Hindawi
Modelling and Simulation in Engineering
Volume 2022, Article ID 5830945, 13 pages
https://doi.org/10.1155/2022/5830945

Research Article

Modeling and Bayesian Analysis of Time between the Breakdown of Electric Feeders

Kahkashan Ateeq[1], **Saima Altaf**[2], **and Muhammad Aslam**[2]

[1]*Department of Statistics, The Women University, Multan, Pakistan*
[2]*Department of Statistics, Bahauddin Zakaria University, Multan, Pakistan*

Correspondence should be addressed to Kahkashan Ateeq; kahkashan.102@wum.edu.pk

Received 20 September 2021; Revised 1 May 2022; Accepted 8 May 2022; Published 30 May 2022

Academic Editor: Noé López Perrusquia

The failure of electric feeders is a common problem in the summer season in Pakistan. In this article, one of the troubling aspects of the electric power system of Pakistan (Multan city) has been studied. The time lapses between the breakdown of electric feeders of the city have been modeled by suggesting an inverse Rayleigh-exponential distribution. The parameters of the distribution are estimated in both the frequentist and Bayesian paradigms. Since the Bayes estimators under informative priors are not attained in the closed form, this paper provides a comparative analysis of the Bayes estimators under Lindley and Tierney–Kadane approximation methods. The simulation study and the real-life data set assessed the validity of the model and the superiority of the Bayes estimators over the maximum likelihood estimators.

1. Introduction

The world we live in relies on the robust and proper functioning of the systems which is normally at risk because of the vulnerabilities and susceptibilities in the systems. Regarding risk analysis in a system, it is a reality that a system must one day fail regardless of how evidently flawless and impeccable the functioning might be and regardless of how intensive the development might be. This is the reason that the failure of machinery, structures, and systems is regular phenomena with which we have to deal every day.

In the engineered systems, such as the electrical power grid, telecommunications, and different assembly lines, the failure rate is one of the parametric indices utilized for the evaluation of the system's efficiency [1]. It is characterized as the inability of the system to perform its assigned capacity satisfactorily without interference over a time frame. An analysis of a system failure helps to emphasize the root cause of the failure and the impact it has on the entire operational process.

The reliability of a system and equipment is expressed by its successful working under the provided circumstances for a specific time. As the failure of machines and electronic systems, etc. is a common phenomenon, one of the main problems is to detect how frequently a system fails at a specific time. To model risk, failure, and uncertainty in different situations, probability theory and survival analysis have been playing a significant role over the years. To define the reliability of a system, many failure-time statistical distributions are widely used. A new modified Weibull [2], Beta Sarhan-Zaindin modified Weibull distribution [3], Odd Lindley-half logistic distribution [4], etc. are some of them.

The inverse Rayleigh (IR) distribution as proposed by [5] has great importance in ecology, biomedicine, survival, and reliability analysis. Especially in engineering, to model the lifetime of a system, the IR distribution is widely used. The weighted inverse Rayleigh distribution is used to discuss two real-life examples, one of which was about the exceedances of flood peaks of the Wheaton River in Canada and the other was about the remission time of bladder cancer [6]. The exponentiated inverse Rayleigh distribution is used for the model validity, and two data sets were taken [7]. The type II Topp-Leone inverse Rayleigh distribution is used to assess the flexibility of the distribution for the data set about the failure time of aircraft windshield [8]. The half-logistic inverse Rayleigh distribution is used for modeling the data set about the taxes revenue in Egypt [9].

Bayesian statistics has emerged as a relatively new and unique branch of statistics in recent years. In the Bayesian paradigm, along with the current information modeled as the probability distribution, the prior information of the parameters of the probability model is also taken into account. Bayesian strategies are regularly prescribed as the best possible approach to make formal utilization of subjective data, for example, professional opinion and individual judgments [10]. This is the reason that many scholars, e.g., [11–13], have made use of Bayesian statistics to model risk, failure, and reliability assessment in different situations and systems.

Energy crises in Pakistan are increasing day by day. The failure rate in Pakistan's power distribution system cannot be determined with certainty or complete knowledge. Mostly failure of the grid power system depends upon climate variables like rain, wind, and temperature. However, the failure rate in such systems may also depend on numerous other factors, such as lack of maintenance, old and faulty systems, and government policies.

The advancements led by many statisticians and Bayesian statisticians have ushered us to study one of the problems related to the energy crisis in Pakistan in the Bayesian framework using a new probability model. We have proposed the inverse Rayleigh-exponential (IRE) distribution using the Transformed-Transformer technique proposed by [14], to model the time lapse between the breakdowns of electric feeders.

Electric feeders are a set of electric conductors that originate at a primary distribution center and supply power to one or more secondary distribution centers, branch distribution centers, or a combination of these. Occurrences of faults in the electricity distribution system can be categorized as momentary or stable. Momentary faults, which can be adjusted when the system is deenergized and then reenergized within 5 to 15 minutes, are called trips. On the other hand, the stable fault which sustains until proper repairing is done by human interference is called breakdown.

In this study, the basic aim is to model the time lapses between the breakdown of electric feeders in a power distribution system and then to find out the estimates. The analysis of the grid system operation (GSO) data has been carried out which were taken from randomly selected 11 kV outgoing urban feeders of 2 grid stations, which were also conveniently selected from nine grid stations of the city of Multan (famous for its hot weather), Pakistan. The data consist of the time lapses between these breakdowns. All the relevant data have been collected in June, July, and August 2018. In Pakistan, the power crisis upsurges to its peak in the summer season, and the situation becomes even worse as the numbers of trips and breakdowns increase because of hard weather.

The parameters of the proposed distribution are estimated using the maximum likelihood estimation technique in classical statistics. The Bayes estimators are estimated using five different loss functions, square error loss function (SELF), weighted loss function (WLF), quadratic loss function (QLF), precautionary loss function (PLF), and modified II loss function (MIILF). The posterior distribution of the parameters of the IRE distribution under informative prior (IR and exponential distributions) is not in the closed form. Two approximation techniques are utilized for the Bayes estimators and their associated risks. The MLEs and Bayes estimators are compared based on minimum values of risks. The rest of the study is organized as follows;

In Section 2, the IRE distribution is derived. In Section 3, the MLEs of the parameters of IRE distribution are estimated, and asymptotic confidence intervals are derived. In Section 4, The Lindley and Tierney–Kadane (T-K) approximation methods are utilized for the estimation of Bayes estimators using informative priors. Sections 5 and 6 deal with the simulation study and real-life data set of time lapses between the breakdowns of the electric feeders for illustrative purposes. Finally, the study is concluded in Section 7.

2. The Inverse Rayleigh-Exponential (IRE) Distribution

[14] proposed the Transformed-Transformer technique and suggested various functional forms of the transformer of rv X depending on the support of another continuous rv T. Keeping in mind the importance of the IR distribution in the different phenomena of engineering, we have derived its generalization, the IRE distribution, using the same technique as suggested by [14]. This IRE distribution is used for modeling the data of the time lapses between the respective breakdowns of electric feeders

In this study, the rv T follows the IR distribution, and then it is transformed into generalized IRE distribution using a function $\mathrm{W}[F(x)] = F(x)/1 - F(x)$. Here, $W[F(x)]$ is the functional form of the CDF of rv X which follows the exponential distribution. The PDF and CDF of IRE distribution are as follows:

$$\begin{aligned} g(x) &= \frac{2\theta\lambda e^{\lambda x}}{\left(e^{\lambda x}-1\right)^3}\ \exp\left(\frac{-\theta}{\left(e^{\lambda x}-1\right)^2}\right), x, \theta, \lambda > 0, \\ G(x) &= \exp\left(\frac{-\theta}{\left(e^{\lambda x}-1\right)^2}\right), x, \theta, \lambda > 0, \end{aligned} \tag{1}$$

where λ is the inverse scale parameter of exponential distribution, and θ is the scale parameter of IR distribution. The flexibility of the IRE distribution is observed by changing the values of parameters. Figure 1 shows the PDF and CDF plots of the distribution, and it is evident that the distribution is unimodal and positively skewed.

2.1. Reliability Analysis. In engineering, reliability is an aspect that deals with the failure of an object or component in a particular duration of time. Let an rv T represent the time until an event of interest occurs. Then, the reliability function is the probability of the non occurrence of the event

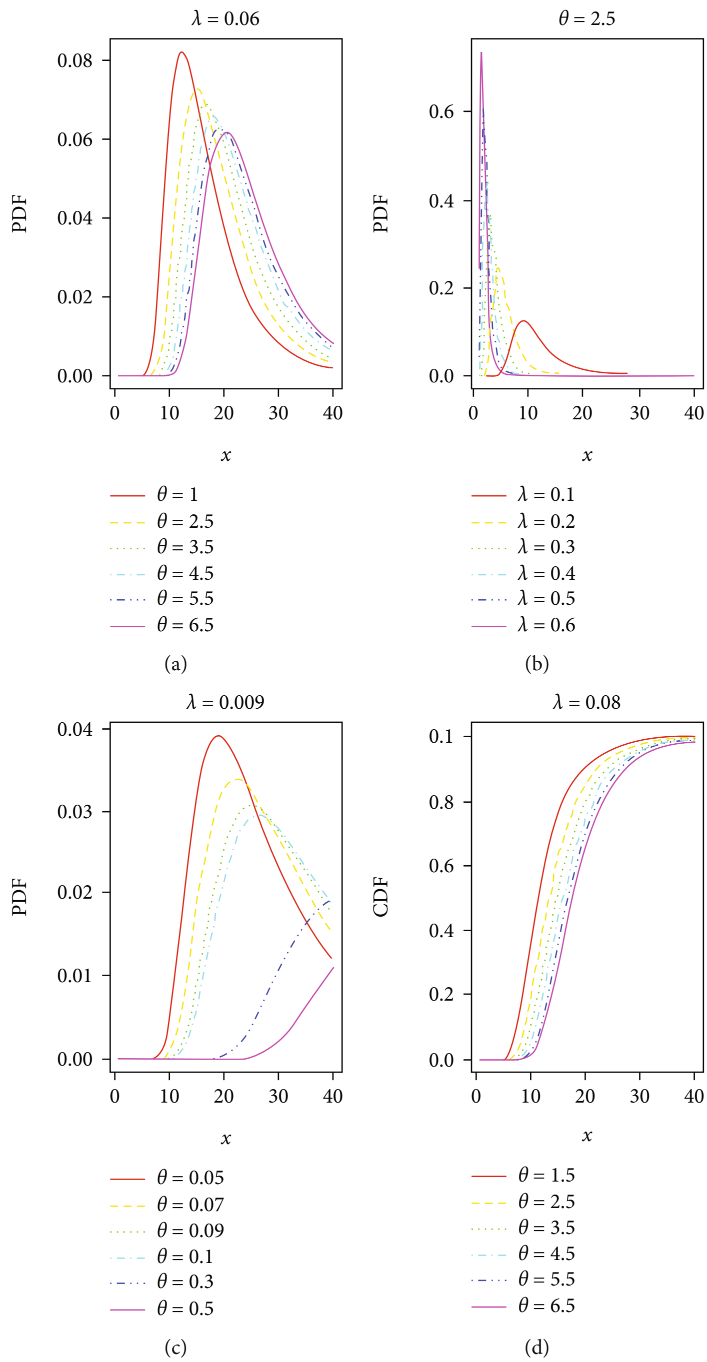


Figure 1: Continued.

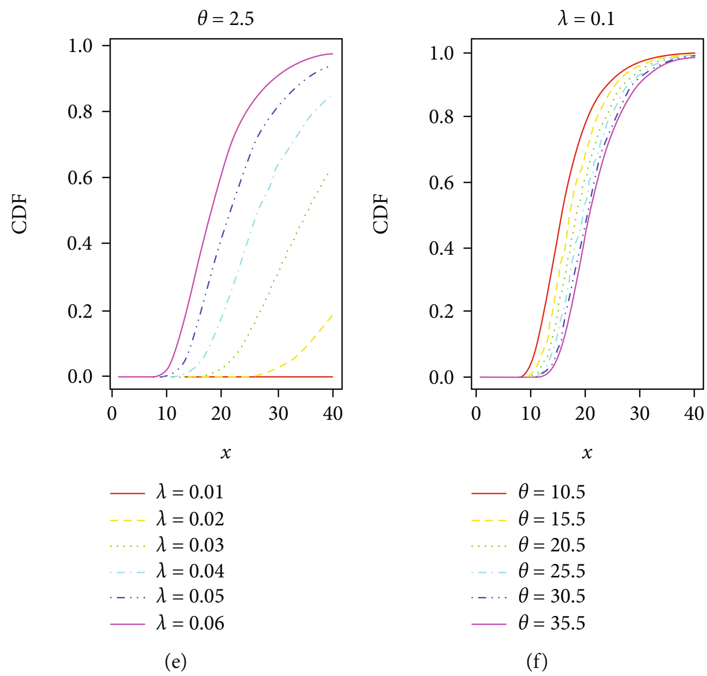


FIGURE 1: PDF and CDF plots of IRE distribution for different values of parameters.

in time t. Then, reliability function of the IRE distribution is as follows:

$$R(t) = 1 - G(t) = 1 - \exp\left(\frac{-\theta}{\left(e^{\lambda t} - 1\right)^2}\right), t, \theta, \lambda > 0. \quad (2)$$

The electrical items are usually at high risk of failure over time. At a specific time, the instantaneous rate of failure is called hazard rate denoted as $h(t)$. For the IRE distribution,

$$h(t) = \frac{g(t)}{S(t)} = \frac{2\theta\lambda e^{\lambda t}}{\left(e^{\lambda t} - 1\right)^3} \exp\left(\frac{-\theta}{\left(e^{\lambda t} - 1\right)^2}\right) \frac{1}{1 - \exp\left(-\theta/\left(e^{\lambda t} - 1\right)^2\right)}, t, \theta, \lambda > 0. \quad (3)$$

The commutative hazard function $(H(t))$ computes the expected failure time for a specific period. For the IRE distribution,

$$H(t) = -\log S(t) = -\log\left[1 - \exp\left(\frac{-\theta}{\left(e^{\lambda t} - 1\right)^2}\right)\right]. \quad (4)$$

Figure 2 shows the reliability, hazard, and cumulative hazard function of the IRE distribution for different values of the parameters.

3. Maximum Likelihood Estimators

Let $X_1, X_2, X_3, \cdots, X_n$ be the random sample follows the IRE distribution. The likelihood and log-likelihood function of the distribution for the parameters $\Delta = (\theta, \lambda)'$ are as follows:

$$L(\Delta) = L(\theta, \lambda|\underline{x}) = \frac{(2\lambda\theta)^n e^{\lambda \sum x}}{\prod\left(e^{\lambda x_i} - 1\right)^3} \exp\left(-\theta \sum_{i=1}^{n} \frac{1}{\left(e^{\lambda x_i} - 1\right)^2}\right), \quad (5)$$

$$\ell(\Delta) = l(\theta, \lambda|\underline{x}) = n \log(2\lambda\theta) + \lambda \sum x - 3\sum_{i=1}^{n} \log\left(e^{\lambda x_i} - 1\right) - \theta \sum_{i=1}^{n} \frac{1}{\left(e^{\lambda x_i} - 1\right)^2}. \quad (6)$$

The score functions of IRE distribution are not in the closed form; hence, maximum likelihood estimators (MLEs) $\widehat{\theta}$ and $\widehat{\lambda}$ of the parameters of IRE distribution are obtained using the Newton Raphson iterative procedure, R package maxLik [15] is used for this purpose.

The asymptotic distributions of the $\widehat{\lambda}$ and $\widehat{\theta}$ are as follows:

$$\sqrt{n}\left(\widehat{\lambda}\right) \xrightarrow{d} N\left(\lambda, I_{\lambda\lambda}^{-1}\right) \sqrt{n}\left(\widehat{\theta}\right) \xrightarrow{d} N\left(\theta, I_{\theta\theta}^{-1}\right), \quad (7)$$

where $I_{\lambda\lambda}^{-1}$ and $I_{\theta\theta}^{-1}$ are the diagonal elements of the inverse of the Fisher information matrix (FIM), which is defined as

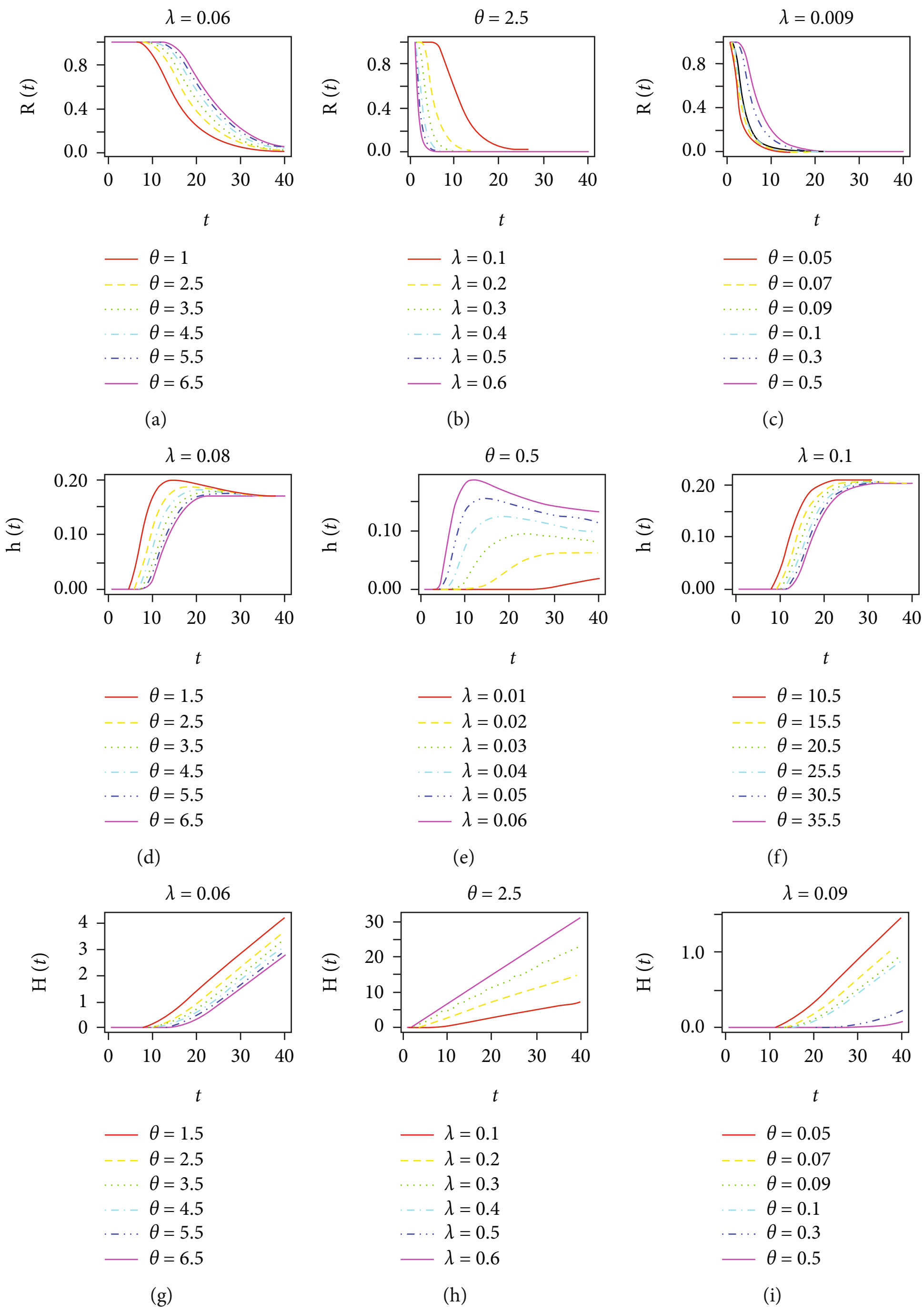


Figure 2: The reliability, hazard, and cumulative hazard function plots of IRE distribution for different values of parameters.

$$\mathrm{FIM} = I = \begin{bmatrix} I_{\lambda\lambda} & I_{\lambda\theta} \\ I_{\theta\lambda} & I_{\theta\theta} \end{bmatrix} = -E \begin{bmatrix} \dfrac{\partial^2 \ell(\Delta)}{\partial \lambda^2} & \dfrac{\partial^2 \ell(\Delta)}{\partial \lambda \partial \theta} \\ \dfrac{\partial^2 \ell(\Delta)}{\partial \lambda \partial \theta} & \dfrac{\partial^2 \ell(\Delta)}{\partial \theta^2} \end{bmatrix},$$

$$\frac{\partial^2 \ell(\Delta)}{\partial \lambda^2} = -\frac{n}{\lambda^2} - 3\sum \frac{x^2 e^{\lambda x}}{\left(e^{\lambda x}-1\right)^1} + 3\sum \frac{x^2 e^{2\lambda x}}{\left(e^{\lambda x}-1\right)^2} + 2\theta \sum \frac{x^2 e^{\lambda x}}{\left(e^{\lambda x}-1\right)^3} - 6\theta \sum \frac{x^2 e^{2\lambda x}}{\left(e^{\lambda x}-1\right)^4},$$

$$\frac{\partial^2 \ell(\Delta)}{\partial \theta^2} = -\frac{n}{\theta^2}, \quad \frac{\partial^2 \ell(\Delta)}{\partial \lambda \partial \theta} = 2\sum \frac{x e^{\lambda x}}{\left(e^{\lambda x}-1\right)^3}. \tag{8}$$

The asymptotic behavior remains valid if information matrix I is replaced by the observed information matrix. So, the approximate $100(1-\alpha)\%$ two-sided confidence interval for the parameters λ and θ of IRE distribution is, respectively, given as

$$\widehat{\lambda} \pm Z_{\alpha/2}\sqrt{I_{\lambda\lambda}^{-1}} \text{ and } \widehat{\theta} \pm Z_{\alpha/2}\sqrt{I_{\theta\theta}^{-1}}, \tag{9}$$

where Z_α is the $\alpha-$th percentile of the standard normal distribution.

4. Bayesian Estimation Using Informative Prior

Bayesian statistics is a technique that updates the belief of a person in the evidence of new data. The parameter is summarized by an entire distribution, known as a prior distribution which is based on past studies or the opinions of experts. A formal rule to combine the prior distribution with the sample information (likelihood function) is provided by the Bayes theorem. This gives the posterior distribution that contains all the updated and probabilistic information about the parameters.

The informative prior which gives specific and definite information about the parameters may lead to efficient Bayes estimates accompanied by low posterior risk. In this study, the informative prior distribution of λ is taken to be the IR distribution and for the parameter θ, and the exponential distribution is taken as the prior. The joint prior distribution of λ and θ after assuming the independence of prior distributions is defined as

$$\pi(\lambda, \theta) = \pi(\lambda)\pi(\theta) \propto \frac{1}{\lambda^3} \exp\left(-a\theta - \frac{b}{\lambda^2}\right), \tag{10}$$

where a and b are the hyperparameters of exponential distribution and IR distribution, respectively.

Using the Bayes theorem, the joint posterior distribution of the parameters of IRE distribution is obtained by combining the likelihood function and the prior distributions given in equations (5) and (10), which is,

$$\pi(\theta, \lambda \mid \underline{x}) = \frac{\left(2^n\lambda^n\theta^n e^{\lambda\Sigma x}/\prod\left(e^{\lambda x_i}-1\right)^3\right) \exp\left[-\theta\Sigma\left(1/\left(e^{\lambda x_i}-1\right)^2\right)\right]\left(1/\lambda^3\right) \exp\left[-a\theta - b/\lambda^2\right]}{\iint\left(2^n\lambda^n\theta^n e^{\lambda\Sigma x}/\prod\left(e^{\lambda x_i}-1\right)^3\right) \exp\left[-\theta\Sigma\left(1/\left(e^{\lambda x_i}-1\right)^2\right)\right]\left(1/\lambda^3\right) \exp\left[-a\theta - b/\lambda^2\right] d\theta\, d\lambda}. \tag{11}$$

The loss function is another specification in Bayesian analysis. It is the deviation of the observed event from the true event, and the expectation of the loss function is the posterior risk. The minimizer of posterior risk is known as the Bayes estimator. In this study, we considered five different types of loss functions that are SELF, WLF, QLF, PLF, and MIILF.

The marginal posterior distributions of θ and λ, obtained by integrating the equation (11) for nuisance parameters, are not in a closed-form expression. The Bayes estimators and associated risk, for the functional form of parameters $U(\Delta)$, take the form:

$$E_{\Delta|\underline{x}}[U(\Delta)] = \tilde{U}(\Delta) = \frac{\iint U(\Delta)L(\Delta)\pi(\theta,\lambda)\, d\theta\, d\lambda}{\iint L(\Delta)\pi(\theta,\lambda)\, d\theta\, d\lambda}, \tag{12}$$

$$= \frac{\iint U(\Delta)\left(2^n\lambda^n\theta^n e^{\lambda\Sigma x}/\prod\left(e^{\lambda x_i}-1\right)^3\right) \exp\left[-\theta\Sigma\left(1/\left(e^{\lambda x_i}-1\right)^2\right)\right]\left(1/\lambda^3\right) \exp\left[-a\theta - b/\lambda^2\right] d\theta d\lambda}{\iint\left(2^n\lambda^n\theta^n e^{\lambda\Sigma x}/\prod\left(e^{\lambda x_i}-1\right)^3\right) \exp\left[-\theta\Sigma\left(1/\left(e^{\lambda x_i}-1\right)^2\right)\right]\left(1/\lambda^3\right) \exp\left[-a\theta - b/\lambda^2\right] d\theta\, d\lambda}. \tag{13}$$

The expression (13) is not tractable to evaluate. Hence, two approximation techniques by Lindley and Tierney and Kadane (T-K) are utilized for the evaluation of Bayes estimators and associated posterior risks.

4.1. Bayes Estimators and Posterior Risk Using Lindley's Method. [16] proposed a simple technique, which evaluates the ratio of two integrals and produces a single numerical result. In Bayesian statistics, this method is widely used; see among others, [17–21], etc., and the references cited therein.

The expression given in equation (12) can be expressed as

$$E_{\Delta|\underline{x}}[U(\Delta)] = \tilde{U}(\Delta) = \frac{\iint U(\Delta) \exp\left(Q(\Delta)\right) d\theta\, d\lambda}{\iint \exp\left(Q(\Delta)\right) d\theta\, d\lambda}. \tag{14}$$

Here, $Q(\Delta) = \ln L(\Delta) + \ln \pi(\Delta) = \ell(\Delta) + \rho$ and $\rho = \log[\pi(\lambda, \theta)]$.

Table 1: The MLEs and Bayes estimators of θ with posterior risks under different loss functions.

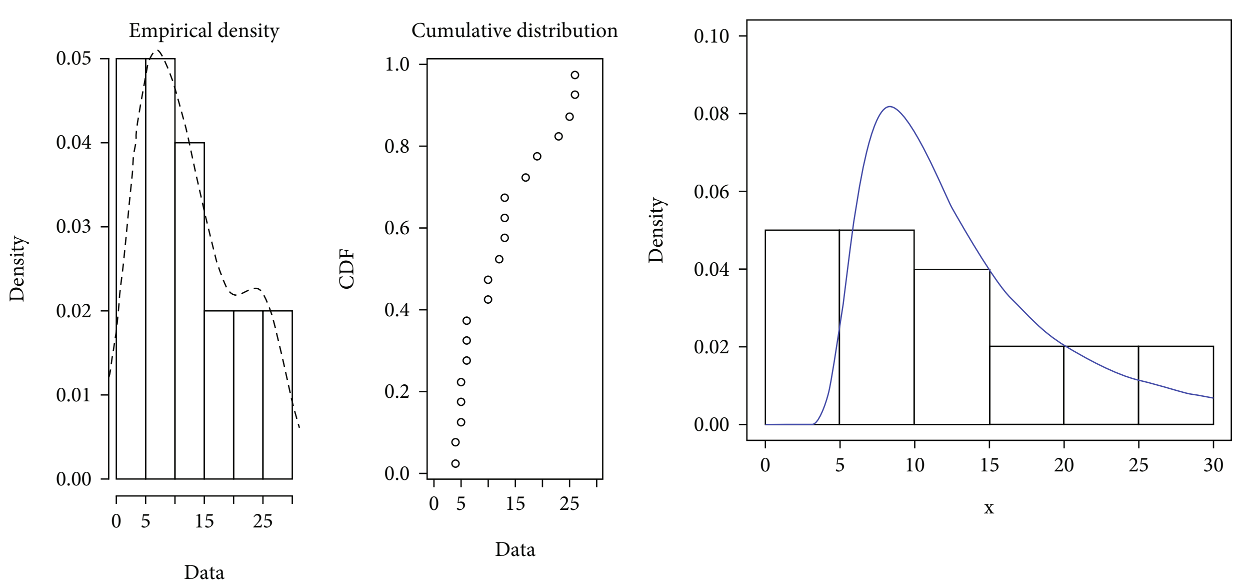
Sample size	MLEs	Bayes estimators									
		Lindley approximation					T-K approximation				
		SELF	WLF	QLF	PLF	MIILF	SELF	WLF	QLF	PLF	MIILF
	$\widehat{\theta}$	$\widetilde{\theta}_S$	$\widetilde{\theta}_w$	$\widetilde{\theta}_Q$	$\widetilde{\theta}_P$	$\widetilde{\theta}_{MII}$	$\widetilde{\theta}_S$	$\widetilde{\theta}_w$	$\widetilde{\theta}_Q$	$\widetilde{\theta}_P$	$\widetilde{\theta}_{MII}$
50	1.0745	0.8938	0.9099	0.9232	0.8837	0.8737	0.7400	0.7345	0.7290	0.7428	0.7456
	0.0581	0.0183	0.0161	0.0048	0.2002	0.1599	0.0055	0.0055	0.0216	0.0055	0.0208
100	1.1586	1.0716	1.0727	1.0740	1.0713	1.0710	1.5259	1.5197	1.5135	1.5291	1.5322
	0.0319	0.0009	0.0011	0.0008	0.0007	0.6891	0.0035	0.0063	0.0117	0.0062	0.0115
200	1.2486	1.2067	1.2057	1.2048	1.2073	1.2079	1.4788	1.4760	1.4732	1.4803	1.4817
	0.0175	0.0013	0.0010	0.0007	0.0012	1.8066	0.0016	0.0028	0.0058	0.0029	0.0058
300	1.3171	1.2903	1.2893	1.2885	1.2907	1.2912	1.5946	1.5922	1.5898	1.5958	1.5971
	0.0125	0.0012	0.0009	0.0004	0.0009	2.6552	0.0015	0.0024	0.0040	0.0025	0.0041
500	1.3963	1.3809	1.3803	1.3797	1.3812	1.3815	1.5816	1.5803	1.5789	1.5824	1.5831
	0.0081	0.0008	0.0006	0.0003	0.0006	3.9292	0.0008	0.0014	0.0023	0.0014	0.0024
700	1.4373	1.4268	1.4263	1.4259	1.4270	1.4272	1.5595	1.5586	1.5576	1.5600	1.5605
	0.0059	0.0006	0.0004	0.0002	0.0004	4.2820	0.0005	0.0009	0.0017	0.0009	0.0016
1000	1.4962	1.4892	1.4889	1.4886	1.4893	1.4894	1.5648	1.5641	1.5634	1.5651	1.5654
	0.0043	0.0004	0.0003	0.0002	0.0003	5.3979	0.0004	0.0006	0.0012	0.0007	0.0012
1500	1.567	1.5628	1.5626	1.5625	1.5629	1.5630	1.5352	1.5348	1.5344	1.5353	1.5356
	0.0030	0.0003	0.0002	0.0001	0.0002	7.1255	0.0002	0.0004	0.0007	0.0004	0.0007

Table 2: The MLEs and Bayes estimators of λ with posterior risks under different loss functions.

Sample size	MLEs	Bayes estimators									
		Lindley approximation					T-K approximation				
		SELF	WLF	QLF	PLF	MIILF	SELF	WLF	QLF	PLF	MIILF
	$\widehat{\lambda}$	$\tilde{\lambda}_S$	$\tilde{\lambda}_w$	$\tilde{\lambda}_Q$	$\tilde{\lambda}_P$	$\tilde{\lambda}_{M2}$	$\tilde{\lambda}_S$	$\tilde{\lambda}_w$	$\tilde{\lambda}_Q$	$\tilde{\lambda}_P$	$\tilde{\lambda}_{M2}$
50	4.1561	3.8308	3.8437	3.8554	3.8236	3.8165	3.0096	3.0096	3.0079	3.9858	3.9626
	0.0149	0.0560	0.0129	0.0030	0.0144	0.0037	0.0472	0.0736	0.0031	0.04757	0.0554
100	4.2354	4.0632	4.0634	4.0638	4.0633	4.0633	3.9006	3.9016	3.8964	3.8843	3.8682
	0.0067	0.0001	0.0002	0.0001	0.0001	0.0001	0.0287	0.0009	0.0072	0.0329	0.0397
200	4.3412	4.2503	4.2484	4.2467	4.2513	4.2523	3.9339	3.9244	3.9323	3.9261	3.9182
	0.0031	0.0086	0.0019	0.0004	0.0020	0.0005	0.0142	0.0004	0.0028	0.0159	0.0186
300	4.3994	4.3368	4.3350	4.3333	4.3377	4.3387	3.9806	3.9818	3.9775	3.9735	3.9666
	0.0019	0.0082	0.0018	0.0003	0.0019	0.0004	0.0136	0.0013	0.0047	0.0140	0.0147
500	4.4638	4.4249	4.4235	4.4221	4.4256	4.4263	4.0027	4.0034	4.0010	4.9987	4.9949
	0.0011	0.0064	0.0014	0.0003	0.0014	0.0003	0.0077	0.0007	0.0027	0.0078	0.0081
700	4.4942	4.4666	4.4655	4.4644	4.4671	4.4677	4.022	4.0229	4.0213	4.0197	4.017
	0.0007	0.0051	0.0011	0.0002	0.0011	0.0002	0.0053	0.0006	0.0019	0.0053	0.0055
1000	4.5467	4.5271	4.5263	4.5255	4.5275	4.5279	1.0263	4.0265	4.0257	4.0244	4.0226
	0.0005	0.0039	0.0008	0.0002	0.0008	0.0002	0.0037	0.0002	0.0009	0.0037	0.0037
1500	4.5872	4.5739	4.5733	4.5727	4.5742	4.5745	4.0687	4.0687	4.0685	4.0676	4.066
	0.0003	0.0028	0.0006	0.0001	0.0006	0.0001	0.0023	0.0002	0.0004	0.0022	0.0021

Table 3: Summary statistics of the data set about the time between break down of electric feeders.

Sample size	Minimum	1^{st} quartile	Median	Mean	3^{rd} quartile	Maximum
20	4.00	5.75	11.00	12.40	17.50	26.00



(a) (b)

Figure 3: (a) Empirical and cumulative distribution of the data. (b) Goodness of fit curve of the IRE distribution to the data.

Then, the Lindley approximations take the form:

$$\tilde{U}(\Delta) \approx \mathrm{U}\left(\widehat{\Delta}\right) + \frac{1}{2}[u_{11}\sigma_{11} + u_{12}\sigma_{12} + u_{21}\sigma_{21} + u_{22}\sigma_{22}] + U_1\rho_1 + U_2\rho_2 + \frac{1}{2}L_{30}\sigma_{11}U_1 + \frac{1}{2}L_{21}(2\sigma_{12}U_1 + \sigma_{11}U_2) + \frac{1}{2}L_{12}(\sigma_{22}U_1 + 2\sigma_{12}U_2) + \frac{1}{2}L_{03}\sigma_{22}U_2, \quad (15)$$

$$\mathrm{U}_1 = \mathrm{u}_1\sigma_{11} + \mathrm{u}_2\sigma_{12}, \quad (16)$$

$$U_2 = u_1\sigma_{21} + u_2\sigma_{22}, \quad (17)$$

$$u_i = \frac{\partial\, U(\Delta)}{\partial \Delta_i}, u_{ij} = \frac{\partial^2\, U(\Delta)}{\partial \Delta_i \partial \Delta_j}. \quad (18)$$

σ_{ij} is the ij -th element of the inverse of the Fisher information matrix. For the IRE distribution,

$$\rho = -a\theta - \frac{b}{\lambda^2} - 3\log\lambda,$$

$$\rho_1 = \frac{\partial\rho}{\partial\theta} = -a, \rho_2 = \frac{\partial\rho}{\partial\lambda} = \frac{2\mathrm{b}}{\lambda^3} - \frac{3}{\lambda},$$

$$L_{30} = \frac{\partial^3 \ell(\Delta)}{\partial^3\theta} = \frac{2n}{\theta^3}, L_{21} = \frac{\partial^3 \ell(\Delta)}{\partial^2\theta\partial\lambda} = 0,$$

$$L_{12} = \frac{\partial^3 \ell(\Delta)}{\partial\theta\partial^2\lambda} = 2\sum x_i^2 e^{\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-3} - 6\sum x_i^2 e^{2\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-4},$$

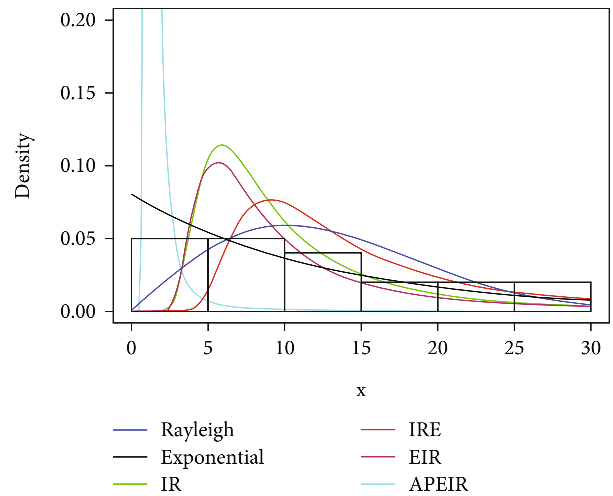


Figure 4: Goodness of fit curve of the IRE distribution with competitive distributions.

$$L_{03} = \frac{\partial^3 \ell(\Delta)}{\partial^3\lambda} = \frac{2n}{\lambda^3} - 3\sum x_i^3 e^{\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-1} + 9\sum x_i^3 e^{2\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-2} - 6\sum x_i^3 e^{3\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-3} + 2\theta\sum x_i^3 e^{\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-3} - 18\theta\sum x_i^3 e^{2\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-4} + 24\theta\sum x_i^3 e^{3\lambda x_i}\left(e^{\lambda x_i} - 1\right)^{-5}. \quad (19)$$

The Bayes estimators of θ under SELF, WLF, QLF, PLF, and MIILF, using expression (15), are

$$\begin{aligned}
\tilde{\theta}_{\mathrm{S}} &= \widehat{\theta} + \sigma_{11}\rho_1 + \sigma_{21}\rho_2 + \frac{1}{2}L_{30}{\sigma_{11}}^2 + \frac{3}{2}L_{21}\sigma_{12}\sigma_{11} + \frac{1}{2}L_{12}\left(\sigma_{22}\sigma_{11} + 2{\sigma_{12}}^2\right) + \frac{1}{2}L_{03}\sigma_{22}\sigma_{21},\\
\tilde{\theta}_{\mathrm{W}} &= \left[\widehat{\theta}^{-1} + \widehat{\theta}^{-3}\sigma_{11} - \widehat{\theta}^{-2}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) - \frac{1}{2}\widehat{\theta}^{-2}\left[L_{30}{\sigma_{11}}^2 + 3\mathrm{L}_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2{\sigma_{12}}^2\right) + L_{03}\sigma_{22}\sigma_{12}\right]\right]^{-1},\\
\tilde{\theta}_{\mathrm{Q}} &= \frac{\widehat{\theta}^{-1} + \widehat{\theta}^{-3}\sigma_{11} - \widehat{\theta}^{-2}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) - 1/2\widehat{\theta}^{-2}\left[L_{30}{\sigma_{11}}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2{\sigma_{12}}^2\right) + L_{03}\sigma_{22}\sigma_{12}\right]}{\widehat{\theta}^{-2} + 3\widehat{\theta}^{-4}\sigma_{11} - 2\widehat{\theta}^{-3}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) - \widehat{\theta}^{-3}\left[L_{30}{\sigma_{11}}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}(\sigma_{22}\sigma_{11} + 2{\sigma_{12}}^2) + L_{03}\sigma_{22}\sigma_{12}\right]},\\
\tilde{\theta}_{\mathrm{P}} &= \left[\widehat{\theta}^2 + \sigma_{11} + 2\widehat{\theta}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + \widehat{\theta}\left[L_{30}{\sigma_{11}}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2{\sigma_{12}}^2\right) + L_{03}\sigma_{22}\sigma_{21}\right]\right]^{1/2},\\
\tilde{\theta}_{\mathrm{MII}} &= \frac{\widehat{\theta}^2 + \sigma_{11} + 2\widehat{\theta}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + \widehat{\theta}\left[L_{30}{\sigma_{11}}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2{\sigma_{12}}^2\right) + L_{03}\sigma_{22}\sigma_{21}\right]}{\widehat{\theta} + \sigma_{11}\rho_1 + \sigma_{21}\rho_2 + 1/2L_{30}{\sigma_{11}}^2 + 3/2L_{21}\sigma_{12}\sigma_{11} + 1/2L_{12}(\sigma_{22}\sigma_{11} + 2{\sigma_{12}}^2) + 1/2L_{03}\sigma_{22}\sigma_{21}}.
\end{aligned} \quad (20)$$

The associated posterior risks are obtained to be the following:

$$
\begin{aligned}
\tilde{\mathscr{R}}_{\theta(\mathrm{S})} &= \left[\hat{\theta}^2 + \sigma_{11} + 2\hat{\theta}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + \hat{\theta}\left[L_{30}\sigma_{11}{}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + L_{03}\sigma_{22}\sigma_{21}\right]\right] \\
&\quad - \left[\hat{\theta} + \sigma_{11}\rho_1 + \sigma_{21}\rho_2 + \frac{1}{2}L_{30}\sigma_{11}{}^2 + \frac{3}{2}L_{21}\sigma_{12}\sigma_{11} + \frac{1}{2}L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + \frac{1}{2}L_{03}\sigma_{22}\sigma_{21}\right]^2, \\
\tilde{\mathscr{R}}_{\theta(\mathrm{W})} &= \left[\hat{\theta} + \sigma_{11}\rho_1 + \sigma_{21}\rho_2 + \frac{1}{2}L_{30}\sigma_{11}{}^2 + \frac{3}{2}L_{21}\sigma_{12}\sigma_{11} + \frac{1}{2}L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + \frac{1}{2}L_{03}\sigma_{22}\sigma_{21}\right] \\
&\quad - \left[\hat{\theta}^{-1} + \hat{\theta}^{-3}\sigma_{11} - \hat{\theta}^{-2}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) - \frac{1}{2}\hat{\theta}^{-2}\left[L_{30}\sigma_{11}{}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + L_{03}\sigma_{22}\sigma_{12}\right]\right]^{-1}, \\
\tilde{\mathscr{R}}_{\theta(\mathrm{Q})} &= 1 - \frac{\left[\hat{\theta}^{-1} + \hat{\theta}^{-3}\sigma_{11} - \hat{\theta}^{-2}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) - 1/2\hat{\theta}^{-2}\left[L_{30}\sigma_{11}{}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + L_{03}\sigma_{22}\sigma_{12}\right]\right]^2}{\hat{\theta}^{-2} + 3\hat{\theta}^{-4}\sigma_{11} - 2\hat{\theta}^{-3}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) - \hat{\theta}^{-3}\left[L_{30}\sigma_{11}{}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2) + L_{03}\sigma_{22}\sigma_{12}\right]}, \\
\tilde{\mathscr{R}}_{\theta(\mathrm{P})} &= 2\left[\left[\hat{\theta}^2 + \sigma_{11} + 2\hat{\theta}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + \hat{\theta}\left[L_{30}\sigma_{11}{}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + L_{03}\sigma_{22}\sigma_{21}\right]\right]^{1/2}\right. \\
&\quad \left. - \left[\hat{\theta} + \sigma_{11}\rho_1 + \sigma_{21}\rho_2 + \frac{1}{2}L_{30}\sigma_{11}{}^2 + \frac{3}{2}L_{21}\sigma_{12}\sigma_{11} + \frac{1}{2}L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + \frac{1}{2}L_{03}\sigma_{22}\sigma_{21}\right]\right], \\
\tilde{\mathscr{R}}_{\theta(\mathrm{MII})} &= 1 - \frac{\left[\hat{\theta} + \sigma_{11}\rho_1 + \sigma_{21}\rho_2 + 1/2L_{30}\sigma_{11}{}^2 + 3/2L_{21}\sigma_{12}\sigma_{11} + 1/2L_{12}\left(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2\right) + 1/2L_{03}\sigma_{22}\sigma_{21}\right]^2}{\hat{\theta}^2 + \sigma_{11} + 2\hat{\theta}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + \hat{\theta}\left[L_{30}\sigma_{11}{}^2 + 3L_{21}\sigma_{12}\sigma_{11} + L_{12}(\sigma_{22}\sigma_{11} + 2\sigma_{12}{}^2) + L_{03}\sigma_{22}\sigma_{21}\right]}.
\end{aligned}
\tag{21}
$$

Similarly, the Bayes estimators and associated posterior risk of parameter λ are evaluated.

4.2. Bayes Estimators and Posterior Risk Using T–K Approximation. For the evaluation of the ratio of two integrals, an approximate method is T-K approximation [22]. Although the Lindley approximation method is also used for this purpose, it required a third derivative of the log-likelihood function, which is sometimes tedious to evaluate.

In the Bayesian analysis, the T-K technique is frequently used, and some are [23–26].

For the IRE distribution, it is supposed that

$$
\begin{aligned}
\mathscr{L}(\Delta) &= \frac{1}{n}[\rho(\Delta) + \ell(\Delta)], \\
\mathscr{L}^*(\Delta) &= \frac{1}{n}U(\Delta) + \mathscr{L}(\Delta).
\end{aligned}
\tag{22}
$$

The expression for the Bayes estimators and associated posterior risk provided in equation (14) can be expressed as

$$
\begin{aligned}
E_{\Delta|\underline{x}}[U(\Delta)] = \tilde{U}(\Delta) &= \frac{\iint \exp\left(\mathscr{L}^*(\Delta)\right)d\theta\, d\lambda}{\iint \exp\left(\mathscr{L}(\Delta)\right)d\theta\, d\lambda} \\
&\approx \sqrt{\frac{\det(\Sigma^*)}{\det(\Sigma)}}\exp\left[n\mathscr{L}^*\left(\hat{\Delta}_{\mathscr{L}^*}\right) - n\mathscr{L}\left(\hat{\Delta}_{\mathscr{L}}\right)\right],
\end{aligned}
\tag{23}
$$

where $\hat{\Delta}_{\mathscr{L}^*}$ and $\hat{\Delta}_{\mathscr{L}}$maximize the $\mathscr{L}^*(\Delta)$ and $\mathscr{L}(\Delta)$, and Σ^* and Σ are the inverse of the Fisher information matrix of $\mathscr{L}^*(\Delta)$ and $\mathscr{L}(\Delta)$ at $\hat{\Delta}_{\mathscr{L}^*}$ and $\hat{\Delta}_{\mathscr{L}}$, respectively.

All the Bayes estimators and associated posterior risks of the parameters θ and λ under SELF, WLF, QLF, PLF, and MIILF are evaluated using the expression (23).

5. Simulation Study

In this section, the Monte Carlo simulation scheme is used to study the behavior of MLEs and the Bayes estimators of the parameters of IRE distribution. For this purpose, random samples of sizes 50, 100, 200, 300, 500, 700, 1000, and1500 are drawn from the IRE distribution using a random number generator $X = (1/\lambda)\log\left[\sqrt{-\theta/\log U} + 1\right]$, where U is a uniform random variate over the interval (0, 1). From these samples, MLEs of the parameters of IRE distribution are estimated using the Newton Raphson iterative procedure. The R package *maxLik* is used for this purpose. The Bayes estimates and the associated posterior risks under Lindley and T-K approximation methods using informative priors (exponential and inverse Rayleigh priors) are evaluated. The values of the parameters are taken to be $\theta = 1.5$ and $\lambda = 4.5$. The elicited values of the hyperparameters are $a = 5.5$ and $b = 2.08$. The computation is executed making programming routines in R-language. The simulation size is set to be 1000. The best estimators are assessed based on minimum values of risk.

Table 4: The MLEs and Bayes estimates of θ and λ under different loss functions using electricity data set.

Parameters	MLEs	Bayes estimators									
		Lindley approximation					T-K approximation				
		SELF	WLF	QLF	PLF	MIILF	SELF	WLF	QLF	PLF	MIILF
θ	0.5420	0.4713	0.4793	0.4856	0.4661	0.4661	1.0319	0.9335	0.8433	1.0799	1.1303
	0.0390	0.0049	0.0079	0.0013	0.0104	0.9483	0.1016	0.0983	0.0967	0.0962	0.0871
λ	0.3217	0.3778	0.3906	0.3763	0.3545	0.3139	0.3568	0.3835	0.3304	0.3298	0.3048
	0.1002	0.1648	0.2872	0.0095	0.3535	0.5358	0.0185	0.0266	0.1383	0.0540	0.1706

Table 5: Mean and standard deviation of data and IRE distribution using MLEs and Bayes estimates.

	Data	IRE distribution									
		MLEs	Lindley approximation					T-K approximation			
			SELF	WLF	QLF	PLF	MIILF	SELF	WLF	QLF	PLF	MIILF
Mean time	12.40	14.55	11.98	11.54	12.98	11.32	11.77	15.54	15.03	14.99	14.01	13.95
s.d.	7.769	5.03	3.92	3.88	2.04	2.67	2.55	4.08	3.99	3.04	2.93	2.55

The results of the simulation study are reported in Tables 1 and 2. For each sample size, the minimum value of risk is shown in bold font.

From Tables 1 and 2, it is observed that the performance of the Bayes estimators under all the loss functions is better than the MLEs for the parameters θ and λ due to minimum values of risks. With increasing sample sizes, all the values of risks decrease and move towards zero. The estimators attained using the Lindley approximation method perform better than those of the T-K approximation technique. While comparing the loss functions, the estimators under QLF proved to be efficient for both parameters.

6. Illustrative Real Life Example

This section illustrates the appropriateness of the IRE distribution using a real-life data set. The estimates of the parameters are attained both in the classical and Bayesian paradigms and are compared. As mentioned earlier, the problem of electric feeders' breakdown and tripping worsens in the summer season in Pakistan. Therefore, we tried to estimate the average time a feeder endures a breakdown after it has suffered a breakdown already and has been repaired, using the IRE distribution. In other words, how long will an average feeder take for another breakdown when it has been repaired after one breakdown in the extremely hot months of summer? For this purpose, a data set of the time-lapses (in days) between one breakdown to the next one is taken from Multan Electric Power Company (MEPCO), Pakistan, for June, July, and August 2018. From two grid stations, 132 kV MESCO and 132 kV Qasim Pur, twenty 11 kV outgoing urban feeders are selected randomly, and the average time between one breakdown to the next one of these feeders is recorded (duration between two breakdowns). The summary statistics of the data are provided in Table 3.

The empirical and cumulative distribution functions of the data set are shown in Figure 3(a), and it is shown in Figure 3(b) that the IRE distribution adequately fits the data.

In Figure 4, it is depicted that the IRE model fits the data appropriately and turns out to be more flexible than the exponential distribution, Rayleigh distribution [27], inverse Rayleigh (IR) distribution [5], exponentiated inverse Rayleigh (EIR) distribution [7], and alpha power exponentiated inverse Rayleigh (APEIR) distribution [28].

For the data set of the time between the breakdowns of electricity feeders, the MLEs and Bayes estimate with the associated posterior risks of the parameters of IRE distribution are evaluated. The MLEs $\widehat{\theta}$ and $\widehat{\lambda}$ are estimated using the Newton Raphson iterative procedure. The Bayes estimates and associated risks under Lindley and T-K methods are evaluated using the theoretical results. The results are shown in Table 4. The results of Table 4 depict the performance of the Bayes estimators that prove to be better than the MLEs for the given data set, as they have minimum values of risks. The Bayes estimators obtained using the Lindley approximation method are even better than the T-K method. The estimators under QLF are found to be better than other loss functions for both the parameters θ and λ. The results for the data set are close to the findings of the simulation study.

Table 5 shows the means and standard deviation (sd) calculated by the data set, MLEs, and Bayes estimate using Lindley and T-K methods provided in Table 4.

Hence, the average time that an electrical feeder takes to endure after one break down to the other one in the summer season of Pakistan is approximately 13 days. Again, it is obvious that the Bayesian estimators obtained using the Lindley approximation method under QLF prove to be the most efficient of all including classical and other Bayesian estimators.

7. Conclusion

In this study, the data about the time between the breakdowns of electricity feeders has been modeled using the IRE distribution. For this purpose, the data set of electricity feeders is taken from MEPCO, Pakistan. The parameters of the distribution are estimated through classical and Bayesian estimation techniques. The MLEs are not in closed form; so, the Newton Raphson iterative method is used in the classical paradigm. To evaluate the Bayesian estimators, two approximation estimation techniques are used. Both the classical and Bayes estimators are compared based on minimum values of risks. The results of the simulation study and real-life data set show that the Bayes estimators are better than the MLEs. While comparing both the approximation techniques, Lindley's method proves to be better than the other one. Since the IRE distribution turns out to be a better fit for the data, it is suggested to the worldwide practitioners, engineers, and policymakers to use the proposed distribution and Bayesian estimation technique for the better prediction of engineering and electricity data set.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request

Ethical Approval

The research study meets all ethical regulations as required by the departmental ethics committee of Bahauddin Zakariya University, Multan, Pakistan.

Consent

Informed consent was obtained from all the participants.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] G. A. Ajenikoko, A. A. Olaomi, and D. O. Aborisade, "A failure rate model for reliability assessment of Ibadan distribution systems," *Computer Engineering and Intelligent Systems*, vol. 5, pp. 21–33, 2014.

[2] S. J. Almalki and J. Yuan, "A new modified Weibull distribution," *Reliability Engineering & System Safety*, vol. 111, pp. 164–170, 2013.

[3] A. Saboor, H. S. Bakouch, and M. N. Khan, "Beta Sarhan-Zaindin modified Weibull distribution," *Applied Mathematical Modelling*, vol. 40, no. 13-14, pp. 6604–6621, 2016.

[4] M. Eliwa, E. Altun, Z. A. Alhussain et al., "A new one-parameter lifetime distribution and its regression model with applications," *PLoS One*, vol. 16, no. 2, article e0246969, 2021.

[5] V. G. Vodˇa, "On the inverse Rayleigh distributed random variable," *Union of Japanese Scientists and Engineers*, vol. 19, pp. 13–21, 1972.

[6] K. Fatima and S. Ahmad, "Weighted inverse Rayleigh distribution," *International Journal of Statistics and Systems*, vol. 12, pp. 119–137, 2017.

[7] G. S. Rao and S. Mbwambo, "Exponentiated inverse Rayleigh distribution and an application to coating weights of iron sheets data," *Journal of Probability and Statistics*, vol. 2019, Article ID 7519429, 13 pages, 2019.

[8] H. Mohammed and N. Yahia, "On type II Topp Leone inverse Rayleigh distribution," *Applied Mathematical Sciences*, vol. 13, no. 13, pp. 607–615, 2019.

[9] A. M. Almarashi, M. M. Badr, M. Elgarhy, F. Jamal, and C. Chesneau, "Statistical inference of the half-logistic inverse Rayleigh distribution," *Entropy*, vol. 22, no. 4, p. 449, 2020.

[10] L. Gao, Y. Zhou, C. Li, and L. Huo, "Reliability assesment of distribution systems with distributed generation based on Bayesian networks," *Engineering Review*, vol. 34, pp. 55–62, 2014.

[11] C. B. Guure, N. A. Ibrahim, M. B. Adam, S. Bosomprah, and A. O. Ahmed, "Bayesian parameter and reliability estimate of Weibull failure time distribution," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 2, no. 37, pp. 611–632, 2014.

[12] A. Moradkhani, M. R. Haghifam, and M. Mohammadzadeh, "Failure rate estimation of overhead electric distribution lines considering data deficiency and population variability," *International Transactions on Electrical Energy Systems*, vol. 25, no. 8, pp. 1452–1465, 2015.

[13] R. Pan, X. Xu, and J. Juarez, "Bayesian analysis of low-cycle fatigue failure in printed wiring boards," *Case Studies in Engineering Failure Analysis*, vol. 7, pp. 65–70, 2016.

[14] A. Alzaatreh, C. Lee, and F. Famoye, "A new method for generating families of continuous distributions," *Metron*, vol. 71, no. 1, pp. 63–79, 2013.

[15] A. Henningsen and O. Toomet, "maxLik: a package for maximum likelihood estimation in R," *Computational Statistics*, vol. 26, no. 3, pp. 443–458, 2011.

[16] D. V. Lindley, "Approximate bayesian methods," *Trabajos de Estadística y de Investigación Operativa*, vol. 31, no. 1, pp. 223–245, 1980.

[17] P. Singh, S. K. Singh, and U. Singh, "Bayes estimator of inverse Gaussian parameters under general entropy loss function using Lindley's approximation," *Communications in Statistics–Simulation and Computation®*, vol. 37, no. 9, pp. 1750–1762, 2008.

[18] S. Ali, M. Aslam, and S. M. A. Kazmi, "A study of the effect of the loss function on Bayes estimate, posterior risk and hazard function for Lindley distribution," *Applied Mathematical Modelling*, vol. 37, no. 8, pp. 6068–6078, 2013.

[19] A. Ahmad, K. Fatima, and S. Ahmad, "Lindley approximation technique for the parameters of Lomax distribution," *International Journal of Statistics in Medical Research*, vol. 6, no. 4, pp. 162–168, 2017.

[20] V. K. Sharma, S. K. Singh, U. Singh, and K. Ul-Farhat, "Bayesian estimation on interval censored Lindley distribution using Lindley's approximation," *International Journal of System Assurance Engineering and Management*, vol. 8, no. S2, pp. 799–810, 2017.

[21] A. A. Ameen and U. A. Akkash, "Maximum likelihood estimation and Bayesian estimation of three-parameter Weibull distribution based on interval-censored data," *Journal of Physics: Conference Series*, vol. 1818, no. 1, article 012199, 2021.

[22] L. Tierney and J. B. Kadane, "Accurate approximations for posterior moments and marginal densities," *Journal of the American Statistical Association*, vol. 81, no. 393, pp. 82–86, 1986.

[23] M. Y. Danish and M. Aslam, "Bayesian estimation for randomly censored generalized exponential distribution under asymmetric loss functions," *Journal of Applied Statistics*, vol. 40, no. 5, pp. 1106–1119, 2013.

[24] H. Sultan and P. Ahmad, “Bayesian approximation techniques of Topp-leone distribution,” *International Journal of Mathematics and Statistics*, vol. 2, pp. 066–072, 2015.

[25] Y. Arı, “Bayesian estimation of GARCH (1, 1) model using Tierney-Kadane’s approximation,” in *International Conference on Applied Economics*, pp. 355–364, Springer, Cham, 2018.

[26] G. Gencer and K. Gencer, “Comparison of maximum likelihood and Bayes estimators under symmetric and asymmetric loss functions by means of Tierney Kadane’s approximation for Weibull distribution,” *Turkish Journal of Science and Technology*, vol. 14, pp. 69–78, 2019.

[27] L. Rayleigh, “XII. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase,” *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 10, no. 60, pp. 73–78, 1880.

[28] M. Ali, A. Khalil, M. Ijaz, and N. Saeed, “Alpha-power exponentiated inverse Rayleigh distribution and its applications to real and simulated data,” *PLoS One*, vol. 16, no. 1, article e0245253, 2021.