\Lambda^{-1}\right.\right.\right.\\
&\quad\left.\left.\left.\cdot\left(\Lambda-d_{\text{ML}}I_p\right)\left(\Lambda+I_p\right)^{-1}\right)\right]^{-1}c_2\right)=0.0163,\\
&\operatorname{tr}\left(c_2'\left[\sigma^2\left(\left(\Lambda+kI_p\right)^{-1}\Lambda\left(\Lambda+kI_p\right)^{-1}\right.\right.\right.\\
&\quad-\left(\Lambda+I_p\right)^{-1}\left(\Lambda-d_{\text{ML}}I_p\right)\Lambda^{-1}\left(\Lambda-d_{\text{ML}}I_p\right)\\
&\quad\left.\left.\left.\cdot\left(\Lambda+I_p\right)^{-1}\right)+c_1c_1'\right]^{-1}c_2\right)=0.0185,\\
&\operatorname{tr}\left(c_2'\left[\sigma^2\left(\Lambda+I_p\right)^{-1}\left(\Lambda+dI_p\right)\Lambda^{-1}\left(\Lambda+dI_p\right)\left(\Lambda+I_p\right)^{-1}\right.\right.\\
&\quad-\left(\Lambda+I_p\right)^{-1}\left(\Lambda-d_{\text{ML}}I_p\right)\Lambda^{-1}\left(\Lambda-d_{\text{ML}}I_p\right)\\
&\quad\left.\left.\cdot\left(\Lambda+I_p\right)^{-1}+c_1c_1'\right]^{-1}c_2\right)=0.0007.
\end{aligned} \tag{33}$$

5. Some Concluding Remarks

Both ridge regression and Liu estimators are widely accepted in the linear regression model as an alternative to the OLS estimator to circumvent the problem of multicollinearity. In this study, we proposed a modified Liu estimator, which possesses a single parameter which places it in the class of the ridge and Liu estimators. Theoretical comparisons, simulation study, and real-life applications evidently show that the proposed estimator consistently dominates the existing Liu estimator and ridge regression estimator under some conditions. We recommend the use of this estimator for the linear regression model with multicollinearity problem. We note that the proposed estimator can be extended to other regression models, for example, logistic regression, Poisson, ZIP, gamma, and related models, and these possibilities are under current investigation.

Data Availability

Data will be made available on request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] A. E. Hoerl and R. W. Kennard, “Ridge regression: biased estimation for nonorthogonal problems,” *Technometrics*, vol. 12, no. 1, pp. 55–67, 1970.

[2] A. F. Lukman, K. Ayinde, S. Binuomote, and O. A. Clement, “Modified ridge-type estimator to combat multicollinearity: application to chemical data,” *Journal of Chemometrics*, vol. 33, no. 5, article e3125, 2019.

[3] L. Kejian, “A new class of blased estimate in linear regression,” *Communications in Statistics - Theory and Methods*, vol. 22, no. 2, pp. 393–402, 1993.

[4] M. R. Özkale and S. Kaçiranlar, “The restricted and unrestricted two-parameter estimators,” *Communications in Statistics - Theory and Methods*, vol. 36, no. 15, pp. 2707–2725, 2007.

[5] A. F. Lukman, K. Ayinde, S. Siok Kun, and E. T. Adewuyi, “A modified new two-parameter estimator in a linear regression model,” *Modelling and Simulation in Engineering*, vol. 2019, Article ID 6342702, 10 pages, 2019.

[6] R. W. Farebrother, “Further results on the mean square error of ridge regression,” *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 38, no. 3, pp. 248–250, 1976.

[7] G. Trenkler and H. Toutenburg, “Mean squared error matrix comparisons between biased estimators—an overview of recent results,” *Statistical Papers*, vol. 31, no. 1, pp. 165–179, 1990.

[8] B. M. G. Kibria, “Performance of some new ridge regression estimators,” *Communications in Statistics: Simulation and Computation*, vol. 32, no. 2, pp. 419–435, 2003.

[9] B. M. G. Kibria and S. Banik, “Some ridge regression estimators and their performances,” *Journal of Modern Applied Statistical Methods*, vol. 15, no. 1, pp. 206–238, 2016.

[10] A. F. Lukman and K. Ayinde, “Review and classifications of the ridge parameter estimation techniques,” *Hacettepe Journal of Mathematics and Statistics*, vol. 46, no. 5, pp. 953–967, 2017.

[11] M. Qasim, M. Amin, and T. Omer, “Performance of some new Liu parameters for the linear regression model,” *Communications in Statistics - Theory and Methods*, vol. 49, no. 17, pp. 4178–4196, 2019.

[12] D. G. Gibbons, “A simulation study of some ridge estimators,” *Journal of the American Statistical Association*, vol. 76, no. 373, pp. 131–139, 1981.

[13] J. P. Newhouse and S. D. Oman, *An Evaluation of Ridge Estimators. A report prepared for United States Air Force project RAND*, 1971.

[14] A. F. Lukman, K. Ayinde, and A. S. Ajiboye, “Monte Carlo study of some classification-based ridge parameter estimators,” *Journal of Modern Applied Statistical Methods*, vol. 16, no. 1, pp. 428–451, 2017.

[15] A. F. Lukman, K. Ayinde, B. Aladeitan, and R. Bamidele, “An unbiased estimator with prior information,” *Arab Journal of Basic and Applied Sciences*, vol. 27, no. 1, pp. 45–55, 2020.

[16] H. Woods, H. H. Steinour, and H. R. Starke, "Effect of composition of Portland cement on heat evolved during hardening," *Industrial and Engineering Chemistry*, vol. 24, no. 11, pp. 1207–1214, 1932.

[17] S. Kaciranlar, S. Sakallioglu, F. Akdeniz, G. P. H. Styan, and H. J. Werner, "A new biased estimator in linear regression and a detailed analysis of the widely-analysed dataset on Portland cement," *Sankhyā: The Indian Journal of Statistics, Series B*, vol. 61, no. 3, pp. 443–459, 1999.

[18] Y. Li and H. Yang, "A new Liu-type estimator in linear regression model," *Statistical Papers*, vol. 53, no. 2, pp. 427–437, 2012.

[19] A. F. Lukman, *Classification-Based Ridge*, Lambert Academic Publishing, Mauritius, 2018.

[20] G. James, D. Witten, T. Hastie, and R. Tibshirani, *An Introduction to Statistical Learning with Application to R*, vol. 103 of Springer Texts in Statistics, 2013.

[21] S. Chatterjee and A. S. Price, *Regression Analysis by Example*, Wiley, New York, 1977.

[22] E. Malinvard, *Statistical Methods of Econometrics*, North Holland, Amsterdam, 3rd edition, 1980.